The Absolute Anabelian Geometry of Hyperbolic Curves

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Contents:

Introduction

Let X_K be a *hyperbolic curve* (cf. §0 below) over a *field K of characteristic* 0. Denote its *algebraic fundamental group* by Π_{X_K}. Thus, we have a *natural surjection*

$$\Pi_{X_K} \twoheadrightarrow G_K$$

of Π_{X_K} onto the *absolute Galois group* G_K of K.

When K is a *finite extension of* $\mathbb{Q}$ *or* $\mathbb{Q}_p$, and one *holds* G_K *fixed*, then it is known (cf. [Tama], [Mzk6]; Theorem 1.3.4 of the present manuscript) that one may *recover the curve* X_K *in a functorial fashion* from Π_{X_K}. This sort of result may be thought of as a *"relative result"* (i.e., over G_K). Then the question naturally arises:

> *To what extent are the* **"absolute analogues"** *of this result valid — i.e., what if one does not hold* G_K *fixed?*

If K is a *number field*, then it is *still possible to recover* X_K from Π_{X_K} (cf. Theorem 1.3.5), by applying the *theorem of Neukirch-Uchida* (cf. Theorem 1.1.3). On the other hand, when K is a *p-adic local field* (i.e., a finite extension of $\mathbb{Q}_p$), the analogue of the theorem of Neukirch-Uchida *fails to hold*, and indeed, it is the opinion of the author at the time of writing that it is *unlikely* (in the p-adic local case) that one can recover X_K in general (i.e., in the fashion of Theorem 1.3.4) from Π_{X_K}.

In the present manuscript, we begin by *reviewing/surveying* in §1 the anabelian geometry of number fields, p-adic local fields, and hyperbolic curves from the point of view of the *goal of understanding to what extent the anabelian geometry of hyperbolic curves over p-adic local fields can be made* **"absolute."** Our main result (Theorem 2.7), given in §2, states that when K is a *p-adic local field*, (although we may be unable to recover X_K itself) *one may recover (in a functorial fashion) the* **special fiber** *of* X_K*, together with its natural* **log structure***, in an* **absolute fashion**, i.e., solely from the isomorphism class of the profinite group Π_{X_K}.

Acknowledgements: I would like to thank *A. Tamagawa* for the time that he so generously shared with me in numerous stimulating discussions, and especially for the following: (i) informing me of the arguments used to prove Lemma 1.1.4 in §1.1; (ii) explaining to me the utility of a theorem of Raynaud in the context of §2 (cf. Lemma 2.4).

Section 0: Notations and Conventions

Numbers:

We will denote by $\mathbb{N}$ the set of *natural numbers*, by which we mean the set of integers $n \geq 0$. A *number field* is defined to be a finite extension of the field of rational numbers $\mathbb{Q}$.

Topological Groups:

Let G be a *topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall \ h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \bigcap H \text{ has finite index in } H, \, g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions*

$$H, \; Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$$

and (iii) H is *normal* in $N_G(H)$.

Note that $Z_G(H)$, $N_G(H)$ are *always closed in* G, while $C_G(H)$ is *not necessarily closed in* G. Indeed, it is not difficult to show that if one takes G to be the semi-direct product of $\prod_{\mathbb{N}} \mathbb{Z}_p$ with $\mathrm{Aut}(\prod_{\mathbb{N}} \mathbb{Z}_p)$, and H to be

$$\prod_{n \in \mathbb{N}} p^n \cdot \mathbb{Z}_p \subseteq \prod_{\mathbb{N}} \mathbb{Z}_p$$

then $C_G(H)$ is not closed in G. For instance, if one denotes by $e_n \in \prod_{\mathbb{N}} \mathbb{Z}_p$ the vector with a 1 in the n-th place and zeroes elsewhere, then the *limit* A_∞ (where

$$A_\infty(e_n) \stackrel{\mathrm{def}}{=} e_n + e_{n+1}$$

for all $n \in \mathbb{N}$) *of the automorphisms* $A_m \in C_G(H)$ (where $A_m(e_n) \stackrel{\mathrm{def}}{=} e_n + e_{n+1}$ if $n \leq m$, $A_m(e_n) \stackrel{\mathrm{def}}{=} e_n$ if $n > m$) *is not contained in* $C_G(H)$.

Definition 0.1.

(i) Let G be a profinite group. Then we shall say that G is *slim* if the centralizer $Z_G(H)$ of any open subgroup $H \subseteq G$ in G is trivial.

(ii) We shall say that a continuous homomorphism of profinite groups $G \to H$ is *relatively slim* if the centralizer in H of the image of every open subgroup of G is trivial.

(iii) We shall say that a closed subgroup $H \subseteq G$ of a profinite group G is *commensurably* (respectively, *normally*) *terminal* if the commensurator $C_G(H)$ (respectively, normalizer $N_G(H)$) is equal to H.

Remark 0.1.1. Thus, a profinite group G is slim if and only if the identity morphism $G \to G$ is relatively slim.

Remark 0.1.2. It is a formal consequence of the definitions that:

$$\text{commensurably terminal} \implies \text{normally terminal}$$

and that (if $H \subseteq G$ is a closed subgroup of a profinite group G, then):

$$H \subseteq G \text{ commensurably terminal, } H \text{ slim} \implies \text{the inclusion } H \hookrightarrow G \text{ is relatively slim}$$

Curves:

Suppose that $g \geq 0$ is an *integer*. Then a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. Then a *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that the pair (Y, D) is unique up to canonical isomorphism. We shall refer to Y (respectively, D) as the *compactification* (respectively, *divisor at infinity*, or *divisor of cusps*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above.

We shall denote the *moduli stack of* r*-pointed stable curves of genus* g (where we assume the points to be *unordered*) by $\overline{\mathcal{M}}_{g,r}$ (cf. [Knud] for an exposition of the theory of such curves). The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will also be referred to as the *moduli stack of hyperbolic curves of type* (g, r). The pair consisting of the *tautological curve* over $\overline{\mathcal{M}}_{g,r}$ (respectively, $\mathcal{M}_{g,r}$), together with the *divisor of marked points*, will be denoted

$$(\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r};\ \overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{C}}_{g,r})$$

(respectively, $(\mathcal{C}_{g,r} \to \mathcal{M}_{g,r};\ \mathcal{D}_{g,r} \subseteq \mathcal{C}_{g,r})$). The complement of $\mathcal{M}_{g,r}$ in $\overline{\mathcal{M}}_{g,r}$, as well as the union of $\overline{\mathcal{D}}_{g,r}$ with the complement of $\mathcal{C}_{g,r}$ in $\overline{\mathcal{C}}_{g,r}$, form *divisors with normal crossings*, hence determine natural *log structures* (cf. [Kato1]) on $\overline{\mathcal{M}}_{g,r}$, $\overline{\mathcal{C}}_{g,r}$. Denote the resulting log stacks by $\overline{\mathcal{M}}_{g,r}^{\log}$, $\overline{\mathcal{C}}_{g,r}^{\log}$. Thus, $\overline{\mathcal{C}}_{g,r}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ is *log smooth*. A morphism of log schemes

$$X^{\log} \to S^{\log}$$

isomorphic to the pull-back of $(\overline{\mathcal{C}}_{g,r} \backslash \overline{\mathcal{D}}_{g,r})^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ via a log morphism $S^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ will be referred to as a *family of hyperbolic log curves* over $S^{\log}$.

Section 1: Review of Anabelian Geometry

§1.1. The Anabelian Geometry of Number Fields

In this §, we review well-known *anabelian* (and related) properties of the *Galois groups of number fields* and (mainly p-adic) *local fields*.

Let F be a *number field*. Fix an *algebraic closure* $\overline{F}$ of F and denote the resulting absolute *Galois group* of F by G_F. Let $\mathfrak{p}$ be a (not necessarily nonarchimedean!) *prime* of F. Write $G_{\mathfrak{p}} \subseteq G_F$ for the *decomposition group* (well-defined up to conjugacy) associated to $\mathfrak{p}$ and $F_{\mathfrak{p}}$ for the *completion* of F at $\mathfrak{p}$.

Theorem 1.1.1. (Slimness and Commensurable Terminality)

(i) Suppose that $\mathfrak{p}$ is **not complex**. *Then the closed subgroup $G_{\mathfrak{p}} \subseteq G_F$ is* **commensurably terminal**.

(ii) Suppose that $\mathfrak{p}$ is **nonarchimedean**. *Then $G_{\mathfrak{p}}$, G_F are* **slim**, *and the inclusion $G_{\mathfrak{p}} \hookrightarrow G_F$ is* **relatively slim**.

Proof. Assertion (i) is a formal consequence of [NSW], Corollary 12.1.3. As for assertion (ii), the slimness of G_F is a formal consequence of [NSW], Proposition 12.1.5. The slimness of $G_{\mathfrak{p}}$ follows from *local class field theory* (cf., e.g., [Serre2]). (That is, if $\sigma \in G_{\mathfrak{p}}$ commutes with an open subgroup $H \subseteq G_{\mathfrak{p}}$, then σ induces the trivial action on the abelianization H^{ab}. But, by local class field theory, H^{ab} may be identified with the profinite completion of $K^\times$, where K is the finite extension of $F_\mathfrak{p}$ determined by H. Thus, σ acts trivially on all sufficiently large finite extensions K of $F_\mathfrak{p}$, so $\sigma = 1$, as desired.) Relative slimness thus follows formally from the slimness of $G_\mathfrak{p}$ and (i) (cf. Remark 0.1.2). ○

Theorem 1.1.2. (Topologically Finitely Generated Closed Normal Subgroups) *Every topologically finitely generated closed normal subgroup of G_F is trivial.*

Proof. This follows from [FJ], Theorem 15.10. ○

Theorem 1.1.3. (The Neukirch-Uchida Theorem on the Anabelian Nature of Number Fields) *Let F_1, F_2 be number fields. Let $\overline{F}_1$ (respectively, $\overline{F}_2$) be an algebraic closure of F_1 (respectively, F_2). Write* $\mathrm{Isom}(\overline{F}_2/F_2, \overline{F}_1/F_1)$ *for the set of field isomorphisms $\overline{F}_2 \xrightarrow{\sim} \overline{F}_1$ that map F_2 onto F_1. Then the natural map*

$$\mathrm{Isom}(\overline{F}_2/F_2, \overline{F}_1/F_1) \to \mathrm{Isom}(\mathrm{Gal}(\overline{F}_1/F_1), \mathrm{Gal}(\overline{F}_2/F_2))$$

is bijective.

Proof. This is the content of [NSW], Theorem 12.2.1. ○

Remark 1.1.3.1. It is important to note, however, that the analogue of Theorem 1.1.3 for finite extensions of $\mathbb{Q}_p$ is *false* (cf. [NSW], p. 674). Nevertheless, by considering isomorphisms of Galois groups that *preserve the higher ramification filtration*, one may obtain *a partial analogue of this theorem for p-adic local fields* (cf. [Mzk5]).

Next, we would like to consider a situation that arises frequently in anabelian geometry. Suppose that G is *equal to* G_F or $G_\mathfrak{p}$ (where we assume now that $\mathfrak{p}$ is *nonarchimedean*!), and that we are given an *exact sequence of profinite groups*:

$$1 \to \Delta \to \Pi \to G \to 1$$

Suppose, moreover, that Δ is *topologically finitely generated.* The following result was related to the author by *A. Tamagawa*:

Lemma 1.1.4. (Intrinsic Characterization of Arithmetic Quotients)

(i) Suppose that $G = G_F$. Let $\Pi' \subseteq \Pi$ be an open subgroup. Then the kernel of the homomorphism $\Pi' \to G$ may be characterized as the unique maximal closed normal subgroup of Π' which is topologically finitely generated.

(ii) Suppose that $G = G_{\mathfrak{p}}$. Assume further that for every open subgroup $\Pi'' \subseteq \Pi$, the abelianization $(\Delta'')^{\mathrm{ab}}$ of Δ'' (where $\Delta'' \stackrel{\mathrm{def}}{=} \Pi'' \bigcap \Delta$) satisfies the following property:

> *(∗) The maximal torsion-free quotient $(\Delta'')^{\mathrm{ab}} \twoheadrightarrow Q''$ of $(\Delta'')^{\mathrm{ab}}$ on which the action of $G'' \stackrel{\mathrm{def}}{=} \Pi''/\Delta''$ (by conjugation) is trivial is a finite* **free** *$\widehat{\mathbb{Z}}$-module.*

Let $\Pi' \subseteq \Pi$ be an arbitrary open subgroup. Then:

$$[G : G'] \cdot [F_{\mathfrak{p}} : \mathbb{Q}_p] = \dim_{\mathbb{Q}_p}((\Pi')^{\mathrm{ab}} \otimes \mathbb{Q}_p) - \dim_{\mathbb{Q}_l}((\Pi')^{\mathrm{ab}} \otimes \mathbb{Q}_l)$$

(where $\Delta' \stackrel{\mathrm{def}}{=} \Delta \bigcap \Pi'$; $G' \stackrel{\mathrm{def}}{=} \Pi'/\Delta'$; p is the rational prime that $\mathfrak{p}$ divides; and l is any prime number distinct from p). (In fact, p may also be characterized as the unique prime number for which the difference on the right is nonzero for infinitely many prime numbers l.) In particular, the subgroup $\Delta \subseteq \Pi$ may be characterized as the intersection of those open subgroups $\Pi' \subseteq \Pi$ such that:

$$[G : G'] = [\Pi : \Pi']$$

(i.e., such that $[G : G'] \cdot [F_{\mathfrak{p}} : \mathbb{Q}_p] = [\Pi : \Pi'] \cdot ([G : G] \cdot [F_{\mathfrak{p}} : \mathbb{Q}_p])$).

Proof. Assertion (i) is a formal consequence of Theorem 1.1.2.

Now we turn to assertion (ii). Denote by K' the finite extension of $F_{\mathfrak{p}}$ determined by G'. Then:

$$[G : G'] \cdot [F_{\mathfrak{p}} : \mathbb{Q}_p] = [K' : \mathbb{Q}_p]$$

On the other hand, it follows formally from (∗) that:

$$\dim_{\mathbb{Q}_p}((\Pi')^{\mathrm{ab}} \otimes \mathbb{Q}_p) - \dim_{\mathbb{Q}_l}((\Pi')^{\mathrm{ab}} \otimes \mathbb{Q}_l) = \dim_{\mathbb{Q}_p}((G')^{\mathrm{ab}} \otimes \mathbb{Q}_p) - \dim_{\mathbb{Q}_l}((G')^{\mathrm{ab}} \otimes \mathbb{Q}_l)$$

Thus, to complete the proof of Lemma 1.1.4, it suffices to prove that:

$$[K' : \mathbb{Q}_p] = \dim_{\mathbb{Q}_p}((G')^{\mathrm{ab}} \otimes \mathbb{Q}_p) - \dim_{\mathbb{Q}_l}((G')^{\mathrm{ab}} \otimes \mathbb{Q}_l)$$

But this is a formal consequence of *local class field theory* (cf., e.g., [Serre2]; §1.2 below), i.e., the fact that $(G')^{\mathrm{ab}}$ is isomorphic to the profinite completion of $(K')^{\times}$. ○

Typically, in applications involving hyperbolic curves, one shows that the condition (∗) of Lemma 1.1.4 is satisfied by applying the following:

Lemma 1.1.5. **(Tate Modules of Semi-abelian Varieties)** *Let K be a finite extension of $\mathbb{Q}_p$. Fix an algebraic closure $\overline{K}$ of K; write $G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$. Let A be a semi-abelian variety over K. Denote the resulting (profinite)* **Tate module** *of A by:*

$$T(A) \stackrel{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, A(\overline{K}))$$

Then the maximal torsion-free quotient $T(A) \twoheadrightarrow Q$ on which G_K acts trivially is a finite free $\widehat{\mathbb{Z}}$-module.

Proof. A semi-abelian variety is an extension of an abelian variety by a torus. Thus, $T(A)$ is the extension of the Tate module of an abelian variety by the Tate module of a torus. Moreover, since (after restricting to some open subgroup of G_K) the Tate module of a torus is isomorphic to the direct sum of a finite number of copies of $\widehat{\mathbb{Z}}(1)$, we thus conclude that the image of the Tate module of the torus in Q is necessarily zero. In particular, we may assume for the remainder of the proof without loss of generality that A is an *abelian variety.*

Now it follows from the theory of [FC](cf., in particular, [FC], Chapter III, Corollary 7.3), that $T(A)$ fits into *exact sequences* (of G_K-modules)

$$0 \to T_{\text{good}} \to T(A) \to T_{\text{com}} \to 0$$

$$0 \to T_{\text{tor}} \to T_{\text{good}} \to T(B) \to 0$$

where $T(B)$ is the Tate module of an abelian variety B over K with *potentially good reduction*; and $T_{\text{com}} = M_{\text{com}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $T_{\text{tor}} = M_{\text{tor}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}(1)$ for finite free $\mathbb{Z}$-modules M_{com}, M_{tor} on which G_K acts via a *finite quotient.* It is thus evident that T_{tor} maps to 0 in Q. Moreover, by [Mzk4], Lemma 8.1 (the proof of which is valid for arbitrary B, even though in *loc. cit.*, this result is only stated in the case of a Jacobian), and the *Riemann Hypothesis for abelian varieties over finite fields* (cf., e.g., [Mumf], p. 206), it follows that $T(B)$ also maps to 0 in Q. Thus, we conclude that Q is equal to the maximal torsion-free quotient of T_{com} on which G_K acts trivially. Since $\widehat{\mathbb{Z}}$ is $\mathbb{Z}$-flat, however, this implies that Q is equal to the result of applying $\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ to the maximal torsion-free quotient of M_{com} on which G_K acts trivially. But this last quotient is manifestly finite and free over $\mathbb{Z}$. This completes the proof. ○

§1.2. The Anabelian Geometry of p-adic Local Fields

In this §, we review certain well-known *"group-theoretic" properties of Galois groups of p-adic local fields*, i.e., properties preserved by *arbitrary isomorphisms* between such Galois groups.

For $i = 1, 2$, let p_i be a *prime number*. Let K_i be a finite extension of $\mathbb{Q}_{p_i}$. We denote the *ring of integers* (respectively, *maximal ideal*; *residue field*) of K_i by $\mathcal{O}_{K_i}$ (respectively, $\mathfrak{m}_{K_i}$; k_i). Also, we assume that we have chosen an *algebraic closures* $\overline{K}_i$ of K_i and write

$$G_{K_i} \stackrel{\text{def}}{=} \text{Gal}(\overline{K}_i/K_i)$$

for the corresponding absolute Galois group of K_i. Thus, by *local class field theory* (cf., e.g., [Serre2]), we have a natural isomorphism

$$(K_i^{\times})^{\wedge} \xrightarrow{\sim} G_{K_i}^{\text{ab}}$$

(where the "$\wedge$" denotes the profinite completion of an abelian group; "$\times$" denotes the group of units of a ring; and "ab" denotes the maximal abelian quotient of a group). In particular, $G_{K_i}^{\text{ab}}$ fits into an *exact sequence*

$$0 \to \mathcal{O}_{K_i}{}^{\times} \to G_{K_i}^{\text{ab}} \to \widehat{\mathbb{Z}} \to 0$$

(arising from a similar exact sequence for $(K_i^{\times})^{\wedge}$). Moreover, we obtain natural inclusions

$$k_i^{\times} \hookrightarrow \mathcal{O}_{K_i}{}^{\times} \subseteq K_i^{\times} \hookrightarrow G_{K_i}^{\text{ab}}$$
$$K_i^{\times}/\mathcal{O}_{K_i}{}^{\times} \xrightarrow{\sim} \mathbb{Z} \hookrightarrow G_{K_i}^{\text{ab}}/\text{Im}(\mathcal{O}_{K_i}{}^{\times})$$

(where "$\xrightarrow{\sim}$" denotes the morphism induced by the valuation on $K_i^{\times}$) by considering the Teichmüller representatives of elements of $k_i^{\times}$ and the Frobenius element, respectively. Also, in the following discussion we shall write:

$$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i) \stackrel{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, \overline{K}_i^{\times}); \quad \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_i) \stackrel{\text{def}}{=} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i) \otimes_{\widehat{\mathbb{Z}}} \widehat{\mathbb{Z}}';$$
$$\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_i) \stackrel{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))$$

(where $\widehat{\mathbb{Z}}' \stackrel{\text{def}}{=} \widehat{\mathbb{Z}}/\mathbb{Z}_p$). Finally, we denote the *cyclotomic character* of G_{K_i} by:

$$\chi_i : G_{K_i} \to \widehat{\mathbb{Z}}^{\times}$$

Proposition 1.2.1. (Invariants of Arbitrary Isomorphisms of Galois Groups of Local Fields) *Suppose that we are given an isomorphism of profinite groups:*

$$\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$$

Then:

(i) We have: $p_1 = p_2$. Thus, (in the remainder of this proposition and its proof) we shall write $p \stackrel{\text{def}}{=} p_1 = p_2$.

(ii) α induces an isomorphism $I_{K_1} \xrightarrow{\sim} I_{K_2}$ between the respective **inertia subgroups** *of G_{K_1}, G_{K_2}.*

(iii) The isomorphism $\alpha^{\text{ab}} : G^{\text{ab}}_{K_1} \xrightarrow{\sim} G^{\text{ab}}_{K_2}$ induced by α preserves the images $\text{Im}(\mathcal{O}_{K_i}{}^{\times})$, $\text{Im}(k_i^{\times})$, $\text{Im}(K_i^{\times})$ of the natural morphisms discussed above.

(iv) The morphism induced by α between the respective quotients $G^{\text{ab}}_{K_i}/\text{Im}(\mathcal{O}_{K_i}{}^{\times})$ preserves the respective **Frobenius elements**.

(v) $[K_1 : \mathbb{Q}_p] = [K_2 : \mathbb{Q}_p]$; $[k_1 : \mathbb{F}_p] = [k_2 : \mathbb{F}_p]$. In particular, the ramification indices of K_1, K_2 over $\mathbb{Q}_p$ coincide.

(vi) The morphisms induced by α on the abelianizations of the various open subgroups of the G_{K_i} induce an isomorphism

$$\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_1) \xrightarrow{\sim} \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_2)$$

In particular, α preserves the **cyclotomic characters** *χ_i.*

(vii) The morphism $H^2(K_1, \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_1)) \xrightarrow{\sim} H^2(K_2, \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_2))$ induced by α (cf. (vi)) preserves the **"residue map"**

$$H^2(K_i, \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_i)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

of local class field theory (cf. [Serre2], §1.1).

Proof. Property (i) follows by considering the *ranks* of $G^{\text{ab}}_{K_i}$ over various $\mathbb{Z}_l$ (cf. Lemma 1.1.4, (ii)). Property (iii) for $\text{Im}(k_i^{\times})$ follows from the fact that $\text{Im}(k_i^{\times})$ may be recovered as the prime-to-p torsion subgroup of $G^{\text{ab}}_{K_i}$. Property (v) follows for $[K_i : \mathbb{Q}_p]$ by considering the $\mathbb{Z}_p$-rank of $G^{\text{ab}}_{K_i}$ (minus 1) and for $[k_i : \mathbb{F}_p]$ by considering the cardinality of $\text{Im}(k_i^{\times})$ (plus 1) — cf. (i), (iii). Property (ii) follows from the fact that whether or not a finite extension is unramified may be determined group-theoretically by considering the variation of the ramification index over $\mathbb{Q}_p$ (cf. (v)). Property (iii) for $\text{Im}(\mathcal{O}_{K_i}{}^{\times})$ follows formally from (ii) (since this image is equal to the image in $G^{\text{ab}}_{K_i}$ of I_{K_i}). Property (iv) follows by applying (iii) for $\text{Im}(k_i^{\times})$ to the various open subgroups of G_{K_i} that correspond to *unramified* extensions of K_i and using the fact the Frobenius element is the unique element that acts as *multiplication by* $|k_1| = |k_2|$ on $\overline{k}_i^{\times}$ (where $\overline{k}_i$ denotes the algebraic closure of k_i induced by $\overline{K}_i$). Here, we note that if L_i is a finite extension of K_i, then the inclusion

$$G_{K_i} \xrightarrow{\sim} (K_i^\times)^\wedge \hookrightarrow (L_i^\times)^\wedge \xrightarrow{\sim} G_{L_i}$$

may be reconstructed group-theoretically by considering the *Verlagerung*, or transfer, map (cf. [Serre2], §2.4). Property (iii) for $\mathrm{Im}(K_i^\times)$ follows formally from (iv). Property (vi) follows formally from (iii). Finally, property (vii) follows (cf. the theory of the Brauer group of a local field, as exposed, for instance, in [Serre2], §1) from the fact that the morphism $H^2(K_i, \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_i)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ may be constructed as the composite of the natural isomorphism

$$H^2(K_i, \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_i)) = H^2(G_{K_i}, \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_i)) \xrightarrow{\sim} H^2(G_{K_i}, \overline{K}_i^\times)$$

— which is group-theoretic, by (iii) — with the inverse of the isomorphism

$$H^2(\mathrm{Gal}(K_i^{\mathrm{unr}}/K_i), (K_i^{\mathrm{unr}})^\times) \xrightarrow{\sim} H^2(G_{K_i}, \overline{K}_i^\times)$$

(where K_i^{unr} denotes the maximal unramified extension of K_i) — which is group-theoretic, by (ii), (iii) — followed by the natural isomorphism

$$H^2(\mathrm{Gal}(K_i^{\mathrm{unr}}/K_i), (K_i^{\mathrm{unr}})^\times) \xrightarrow{\sim} H^2(\mathrm{Gal}(K_i^{\mathrm{unr}}/K_i), \mathbb{Z}) = H^2(\widehat{\mathbb{Z}}, \mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$

— which is group-theoretic, by (ii), (iii), (iv). ○

Before proceeding, we observe that Proposition 1.2.1, (i), may be extended as follows: Write

$$\mathfrak{A}_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} |\mathrm{Spec}(\mathbb{Z})| \bigcup \{\infty\}$$

(where "$| - |$" denotes the underlying set of a scheme) for the set of *"all arithmetic primes of* $\mathbb{Q}$*."* If $v \in \mathfrak{A}_{\mathbb{Q}}$ is equal to $\{(0)\} \in |\mathrm{Spec}(\mathbb{Z})|$ (respectively, ∞), set $G_v \stackrel{\mathrm{def}}{=} G_{\mathbb{Q}}$ (respectively, $G_v \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\mathbb{C}/\mathbb{R})$). If $v \in |\mathrm{Spec}(\mathbb{Z})| \subseteq \mathfrak{A}_{\mathbb{Q}}$ is equal to the prime determined by a prime number p, set $G_v \stackrel{\mathrm{def}}{=} G_{\mathbb{Q}_p}$.

Proposition 1.2.2. **(Intrinsicity of Arithmetic Types)** *For* $i = 1, 2$, *let* $v_i \in \mathfrak{A}_{\mathbb{Q}}$. *Suppose that* H_i *is an open subgroup of* G_{v_i}. *Then* $H_1 \cong H_2$ *implies* $v_1 = v_2$.

Proof. Indeed, open subgroups of $G_{\mathbb{Q}}$ may be distinguished by the fact that their abelianizations *fail to be topologically finitely generated.* (Indeed, consider the abelian extensions arising from adjoining roots of unity.) By contrast, abelianizations of open subgroups of $G_{\mathbb{R}}$ or $G_{\mathbb{Q}_p}$ (cf. the above discussion) are topologically finitely generated. Next, open subgroups of $G_{\mathbb{R}}$ may be distinguished from those of some $G_{\mathbb{Q}_p}$ by the fact they are *finite*. The remainder of Proposition 1.2.2 follows from Proposition 1.2.1, (i). $\bigcirc$

Next, let us write $\mathrm{Spec}(\mathcal{O}_{K_i})^{\log}$ for the log scheme obtained by equipping the scheme $\mathrm{Spec}(\mathcal{O}_{K_i})$ with the *log structure* defined by the divisor $V(\mathfrak{m}_{K_i})$. Thus, by pulling back this log structure via the natural morphism $\mathrm{Spec}(k_i) \hookrightarrow \mathrm{Spec}(\mathcal{O}_{K_i})$, we obtain a log scheme $\mathrm{Spec}(k_i)^{\log}$, which we denote by

$$k_i^{\log}$$

for short. Note that the "étale monoid" that defines the log structure on $k_i^{\log}$ "admits a global chart" in the sense that it is defined by a *single constant monoid* (in the Zariski topology of $\mathrm{Spec}(k_i)$) $M_{k_i^{\log}}$, which fits into a natural *exact sequence* (of monoids):

$$1 \to k_i^{\times} \to M_{k_i^{\log}} \to \mathbb{N} \to 0$$

Thus, the $k_i^{\times}$-torsor U_i determined by considering the inverse image of $1 \in \mathbb{N}$ in this sequence may be identified with the $k_i^{\times}$-torsor of *uniformizers* $\in \mathfrak{m}_{K_i}$ *considered modulo* $\mathfrak{m}_{K_i}^2$.

Next, let us write

$$G_{K_i} \twoheadrightarrow G_{k_i}^{\log}$$

for the quotient defined by the *maximal tamely ramified extension* K_i^{tame} of K_i. Thus, $G_{k_i}^{\log}$ may also be thought of as the *"logarithmic fundamental group"* $\pi_1(k_i^{\log})$ of the log scheme $k_i^{\log}$. Moreover, $G_{k_i}^{\log}$ fits into a natural *exact sequence*:

$$1 \to \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i) \to G_{k_i}^{\log} \to \widehat{\mathbb{Z}} \to 1$$

where, just as in the case of K_i, we write:

$$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i) = \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_i) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \overline{k}_i^{\times});$$
$$\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{k}_i) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))$$

The *"abelianization"* of this exact sequence yields an exact sequence:

$$1 \to k_i^\times \to (G_{k_i}^{\log})^{\mathrm{ab}} \to \widehat{\mathbb{Z}} \to 1$$

Now we have the following:

Proposition 1.2.3.

(i) Any isomorphism $\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$ (as in Proposition 1.2.1) induces an isomorphism $G_{k_1}^{\log} \xrightarrow{\sim} G_{k_2}^{\log}$ between the respective quotients.

(ii) There is a natural isomorphism of $k_i^\times$-torsors between the **torsor of uniformizers** *U_i discussed above and the $H^1(\widehat{\mathbb{Z}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i)) = k_i^\times$-***torsor** *$H^1(G_{k_i}^{\log}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))^{[1]}$ of elements of $H^1(G_{k_i}^{\log}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))$ that map to the identity element in $H^1(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i)) = \mathrm{Hom}_{\widehat{\mathbb{Z}}'}(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))$. This isomorphism is defined by associating to a uniformizer $\pi \in U_i$ the $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i)$-torsor over $k_i^{\log}$ determined by the roots $\pi^{1/N}$, as N ranges (multiplicatively) over the positive integers prime to p_i.*

(iii) The profinite group $G_{k_i}^{\log}$ is **slim**.

Proof. Property (i) follows from Proposition 1.2.1, (ii), together with the fact that the quotient $G_{K_i} \twoheadrightarrow G_{k_i}^{\log}$ may be identified with the quotient of G_{K_i} by the (unique) maximal pro-p subgroup of I_{K_i}. Next, since *any* morphism of $k_i^\times$-torsors is *necessarily* an isomorphism, property (ii) follows by observing that the stated association of coverings to uniformizers is indeed a morphism of $k_i^\times$-torsors — a *tautology*, which may by verified by considering the case $N = q_i - 1$ (where q_i is the cardinality of k_i), in which case this tautology amounts to the *computation*: $(\zeta^{1/N})^{q_i} = \zeta \cdot (\zeta^{1/N})$ (for $\zeta \in k_i^\times$). Finally, property (iii) follows formally from the fact that the quotient $G_{k_i}^{\log}/\mathrm{Im}(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))$ acts *faithfully* on all open subgroups of the closed subgroup $\mathrm{Im}(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i)) \subseteq G_{k_i}^{\log}$. ○

In the following, let us denote by $(k_i^{\log})^\sim \to k_i^{\log}$ the *"universal covering"* of $k_i^{\log}$ defined by the extension K_i^{tame} of K_i. Thus, $G_{k_i}^{\log}$ acts naturally as the *group of covering transformations* of $(k_i^{\log})^\sim \to k_i^{\log}$.

Proposition 1.2.4. (The "Grothendieck Conjecture" for the Logarithmic Point) *Suppose that we are given an isomorphism of profinite groups:*

$$\lambda : G_{k_1}^{\log} \xrightarrow{\sim} G_{k_2}^{\log}$$

Then:

(i) We have: $|k_1| = |k_2|$; $p_1 = p_2$. Thus, (in the remainder of this proposition and its proof) we shall write $p \stackrel{\mathrm{def}}{=} p_1 = p_2$.

(ii) λ *preserves the subgroups* $\mathrm{Im}(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i)) \subseteq G_{k_i}^{\log}$ *as well as the Frobenius elements in the quotients* $G_{k_i}^{\log}/\mathrm{Im}(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))$.

(iii) Assume further that the morphism

$$\overline{k}_1^{\times} \xrightarrow{\sim} \overline{k}_2^{\times}$$

induced by λ *(by thinking of* $\overline{k}_i^{\times}$ *as* $\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{k}_i)$*) arises from an* **isomorphism of fields** $\overline{\sigma} : \overline{k}_1 \xrightarrow{\sim} \overline{k}_2$. *Then there exists a unique commutative diagram*

$$\begin{array}{ccc} (k_1^{\log})^{\sim} & \xrightarrow{(\sigma^{\log})^{\sim}} & (k_2^{\log})^{\sim} \\ \downarrow & & \downarrow \\ k_1^{\log} & \xrightarrow{\sigma^{\log}} & k_2^{\log} \end{array}$$

of log schemes, compatible with the natural action of $G_{k_i}^{\log}$ *on* $(k_i^{\log})^{\sim}$ *(for* $i = 1, 2$*), in which the vertical morphisms are the natural morphisms, and the horizontal morphisms are* **isomorphisms** *for which the morphisms on the underlying schemes are those induced by* $\overline{\sigma}$.

Proof. Property (i) follows by observing that p_i is the unique prime number such that 1 plus the cardinality of the torsion subgroup of $(G_{k_i}^{\log})^{\mathrm{ab}}$ — i.e., the cardinality of k_i — is equal to a power of p_i. Property (ii) follows by thinking of the quotients $G_{k_i}^{\log}/\mathrm{Im}(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))$ as the quotients of $G_{k_i}^{\log}$ obtained by forming the quotient of $(G_{k_i}^{\log})^{\mathrm{ab}}$ by its torsion subgroup, and then using that the Frobenius element is the unique element that acts on the abelian group $\mathrm{Im}(\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))$ via multiplication by $|k_1| = |k_2|$. As for (iii), the morphism $\sigma^{\log}$ is the unique logarithmic extension of σ whose induced morphism $U_1 \xrightarrow{\sim} U_2$ is the morphism obtained (cf. Proposition 1.2.3, (ii)) by considering the morphism induced by λ between the $H^1(\widehat{\mathbb{Z}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i)) = k_i^{\times}$**-torsors** $H^1(G_{k_i}^{\log}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i))^{[1]}$ (for $i = 1, 2$) — which are preserved by λ, by (ii). Note that here we also use (cf. (ii)) that the *Frobenius element* $\in \widehat{\mathbb{Z}}$ is preserved, since this element is necessary to ensure the compatibility of the identifications

$$H^1(\widehat{\mathbb{Z}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{k}_i)) = k_i^{\times}$$

(cf. Proposition 1.2.3, (ii)). The morphism $(\sigma^{\log})^{\sim}$ is obtained by applying this construction of "$\sigma^{\log}$" to the various finite log étale coverings of $k_i^{\log}$ obtained by considering various open subgroups of $G_{k_i}^{\log}$. Here, the transition morphisms among coverings are induced by the *Verlagerung*, as in the proof of Proposition 1.2.1. Finally, the uniqueness of the lifting $(\sigma^{\log})^{\sim}$ of $\sigma^{\log}$ is a formal consequence of the fact that $G_{k_i}^{\log}$ is *center-free* (cf. Proposition 1.2.3, (iii)). ○

§1.3. The Anabelian Geometry of Hyperbolic Curves

Characteristic Zero:

Let K be a *field of characteristic* 0 whose absolute Galois group is *slim*. Let X be a *hyperbolic curve* of type (g, r) over $S \stackrel{\text{def}}{=} \text{Spec}(K)$. Fix an *algebraic closure* $\overline{K}$ of K and write $\overline{s} : \text{Spec}(\overline{K}) \to S$; $G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$. Let $\overline{x} \in X(\overline{K})$ be a $\overline{K}$-valued point of X lying over $\overline{s}$. Then, setting $\Pi_X \stackrel{\text{def}}{=} \pi_1(X, \overline{x})$, we obtain an *exact sequence*

$$1 \to \Delta_X \to \Pi_X \to G_K \to 1$$

which determines a well-defined (up to composition with an inner automorphism of the range) *continous homomorphism*

$$G_K \to \text{Out}(\Delta_X)$$

to the *group of outer automorphisms* $\text{Out}(\Delta_X)$ *of* Δ_X.

Lemma 1.3.1. (Slimness of Geometric and Arithmetic Fundamental Groups) *The profinite groups* Δ_X, Π_X *are* **slim**.

Proof. The slimness of Π_X is a formal consequence of the slimness of Δ_X and our *assumption that* G_K *is slim*. Thus, it remains to prove that Δ_X is slim. Let $H \subseteq \Delta_X$ be an *open normal subgroup* for which the associated covering $X_H \to X_{\overline{K}} \stackrel{\text{def}}{=} X \times_K \overline{K}$ is such that X_H is a *curve of genus* ≥ 2. Thus, H^{ab} may be thought of as the profinite Tate module associated to the *generalized Jacobian* of the singular curve obtained from the unique smooth compactification of X_H by identifying the various *cusps* (i.e., points of the compactification not lying in X_H) to a *single point*. In particular, if conjugation by an element $\delta \in \Delta_X$ induces the *trivial action* on H^{ab}, then we conclude that the image of δ in Δ_X/H induces the trivial action on the generalized Jacobian just discussed, hence on X_H itself. But this implies that $\delta \in H$. By taking H to be sufficiently small, we thus conclude that $\delta = 1$. ○

In particular, it follows formally from Lemma 1.3.1 that:

Corollary 1.3.2. (A Natural Exact Sequence) *We have a natural exact sequence of profinite groups:*

$$1 \to \Delta_X \to \text{Aut}(\Delta_X) \to \text{Out}(\Delta_X) \to 1$$

In particular, by pulling back this exact sequence, one may recover the exact sequence $1 \to \Delta_X \to \Pi_X \to G_K \to 1$ *entirely group-theoretically from the outer Galois representation* $G_K \to \mathrm{Out}(\Delta_X)$.

One example of the sort of "K" under consideration is the case of a *"sub-p-adic field"*:

Corollary 1.3.3. **(Slimness of Sub-p-adic Fields)** *The absolute Galois group of a* **sub-p-adic field** *(i.e., a field isomorphic to a subfield of a finitely generated field extension of* $\mathbb{Q}_p$*, for some prime number* p*) is* **slim**.

Proof. This fact is implied by the argument of the proof of [Mzk6], Lemma 15.8. ○

In [Mzk6], the author (essentially) proved the following result (cf. [Mzk6], Theorem A):

Theorem 1.3.4. **("Sub-p-adic Profinite Grothendieck Conjecture")** *Suppose that* K *is a* **sub-p-adic field***, and that* X *and* Y *are hyperbolic curves over* K*. Denote by* $\mathrm{Isom}_K(X,Y)$ *the set of* K*-isomorphisms* $X \xrightarrow{\sim} Y$*; by* $\mathrm{Isom}^{\mathrm{Out}}_{G_K}(\Delta_X, \Delta_Y)$ *the set of outer isomorphisms between the two profinite groups in parentheses that are compatible with the respective outer actions of* G_K*. Then the natural map*

$$\mathrm{Isom}_K(X_K, Y_K) \to \mathrm{Isom}^{\mathrm{Out}}_{G_K}(\Delta_X, \Delta_Y)$$

is bijective.

Thus, by combining Theorems 1.1.3; 1.3.4; Lemma 1.1.4, (i), we obtain the following:

Corollary 1.3.5. **(Absolute Grothendieck Conjecture over Number Fields)** *In the situation of Theorem 1.3.4, suppose that* K *is a* **number field***. Denote by* $\mathrm{Isom}(X,Y)$ *the set of isomorphisms* $X \xrightarrow{\sim} Y$*; by* $\mathrm{Isom}^{\mathrm{Out}}(\Pi_{X_K}, \Pi_{Y_K})$ *the set of outer isomorphisms between the two profinite groups in parentheses. Then the natural map*

$$\mathrm{Isom}(X_K, Y_K) \to \mathrm{Isom}^{\mathrm{Out}}(\Pi_{X_K}, \Pi_{Y_K})$$

is bijective.

Remark 1.3.5.1. Since the analogue of Theorem 1.1.3 in the p-adic local case is false, it seems *unlikely* to the author at the time of writing that the analogue of Corollary 1.3.5 should hold over p-adic local fields.

One interesting result in the present context is the following, due to *M. Matsumoto* (cf. [Mtmo], Theorems 2.1, 2.2):

Theorem 1.3.6. **(Injectivity of Galois)** *Let X be an* **affine hyperbolic curve** *over a* **sub-complex field** *K — i.e., a field isomorphic to a subfield of the field of complex numbers. Then the resulting outer Galois representation*

$$G_K \to \mathrm{Out}(\Delta_X)$$

is **injective**.

Remark 1.3.6.1. This injectivity was first proven by Belyi in the case of hyperbolic curves of type $(g, r) = (0, 3)$. It was then conjectured by Voevodskii to be true for all (hyperbolic) (g, r) and proven by Voevodskii to be true for $g = 1$. Finally, it was proven by Matsumoto to hold for all (g, r) such that $r > 0$. To the knowledge of the author, the proper case remains open at the time of writing. We refer to the discussion surrounding [Mtmo], Theorem 2.1, for more details on this history.

Remark 1.3.6.2. One interesting aspect of the homomorphism appearing in Theorem 1.3.6 is that it allows one to interpret Theorem 1.3.4 as a *computation of the centralizer of the image of this homomorphism* $G_K \to \mathrm{Out}(\Delta_X)$.

Next, we would like to discuss various properties of the *inertia groups of the cusps* of a hyperbolic curve. For every *cusp* x of $X_{\overline{K}} \stackrel{\mathrm{def}}{=} X \times_K \overline{K}$ — i.e., point of the unique smooth compactification of $X_{\overline{K}}$ over $\overline{K}$ that does not lie in $X_{\overline{K}}$ — we have an associated *inertia group* (abstractly isomorphic to $\widehat{\mathbb{Z}}$)

$$I_x \subseteq \Delta_X$$

(well-defined, up to conjugation). If l is any *prime number*, then let us denote the *maximal pro-l quotient* of a profinite group by means of a superscript "(l)." Thus, we also obtain an inertia group $I_x^{(l)} \subseteq \Delta_X^{(l)}$ (abstractly isomorphic to $\mathbb{Z}_l$).

Lemma 1.3.7. **(Commensurable Terminality of Inertia Groups)** *The subgroups $I_x^{(l)} \subseteq \Delta_X^{(l)}$, $I_x \subseteq \Delta_X$ are commensurably terminal.*

Proof. Indeed, let σ be an element of the commensurator. If the asserted commensurable terminality is *false*, then by projecting to a finite quotient, we may assume that we have a *finite Galois covering*

$$Z \to X_{\overline{K}}$$

(of degree a power of l in the pro-l case), together with a *cusp* z of Z such that: (i) z maps to x; (ii) $z \neq z^\sigma$; (iii) z, z^σ have conjugate inertia groups in Δ_Z. We may also assume (by taking $\Delta_Z \subseteq \Delta_X$ to be sufficiently small) that Z has *genus* ≥ 2 and admits a cusp $z' \neq z, z^\sigma$. Then it is easy to see that Z admits an infinite abelian (pro-l, in the pro-l case) covering which is totally ramified at z, z', but not at z^σ. But this contradicts property (iii). ○

Now, let us assume that we are given *two hyperbolic curves* $(X_i)_{K_i}$ (for $i = 1, 2$), each defined over a finite extension K_i of $\mathbb{Q}_{p_i}$. Let us write q_i for the *cardinality of the residue field of* K_i. Choose basepoints for the $(X_i)_{K_i}$ and denote the resulting fundamental groups by $\Pi_{(X_i)_{K_i}}$. Also, let us denote the unique proper curve over K_i that *compactifies* $(X_i)_{K_i}$ by $(Y_i)_{K_i}$. Suppose, moreover, that we are given an *isomorphism*

$$\alpha_X : \Pi_{(X_1)_{K_1}} \xrightarrow{\sim} \Pi_{(X_2)_{K_2}}$$

of profinite groups.

Lemma 1.3.8. **(Group-Theoreticity of Arithmetic Quotients)** *The isomorphism α_X is necessarily compatible with the quotients $\Pi_{(X_i)_{K_i}} \twoheadrightarrow G_{K_i}$.*

Proof. This follows formally from Lemmas 1.1.4, 1.1.5. ○

Thus, Lemma 1.3.8, Proposition 1.2.1, (v), imply that $q_1 = q_2$.

Lemma 1.3.9. **(Group-Theoreticity of the Cusps)** *The types (g_i, r_i) of the hyperbolic curves $(X_i)_{K_i}$ coincide. In particular, for any prime number l, α_X maps inertia groups of cusps in Δ_{X_1} (respectively, $\Delta^{(l)}_{X_1}$) to inertia groups of cusps in Δ_{X_2} (respectively, $\Delta^{(l)}_{X_2}$).*

Proof. Whether or not $r_i = 0$ may be determined by considering whether or not Δ_{X_i} is *free* as a profinite group. When $r_i > 0$, one may compute r_i by considering the *weight* — i.e., the number w such that the eigenvalues of the action are algebraic numbers of archimedean absolute value q_i^w — of the action of the Frobenius element $\in G_{k_i}$ (cf. Proposition 1.2.1, (iv)) as follows: First, we observe that (as is well-known) the weights of the action of Frobenius on $\Delta^{\mathrm{ab}}_{X_i} \otimes \mathbb{Q}_l$ (where l is a prime number distinct from p_1, p_2)

belong to the set $\{0, 1, \frac{1}{2}\}$. Now if M is a $\mathbb{Q}_l$-vector space on which Frobenius acts, let us write

$$M^{\mathrm{wt}\ w}$$

for the $\mathbb{Q}_l$-subspace of M on which Frobenius acts with eigenvalues of weight w. Then, setting $I_i \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l \twoheadrightarrow \Delta_{Y_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l)$, we have:

$$\begin{aligned} r_i - 1 &= \dim_{\mathbb{Q}_l}(I_i) = \dim_{\mathbb{Q}_l}(I_i^{\mathrm{wt}\ 1}) \\ &= \dim_{\mathbb{Q}_l}(\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l)^{\mathrm{wt}\ 1} - \dim_{\mathbb{Q}_l}(\Delta_{Y_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l)^{\mathrm{wt}\ 1} \\ &= \dim_{\mathbb{Q}_l}(\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l)^{\mathrm{wt}\ 1} - \dim_{\mathbb{Q}_l}(\Delta_{Y_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l)^{\mathrm{wt}\ 0} \\ &= \dim_{\mathbb{Q}_l}(\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l)^{\mathrm{wt}\ 1} - \dim_{\mathbb{Q}_l}(\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l)^{\mathrm{wt}\ 0} \end{aligned}$$

(where the fourth equality follows from the auto-duality (up to a Tate twist) of $\Delta_{Y_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l$; and the second and fifth equalities follow from the fact that Frobenius acts on I_i with weight 1). On the other hand, the quantities appearing in the final line of this sequence of equalities are all "group-theoretic." Thus, we conclude that $r_1 = r_2$. Since $\dim_{\mathbb{Q}_l}(\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Q}_l) = 2g_i - 2 + r_i$, this implies that $g_1 = g_2$, as desired.

Finally, the statement concerning preservation of inertia groups *follows formally* from the fact that "r_i is group-theoretic" (by applying this fact to coverings of X_i). Indeed, let l be a *prime number* (possibly equal to p_1 or p_2). Since r_i may be recovered group-theoretically, given any finite étale coverings

$$Z_i \to V_i \to X_i$$

such that Z_i is Galois, of degree a *power of l*, over V_i, one may determine group-theoretically whether or not $Z_i \to V_i$ is "*totally ramified* at a single point of Z_i and unramified elsewhere," since this condition is easily verified to be equivalent to the equality:

$$r_{Z_i} = \deg(Z_i/V_i) \cdot (r_{V_i} - 1) + 1$$

Moreover, the group-theoreticity of this condition extends immediately to the case of *pro-l* coverings $Z_i \to V_i$. Thus, by Lemma 1.3.7, we conclude that *the inertia groups of cusps in* $(\Delta_{X_i})^{(l)}$ (i.e., the maximal pro-l quotient of Δ_{X_i}) *may be characterized (group-theoretically!)* as the maximal subgroups of $(\Delta_{X_i})^{(l)}$ that correspond to (profinite) coverings satisfying this condition.

In particular, (by Lemma 1.3.7) the *set of cusps of* X_i may be reconstructed (group-theoretically!) as the set of $(\Delta_{X_i})^{(l)}$-orbits (relative to the action via conjugation) of such inertia groups in $(\Delta_{X_i})^{(l)}$. Thus, by applying this observation to arbitrary finite étale coverings of X_i, we recover the inertia subgroups of the cusps of Δ_{X_i} as the subgroups that fix *some* cusp of the universal covering $\widetilde{X}_i \to X_i$ of X_i determined by the basepoint in question. This completes the proof. ○

Positive Characteristic:

For $i = 1, 2$, let k_i be a *finite field* of characteristic p; X_i a *hyperbolic curve* over k_i. Choose a *universal tamely ramified* (i.e., at the punctures of X_i) *covering* $\widetilde{X}_i \to X_i$ of X_i; write

$$\Pi_{X_i}^{\text{tame}} \stackrel{\text{def}}{=} \text{Gal}(\widetilde{X}_i/X_i)$$

for the corresponding *fundamental groups*. Thus, we obtain *exact sequences*:

$$1 \to \Delta_{X_i}^{\text{tame}} \to \Pi_{X_i}^{\text{tame}} \to G_{k_i} \to 1$$

(where G_{k_i} is the absolute Galois group of k_i determined by $\widetilde{X}_i$). As is well-known, the *Frobenius element* determines a natural isomorphism $\widehat{\mathbb{Z}} \cong G_{k_i}$.

Lemma 1.3.10. **(Slimness of Fundamental Groups)** *For $i = 1, 2$, the profinite groups $\Delta_{X_i}^{\text{tame}}$, $\Pi_{X_i}^{\text{tame}}$ are* **slim**.

Proof. The slimness of $\Delta_{X_i}^{\text{tame}}$ follows by exactly the same argument — i.e., by considering the action of $\Delta_{X_i}^{\text{tame}}$ on *abelianizations of open subgroups* — as that given in the proof of Lemma 1.3.1. By a similar argument, the slimness of $\Pi_{X_i}^{\text{tame}}$ follows formally from:

(i) the slimness of $\Delta_{X_i}^{\text{tame}}$;

(ii) the *Riemann Hypothesis for abelian varieties over finite fields* (cf., e.g., [Mumf], p. 206); and

(iii) the fact that the covering of X_i determined by a sufficiently small open subgroup $H \subseteq \Pi_{X_i}^{\text{tame}}$ has *p-rank* ≥ 1 (cf. [Tama], Lemma 1.9).

(Here, we note that while (ii) is sufficient to deal with the "l-primary portion" of $\widehat{\mathbb{Z}} \cong G_{k_i}$ (for $l \neq p$); one needs *both* (ii), (iii) to cover the *"p-primary portion"* of $\widehat{\mathbb{Z}} \cong G_{k_i}$.) ○

The following fundamental result is due to *A. Tamagawa* (cf. [Tama], Theorem 4.3):

Theorem 1.3.11. (The Grothendieck Conjecture for Affine Hyperbolic Curves over Finite Fields) *Assume, for $i = 1, 2$, that X_i is* **affine**. *Then the natural map*

$$\text{Isom}(\widetilde{X}_1/X_1, \widetilde{X}_2/X_2) \to \text{Isom}(\Pi^{\text{tame}}_{X_1}, \Pi^{\text{tame}}_{X_2})$$

(from scheme-theoretic isomorphisms $\widetilde{X}_1 \xrightarrow{\sim} \widetilde{X}_2$ lying over an isomorphism $X_1 \xrightarrow{\sim} X_2$ to isomorphisms of profinite groups $\Pi^{\text{tame}}_{X_1} \xrightarrow{\sim} \Pi^{\text{tame}}_{X_2}$) is bijective.

Finally, we observe that, just as in the characteristic zero case, *inertia groups of cusps are commensurably terminal*: If x_i is a *cusp* of $(X_i)_{\overline{k}_i} \stackrel{\text{def}}{=} X_i \times_{k_i} \overline{k}_i$, then we have an associated *inertia group* (abstractly isomorphic to $\widehat{\mathbb{Z}}'$)

$$I_{x_i} \subseteq \Delta^{\text{tame}}_{X_i}$$

(well-defined, up to conjugation). If l is any *prime number distinct from* p, then we also obtain an inertia group $I^{(l)}_{x_i} \subseteq (\Delta^{\text{tame}}_{X_i})^{(l)}$ (abstractly isomorphic to $\mathbb{Z}_l$).

Lemma 1.3.12. (Commensurable Terminality of Inertia Groups) *The subgroups $I^{(l)}_{x_i} \subseteq (\Delta^{\text{tame}}_{X_i})^{(l)}$, $I_{x_i} \subseteq \Delta^{\text{tame}}_{X_i}$ are commensurably terminal.*

Proof. The proof is entirely similar to that of Lemma 1.3.7. ○

Section 2: Reconstruction of the Logarithmic Special Fiber

For $i = 1, 2$, let K_i be a *finite extension* of $\mathbb{Q}_{p_i}$ (cf. §1.2), and suppose that we are given a *hyperbolic curve* $(X_i)_{K_i}$ over K_i. Let us fix a $\overline{K}_i$-valued basepoint for $(X_i)_{K_i}$ and denote the resulting fundamental $\pi_1((X_i)_{K_i})$ by $\Pi_{(X_i)_{K_i}}$. Suppose, moreover, that we are given an *isomorphism* $\alpha_X : \Pi_{(X_1)_{K_1}} \xrightarrow{\sim} \Pi_{(X_2)_{K_2}}$, which, by Lemma 1.3.8, necessarily fits into a *commutative diagram*

$$\begin{array}{ccc} \Pi_{(X_1)_{K_1}} & \xrightarrow{\alpha_X} & \Pi_{(X_2)_{K_2}} \\ \downarrow & & \downarrow \\ G_{K_1} & \xrightarrow{\alpha_K} & G_{K_2} \end{array}$$

where the vertical morphisms are the natural ones, and the horizontal morphisms are assumed to be *isomorphisms*. Note that by Proposition 1.2.1, (i), this already implies that $p_1 = p_2$; set $p \stackrel{\mathrm{def}}{=} p_1 = p_2$. That such a diagram *necessarily arises "geometrically"* follows from the main theorem of [Mzk6] (cf. Theorem 1.3.4) — *if one assumes that* α_K *arises geometrically* (i.e., from an isomorphism of fields $K_1 \xrightarrow{\sim} K_2$). In this §, we would like to investigate what one can say *in general* (i.e., without assuming that α_K arises geometrically) concerning this sort of commutative diagram. In some sense, all the key arguments that we use here are *already present in [Mzk4]*, except that there, these arguments were applied to prove *different theorems*. Thus, in the following discussion, we explain how the *same arguments* may be used to prove Theorem 2.7 below.

Let us denote the *type* of the hyperbolic curve $(X_i)_{K_i}$ by (g_i, r_i). Also, we shall denote the *geometric fundamental group* by

$$\Delta_{X_i} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{(X_i)_{K_i}} \to G_{K_i})$$

and the unique proper curve over K_i that *compactifies* $(X_i)_{K_i}$ by $(Y_i)_{K_i}$.

Lemma 2.1. **(Group-Theoreticity of Stability)** $(X_1)_{K_1}$ *has stable reduction if and only if* $(X_2)_{K_2}$ *does.*

Proof. This follows (essentially) from the well-known criterion of *Serre-Tate*: That is to say, $(X_i)_{K_i}$ has stable reduction if only if the actions of G_{K_i} on $\Delta^{\mathrm{ab}}_{Y_i} \otimes \widehat{\mathbb{Z}}'$ and on the (finite) *set of conjugacy classes of inertia groups of cusps in* Δ_{X_i} (i.e., the set of cusps of $(X_i)_{K_i} \otimes_{K_i} \overline{K}_i$ — cf. Lemma 1.3.9) is *unramified* (a condition which is group-theoretic, by Proposition 1.2.1, (ii)). ○

Now let us assume that $(X_i)_{K_i}$ has *stable reduction* over $\mathcal{O}_{K_i}$. Denote the *stable model* of $(X_i)_{K_i}$ over $\mathcal{O}_{K_i}$ by:

$$(\mathcal{X}_i)_{\mathcal{O}_{K_i}} \to \mathrm{Spec}(\mathcal{O}_{K_i})$$

Here, in the case where $r_i > 0$, we mean by the term *"stable model"* the complement of the marked points in the unique stable pointed curve over $\mathcal{O}_{K_i}$ that extends the pointed curve over K_i determined by $(X_i)_{K_i}$. Then, by the theory of [Mzk4], §2, 8, there exists a well-defined *quotient*

$$\Pi_{(X_i)_{K_i}} \twoheadrightarrow \Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$$

whose finite quotients correspond to (subcoverings of) admissible coverings of the result of base-changing $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ to rings of integers of tamely ramified extensions of K_i. In particular, we have a natural exact sequence:

$$1 \to \Delta^{\mathrm{adm}}_{X_i} \to \Pi^{\mathrm{adm}}_{(X_i)_{K_i}} \to G^{\log}_{k_i} \to 1$$

(where $\Delta^{\mathrm{adm}}_{X_i}$ is defined so as to make the sequence exact). Moreover, $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$ itself admits a natural quotient (cf. [Mzk4], §3)

$$\Pi_{(X_i)_{K_i}} \twoheadrightarrow \Pi^{\mathrm{adm}}_{(X_i)_{K_i}} \twoheadrightarrow \Pi^{\mathrm{et}}_{(X_i)_{K_i}}$$

whose finite quotients correspond to coverings of $(X_i)_{K_i}$ that extend to *finite étale coverings of* $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ *which are tamely ramified at the cusps.* In particular, we have a natural exact sequence:

$$1 \to \Delta^{\mathrm{et}}_{X_i} \to \Pi^{\mathrm{et}}_{(X_i)_{K_i}} \to G_{k_i} \to 1$$

(where $\Delta^{\mathrm{et}}_{X_i}$ is defined so as to make the sequence exact).

Lemma 2.2. **(Admissible and Étale Quotients)**

(i) The profinite groups $\Pi_{(X_i)_{K_i}}$, $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$, *and* $\Pi^{\mathrm{et}}_{(X_i)_{K_i}}$ *are all* **slim**.

(ii) The morphism α_X *is* **compatible** *with the quotients*

$$\Pi_{(X_i)_{K_i}} \twoheadrightarrow \Pi^{\mathrm{adm}}_{(X_i)_{K_i}} \twoheadrightarrow \Pi^{\mathrm{et}}_{(X_i)_{K_i}}$$

of $\Pi_{(X_i)_{K_i}}$.

Proof. The portion of assertion (i) concerning $\Pi_{(X_i)_{K_i}}$ (respectively, $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$, $\Pi^{\mathrm{et}}_{(X_i)_{K_i}}$) follows formally from Theorem 1.1.1, (ii); Lemma 1.3.1 (respectively, Proposition 1.2.3, (iii); the proof of Lemma 1.3.10).

Next, we turn to assertion (ii). For $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$, this follows (essentially) from Proposition 8.4 of [Mzk4]. Of course, in [Mzk4], $K_1 = K_2$ and α_K is the identity, but in fact, the only property of α_K necessary for the proof of [Mzk4], Proposition 8.4 — which is, in essence, a formal consequence of [Mzk4], Lemma 8.1 (concerning *unramified quotients* of the p-adic Tate module of a semi-abelian variety over a p-adic local field) — is that α_K *preserve the inertia and wild inertia groups* (which we know, by Proposition 1.2.1, (ii); Proposition 1.2.3, (i), of the present paper).

Similarly, the portion of assertion (ii) concerning $\Pi^{\mathrm{et}}_{(X_i)_{K_i}}$ follows (essentially) from [Mzk4], Proposition 3.2. That is to say, even though α_K is not necessarily the identity in the present discussion, the only properties of α_K that are necessary for the proof of

[Mzk4], Proposition 3.2, are Proposition 1.2.3, (i); Proposition 1.2.4, (ii) (of the present paper).

Finally, we remark that although in [Mzk4], we only treated the case where $r_i = 0$, one verifies easily that the arguments there extend immediately to the case of arbitrary $r_i \geq 0$: Indeed, tame ramification at the marked points may be *distinguished* from tame ramification at the nodes by considering the action of the Galois group of a covering on the semi-graphs of Lemma 2.3 below. ○

Lemma 2.3. **(Group-Theoreticity of Dual Semi-Graphs of the Special Fiber)** *The morphism α_X induces an isomorphism*

$$\alpha_{X,\Gamma^c} : \Gamma^c_{(\mathcal{X}_1)_{k_1}} \xrightarrow{\sim} \Gamma^c_{(\mathcal{X}_2)_{k_2}}$$

between the "dual semi-graphs with compact structure" (i.e., the usual dual graphs $\Gamma_{(\mathcal{X}_i)_{k_i}}$, together with extra edges corresponding to the cusps — cf. the Appendix) of the special fibers $(\mathcal{X}_i)_{k_i}$ of $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$. Moreover, α_{X,Γ^c} is functorial with respect to passage to finite étale coverings of the $(X_i)_{K_i}$.

Proof. Indeed, if one *forgets about the "compact structure,"* then this is a consequence of Lemma 1.3.9, Lemma 2.2 (in the *proper case*), and the theory of [Mzk4], §1 – 5, summarized in [Mzk4], Corollary 5.3. Even though α_K is not necessarily the identity in the present discussion, the only properties of α_K that are necessary for the proof of [Mzk4], Corollary 5.3 are Proposition 1.2.3, (i); Proposition 1.2.4, (ii) (of the present paper). That is to say, the point is that the *Frobenius element is preserved*, which means that the *weight filtrations* on l-adic cohomology (where l is a prime distinct from p) are, as well. The *compatibility with the "compact structure"* follows from the pro-l (where $l \neq p$) portion of Lemma 1.3.9, together with the easily verified fact (cf. the proof of Lemma 1.3.7) that the inertia group of a cusp in $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$ is contained (up to conjugacy) in the decomposition group of a *unique* irreducible component of $(\mathcal{X}_i)_{k_i}$. ○

Next, we would like to show that α_X is necessarily *"of degree 1."* This is essentially the argument of [Mzk4], Lemma 9.1, but we present this argument in detail below since we are working here under the assumption that α_K is *arbitrary.* For simplicity, *we assume until further notice is given* that $r_i = 0$ and that the special fiber $(\mathcal{X}_i)_{k_i}$ of $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ is *singular* and *sturdy* (cf. [Mzk4], Definition 1.1) — i.e., the normalizations of all the geometric irreducible components of $(\mathcal{X}_i)_{k_i}$ have genus ≥ 2 — and has a *noncontractible dual semi-graph* $\Gamma^c_{(\mathcal{X}_i)_{k_i}}$ — i.e., this semi-graph is *not a tree.* (These conditions may always be achieved by replacing $(X_i)_{K_i}$ by a finite étale covering of $(X_i)_{K_i}$ — cf. [Mzk4], Lemma 2.9; [Mzk4], the first two paragraphs of the proof of Theorem 9.2.)

We begin by introducing some notation. Write:

$$V_i \stackrel{\text{def}}{=} \Delta^{\text{ab}}_{X_i};$$
$$H_i \stackrel{\text{def}}{=} H_1^{\text{sing}}(\Gamma^{\text{c}}_{(\mathcal{X}_i)_{k_i}}, \mathbb{Z}) = H_1^{\text{sing}}(\Gamma_{(\mathcal{X}_i)_{k_i}}, \mathbb{Z})$$

(where "H_1^{sing}" denotes the first singular homology group). Thus, by considering the coverings of $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ induced by unramified coverings of the graph $\Gamma_{(\mathcal{X}_i)_{k_i}}$, we obtain natural (*group-theoretic!*) *"combinatorial quotients"*:

$$V_i \twoheadrightarrow (H_i)_{\widehat{\mathbb{Z}}} \stackrel{\text{def}}{=} H_i \otimes \widehat{\mathbb{Z}}$$

Lemma 2.4. (Ordinary New Parts, after Raynaud) *For a "sufficiently large prime number l" (where "sufficiently large" depends only on p, g_i), there exists a cyclic étale covering $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}} \to (\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ of $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ of degree l such that the "new part" $V_i^{\text{new}} \stackrel{\text{def}}{=} \Delta^{\text{ab}}_{(Z_i)_{K_i}}/\Delta^{\text{ab}}_{(X_i)_{K_i}}$ of the abelianized geometric fundamental group of $(Z_i)_{K_i}$ satisfies the following:*

(i) We have an exact sequence:

$$0 \to V_i^{\text{mlt}} \to (V_i^{\text{new}})_{\mathbb{Z}_p} \stackrel{\text{def}}{=} V_i^{\text{new}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p \to V_i^{\text{etl}} \to 0$$

— where V_i^{etl} is an **unramified** *G_{K_i}-module, and V_i^{mlt} is the* **Cartier dual** *of an unramified G_{K_i}-module.*

(ii) The "combinatorial quotient" of $\Delta^{\text{ab}}_{(Z_i)_{K_i}}$ (arising from the coverings of $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$ induced by unramified coverings of the dual semi-graph of the special fiber of $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$) induces a **nonzero** *quotient $V_i^{\text{new}} \twoheadrightarrow (H_i^{\text{new}})_{\widehat{\mathbb{Z}}}$ of V_i^{new}.*

Here, the injection $\Delta^{\text{ab}}_{(X_i)_{K_i}} \hookrightarrow \Delta^{\text{ab}}_{(Z_i)_{K_i}}$ is the injection induced by pull-back via $(Z_i)_{K_i} \to (X_i)_{K_i}$ and Poincaré duality (or, alternatively, by the "Verlagerung").

Proof. Note that since both conditions (i), (ii) are *group-theoretic*, they may be realized *simultaneously* for $i = 1, 2$. Now to satisfy *condition (i)*, it suffices — cf., e.g., the discussion in [Mzk4], §8, of "V_G," "$V_{G^{\text{ord}}}$" — to choose the covering so that the "new parts" of the Jacobians of the irreducible components of the special fiber of $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$ are all *ordinary*. That this is possible for l sufficiently large is a consequence of a *theorem of Raynaud* (as formulated, for instance, in [Tama], Lemma 1.9). Then let us observe that, so long as we choose the étale covering $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}} \to (\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ so that it is *nontrivial* over every irreducible component of $(\mathcal{X}_i)_{k_i}$, *condition (ii) is automatically satisfied*: Indeed, if we write $h_i \stackrel{\text{def}}{=} \text{rk}_{\mathbb{Z}}(H_i)$ — so $h_i > 0$ since $\Gamma^{c}_{(\mathcal{X}_i)_{k_i}}$ is assumed to be *noncontractible* — then to assert that condition (ii) fails to hold — i.e., that there are *"no new cycles in the dual graph"* — is to assert that we have an *equality of Euler characteristics*:

$$\Big(\sum_j g_{Z,j}\Big) + h_i - 1 = l\Big\{\Big(\sum_j g_{X,j}\Big) + h_i - 1\Big\}$$

(where the first (respectively, second) sum is the sum of the genera of the irreducible components of the geometric special fiber of $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$ (respectively, $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$)). But, since

$$\sum_j (g_{Z,j} - 1) = \sum_j l(g_{X,j} - 1)$$

we thus conclude that $(l-1) = \Big\{\sum_j (l-1)\Big\} + h_i(l-1)$, hence that $1 = (\sum_j 1) + h_i$ — which is *absurd*, since both the sum and h_i are ≥ 1. This completes the proof. ○

Remark 2.4.1. The author would like to thank A. Tamagawa for explaining to him the utility of Raynaud's theorem in this sort of situation.

In the following discussion, to keep the notation simple, *we shall replace* $(X_i)_{K_i}$ by some $(Z_i)_{K_i}$ as in Lemma 2.4. Thus, V_i^{new} is a G_{K_i}*-quotient module* of V_i. Moreover, we have a *surjection*

$$V_i^{\text{new}} \twoheadrightarrow (H_i^{\text{new}})_{\widehat{\mathbb{Z}}}$$

such that the quotient $(H_i)_{\widehat{\mathbb{Z}}} \twoheadrightarrow (H_i^{\text{new}})_{\widehat{\mathbb{Z}}}$ is *defined over* $\mathbb{Z}$, i.e., arises from a quotient $H_i \twoheadrightarrow H_i^{\text{new}}$. (Indeed, this last assertion follows from the fact that the quotient $H_i \twoheadrightarrow H_i^{\text{new}}$ arises as the cokernel (modulo torsion) of the morphism induced on first singular cohomology modules by a finite (ramified) covering of graphs — i.e., the covering induced on dual graphs by the covering $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}} \to (\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ of Lemma 2.4.)

On the other hand, the *cup product* on group cohomology gives rise to a *nondegenerate (group-theoretic!) pairing*

$$V_i^{\vee} \otimes_{\widehat{\mathbb{Z}}} V_i^{\vee} \otimes_{\widehat{\mathbb{Z}}} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i) \to M_i \stackrel{\text{def}}{=} H^2(\Delta_{X_i}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i)) \ (\cong \widehat{\mathbb{Z}})$$

(where we think of $V_i^{\vee} \stackrel{\text{def}}{=} \text{Hom}(V_i, \widehat{\mathbb{Z}})$ as $H^1(\Delta_{X_i}, \widehat{\mathbb{Z}})$), hence, by restriction to $(V_i^{\text{new}})^{\vee} \hookrightarrow V_i^{\vee}$, a *pairing*

$$(V_i^{\text{new}})^{\vee} \otimes_{\widehat{\mathbb{Z}}} (V_i^{\text{new}})^{\vee} \otimes_{\widehat{\mathbb{Z}}} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i) \to M_i \stackrel{\text{def}}{=} H^2(\Delta_{X_i}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i)) \ (\cong \widehat{\mathbb{Z}})$$

which is still *nondegenerate* (over $\mathbb{Q}$), since it arises from an *ample line bundle* — namely, the restriction of the polarization determined by the theta divisor on the Jacobian of $(X_i)_{K_i}$ to the "new part" of $(X_i)_{K_i}$. This pairing determines an *"isogeny"* (i.e., a morphism which is an isomorphism over $\mathbb{Q}$):

$$(V_i^{\mathrm{new}})^\vee \otimes_{\widehat{\mathbb{Z}}} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i) \otimes_{\widehat{\mathbb{Z}}} M_i^\vee \hookrightarrow V_i^{\mathrm{new}}$$

Thus, if we take the dual of the surjection discussed in the preceding paragraph, then we obtain an inclusion

$$(H_i^{\mathrm{new}})^\vee_{\widehat{\mathbb{Z}}} \otimes \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i) \otimes M_i^\vee \hookrightarrow (V_i^{\mathrm{new}})^\vee \otimes \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i) \otimes M_i^\vee \hookrightarrow V_i^{\mathrm{new}}$$

which (as one sees, for instance, by applying the fact that $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i)^{G_{K_i}} = 0$) maps into the kernel of the surjection $V_i^{\mathrm{new}} \twoheadrightarrow (H_i^{\mathrm{new}})_{\widehat{\mathbb{Z}}}$.

Next, let us observe that the *kernel* N_i of the *surjection of unramified* G_{K_i}*-modules* (i.e., G_{k_i}-modules)

$$V_i^{\mathrm{etl}} \twoheadrightarrow (H_i^{\mathrm{new}})_{\mathbb{Z}_p}$$

satisfies:

$$H^0(G_{k_i}, N_i \otimes \mathbb{Q}_p) = H^1(G_{k_i}, N_i \otimes \mathbb{Q}_p) = 0$$

(Indeed, N_i arises as a submodule of the module of p-power torsion points of an abelian variety over k_i, so the vanishing of these cohomology groups follows from the *Riemann Hypothesis for abelian varieties over finite fields* (cf., e.g., [Mumf], p. 206), i.e., the fact that (some power of) the Frobenius element of G_{k_i} acts on N_i with eigenvalues which are algebraic numbers with complex absolute values equal to a *nonzero* rational power of p.) In particular, we conclude that the above surjection admits a *unique* G_{K_i}*-equivariant splitting* $(H_i^{\mathrm{new}})_{\mathbb{Z}_p} \hookrightarrow (V_i^{\mathrm{etl}})_{\mathbb{Q}_p}$. Similarly, (by taking Cartier duals) the injection $(H_i^{\mathrm{new}})^\vee_{\mathbb{Z}_p} \otimes \boldsymbol{\mu}_{\mathbb{Z}_p}(\overline{K}_i) \otimes M_i^\vee \hookrightarrow V_i^{\mathrm{mlt}}$ also admits a unique G_{K_i}-equivariant splitting over $\mathbb{Q}_p$. Thus, *by applying these splittings*, we see that the G_{K_i}-action on $(V_i^{\mathrm{new}})_{\mathbb{Z}_p}$ determines a *p-adic extension class*

$$(\eta_i)_{\mathbb{Z}_p} \in \{(H_i^{\mathrm{new}})^\vee_{\mathbb{Q}_p}\}^{\otimes 2} \otimes M_i^\vee \otimes (H^1(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))/H^1_f(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))) = \{(H_i^{\mathrm{new}})^\vee_{\mathbb{Q}_p}\}^{\otimes 2} \otimes M_i^\vee$$

where (by Proposition 1.2.1, (vii)) $H^1(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))$ may be identified with $(K_i^\times)^\wedge$, and we define

$$H^1_f(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i)) \stackrel{\text{def}}{=} \mathcal{O}_{K_i} \subseteq (K_i^\times)^\wedge \xrightarrow{\sim} H^1(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))$$

so the quotient group $(H^1(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))/H^1_f(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i)))$ may be identified with $\widehat{\mathbb{Z}}$.

Next, let us observe that the kernel N_i' of $(V_i^{\text{new}})_{\widehat{\mathbb{Z}}'} \twoheadrightarrow (H_i^{\text{new}})_{\widehat{\mathbb{Z}}'}$ is an *unramified* representation of G_{K_i} (since it arises from the module of prime-to-p-power torsion points of a semi-abelian variety over k_i). Moreover, the injection of *unramified* G_{K_i}*-modules*

$$(H_i^{\text{new}})^\vee_{\widehat{\mathbb{Z}}'} \otimes \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_i) \otimes M_i^\vee \hookrightarrow N_i'$$

splits uniquely over $\mathbb{Q}$, since (by the Riemann Hypothesis for abelian varieties over finite fields — cf., e.g., [Mumf], p. 206) the Frobenius element of G_{k_i} acts on the smaller module (respectively, quotient by this smaller module) with *weight* 1 (respectively, $\frac{1}{2}$). Thus, just as in the p-adic case, we may construct a *prime-to-p-adic extension class* $(\eta_i)_{\widehat{\mathbb{Z}}'}$ from the G_{K_i}-action on $(V_i^{\text{new}})_{\widehat{\mathbb{Z}}'}$, which, together with $(\eta_i)_{\mathbb{Z}_p}$, yields an *extension class* (cf. [FC], Chapter III, Corollary 7.3):

$$\eta_i \in \{(H_i^{\text{new}})^\vee_{\widehat{\mathbb{Z}}}\}^{\otimes 2} \otimes M_i^\vee \otimes \{H^1(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))/H^1_f(K_i, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))\} \otimes \mathbb{Q} = \{(H_i^{\text{new}})^\vee_{\widehat{\mathbb{Z}}}\}^{\otimes 2} \otimes M_i^\vee \otimes \mathbb{Q}$$

That is to say, η_i may be thought of as a (*group-theoretically reconstructible!*) bilinear form:

$$\langle -, - \rangle_i : (H_i^{\text{new}})^{\otimes 2}_{\widehat{\mathbb{Z}}} \to (M_i^\vee)_{\mathbb{Q}} \stackrel{\text{def}}{=} M_i^\vee \otimes \mathbb{Q}$$

Moreover:

Lemma 2.5. *Assume that* $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ *arises as some "*$(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$*" as in Lemma 2.4. Then:*

(i) **(Positive Rational Structures)** *The image of* $(H_i^{\text{new}})^{\otimes 2}$ *under the morphism* $(H_i^{\text{new}})^{\otimes 2}_{\widehat{\mathbb{Z}}} \to (M_i^\vee)_{\mathbb{Q}}$ *forms a rank one* $\mathbb{Z}$*-submodule of* $(M_i^\vee)_{\mathbb{Q}}$*. Moreover, for any two nonzero elements* $a, b \in H_i$*,* $\langle a, a \rangle_i$ *differs from* $\langle b, b \rangle_i$ *by a factor in* $\mathbb{Q}_{>0}$ *(i.e., a positive rational number). In particular, this image determines a "*$\mathbb{Q}_{>0}$*-structure" on* $(M_i^\vee)_{\mathbb{Q}}$*, i.e., a* $\mathbb{Q}$*-rational structure on* $(M_i^\vee)_{\mathbb{Q}}$*, together with a collection of generators of this* $\mathbb{Q}$*-rational structure that differ from one another by factors in* $\mathbb{Q}_{>0}$*. Finally, this* $\mathbb{Q}_{>0}$*-structure is the same as the* $\mathbb{Q}_{>0}$*-structure on* $M_i^\vee$ *determined by the first Chern class of an ample line bundle on* $(X_i)_{K_i}$ *in* $M_i = H^2(\Delta_{X_i}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_i))$*.*

(ii) **(Preservation of Degree)** *The isomorphism*

$$M_1 = H^2(\Delta_{X_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_1)) \xrightarrow{\sim} H^2(\Delta_{X_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}_2)) = M_2$$

induced by α_X *preserves the elements on both sides determined by the first Chern class of a line bundle on* $(X_i)_{K_i}$ *of degree* 1.

Proof. Indeed, assertion (i), follows formally from [FC], Chapter III, Corollary 7.3, and Theorem 10.1, (iii) (by considering "new part" of the Jacobian of $(X_i)_{K_i}$ equipped with the polarization induced by the theta polarization on the Jacobian).

As for assertion (ii), the elements in question are the *unique* elements that, on the one hand, are *rational* and *positive* with respect to the structures discussed in assertion (i), and, on the other hand, *generate* M_i as a $\widehat{\mathbb{Z}}$-module. ○

Remark 2.5.1. Note that *the conclusion of Lemma 2.5, (ii), is valid not just for* $(X_i)_{K_i}$*, but for* **any** *finite étale cover of the original* $(X_i)_{K_i}$, i.e., even if this cover does not arise as some "$(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$" as in Lemma 2.4. Indeed, this follows from the fact that the *crucial "*$\mathbb{Q}_{>0}$*-structure"* of Lemma 2.5, (i), is preserved by pull-back to such a cover, which just multiplies the Chern class at issue in Lemma 2.5, (ii), by the degree of the cover (an element of $\mathbb{Q}_{>0}$!).

Remark 2.5.2. In the discussion of [Mzk4], §9, it was not necessary to be as careful as we were in the discussion above in constructing the p-adic class $(\eta_i)_{\mathbb{Z}_p}$ (i.e., "μ_p" in the notation of *loc. cit.*). This is because in *loc. cit.*, we were working over a *single* p*-adic base-field* "K." In this more restricted context, the extension class $(\eta_i)_{\mathbb{Z}_p}$ may be *extracted much more easily* from V_i by simply forming the quotient by the submodule of $H^1(K_i, \mathrm{Ker}((V_i)_{\mathbb{Z}_p} \twoheadrightarrow (H_i)_{\mathbb{Z}_p}))$ generated by the elements which are *"crystalline,"* or, more simply, of *"geometric origin"* (i.e., arise from $\mathcal{O}_K$-rational points of the formal group associated to the p-divisible group determined — via "Tate's theorem" (cf. Theorem 4 of [Tate]) — by the G_{K_i}-module $\mathrm{Ker}((V_i)_{\mathbb{Z}_p} \twoheadrightarrow (H_i)_{\mathbb{Z}_p})$). Unfortunately, the author omitted a detailed discussion of this aspect of the argument in the discussion of [Mzk4], §9.

Remark 2.5.3. Relative to Remark 2.5.2, we note nevertheless that even in the discussion of [Mzk4], §9, it is still necessary to work (at least until one recovers the "$\mathbb{Q}_{>0}$-structure" — cf. Remark 2.5.1) with $(X_i)_{K_i}$ such that the dual graph of the special fiber $(\mathcal{X}_i)_{k_i}$ is *noncontractible*. This minor technical point was omitted in the discussion of [Mzk4], §9.

Next, let us write $(\mathcal{X}_i^{\log})_{\mathcal{O}_{K_i}}$ for the log scheme obtained by equipping $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ with the log structure determined by the monoid of regular functions $\in \mathcal{O}_{(\mathcal{X}_i)_{\mathcal{O}_{K_i}}}$ which are invertible on the open subscheme $(X_i)_{K_i} \subseteq (\mathcal{X}_i)_{\mathcal{O}_{K_i}}$. Thus, in the terminology of [Kato2], $(\mathcal{X}_i^{\log})_{\mathcal{O}_{K_i}}$ is *log regular*. Also, let us write $(\mathcal{X}_i^{\log})_{k_i}$ for the log scheme obtained by equipping $(\mathcal{X}_i)_{k_i}$ with the log structure determined by restricting the log structure of $(\mathcal{X}_i^{\log})_{\mathcal{O}_{K_i}}$. Thus, the quotient $\Pi_{(X_i)_{K_i}} \twoheadrightarrow \Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$ determines a *"universal admissible covering"*

$$(\widetilde{\mathcal{X}}_i^{\log})_{k_i} \to (\mathcal{X}_i^{\log})_{k_i}$$

of $(\mathcal{X}_i^{\log})_{k_i}$.

Now let us choose a *connected component* $\widetilde{\mathcal{I}}_i$ of the k_i-smooth locus (i.e., the complement of the nodes) of $(\widetilde{\mathcal{X}}_i^{\log})_{k_i}$. Write $\mathcal{I}_i \subseteq (\mathcal{X}_i)_{k_i}$ for the image of $\widetilde{\mathcal{I}}_i$ in $(\mathcal{X}_i)_{k_i}$. Thus,

$$\widetilde{\mathcal{I}}_i \to \mathcal{I}_i$$

is a *"tame universal covering"* of $\mathcal{I}_i$ (i.e., a universal covering of the hyperbolic curve $\mathcal{I}_i$ among those finite étale coverings that are tamely ramified at the "cusps" of this hyperbolic curve). In the following discussion, we shall also assume, for simplicity, that $\mathcal{I}_i$ is *geometrically connected* over k_i (a condition that may always be achieved by replacing K_i by a finite unramified extension of K_i).

Now the Galois group $\Pi_{\mathcal{I}_i}$ of this covering may also be thought of as the *quotient of the decomposition group in* $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$ *of* $\widetilde{\mathcal{I}}_i$ *by its inertia group.* In particular, since $\Pi_{\mathcal{I}_i}$ is formed by taking the *quotient* by this inertia group, it follows that the surjection $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}} \twoheadrightarrow G^{\log}_{k_i}$ induces a natural surjection

$$\Pi_{\mathcal{I}_i} \twoheadrightarrow G_{k_i}$$

whose kernel is the *geometric (tame) fundamental group* $\pi_1^{\mathrm{tame}}((\mathcal{I}_i)_{\overline{k}_i})$ of $\mathcal{I}_i$.

Finally, we observe that *it makes sense to speak of* $\widetilde{\mathcal{I}}_1$ *and* $\widetilde{\mathcal{I}}_2$ *as corresponding via* α_X. Indeed, by Lemma 2.3, α_X induces an isomorphism between the *pro-graphs* determined by the $(\widetilde{\mathcal{X}}_i^{\log})_{k_i}$. Thus, the $\widetilde{\mathcal{I}}_i$ may be said to *correspond via* α_X when the vertices that they determine in these pro-graphs correspond. Moreover, when the $\widetilde{\mathcal{I}}_i$ correspond via α_X, it follows (by considering the stabilizer of the vertex determined by $\widetilde{\mathcal{I}}_i$) that α_X induces a bijection between the respective *decomposition groups* $\mathcal{D}_i$ in $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$ of $\widetilde{\mathcal{I}}_i$, as well as between the respective *inertia subgroups* of these decomposition groups $\mathcal{D}_i$ (which may be characterized group-theoretically as the *centers* of the subgroups $\mathcal{D}_i \bigcap \mathrm{Ker}(\Pi^{\mathrm{adm}}_{(X_i)_{K_i}} \twoheadrightarrow G_{k_i})$, since $\Delta^{\mathrm{adm}}_{X_i}$ is *center-free* — cf. the proofs of Lemmas 1.3.1, 1.3.10). Thus, in summary, α_X induces a *commutative diagram*:

$$\begin{array}{ccc} \Pi_{\mathcal{I}_1} & \xrightarrow{\sim} & \Pi_{\mathcal{I}_2} \\ \downarrow & & \downarrow \\ G_{k_1} & \xrightarrow{\sim} & G_{k_2} \end{array}$$

We are now ready (cf. [Mzk4], §7) to apply the *main result* of [Tama]. This result states that commutative diagrams as above are in natural bijective correspondence with commutative diagrams

$$\begin{array}{ccc} \widetilde{\mathcal{I}}_1 & \xrightarrow{\sim} & \widetilde{\mathcal{I}}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 & \xrightarrow{\sim} & \mathcal{I}_2 \end{array}$$

lying over commutative diagrams

$$\begin{array}{ccc} \overline{k}_1 & \xrightarrow{\sim} & \overline{k}_2 \\ \downarrow & & \downarrow \\ k_1 & \xrightarrow{\sim} & k_2 \end{array}$$

(cf. Theorem 1.3.11). In particular, these commutative diagrams induce an isomorphism

$$(\widehat{\mathbb{Z}}' \cong)\ H^2_{\mathrm{c}}((\mathcal{I}_1)_{\overline{k}_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_1)) \xrightarrow{\sim} H^2_{\mathrm{c}}((\mathcal{I}_2)_{\overline{k}_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_2))\ (\cong \widehat{\mathbb{Z}}')$$

(where "H^2_{c}" denotes étale cohomology with compact supports — cf. [Milne], Chapter III, Proposition 1.29; Remark 1.30) which maps the element "1" (i.e., the element determined by the first Chern class of a line bundle of degree 1) on the left to the element "1" on the right. (Indeed, this follows from the fact that the morphism $\mathcal{I}_1 \xrightarrow{\sim} \mathcal{I}_2$ appearing in the above commutative diagram is an isomorphism, hence of degree 1.) Note that the isomorphism $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_1) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_2)$ that we use here is that obtained from the commutative diagram above, i.e., that provided by Theorem 1.3.11.

Lemma 2.6. (Compatibility of Isomorphisms Between Roots of Unity) *Assume that $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ arises as some "$(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$" as in Lemma 2.4. Then the following diagram*

$$\begin{array}{ccc} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_1) & \xrightarrow{\sim} & \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_2) \\ \downarrow & & \downarrow \\ \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_1) & \xrightarrow{\sim} & \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_2) \end{array}$$

— *in which the vertical morphisms are the natural ones (obtained by considering Teichmüller representatives); the upper horizontal morphism is the morphism determined by Theorem 1.3.11; and the lower horizontal morphism is the morphism determined by Proposition 1.2.1, (vi)* — **commutes**.

Proof. Indeed, the diagram in the statement of Lemma 2.6 induces a diagram:

$$\begin{array}{ccc} H^2_{\rm c}((\mathcal{I}_1)_{\overline{k}_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_1)) & \stackrel{\sim}{\rightarrow} & H^2_{\rm c}((\mathcal{I}_2)_{\overline{k}_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_2)) \\ \downarrow & & \downarrow \\ H^2_{\rm c}((\mathcal{I}_1)_{\overline{k}_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_1)) & \stackrel{\sim}{\rightarrow} & H^2_{\rm c}((\mathcal{I}_2)_{\overline{k}_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_2)) \end{array} \qquad \ldots (*_1)$$

Moreover, we have a diagram

$$\begin{array}{ccc} H^2_{\rm c}((\mathcal{I}_1)_{\overline{k}_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_1)) & \stackrel{\sim}{\rightarrow} & H^2_{\rm c}((\mathcal{I}_2)_{\overline{k}_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_2)) \\ \downarrow & & \downarrow \\ H^2((X_1)_{\overline{K}_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_1)) & \stackrel{\sim}{\rightarrow} & H^2((X_2)_{\overline{K}_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_2)) \end{array} \qquad \ldots (*_2)$$

where the horizontal morphisms are induced by α_K (cf. Proposition 1.2.1, (vi)), and the vertical morphisms are induced "group-theoretically" as follows: First, observe that

$$H^2((X_i)_{\overline{K}_i}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_i)) \cong H^2(\Delta_{X_i}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_i)) \cong H^2(\Delta^{\rm adm}_{X_i}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_i))$$

while $H^2_{\rm c}((\mathcal{I}_i)_{\overline{k}_i}, -)$ may be thought of as the *"cohomology of the group* $\Delta_{\mathcal{I}_i} \stackrel{\rm def}{=} \mathrm{Ker}(\Pi_{\mathcal{I}_i} \twoheadrightarrow G_{k_i})$ *with trivializations over the inertia groups in* $\Delta_{\mathcal{I}_i}$ *of the cusps of* $\mathcal{I}_i$*"* (cf. the exact sequences of [Milne], Chapter III, Remark 1.30). On the other hand, let us recall that $\Delta^{\rm adm}_{X_i}$ may be constructed using a *semi-graph of (profinite) groups* (whose underlying semi-graph is the dual semi-graph of $(\mathcal{X}_i)_{k_i}$ with compact structure $\Gamma^{\rm c}_{(\mathcal{X}_i)_{k_i}}$ — cf. the Appendix) in which $\Delta_{\mathcal{I}_i}$ is the *group lying at the vertex* corresponding to $\mathcal{I}_i$, and the *groups lying at the edges meeting this vertex* are all inertia groups in $\Delta_{\mathcal{I}_i}$ of cusps of $\mathcal{I}_i$. Thus, the vertical morphisms of diagram $(*_2)$ may be thought of as being obtained by extending cohomology classes of $\Delta_{\mathcal{I}_i}$ with trivializations over the inertia groups of $\Delta_{\mathcal{I}_i}$ to cohomology classes "over the entire semi-graph of (profinite) groups" (and hence over $\Delta^{\rm adm}_{X_i}$) by *gluing* such classes to the trivial cohomology classes over all the other edges and vertices. In particular, we thus see that the vertical morphisms of diagram $(*_2)$ are *group-theoretic*, i.e., (in other words) *diagram* $(*_2)$ *commutes.*

Now let us compose the above two diagrams $(*_1)$, $(*_2)$ to form a single diagram:

$$\begin{array}{ccc} H^2_{\mathrm{c}}((\mathcal{I}_1)_{\overline{k}_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_1)) & \xrightarrow{\sim} & H^2_{\mathrm{c}}((\mathcal{I}_2)_{\overline{k}_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{k}_2)) \\ \downarrow & & \downarrow \\ H^2((X_1)_{\overline{K}_1}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_1)) & \xrightarrow{\sim} & H^2((X_2)_{\overline{K}_2}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}_2)) \end{array} \quad \ldots (*_3)$$

Note that *this diagram* $(*_3)$ *commutes*, since, by Lemma 2.5, (ii) (applied to the lower horizontal morphism of $(*_3)$), and the discussion immediately preceding the present Lemma 2.6) (applied to the upper horizontal morphism of $(*_3)$), all of the morphisms of this diagram are compatible with the elements "1" determined by the first Chern class of a line bundle of degree 1. But this implies that *diagram* $(*_1)$ *commutes* (since diagram $(*_2)$ has already been shown to be commutative, and all the arrows in both of these diagrams $(*_1)$, $(*_2)$ are isomorphisms between rank one free $\widehat{\mathbb{Z}}'$-modules). On the other hand, since diagram $(*_1)$ was obtained by applying the functors $H^2_{\mathrm{c}}((\mathcal{I}_i)_{\overline{k}_i}, -)$ (which are manifestly faithful) to the diagram appearing in the statement of Lemma 2.6, we thus conclude that the diagram appearing in the statement of Lemma 2.6 is commutative, as desired. ○

The significance of Lemma 2.6 from our point of view is the following: Lemma 2.6 implies that α_K *induces an isomorphism*

$$\alpha_{G^{\log}_k} : G^{\log}_{k_1} \xrightarrow{\sim} G^{\log}_{k_2}$$

which satisfies the hypothesis of Proposition 1.2.4, (iii). Thus, we conclude from Proposition 1.2.4, (iii), that $\alpha_{G^{\log}_k}$ *arises geometrically.* In particular, it follows that *we may apply [Mzk4], Theorem 7.2,* to the commutative diagram

$$\begin{array}{ccc} \Pi^{\mathrm{adm}}_{(X_1)_{K_1}} & \xrightarrow{\sim} & \Pi^{\mathrm{adm}}_{(X_2)_{K_2}} \\ \downarrow & & \downarrow \\ G^{\log}_{k_1} & \xrightarrow{\sim} & G^{\log}_{k_2} \end{array}$$

(where we note that Lemma 2.6 also implies — when translated into the terminology of [Mzk4], §7 — that the *"RT-degree"* associated to this commutative diagram is 1, as is necessary for the application of [Mzk4], Theorem 7.2). Here, we observe that although [Mzk4], Theorem 7.2, is only stated in the proper singular case, it *extends immediately to the affine* (and not necessarily singular) case. In particular, we conclude that the above commutative diagram of fundamental groups *arises geometrically* from a commutative diagram:

$$\begin{array}{ccc} (\mathcal{X}_1^{\log})_{k_1} & \xrightarrow{\sim} & (\mathcal{X}_2^{\log})_{k_2} \\ \downarrow & & \downarrow \\ k_1^{\log} & \xrightarrow{\sim} & k_2^{\log} \end{array}$$

Moreover, the isomorphism exhibited in the upper horizontal arrow of this commutative diagram is easily seen to be *functorial* with respect to *arbitrary finite étale coverings of the* $(X_i)_{K_i}$ (i.e., not just coverings that arise from finite étale coverings of the $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$). Indeed, this functoriality follows formally from the *uniqueness* assertion in Proposition 1.2.4, (iii), and the fact that *dominant* (i.e., not just finite étale) *morphisms between hyperbolic curves in characteristic* p *may be distinguished* by considering the morphisms that they induce between the respective Jacobians, hence, in particular, by the morphisms that they induce between the *l-power torsion points* (where $l \neq p$) of the respective Jacobians. Thus, in summary:

Theorem 2.7. (Group-Theoretic Reconstruction of the Logarithmic Special Fiber of a p-adic Hyperbolic Curve) *Let p be a prime number. For $i = 1, 2$, let K_i be a finite extension of $\mathbb{Q}_p$, and $(X_i)_{K_i}$ a* **hyperbolic curve** *over K_i whose associated pointed stable curve has stable reduction over $\mathcal{O}_{K_i}$. Denote the resulting "stable model" of $(X_i)_{K_i}$ over $\mathcal{O}_{K_i}$ by $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$. Assume that we have chosen basepoints of the $(X_i)_{K_i}$ (which thus induce basepoints of the K_i). Then every isomorphism of profinite groups $\Pi_{(X_1)_{K_1}} \xrightarrow{\sim} \Pi_{(X_2)_{K_2}}$ induces commutative diagrams:*

$$\begin{array}{ccc} \Pi_{(X_1)_{K_1}} & \xrightarrow{\sim} & \Pi_{(X_2)_{K_2}} \\ \downarrow & & \downarrow \\ G_{K_1} & \xrightarrow{\sim} & G_{K_2} \end{array} \qquad \begin{array}{ccc} \Pi^{\mathrm{adm}}_{(X_1)_{K_1}} & \xrightarrow{\sim} & \Pi^{\mathrm{adm}}_{(X_2)_{K_2}} \\ \downarrow & & \downarrow \\ G^{\log}_{k_1} & \xrightarrow{\sim} & G^{\log}_{k_2} \end{array}$$

Moreover, the latter commutative diagram (of admissible quotients $\Pi^{\mathrm{adm}}_{(X_i)_{K_i}}$ of the $\Pi_{(X_i)_{K_i}}$ lying over the tame Galois groups $G^{\log}_{k_i}$ of the K_i) necessarily arises from unique commutative diagrams of log schemes

$$\begin{array}{ccc} (\widetilde{\mathcal{X}}_1^{\log})_{k_1} & \xrightarrow{\sim} & (\widetilde{\mathcal{X}}_1^{\log})_{k_1} \\ \downarrow & & \downarrow \\ (\mathcal{X}_1^{\log})_{k_1} & \xrightarrow{\sim} & (\mathcal{X}_2^{\log})_{k_2} \end{array} \qquad \begin{array}{ccc} (k_1^{\log})^{\sim} & \xrightarrow{\sim} & (k_2^{\log})^{\sim} \\ \downarrow & & \downarrow \\ k_1^{\log} & \xrightarrow{\sim} & k_2^{\log} \end{array}$$

where the commutative diagram on the left lies over the commutative diagram on the right (which is as in Proposition 1.2.4, (iii)). Here, we equip $\mathrm{Spec}(\mathcal{O}_{K_i})$ *(respectively,* $\mathrm{Spec}(k_i)$*; $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$; $(\mathcal{X}_i)_{k_i}$) with the log structure determined by the closed point (respectively, determined by restricting the log structure on* $\mathrm{Spec}(\mathcal{O}_{K_i})$*; determined by the monoid of functions invertible on the open subscheme $(X_i)_{K_i}$; determined by restricting* *the log structure on* $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$*) and denote the resulting log scheme by* $\mathcal{O}_{K_i}^{\log}$ *(respectively,* $k_i^{\log}$*;* $(\mathcal{X}_i^{\log})_{\mathcal{O}_{K_i}}$*;* $(\mathcal{X}_i^{\log})_{k_i}$*); the vertical morphisms in the above commutative diagrams of log schemes are the universal coverings induced by the various basepoints chosen.*

Proof. First, note that the additional assumptions that were used in the course of the above discussion — e.g., that $(\mathcal{X}_i)_{\mathcal{O}_{K_i}}$ arise as some "$(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$" as in Lemma 2.4 — were applied *only to show that the hypotheses of Proposition 1.2.4, (iii) (and [Mzk4], Theorem 7.2) are satisfied*. Thus, (cf. Remark 2.5.1; the paragraph following the proof of Lemma 2.3) one concludes that — except when $(X_i)_{K_i}$ is *proper*, with *good reduction* — one may *reconstruct the logarithmic special fiber in a functorial fashion* (i.e., with respect to finite étale coverings of the $(X_i)_{K_i}$), as desired.

In the case that $(X_i)_{K_i}$ is *proper*, but has *good reduction* over $\mathcal{O}_{K_i}$, we may still reconstruct its logarithmic special fiber (despite the fact that [Tama], Theorem 4.3, is only stated in [Tama] for *affine* hyperbolic curves!) by arguing as follows: First of all, we observe that in the case of good reduction, the log structure of the special fiber of the curve is obtained by simply *pulling back* the log structure of $k_i^{\log}$. Thus, it suffices to construct the (non-logarithmic, scheme-theoretic) special fiber. Next, we observe that (after possibly enlarging K_i) there exist — cf., e.g., [Mzk4], the first two paragraphs of the proof of Theorem 9.2 — *corresponding finite Galois étale coverings* $(Z_i)_{K_i} \to (X_i)_{K_i}$ (for $i = 1, 2$), where $(Z_i)_{K_i}$ is a hyperbolic curve over K_i with *bad stable reduction* $(\mathcal{Z}_i)_{\mathcal{O}_{K_i}}$ over $\mathcal{O}_{K_i}$. Thus, by applying Theorem 2.7 to $(Z_i)_{K_i}$ allows us to reconstruct the logarithmic special fiber $(\mathcal{Z}_i^{\log})_{k_i}$, together with the *action of the Galois group* $G_i \stackrel{\mathrm{def}}{=} \mathrm{Gal}((Z_i)_{K_i}/(X_i)_{K_i})$. Note that irreducible components of $(\mathcal{Z}_i)_{k_i}$ that *dominate* $(\mathcal{X}_i)_{k_i}$ may be distinguished (group-theoretically!) by the fact that their geometric fundamental groups map *surjectively* onto open subgroups of the geometric fundamental group of $(\mathcal{X}_i)_{k_i}$. Let us choose corresponding (closed, proper) irreducible components

$$C_i \subseteq (\mathcal{Z}_i)_{k_i}$$

that *dominate* (hence surject onto) $(\mathcal{X}_i)_{k_i}$. Denote the *decomposition (respectively, inertia) group* associated to C_i by $D_i \subseteq G_i$ (respectively, $I_i \subseteq D_i \subseteq G_i$). Thus, D_i/I_i acts *faithfully* on C_i, and the *order* $|I_i|$ *of* I_i is a power of p, equal to the *degree of inseparability* of the function field of C_i over the function field of $(\mathcal{X}_i)_{k_i}$. Then we may reconstruct $(\mathcal{X}_i)_{k_i}$ as a *finite flat quotient* of C_i by considering the *subsheaf*

$$(\mathcal{O}_{C_i}^{|I_i|})^{D_i} \subseteq \mathcal{O}_{C_i}$$

(i.e., the D_i-invariants of the subalgebra $\mathcal{O}_{C_i}^{|I_i|} \subseteq \mathcal{O}_{C_i}$, where we use that $|I_i|$ is a *power of* p). By applying the functoriality with respect to finite étale coverings of $(X_i)_{K_i}$

observed in the discussion immediately preceding the statement of Theorem 2.7, we conclude that this construction of $(\mathcal{X}_i)_{k_i}$ is *independent* of the choice of $(Z_i)_{K_i}$, C_i, and itself *functorial* with respect to finite étale coverings of $(X_i)_{K_i}$.

This completes our *reconstruction of the logarithmic special fibers of the* $(X_i)_{K_i}$, in a fashion that is *functorial* with respect to finite étale coverings of the $(X_i)_{K_i}$. Thus, we conclude, in particular, (from this functoriality, applied to covering transformations; Lemma 2.2, (i)) that the morphism induced on admissible fundamental groups by the isomorphism constructed between logarithmic special fibers *coincides* with the original given morphism between admissible fundamental groups. This completes the proof of Theorem 2.7. ○

Remark 2.7.1. Given data as in Theorem 2.7, one may consider the *outer Galois representation*

$$G_{K_i} \to \mathrm{Out}(\Delta_{X_i})$$

which is known to be *injective* if $r > 0$ (cf. Theorem 1.3.6). Thus, at least in the case $r > 0$, it is natural to ask:

What is the commensurator of $\mathrm{Im}(G_{K_i})$ *in* $\mathrm{Out}(\Delta_{X_i})$*?*

Although Theorem 2.7 does not give a complete explicit answer to this question, it tells us that, at any rate, *elements of this commensurator* (which define isomorphisms of the sort that are treated in Theorem 2.7) *preserve the logarithmic special fiber.* In particular, (although one does not know whether or not elements of this commensurator induce "isogenies" of K_i, i.e., are "geometric") one obtains that elements of this commensurator *do* induce "isogenies" of $k_i^{\log}$. Moreover, since it follows from Theorem A of [Mzk6] (cf. Theorem 1.3.4, Remark 1.3.6.2) that the *centralizer* of $\mathrm{Im}(G_{K_i})$ in $\mathrm{Out}(\Delta_{X_i})$ consists precisely of those (finitely many) automorphisms that arise *geometrically* (i.e., from automorphisms of $(X_i)_{K_i}$), it follows that an "isogeny" of G_{K_i} induced by an element of this commensurator corresponds to (up to finitely many well-understood possibilities) an essentially *unique* element of this commensurator. This motivates the point of view that:

> *The "isogenies" of* G_{K_i} *defined by elements of this commensurator — which we shall refer to as* **quasi-conformal isogenies of** G_{K_i} *— are natural objects to study in their own right.*

The reason for the choice of the terminology "quasi-conformal" is that those isogenies that are "of geometric origin" — i.e., *"conformal"* — are (by the main theorem of [Mzk5]) precisely those which preserve the higher ramification filtration, which is closely related to the *"canonical p-adic metric"* on the local field in question. Thus, quasi-conformal isogenies do not preserve the "metric (or conformal) structure" but *do* preserve the "logarithmic special fiber" which one may think of as a sort of p-adic analogue of the "topological type" of the objects in question.

Remark 2.7.2. Note that isomorphisms

$$k_1^{\log} \xrightarrow{\sim} k_2^{\log}$$

(such as those arising from "quasi-conformal isomorphisms" $G_{K_1} \xrightarrow{\sim} G_{K_2}$ as in Theorem 2.7) need not be *"geometric"* from the point of view of characteristic zero (i.e., induced by an isomorphism of fields $K_1 \xrightarrow{\sim} K_2$). For instance, such an isomorphism might take the section of the log structure corresponding to p to some multiple of this section by a root of unity (a situation which could never occur if the isomorphism arose from an isomorphism $K_1 \xrightarrow{\sim} K_2$). Whether or not, however, this sort of phenomenon actually takes place in the case of "quasi-conformal isomorphisms" as in Theorem 2.7 is not clear to the author at the time of writing.

Remark 2.7.3. The theory of the present § prompts the question:

> *Do isomorphisms $\Pi_{(X_1)_{K_1}} \xrightarrow{\sim} \Pi_{(X_2)_{K_2}}$ as in Theorem 2.7* **only** *preserve the logarithmic special fiber or do they preserve* **other information** *as well concerning the liftings $(X_i)_{K_i}$ of the respective logarithmic special fibers?*

Although the author is unable to give a complete answer to this question at the time of writing, it does appear that when the lifting in question is in some sense *"canonical,"* then this canonicality is preserved by isomorphisms as in Theorem 2.7. In a future paper, we hope to discuss this sort of phenomenon — which may be observed, for instance, in the following cases:

(1) *Serre-Tate canonical liftings*;

(2) *"arithmetic hyperbolic curves,"* i.e., hyperbolic curves isogenous to a Shimura curve;

(3) *canonical liftings in the sense of "p-adic Teichmüller theory"* (cf. [Mzk1], [Mzk2])

— in more detail. Perhaps this phenomenon should be regarded as a *natural extension* of the phenomenon of preservation of the logarithmic fiber in the sense that *canonical* *liftings* are, in some sense, liftings that are *"defined over* $\mathbb{F}_1$*"* — i.e., a hypothetical (but, of course, fictional!) absolute field of constants sitting inside $\mathbb{Z}_p$.

Appendix: Terminology of Graph Theory

The Notion of a Semi-Graph:

We shall refer to as a *semi-graph* Γ the following collection of data:

(1) a set $\mathcal{V}$ — whose elements we refer to as *"vertices"*;

(2) a set $\mathcal{E}$ — whose elements we refer to as *"edges"* — of (not necessarily distinct) unordered pairs of (not necessarily distinct) elements of the set $\mathcal{V} \coprod \{\emptyset\}$ (i.e., the disjoint union of $\mathcal{V}$ and the symbol "$\emptyset$").

A *graph* Γ is a semi-graph for which all of the edges are unordered pairs of elements of $\mathcal{V}$ itself. We will say that a graph or semi-graph is *finite* if its sets of vertices and edges are finite.

Let $\Gamma = \{\mathcal{V}, \mathcal{E}\}$ be a *semi-graph*. If $e \in \mathcal{E}$ is an *edge* of Γ consisting of two (not necessarily distinct elements) v_1, v_2 of $\mathcal{V} \coprod \{\emptyset\}$, then we shall say that e *joins* v_1 to v_2, that e *meets* v_1, v_2, or that e *abuts* to v_1, v_2. If precisely one (respectively, two) of v_1, v_2 is equal to $\emptyset$, then we shall say that e *abuts to precisely one (respectively, no) vertex.* Thus, an edge of a *graph* always abuts to precisely two (not necessarily disinct) vertices, while an edge of a semi-graph may abut to precisely one vertex, or to no vertices at all.

By thinking of vertices as *points* and edges as *line segments* that join points to points or are "open" at one or both ends, we may think of semi-graphs as defining *topological spaces.* Thus, it makes sense to speak of a semi-graph as being *contractible* (in the sense of algebraic topology). Such a semi-graph will be referred to as a *tree.*

Finally, a *morphism* between semi-graphs is a pair of compatible maps between the respective sets of vertices and the respective sets of edges. Here, we allow an edge that abuts to no (respectively, precisely one) vertex to map to an edge that abuts to any number ≥ 0 (respectively, ≥ 1) of vertices.

Semi-Graphs of Profinite Groups:

We shall refer to the following data G:

(i) a finite *semi-graph* Γ;

(ii) for each vertex v of Γ, a *profinite group* G_v;

(iii) for each edge e of Γ, a *profinite group* G_e, together with, for each vertex v to which e abuts, a continuous homomorphism $e_v : G_e \to G_v$.

as a *semi-graph of profinite groups*. When Γ is a graph, we shall refer to this data G as a *graph of profinite groups*.

Suppose that we are given a *semi-graph of profinite groups* G. Then to G, one may associate (in a natural, functorial fashion) a *profinite group* — namely, the *profinite completion* of the well-known construction of the *fundamental group associated to a (semi-)graph of groups* (cf. [Serre1], I, §5.1).

Pointed Stable Curves:

Let k be an *algebraically closed field of characteristic* 0. Let $g, r \geq 0$ be integers such that $2g - 2 + r > 0$. Let $(\overline{X} \to \mathrm{Spec}(k), D \subseteq \overline{X})$ be an *r-pointed stable curve of genus g* (where $D \subseteq \overline{X}$ is the divisor of marked points) over k, and set:

$$X \stackrel{\text{def}}{=} \overline{X} \backslash D$$

Write Γ_X for the *dual graph* of X. Thus, the *vertices v* of Γ_X correspond to irreducible components I_v of X, while the *edges e* of Γ_X correspond to nodes ν_e of X. Moreover, the node ν_e has two *branches* e_v and e_w, which correspond to the vertices v, w joined by the edge e. Let us write $X_v \subseteq X$ for the open subscheme which is the complement of the nodes in the irreducible component I_v, and X_{e_v} for the scheme-theoretic intersection with X_v of the completion of the branch e_v at the node ν_e. Thus, X_{e_v} is noncanonically isomorphic to $\mathrm{Spec}(k[[t]][t^{-1}])$ (where t is an indeterminate).

In the following discussion, we would like to fix isomorphisms:

$$X_{e_v} \cong X_{e_w}$$

via which we shall identify X_{e_v} with X_{e_w} and denote the resulting object by X_e. In particular, we have natural morphisms $X_e \to X_v$, $X_e \to X_w$. One verifies immediately that the induced morphism on (algebraic) fundamental groups is *independent* of the choice of isomorphism. Thus, the dual graph Γ_X, together with the result of applying "$\pi_1(-)$" to the data $\{X_v; X_e; X_e \to X_v\}$ determines a *graph of profinite groups* $\mathcal{G}_X$ associated to the stable curve X.

When considering the case of a curve with marked points (i.e., $r > 0$), it is useful to consider the following slightly modified *"data with compact structure"*: Let us denote by Γ_X^c the *semi-graph* obtained from Γ_X by appending to Γ_X, for each marked point $x \in \overline{X}$, the following:

an *edge* e_x that *abuts to only one vertex*, namely the vertex v_x corresponding to the irreducible component of $\overline{X}$ that contains x.

We shall refer to the new edges "e_x" that were added to Γ_X to form Γ^{c}_X as the *marked edges* of Γ^{c}_X and to Γ^{c}_X itself as the *dual graph with compact structure* associated to X.

If, moreover, we associate to e_x the scheme X_x (which is noncanonically isomorphic to $\mathrm{Spec}(k[[t]][t^{-1}])$) obtained by removing x from the completion of $\overline{X}$ at x, and apply "$\pi_1(-)$" to the natural morphism $X_{e_x} \to X_{v_x}$, then we obtain a natural *semi-graph of profinite groups* $\mathcal{G}^{\mathrm{c}}_X$ with underlying semi-graph Γ^{c}_X. Moreover, one checks easily that the *profinite group associated* (as described above) to $\mathcal{G}_X$ or $\mathcal{G}^{\mathrm{c}}_X$ is isomorphic to "$\widehat{\Pi}_{g,r}$," i.e., the *profinite completion of the fundamental group of a Riemann surface of genus g with r points removed*.

In fact, if we take $k = \mathbb{C}$ and we think of the X_v as *Riemann surfaces* and of the X_e as "copies of the *circle* $\mathbb{S}^1$," then we see that this construction corresponds quite geometrically to *gluing Riemann surfaces with boundary along copies of the circle*.

Finally, we remark that thinking of coverings of X in terms of coverings of the X_v glued together along the nodal X_e amounts essentially to the notion of an *admissible cover* of a stable curve (cf. [Mzk3], §3).

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