The effect of new Stokes curves in the exact steepest descent method

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1 Introduction — Brief review of exact steepest descent method and several problems of it

The exact steepest descent method was born in [AKT4] by combining the ordinary steepest descent method with the exact WKB analysis. (See, e.g., [AKT2] for the notion and notations of the exact WKB analysis used in this report.) It is a straightforward generalization of the ordinary steepest descent method and provides us with a new powerful tool for the description of Stokes curves as well as for connection problems of ordinary differential equations. Still in [AKT4] some restrictions were imposed for its applicability. In this report, in order that we may remove such restrictions and apply it to more general equations in the future, we discuss the effects of several kinds of new Stokes curves in the exact steepest descent method.

Let us here review the exact steepest descent method briefly. An equation to be discussed is an ordinary differential equation with polynomial coefficients of the following form:

$$P\psi = \sum_{\substack{0\le j\le m\\ 0\le k\le n}} a_{jk} x^k \eta^{m-j} \frac{d^j\psi}{dx^j} = 0, \tag{1}$$

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where a_{jk} is a complex constant and $\eta > 0$ is a large parameter. By the Laplace transformation $\psi = \int \exp(\eta x\xi)\hat{\psi}(\xi)d\xi$ with respect to an independent variable x with a large parameter η (1) is transformed into

$$\hat{P}\hat{\psi} = \sum a_{jk}\eta^{m-k}\left(-\frac{d}{d\xi}\right)^k \xi^j\hat{\psi} = 0. \tag{2}$$

In the exact steepest descent method, following the idea of Berk et al. ([BNR]), we take a WKB solution $\hat{\psi}_k$ (more precisely, the Borel sum of $\hat{\psi}_k$) and consider its inverse Laplace transform

$$\int_{\sigma_k^{(j)}} \exp(\eta x\xi)\hat{\psi}_k d\xi = \int_{\sigma_k^{(j)}} \exp\left(\eta\left(x\xi - \int^{\xi} x_k(\xi)d\xi\right) + \cdots\right) d\xi \tag{3}$$

to discuss a solution of the original equation (1). Here $x_k(\xi)$ is a root (with respect to x) of the characteristic equation

$$p(x,\xi) \overset{\text{def}}{=} \sum_{\substack{0\le j\le m\\ 0\le k\le n}} a_{jk}x^k\xi^j = 0 \tag{4}$$

and $\sigma_k^{(j)}$ is a steepest descent path of $\mathrm{Re}f_k(x,\xi)$ passing through a saddle point of $f_k(x,\xi)$, where $f_k(x,\xi) \overset{\text{def}}{=} x\xi - \int^{\xi} x_k(\xi)d\xi$ denotes the phase function of (3). Note that, since the integrand of (3) is the Borel sum of a WKB solution $\hat{\psi}_k$, the so-called Stokes phenomenon occurs and $\hat{\psi}_k$ becomes a linear combination of $\hat{\psi}_k$ and $\hat{\psi}_{k'}$ (as was first observed by Voros [V]) when the steepest descent path $\sigma_k^{(j)}$ crosses a Stokes curve of type $(k > k')$ for $\hat{P}$. Hence, taking this Stokes phenomenon into account, we find that we should globally consider a linear combination of integrals of the following form:

$$\int_{\sigma_k^{(j)}} \exp(\eta x\xi)\hat{\psi}_k d\xi + c_{k'}\int_{\sigma_{k'}^{(j)}} \exp(\eta x\xi)\hat{\psi}_{k'} d\xi + c_{k''}\int_{\sigma_{k''}^{(j)}} \exp(\eta x\xi)\hat{\psi}_{k''} d\xi + \cdots, \tag{5}$$

where $\sigma_{k'}^{(j)}$ is a steepest descent path of $\mathrm{Re}f_{k'}$ emanating from a crossing point of $\sigma_k^{(j)}$ and a Stokes curve of type $(k > k')$ ($\sigma_{k''}^{(j)}$, $\sigma_{k'''}^{(j)}$, ...are also steepest descent paths obtained by similarly repeated bifurcation procedures), and $c_{k'}$ ($c_{k''}$ ...as well) is a constant determined by the connection formula which describes the Stokes phenomenon at the crossing point. The configuration of these steepest descent paths (the whole of which is called an "exact steepest descent path") is closely related to asymptotic behaviors (including exponentially small terms) of a WKB solution of (1). For example, a Stokes curve of (1) is characterized as a point where an exact steepest descent path passing through a saddle point hits another saddle point ("Exact Steepest Descent Path Ansatz" or "ESDP Ansatz" for short, cf. [AKT4]). The exact steepest descent method is, in a word, a method of investigating global asymptotic behaviors of solutions of (1) by tracing the configuration of exact steepest descent paths.

However, if we try to apply this method to general equations, we may encounter several difficulties mainly because the Laplace transformed equation (2) often has new Stokes curves and/or some singular points. The purpose of this report is to discuss how these difficulties can be overcome by studying a few examples with the aid of a computer. To be more concrete, we investigate the following situations: In [AKT4], to avoid these difficulties, we imposed the restriction that the Laplace transformed operator $\hat{P}$ is of at most second order (i.e., the degree of the coefficients of P is at most two). If we try to remove this restriction, the effect of a new Stokes curve for $\hat{P}$ becomes a problem. In Section 2 we first investigate the effect of a new Stokes curve for $\hat{P}$. Next in Section 3 we consider the case where $\hat{P}$ has a singular point. In particular, when $\hat{P}$ has a singular point of "simple pole type", there appears a Stokes curve emanating from such a singular point ([K1], [K2]). In Section 3 we investigate the effect of a Stokes curve emanating from a singular point of simple pole type. Furthermore, in Section 4 we deal with the case where the characteristic polynomial $p(x,\xi)$ defined by (4) is factorized as

$$p(x,\xi) = (\xi-\alpha)(\xi-p_0(x))(\xi-p_1(x)) \tag{6}$$

with α being a constant. In a generic situation a root $\xi_j(x)$ of (4) with respect to ξ gives a saddle point of the phase function $f_k(x,\xi)$ of (3) and the integral (5) along an exact steepest descent path passing through $\xi_j(x)$ corresponds to (the Borel sum of) a WKB solution ψ_j of (1) with the phase factor $\eta\int^x \xi_j(x)dx$. However, in the situation where $p(x,\xi)$ is factorized as (6) only $p_0(x)$ and $p_1(x)$ give saddle points of (3) and hence the number of saddle points is strictly smaller than that of WKB solutions of (1) (that is, a WKB solution with the phase factor $\eta\int^x \alpha dx = \eta\alpha x$ cannot be expressed in the form of (5)). In section 4, taking up an example which has its origin in the problem of non-adiabatic transition probabilities in quantum mechanics, we consider the case where (4) has a root (with respect to ξ) independent of x. Finally in Section 5 we give a summary and concluding remarks.

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2 The effect of a new Stokes curve

In this section we study the following equation:

$$\begin{aligned} P\psi = &-(12+14i)x\frac{d^2\psi}{dx^2} \\ &+\left((6+3i)x^2+2-11i-(24+28i+4c_1)\eta^{-1}\right)\eta\frac{d\psi}{dx} \\ &-\left(x^3-\frac{15}{4}(1+2i)x-(12+6i-2ic_0)\eta^{-1}x\right)\eta^2\psi = 0, \end{aligned} \tag{7}$$

where c_0 and c_1 are arbitrary complex constants. This equation is an example of Carroll-Hioe type equations discussed in [AKT3]. As a matter of fact, by setting

$$
\begin{gathered}
r_1 = 2 - i, \quad r_2 = -(\frac{1}{2} + 2i), \quad r_3 = 0, \\
\Omega_{12} = -3 + 4i, \quad \Omega_{23} = 1 - 3i
\end{gathered}
\tag{8}
$$

in Equation (CH) of [AKT3, p. 629], we obtain (7). As (7) is of second order, its Stokes geometry can be completed without introducing any new Stokes curve. The result is shown in Figure 1, where and in subsequent Figures 2, 5 and 8 a Stokes curve (with its type being specified by a symbol "$+ < -$" etc.) is designated by a solid line, while a wiggly line designates a cut which is placed to define a characteristic root of (7) as single-valued analytic function.

Figure 1: Stokes curves of (7).

Note that the origin $x = 0$ in Figure 1 is a regular singular point of "double pole type" discussed in [AKT1, Section 3]. In what follows we will observe that a new Stokes curve of the Laplace transform of (7) plays an important role when we try to detect a Stokes curve of (7) by using the exact steepest descent method.

The Laplace transform of (7) is

$$
\begin{aligned}
\hat{P}\hat{\psi} = \eta^{-1} \Bigg[\frac{d^3\hat{\psi}}{d\xi^3} &+ (6 + 3i)\xi\eta \frac{d^2\hat{\psi}}{d\xi^2} \\
&+ \left((12 + 14i)\xi^2 - \frac{15 + 30i}{4} + 2ic_0\eta^{-1} \right) \eta^2 \frac{d\hat{\psi}}{d\xi} \\
&+ \left((2 - 11i) - 4c_1\eta^{-1} \right) \xi\eta^3\hat{\psi} \Bigg] = 0
\end{aligned}
\tag{9}
$$

and the configuration of its ordinary and new Stokes curves is drawn in Figure 2 below:

Figure 2: Stokes curves of the Laplace transformed equation (9).

In Figure 2 (and in Figures 5 and 8 as well) a small dot designates an ordinary or virtual turning point and a broken line means that no Stokes phenomenon occurs on that portion of the curve. Note that $\hat{P}$ itself is a Carroll-Hioe type operator. Hence we readily find that it is transformed into a Laplace type operator by a change of independent variables $z = \xi^2$ and consequently it possesses an integral representation of solutions. Figure 2 can be confirmed to be the correct Stokes geometry of (9) by using the integral representation.

As is observed in Figure 2, (9) has several new Stokes curves. These new Stokes curves are necessary to detect a Stokes curve of (7). For example, we find that, in order to detect a Stokes curve γ in Figure 1, a new Stokes curve $\hat{\gamma}$ (of type $(0 < 2)$) passing through an ordered crossing point $\hat{A}$ in Figure 2 is necessary in the following way: Let us take two points x_0 and x_1 near γ as is shown in Figure 1 and describe the configuration of exact steepest descent paths at these two points x_0 and x_1. We then obtain Figure 3. (In Figure 3 and subsequent figures describing exact steepest descent paths as well a solid line designates a steepest descent path and a dotted line designates a Stokes curve of the Laplace transformed equation.) Figure 3 shows that between x_0 and x_1 a steepest descent path σ, which a steepest descent path passing through a saddle point ξ_+ bifurcates at its crossing point with $\hat{\gamma}$, hits another saddle point ξ_-. This clearly visualizes the necessity of the new Stokes curve $\hat{\gamma}$.

Furthermore, we next let x_0 and x_1 be closer to a turning point a_0 (cf. Figure 1). Then the steepest descent path passing through ξ_+ crosses an ordered crossing point $\hat{A}$ and, for example, at $\tilde{x}_0$ and $\tilde{x}_1$ the configuration of exact steepest descent paths becomes that described in Figure 4. In Figure 4 a bifurcated steepest descent path $\tilde{\sigma}$ obtained by repeated bifurcation from the steepest descent path passing through ξ_+ (that is, the steepest descent path passing through ξ_+ bifurcates another steepest descent path at its crossing point with a Stokes curve of type $(1 < 2)$, and further it bifurcates $\tilde{\sigma}$ at its crossing point with a Stokes curve of type $(0 < 1)$) hits a saddle point ξ_-.

Figure 3: Exact steepest descent paths at $x = x_0$ (a) and $x = x_1$ (b).

Figure 4: Exact steepest descent paths at $x = \tilde{x}_0$ (a) and $x = \tilde{x}_1$ (b).

In this manner a new Stokes curve is built in the exact steepest descent method very exquisitely to the effect that it explains the change of configuration occurring when a steepest descent path crosses an ordered crossing point very well.

3 The effect of a singular point of simple pole type

We next consider the following example in this section:

$$P\psi = x^2 \frac{d^3\psi}{dx^3} - x^2 \eta \frac{d^2\psi}{dx^2} - \theta\eta^2 \frac{d\psi}{dx} - i\theta\eta^3\psi = 0, \tag{10}$$

where $\theta = \exp(2i\pi/5)$. Its Laplace transform is given by

$$(11) \qquad \hat{P}\hat{\psi} = \eta\left[\xi^2(\xi-1)\frac{d^2\hat{\psi}}{d\xi^2} + (6\xi^2 - 4\xi)\frac{d\hat{\psi}}{d\xi} - \left(\theta\xi + i\theta + (-6\xi+2)\eta^{-2}\right)\eta^2\hat{\psi}\right] = 0.$$

The Laplace transformed equation (11) has a singularity of "double pole type" at $\xi = 0$ and of "simple pole type" at $\xi = 1$. In particular, there appears a Stokes curve emanating from the singular point $\xi = 1$ of "simple pole type". In what follows we investigate the effect of such a Stokes curve emanating from a simple pole type singularity.

The configuration of Stokes curves of (10) is shown in Figure 5.

Figure 5: Stokes curves of (10).

Figure 6: Magnification of Figure 5 near the Stokes curves γ.

We first take two points x_0 and x_1 near a Stokes curve γ (cf. Figure 6) and draw the configuration of exact steepest descent paths at these points. The resulting figures are Figures 7(a) and 7(b).

Figure 7: Exact steepest descent paths at $x = x_0$ (a) and $x = x_1$ (b).

A change of the configuration can be readily observed: A steepest descent path σ passing through a saddle point ξ_0 bifurcates another steepest descent path σ_b at a crossing point of σ and a Stokes curve $\hat{\gamma}$ emanating from the singular point $\xi = 1$ of simple pole type, and σ and σ_b simultaneously hit a saddle point ξ_1. This shows the relevance of a Stokes curve emanating from a simple pole type singularity in the exact steepest descent method.

Remark 3.1 In Figure 7(a) a steepest descent path σ_b bifurcated at a crossing point of σ and $\hat{\gamma}$ intersects again with $\hat{\gamma}$. We can verify that this second intersection point of σ_b and $\hat{\gamma}$ is passed also by the original steepest descent path σ. As a matter of fact, letting ξ_* denote the first intersection point of σ and $\hat{\gamma}$ (i.e., the bifurcation point of σ_b), we find that $\hat{\gamma}$, σ and σ_b can be described respectively by

$$\hat{\gamma} : \quad \mathrm{Im} \int_{\xi_*}^{\xi} (x_{k'} - x_k) d\xi = 0, \tag{12}$$

$$\sigma : \quad \mathrm{Im} \left(x(\xi - \xi_*) - \int_{\xi_*}^{\xi} x_k d\xi \right) = 0, \tag{13}$$

$$\sigma_b : \quad \mathrm{Im} \left(x(\xi - \xi_*) - \int_{\xi_*}^{\xi} x_{k'} d\xi \right) = 0 \tag{14}$$

with some indices k and k'. Then the second intersection point ξ_{**} of σ_b and $\hat{\gamma}$ should satisfy

$$\mathrm{Im} \int_{\xi_*}^{\xi_{**}} (x_{k'} - x_k) d\xi = 0 \tag{15}$$

and

$$\text{(16)} \qquad \operatorname{Im}\left(x(\xi_{**} - \xi_*) - \int_{\xi_*}^{\xi_{**}} x_{k'} d\xi \right) = 0.$$

Summing up these two equations, we obtain

$$\text{(17)} \qquad \operatorname{Im}\left(x(\xi_{**} - \xi_*) - \int_{\xi_*}^{\xi_{**}} x_{k} d\xi \right) = 0,$$

which implies that σ also passes through ξ_{**}.

If we further let x_0 and x_1 approach closer to a crossing point B of Stokes curves (cf. Figure 6), we obtain Figure 8.

Figure 8: Exact steepest descent paths at $x = \tilde{x}_0$ (a) and $x = \tilde{x}_1$ (b).

It appears that only σ hits a saddle point ξ_1 and a Stokes curve $\hat{\gamma}$ emanating from $\xi = 1$ is irrelevant in Figure 8. However, the degeneracy in Figure 8 is "multiple-ply"; other steepest descent paths are bifurcated at crossing points $\tilde{\xi}_*$ and $\tilde{\xi}_{**}$ of σ and $\hat{\gamma}$ and these bifurcated steepest descent paths simultaneously hit ξ_1 with overlying σ (cf. Remark 3.2 below). Thus a Stokes curve emanating from a simple pole type singularity is relevant also in Figure 8.

Remark 3.2 The Stokes curve $\hat{\gamma}$ emanating from a singular point $\xi = 1$ of simple pole type can be described by

$$\text{(18)} \qquad \operatorname{Im} \int_1^{\xi} (x_{k'} - x_k) d\xi = 0,$$

or equivalently by

$$\text{(19)} \qquad \operatorname{Im} \int_{\xi^\dagger}^{\xi} (x_{k'} - x_k) d\xi = 0,$$

where $\xi^\dagger$ denotes a point on $\mathcal{R}$, the Riemann surface of x_k and $x_{k'}$ ramified at $\xi = 1$, which has the same projection with ξ but is different itself from ξ. Since $\hat{\gamma}$ passes through $\tilde{\xi}_*$ and $\tilde{\xi}_{**}$, we have

$$\mathrm{Im} \int_{\tilde{\xi}_*^\dagger}^{\tilde{\xi}_*} (x_{k'} - x_k) d\xi = \mathrm{Im} \int_{\tilde{\xi}_{**}^\dagger}^{\tilde{\xi}_{**}} (x_{k'} - x_k) d\xi = 0. \tag{20}$$

Taking these relations into account, we can verify that the steepest descent paths bifurcated at $\tilde{\xi}_*$ and $\tilde{\xi}_{**}$ overlie σ by the same reasoning as in Remark 3.1.

4 The effect of a constant characteristic root

In this section we discuss an example of the form

$$\begin{aligned} P\psi = \frac{d^3\psi}{dx^3} - i\eta \frac{d^2\psi}{dx^2} + \left(x^2 + (i + |c_0|^2 + |c_1|^2)\eta^{-1}\right) \eta^2 \frac{d\psi}{dx} \\ - \left(ix^2 + (-1 + i|c_0|^2 x + i|c_1|^2)\eta^{-1}\right) \eta^3 \psi = 0 \end{aligned} \tag{21}$$

with c_0 and c_1 being arbitrary constants, which is equivalent to the following 3×3 non-adiabatic level crossing problem in quantum mechanics:

$$i\frac{d}{dx}\varphi = \eta \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -x \end{pmatrix} + \eta^{-1/2} \begin{pmatrix} 0 & c_0 & 0 \\ \overline{c_0} & 0 & c_1 \\ 0 & \overline{c_1} & 0 \end{pmatrix} \right] \varphi. \tag{22}$$

The configuration of Stokes curves of (21) is shown in Figure 9.

Figure 9: Stokes curves of (21).

There are three ordinary turning points (all of them are double) and two virtual turning points for (21). (The correctness of Figure 9 can be confirmed by the same reasoning as that employed in [AKT5, Remark 2.1].) Since the characteristic polynomial of (21) can be factorized as $(\xi - i)(\xi + ix)(\xi - ix)$, the difficulty explained in Section 1 appears for (21) due to the existence of a constant characteristic root $\xi_0 = i$. In what follows we investigate the effect of this constant characteristic root $\xi_0 = i$.

Figures 10(a), ..., 10(f) respectively describe the configuration of exact steepest descent paths at points $x_0, \ldots, x_5$ near a crossing point A of Stokes curves in Figure 9.

Figure 10: Exact steepest descent paths at $x = x_0$ (a), $x = x_1$ (b), ..., $x = x_5$ (f).

In Figure 10, besides exact steepest descent paths passing through saddle points ξ_1 and ξ_2, we have added steepest descent paths of

$$\text{Re} f_{\pm} = \text{Re}\left(x\xi - \int^{\xi} x_{\pm} d\xi\right) = \text{Re}\left(x\xi - \int^{\xi} (\pm i\xi) d\xi\right) \tag{23}$$

emanating from the constant characteristic root $\xi_0 = i$ and steepest descent paths bifurcated from them also ("exact steepest descent path emanating from $\xi_0 = i$"). Note that, since $\xi_0 = i$ is a "new" turning point for the Laplace transformed equation in the sense of [K3], a Stokes curve passing through $\xi_0 = i$ is also included in Figure 10. As is clear from Figure 10 (for example, from comparison between

Figures 10(c) and 10(d)), a Stokes phenomenon for Borel resummed WKB solutions of (21) occurs at a point where such an exact steepest descent path emanating from $\xi_0 = i$ hits a saddle point. This example strongly suggests that a constant characteristic root like $\xi_0 = i$ of (21) should be dealt with in the same manner as a saddle point.

Remark 4.1 From the above considerations it may appear that only one of the exact steepest descent paths of $\mathrm{Re} f_{\pm}$ emanating from $\xi_0 = i$ should be relevant. However, generically speaking, both exact steepest descent paths (n exact steepest descent paths in case the Laplace transformed equation is of n-th order) must be taken into account, as is shown by an example in [AKoT, Section 4] (cf. Figures 18 and 19 of [AKoT]). Both exact steepest descent paths being relevant might be related to the fact that $\xi_0 = i$ is a "new" turning point in the sense of [K3].

5 Concluding remarks

As the examples in Section 2 and 3 show, we should deal with a new Stokes curve and a Stokes curve emanating from a singular point of simple pole type as if they were an ordinary Stokes curve in defining exact steepest descent paths. These new Stokes curves and Stokes curves emanating from simple pole type singularities are built in the exact steepest descent method very exquisitely. Furthermore, when the characteristic equation has a root (with respect to ξ) being independent of x, such a constant characteristic root plays the same role with a saddle point.

In order to establish the exact steepest descent method for generic equations, we are required to take into account (and to prove rigorously) these effects. In particular, when there exists a constant characteristic root $\xi_0 = \alpha$, it is an interesting and important problem to find out an exact description of (the Borel sum of) a WKB solution of $P\psi = 0$ with the phase factor $\eta\alpha x$ in terms of the inverse Laplace integrals. (The fact that $\xi_0 = \alpha$ is a "new" turning point in the sense of [K3] might be a key to attack this problem.)

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