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Deconvolution of an L_2-convex Function †

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Abstract

L_2-convex functions, which are the convolution of two L-convex functions, constitute a wide class of discrete convex functions in discrete convex analysis, a unified framework of discrete optimization, proposed by Murota. This paper shows a technical result that any L_2-convex function can be represented by the convolution of two L-convex functions attaining the infimum in the definition of the convolution. This result gives simple proofs for several known results on L_2-convex functions.

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1 Introduction

In the area of discrete optimization, nonlinear optimization problems have been investigated as well as linear optimization problems. "Discrete convex analysis," proposed by Murota [6, 7], is being recognized as a unified framework of discrete optimization problems with reference to existing studies on submodular functions [3], valuated matroids [2, 8] and convex analysis [13]. Discrete convex analysis is not only a general framework but also a fruitful one with applications in the areas of mathematical economics and engineering [1, 4, 9, 10, 11]. The concepts of L-convex and M-convex functions play central roles in discrete convex analysis, and these are extended to wider and important classes of discrete convex functions, called L_2-convex and M_2-convex functions that are relevant to the matroid intersection problem. Given a pair of two matroids defined on a common ground set V, the matroid intersection problem is to find a common independent set of maximum size, and is a variant of M_2-concave function maximization. On the other hand, it is well-known that the maximum size of a common independent set is characterized by the minimum of $\rho_1(X) + \rho_2(V \setminus X)$ over all subsets X of V, where ρ_1 and ρ_2 are rank functions of the given matroids. The function $g(Y) = \min_{X \subseteq Y} (\rho_1(X) + \rho_2(Y \setminus X))$ is an L_2-convex function in disguise. See [7, 9] for details.

This paper focuses on L_2-convex functions, and gives several new results and simple proofs of known results on L_2-convex functions. We introduce the definitions L-convex and L_2-convex functions and our results below.

Let V be a nonempty finite set. We denote by $\mathbf{Z}$ and $\mathbf{R}$ the sets of all integers and reals respectively, and by $\mathbf{Z}^V$ the set of all integral vectors $p = (p(v) : v \in V)$ indexed by V. For any $p, q \in \mathbf{Z}^V$, the vectors $p \wedge q$ and $p \vee q$ in $\mathbf{Z}^V$ are such that

$$(p \wedge q)(v) = \min\{p(v), q(v)\}, \quad (p \vee q)(v) = \max\{p(v), q(v)\} \quad (v \in V).$$

Given a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, the *effective domain* of g is defined by

$$\mathrm{dom}\, g = \{p \in \mathbf{Z}^V \mid g(p) \neq \pm\infty\}.$$

A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and it satisfies the following two conditions:

(SBF) g is *submodular*, i.e.,

$$g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \quad (\forall p, q \in \mathbf{Z}^V),$$

(TRF) there exists $r \in \mathbf{R}$ such that

$$g(p + \mathbf{1}) = g(p) + r \quad (\forall p \in \mathbf{Z}^V),$$

where $\mathbf{1}$ denotes the vector of all ones.

For any functions $g_1, g_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the *infimal convolution* (or simply *convolution*) of g_1 and g_2, denoted by $g_1 \Box g_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, is defined by

$$(g_1 \Box g_2)(p) = \inf\{g_1(p_1) + g_2(p_2) \mid p_1 + p_2 = p,\ p_1, p_2 \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V). \tag{1}$$

It is easy to show that if $g_1 \Box g_2 > -\infty$ then the effective domain of $g_1 \Box g_2$ coincides with the Minkowski sum of the effective domains of g_1 and g_2, i.e.,

$$\mathrm{dom}\,(g_1 \Box g_2) = \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2 \equiv \{p_1 + p_2 \mid p_1 \in \mathrm{dom}\, g_1,\ p_2 \in \mathrm{dom}\, g_2\}.$$

A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L_2-convex* if $\mathrm{dom}\, g \neq \emptyset$ and $g = g_1 \Box g_2$ for some L-convex functions $g_1, g_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. For an L-convex function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $g(p + \mathbf{1}) = g(p) + r$ ($\forall p \in \mathbf{Z}^V$), let us consider an L-convex function $h : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by $h(p) = r\alpha$ if $p = \alpha\mathbf{1}$ for some $\alpha \in \mathbf{Z}$; otherwise $h(p) = +\infty$. Then, we have $g = g \Box h$, and hence, the class of L_2-convex functions contains that of L-convex functions. On the other hand, it is known that the convolution of two L-convex functions is not necessarily L-convex [6]. Thus, the class of L_2-convex functions is properly larger than that of L-convex functions.

Before stating our main result, we give an example.

Example 1 Let $g_1, g_2 : \mathbf{Z}^2 \to \mathbf{R}$ be the functions defined by

$$g_1(p) = \exp(p(2) - p(1)), \quad g_2(p) = \exp(p(1) - p(2)) \quad (p = (p(1), p(2)) \in \mathbf{Z}^2).$$

We can easily show that g_1 and g_2 are L-convex and that L_2-convex function $g = g_1 \Box g_2$ is identically zero. On the other hand, for any $p_1, p_2 \in \mathbf{Z}^2$, $g_1(p_1) + g_2(p_2) > 0$ holds. This says that there do not necessarily exist points attaining the infimum in the right hand side of (1). However, L-convex functions g_1' and g_2' with $g_1'(p) = g_2'(p) = 0$ for all $p \in \mathbf{Z}^2$ satisfy

$$g(p) = \min\{g_1'(p_1) + g_2'(p_2) \mid p_1 + p_2 = p,\ p_1, p_2 \in \mathbf{Z}^2\} \quad (p \in \mathbf{Z}^2),$$

and hence, there exists a "deconvolution" of g into g_1' and g_2' attaining the infimum in (1). ∎

Our main result says that such a deconvolution of an L_2-convex function is generally possible.

Theorem 2 *For any L_2-convex function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, there exist two L-convex functions g_1' and g_2' such that*

$$g(p) = \min\{g_1'(p_1) + g_2'(p_2) \mid p_1 + p_2 = p,\ p_1, p_2 \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V). \tag{2}$$

Theorem 2 affords simple proofs of several known or extended results on L_2-convex functions, e.g., an optimality criterion, a proximity property and a characterization of the set of all minimizers of an L_2-convex function as follows.

For a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we denote by $\arg\min g$ the set of all minimizers of g, i.e.,

$$\arg\min g = \{p \in \mathbf{Z}^V \mid g(p) \leq g(q) \ (\forall q \in \mathbf{Z}^V)\}.$$

A minimizer of an L_2-convex function can be characterized by a local optimal criterion below.

Theorem 3 ([12]) *For an L_2-convex function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $p^* \in \mathrm{dom}\, g$, we have*

$$p^* \in \arg\min g \quad \Longleftrightarrow \quad \begin{cases} g(p^*) \leq g(p^* + \chi_S) \quad (\forall S \subseteq V), \\ g(p^* + \mathbf{1}) = g(p^*), \end{cases}$$

where χ_S denotes the characteristic vector *of S defined by*

$$\chi_S(v) = \begin{cases} 1 & (v \in S) \\ 0 & (v \notin S) \end{cases} \qquad (v \in V).$$

The original proof of Theorem 3 in [12] utilizes an argument similar to the proof of Proposition 8 in Section 2. By applying Theorem 2, we can easily prove it (see Section 3).

The second application of Theorem 2 is a proximity theorem for L_2-convex functions, which guarantees that for a minimal solution p of a "scaled" function, there exists a minimizer p^* of the original function near p.

Theorem 4 *Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an L_2-convex function with $g(p + \mathbf{1}) = g(p)$ $(\forall p \in \mathbf{Z}^V)$ and let $\alpha \in \mathbf{Z}_{++}$. If $p \in \mathrm{dom}\, g$ satisfies*

$$g(p) \leq g(p + \alpha\chi_S) \quad (\forall S \subseteq V),$$

then $\arg\min g \neq \emptyset$ and there exists $p^* \in \arg\min g$ with

$$p \leq p^* \leq p + 2(n-1)(\alpha - 1)\mathbf{1}.$$

Here $n = |V|$ and $\mathbf{Z}_{++}$ denotes the set of all positive integers.

A proximity theorem for L_2-convex functions was first stated in [12], but it is weaker than Theorem 4 in the sense that it assumes an essential boundedness of g, where an L_2-convex function g is said to be *essentially bounded* if $\mathrm{dom}\, g \cap \{p \in \mathbf{Z}^V \mid p(v) = 0\}$ is bounded for some $v \in V$. The proximity theorem free from this restrictive assumption can be proved easily by Theorem 2 (see Section 3).

The third application of Theorem 2 is a characterization of the set of all minimizers of an L$_2$-convex function. It is shown in [7] that for two L-convex functions $g_1, g_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, $\arg\min(g_1 \Box g_2)$ is either an empty set or equal to $\arg\min g_1 + \arg\min g_2$, and furthermore, if $\arg\min(g_1 \Box g_2) \neq \emptyset$ then it is an L$_2$-convex set. Here, a set $P \subseteq \mathbf{Z}^V$ is called an *L-convex set* if it is nonempty and satisfies

$$p,\ q \in P \ \Rightarrow\ (p \wedge q),\ (p \vee q),\ p \pm \mathbf{1} \in P,$$

and the Minkowski sum of two L-convex sets is called an *L_2-convex set.* The above former result relies on the integrality of the range of functions g_1 and g_2. For example, the statement does not hold for two L-convex functions g_1 and g_2 in Example 1 because $\arg\min g = \mathbf{Z}^2$ and $\arg\min g_1 = \arg\min g_2 = \emptyset$. However, Theorem 2 extends the above latter result to the following theorem.

Theorem 5 *For an L_2-convex function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, if* $\arg\min g \neq \emptyset$ *then* $\arg\min g$ *is an L_2-convex set.*

The organization of this paper is: Section 2 shows Theorem 2, and Section 3 proves Theorems 3, 4 and 5, and Section 3 also gives elementary proofs of two results on the convolution of L-convex functions.

2 Proof of Our Main Result

In this section, we give a proof of Theorem 2. Before giving it, we introduce four technical results.

Proposition 6 *Assume that $g = g_1 \Box g_2$ for some functions g_1 and g_2 such that $g_1(q + \mathbf{1}) = g_1(q) + r_1$ and $g_2(q + \mathbf{1}) = g_2(q) + r_2$ for all $q \in \mathbf{Z}^V$. If $g(p) \in \mathbf{R}$ for some $p \in \mathbf{Z}^V$, then $r_1 = r_2$ holds, and furthermore, $g(q + \mathbf{1}) = g(q) + r_1$ for any $q \in \mathbf{Z}^V$.*

Proof. It follows from $g(p) \in \mathbf{R}$ that there exist $p_1 \in \mathrm{dom}\, g_1$ and $p_2 \in \mathrm{dom}\, g_2$ with $p_1 + p_2 = p$. Then, we have

$$\begin{aligned} g(p) &\leq \inf\{g_1(p_1 + \alpha_1 \mathbf{1}) + g_2(p_2 + \alpha_2 \mathbf{1}) \mid \alpha_1 + \alpha_2 = 0,\ \alpha_1, \alpha_2 \in \mathbf{Z}\} \\ &= \inf\{g_1(p_1) + r_1\alpha_1 + g_2(p_2) + r_2\alpha_2 \mid \alpha_1 + \alpha_2 = 0,\ \alpha_1, \alpha_2 \in \mathbf{Z}\} \\ &= g_1(p_1) + g_2(p_2) + \inf\{(r_1 - r_2)\alpha_1 \mid \alpha_1 \in \mathbf{Z}\}. \end{aligned}$$

This says that if $g(p) \in \mathbf{R}$ then $r_1 = r_2$ must hold. Analogously, for any $q \in \mathbf{Z}^V$, we have

$$g(q + \mathbf{1}) \leq \inf\{g_1(q_1 + \alpha_1 \mathbf{1}) + g_2(q_2 + \alpha_2 \mathbf{1}) \mid q_1 + q_2 = q,\ \alpha_1 + \alpha_2 = 1\}$$

$$
\begin{aligned}
&= g(q) + r_1, \\
g(q) &\leq \inf\{g_1(q_1 + \alpha_1 \mathbf{1}) + g_2(q_2 + \alpha_2 \mathbf{1}) \mid q_1 + q_2 = q + \mathbf{1},\ \alpha_1 + \alpha_2 = -1\} \\
&= g(q + \mathbf{1}) - r_1.
\end{aligned}
$$

Hence, $g(q + \mathbf{1}) = g(q) + r_1$ holds for any $q \in \mathbf{Z}^V$. ∎

In the sequel, we assume that $g : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined as the convolution of two L-convex functions g_1 and g_2, and assume that $g(p) \in \mathbf{R}$ for some p, i.e.,

$$g_1(p + \mathbf{1}) = g_1(p) + r, \quad g_2(p + \mathbf{1}) = g_2(p) + r \quad (\forall p \in \mathbf{Z}^V) \tag{3}$$

by Proposition 6. We define the *positive support* of a vector $p \in \mathbf{Z}^V$ by

$$\mathrm{supp}^+(p) = \{v \in V \mid p(v) > 0\}.$$

For any $p, q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ with $p \leq q$, we say that q is *adjacent* to p if $q \neq p$ and there exists no nonempty subset S of $\mathrm{supp}^+(q - p)$ such that

$$q - \chi_S \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2, \quad q - \chi_S \neq p.$$

Proposition 7 *For any $p, q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ with $p \leq q$, there exists a sequence $(q = q_0, q_1, \ldots, q_m = p)$ of points in $\mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ such that q_{i-1} is adjacent to q_i for $i = 1, \ldots, m$.*

Proof. Since there is nothing to prove if q is adjacent to p, we assume that q is not adjacent to p. Thus, there exists a nonempty subset S of $\mathrm{supp}^+(q - p)$ such that $q_1 = q - \chi_S \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ and $q_1 \neq p$. Moreover, if S is minimal among such nonempty subsets of $\mathrm{supp}^+(q - p)$, then q must be adjacent to q_1. By repeating the above process, a required sequence is obtained because $q_1 \geq p$ and $||q_1 - p||_1 < ||q - p||_1$. ∎

In order to show a statement

> "if a property P holds for some $p \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ then P also holds for all points $q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ with $q \geq p$,"

it is enough, by Proposition 7, to show that P holds for q adjacent to p. A necessary condition for adjacency is given as follows.

Since $\mathrm{dom}\, g_1$ and $\mathrm{dom}\, g_2$ are an L-convex set, for two points $p, q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ with $p \leq q$, there exist decompositions of p and q such that

$$
\begin{aligned}
&p_1 + p_2 = p, \quad q_1 + q_2 = q, \quad p_1 \leq q_1, \quad p_2 \geq q_2, \\
&p_1, q_1 \in \mathrm{dom}\, g_1, \quad p_2, q_2 \in \mathrm{dom}\, g_2.
\end{aligned}
\tag{4}
$$

Proposition 8 *For any points $p, q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ with $p \leq q$, if q is adjacent to p then $||q - p||_\infty = 1$ and $q_1(u) - p_1(u) = q_1(v) - p_1(v)$ for any p_1, p_2, q_1 and q_2 satisfying* (4) *and for any $u, v \in \mathrm{supp}^+(q - p)$.*

Proof. Let $\alpha = \max\{q_1(v) - p_1(v) \mid v \in \mathrm{supp}^+(q - p)\}$ and $S = \{v \in \mathrm{supp}^+(q - p) \mid q_1(v) - p_1(v) = \alpha\}$. Since $q \neq p$ holds, we have $\alpha \geq 1$ and $S \neq \emptyset$. We will show the following claim.

Claim: If $\alpha \geq 2$ then $p + \chi_S, q - \chi_S \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$.

By using the claim, we can complete the proof of Proposition 8. If $||q - p||_\infty \geq 2$ or S is a proper subset of $\mathrm{supp}^+(q - p)$, then $\alpha \geq 2$ and $q - \chi_S \neq p$ must hold. By the claim, however, this contradicts that q is adjacent to p.

We now prove the above claim. Assume that $\alpha \geq 2$ and let $\beta = \alpha - 1$. We consider points defined by

$$\begin{array}{lll} p_1' = (p_1 + \beta \mathbf{1}) \vee q_1, & p_2' = (p_2 - \beta \mathbf{1}) \wedge q_2, & p' = p_1' + p_2', \\ q_1' = (p_1 + \beta \mathbf{1}) \wedge q_1, & q_2' = (p_2 - \beta \mathbf{1}) \vee q_2, & q' = q_1' + q_2'. \end{array}$$

Obviously, $p', q' \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ and $p' + q' = p + q$ hold. We will show that

$$p' = p + \chi_S$$

which also implies $q' = q - \chi_S$. By the definitions of p_1' and p_2', we have

$$\begin{array}{lcl} p_1(v) + \beta \geq q_1(v) & \Rightarrow & p_1'(v) = p_1(v) + \beta, \\ p_1(v) + \beta < q_1(v) & \Rightarrow & p_1'(v) = q_1(v), \\ p_2(v) - \beta \leq q_2(v) & \Rightarrow & p_2'(v) = p_2(v) - \beta, \\ p_2(v) - \beta > q_2(v) & \Rightarrow & p_2'(v) = q_2(v) \end{array} \tag{5}$$

for each $v \in V$. Let v be any element of V. We divide into three cases. (i) If $p(v) = q(v)$ holds, then

$$\begin{array}{lcl} p_1(v) + \beta \geq q_1(v) & \Rightarrow & p_2(v) - \beta \leq q_2(v), \\ p_1(v) + \beta < q_1(v) & \Rightarrow & p_2(v) - \beta > q_2(v) \end{array}$$

hold, and therefore, $p'(v) = p(v)$ is satisfied by (4) and (5). (ii) If $p(v) < q(v)$ and $p_1(v) + \beta \geq q_1(v)$, then $v \notin S$ and $p_2(v) - \beta \leq q_2(v)$ must hold, and hence, $p'(v) = p(v)$. (iii) Assume that $p(v) < q(v)$ and $p_1(v) + \beta < q_1(v)$. By the definition of β, the latter implies $p_1(v) + \beta = q_1(v) - 1$, i.e., $v \in S$. Moreover, we have $p_2(v) - \beta \leq q_2(v)$; since otherwise, we would obtain $p_1(v) + p_2(v) > q_1(v) + q_2(v) - 1$ which contradicts the assumption $p(v) < q(v)$. Thus, $p'(v) = q_1(v) + p_2(v) - \beta = p_1(v) + p_2(v) + 1 = p(v) + 1$ holds. From the above discussion, $p' = p + \chi_S$. ∎

We emphasize that in Proposition 9 below we do not exclude the possibility that g is equal to $-\infty$ at some point of $\mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$.

Proposition 9 *For two points $p \in \mathrm{dom}\, g$ and $q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ with $p \leq q$, we assume that q is adjacent to p. If points p_1, p_2 and a positive number γ satisfy*

$$g(p) + \gamma \geq g_1(p_1) + g_2(p_2) \geq g(p), \quad p_1 + p_2 = p, \quad p_1 \in \mathrm{dom}\, g_1, \quad p_2 \in \mathrm{dom}\, g_2,$$

then, for any q_1 and q_2 with

$$q_1 + q_2 = q, \quad q_1 \in \mathrm{dom}\, g_1, \quad q_2 \in \mathrm{dom}\, g_2,$$

there exist q_1'' and q_2'' such that

$$\begin{array}{l} g_1(q_1) + g_2(q_2) + 2\gamma \geq g_1(q_1'') + g_2(q_2''), \\ q_1'' + q_2'' = q, \quad q_1'' \in \mathrm{dom}\, g_1, \quad q_2'' \in \mathrm{dom}\, g_2, \\ \| p_1 - q_1'' \|_\infty \leq 1, \quad \| p_2 - q_2'' \|_\infty \leq 1. \end{array} \tag{6}$$

Proof. By (3), we can assume that $p_1 \leq q_1$ and $p_2 \geq q_2$ without loss of generality. Let $\alpha = \max\{q_1(v) - p_1(v) \mid v \in \mathrm{supp}^+(q-p)\}$ and $S = \{v \in \mathrm{supp}^+(q-p) \mid q_1(v) - p_1(v) = \alpha\}$. It follows from Proposition 8 that $\| q - p \|_\infty = 1$ and $S = \mathrm{supp}^+(q-p)$.

If $\alpha = 1$ then we put $q_1' = q_1$ and $q_2' = q_2$; otherwise, we construct q_1' and q_2' as below. Since $S = \mathrm{supp}^+(q-p)$ and $\alpha \geq 2$, we have $q_1(v) - p_1(v) \geq 2$ for any $v \in S$ and $\beta = \alpha - 1 \geq 1$. We consider points defined by

$$\begin{array}{ll} p_1' = p_1 \wedge (q_1 - \beta\mathbf{1}), & p_2' = p_2 \vee (q_2 + \beta\mathbf{1}), \\ q_1' = p_1 \vee (q_1 - \beta\mathbf{1}), & q_2' = p_2 \wedge (q_2 + \beta\mathbf{1}). \end{array} \tag{7}$$

Obviously, $p_1', q_1' \in \mathrm{dom}\, g_1$ and $p_2', q_2' \in \mathrm{dom}\, g_2$. We will show that

$$q_1' + q_2' = q, \quad p_1 \leq q_1', \quad p_2 \geq q_2', \quad q_1'(v) = p_1(v) + 1 \ (\forall v \in S). \tag{8}$$

Trivially, $p_1 \leq q_1'$ and $p_2 \geq q_2'$ are satisfied. We divide into two cases. If $v \notin S$ then $p_1(v)+p_2(v) = (q_1(v)-\beta)+(q_2(v)+\beta)$ holds, and hence, $q_1'(v)+q_2'(v) = p(v) = q(v)$. If $v \in S$ then $q_1'(v) = p_1(v)+1$ and $q_2'(v) = p_2(v)$ hold because $q_1(v) = p_1(v)+\beta+1$ and $q_2(v) = p_2(v) - \beta$. Thus, (8) is satisfied. Furthermore, we have $p_1' + p_2' = p$ because $p_1' + p_2' + q_1' + q_2' = p + q$ by (7) and because $q_1' + q_2' = q$ by (8). On the other hand, the L-convexity of g_1, g_2 and (3) say that

$$\begin{array}{rcl} g_1(p_1) + g_1(q_1) + g_2(p_2) + g_2(q_2) & = & g_1(p_1) + g_1(q_1 - \beta\mathbf{1}) + g_2(p_2) + g_2(q_2 + \beta\mathbf{1}) \\ & \geq & g_1(p_1') + g_1(q_1') + g_2(p_2') + g_2(q_2'). \end{array} \tag{9}$$

By (9) and the hypothesis, we obtain

$$\begin{array}{rcl} g(p) + g_1(q_1) + g_2(q_2) + \gamma & \geq & g_1(p_1') + g_2(p_2') + g_1(q_1') + g_2(q_2') \\ & \geq & g(p) + g_1(q_1') + g_2(q_2'), \end{array}$$

where the second inequality follows from $p_1' + p_2' = p$. Therefore,

$$g_1(q_1) + g_2(q_2) + \gamma \geq g_1(q_1') + g_2(q_2') \tag{10}$$

must hold. From the above discussion, q_1' and q_2' satisfy (8) and (10), whether $\alpha = 1$ or $\alpha \geq 2$.

If $q_1'(v) - p_1(v) \leq 1$ for any $v \in V \setminus S$, then $q_1'' = q_1'$ and $q_2'' = q_2'$ satisfy (6). In the sequel, we assume that $\max\{q_1'(v) - p_1(v) \mid v \in V \setminus S\} \geq 2$. We now consider points p_1'', p_2'', q_1'' and q_2'' defined by

$$\begin{aligned} p_1'' &= (p_1 + \mathbf{1}) \vee q_1', \quad & p_2'' &= (p_2 - \mathbf{1}) \wedge q_2', \\ q_1'' &= (p_1 + \mathbf{1}) \wedge q_1', \quad & q_2'' &= (p_2 - \mathbf{1}) \vee q_2'. \end{aligned}$$

Obviously, $p_1'', q_1'' \in \mathrm{dom}\, g_1$ and $p_2'', q_2'' \in \mathrm{dom}\, g_2$. We will show that

$$p_1'' + p_2'' = p, \quad q_1'' + q_2'' = q. \tag{11}$$

If $v \notin S$ then $(p_1(v)+1)+(p_2(v)-1) = q_1'(v)+q_2'(v)$ holds, and hence, $p_1''(v)+p_2''(v) = q_1''(v) + q_2''(v) = p(v) = q(v)$. If $v \in S$ then $p_1''(v) = q_1''(v) = q_1'(v)$, $p_2''(v) = p_2(v) - 1$ and $q_2''(v) = q_2'(v)$ hold by (8), and hence, (11) holds. Furthermore, the L-convexity of g_1, g_2 and (3) yield

$$g_1(p_1) + g_1(q_1') + g_2(p_2) + g_2(q_2') \geq g_1(p_1'') + g_1(q_1'') + g_2(p_2'') + g_2(q_2''). \tag{12}$$

By (10) and (12), we have

$$\begin{aligned} g(p) + g_1(q_1) + g_2(q_2) + 2\gamma &\geq g_1(p_1'') + g_2(p_2'') + g_1(q_1'') + g_2(q_2'') \\ &\geq g(p) + g_1(q_1'') + g_2(q_2''). \end{aligned}$$

Thus, we have

$$g_1(q_1) + g_2(q_2) + 2\gamma \geq g_1(q_1'') + g_2(q_2'').$$

Moreover, by the definitions of q_1'' and q_2'' and (8), we have $\| p_1 - q_1'' \|_\infty \leq 1$ and $\| p_2 - q_2'' \|_\infty \leq 1$. Therefore, q_1'' and q_2'' satisfy (6). ∎

We start a discussion about Theorem 2. In the rest of this section, we assume that an L_2-convex function g is defined by L-convex functions g_1 and g_2. We note that $\mathrm{dom}\, g = \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ holds because $g > -\infty$.

Here we arbitrarily fix a point $p \in \mathrm{dom}\, g$. By the definition of $g(p)$, there exists a sequence of pairs of points $\{(p_1^k, p_2^k)\}_{k \in \mathbf{Z}_{++}}$ such that

$$\begin{aligned} & g(p) + \tfrac{1}{k} \geq g_1(p_1^k) + g_2(p_2^k) \geq g(p), \\ & p_1^k + p_2^k = p, \quad p_1^k \in \mathrm{dom}\, g_1, \quad p_2^k \in \mathrm{dom}\, g_2. \end{aligned} \tag{13}$$

Furthermore, by (3), we can assume, in addition, that

$$p_1^k \leq p_1^{k+1} \quad (k \in \mathbf{Z}_{++}). \tag{14}$$

Since $p_1^k + p_2^k = p$ holds, (14) is equivalent to $p_2^k \geq p_2^{k+1}$ for all $k \in \mathbf{Z}_{++}$. We will give several propositions to prove Theorem 2.

Proposition 10 *There exist a sequence* $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ *and a partition* (V_0, V_∞) *of* V *with* $V_0 \neq \emptyset$ *that satisfy* (13), (14) *and*

$$\begin{array}{ll} p_1^k(v) = p_1^{k+1}(v) & (v \in V_0,\ k \in \mathbf{Z}_{++}), \\ \lim_{k\to\infty} p_1^k(v) = \infty & (v \in V_\infty), \end{array} \tag{15}$$

where the condition (15) *says that* $p_1^k(v)$ *is fixed for* $v \in V_0$ *and diverges for* $v \in V_\infty$.

Proof. Let $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ be a sequence satisfying (13) and (14), and let $\beta_k = \min\{p_1^k(v) - p_1^1(v) \mid v \in V\}$ and $F^k = \left\{w \in V \mid p_1^k(w) - p_1^1(w) = \beta_k\right\}$. Then, there exists $u \in V$ belonging to infinitely many F^ks. We regard the sequence

$$\left\{\left(p_1^k - \beta_k\mathbf{1},\ p_2^k + \beta_k\mathbf{1}\right)\right\}_{k\in\{j\in\mathbf{Z}_{++}\mid u\in F^j\}}$$

as a new $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$, which satisfies (13), $p_1^1 \leq p_1^k$ and $p_1^k(u) = p_1^{k+1}(u)$ for all $k \in \mathbf{Z}_{++}$.

We initially put $V_0 = \{u\}$ and $V_\infty = \emptyset$, and modify these and $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ by repeating the following process: for $v \in V \setminus (V_0 \cup V_\infty)$, if there exists an infinite subsequence of $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ such that $p_1^k(v) \leq M$ holds in the subsequence for some $M \in \mathbf{Z}$, then we add v to V_0 and replace $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ by the subsequence; otherwise, we add v to V_∞. Thus, the sequence $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ finally obtained by the above process has (13). Moreover,

$$\begin{array}{ll} L \leq p_1^k(v) \leq U & (v \in V_0,\ k \in \mathbf{Z}_{++}), \\ \lim_{k\to\infty} p_1^k(v) = \infty & (v \in V_\infty) \end{array}$$

must hold for some $L, U \in \mathbf{Z}$. This guarantees the existence of a subsequence possessing (13), (14) and (15) of $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$. ∎

Proposition 11 *If there exist two sequences satisfying* (13), (14) *and* (15) *for two partitions* (V_0, V_∞) *and* $(\widehat{V}_0, \widehat{V}_\infty)$ *of* V, *respectively, then there also exists such a sequence satisfying* (13), (14) *and* (15) *for* $(V_0 \cup \widehat{V}_0, V_\infty \cap \widehat{V}_\infty)$.

Proof. Let $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ and $\{(\widehat{p}_1^k, \widehat{p}_2^k)\}_{k\in\mathbf{Z}_{++}}$ be sequences having (13), (14) and (15) for (V_0, V_∞) and $(\widehat{V}_0, \widehat{V}_\infty)$, respectively. Since $p_1^k + p_2^k = \widehat{p}_1^k + \widehat{p}_2^k = p$ holds, we have

$$(p_1^k \wedge \widehat{p}_1^k) + (p_2^k \vee \widehat{p}_2^k) = (p_1^k \vee \widehat{p}_1^k) + (p_2^k \wedge \widehat{p}_2^k) = p.$$

Obviously, $p_1^k \wedge \widehat{p}_1^k \leq p_1^{k+1} \wedge \widehat{p}_1^{k+1}$ holds for any $k \in \mathbf{Z}_{++}$. By (14) and (15) for $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ and $\{(\widehat{p}_1^k, \widehat{p}_2^k)\}_{k\in\mathbf{Z}_{++}}$, we have

$$\lim_{k\to\infty}(p_1^k \wedge \widehat{p}_1^k)(v) = \infty \qquad (v \in V_\infty \cap \widehat{V}_\infty),$$
$$(p_1^k \wedge \widehat{p}_1^k)(v) = (p_1^{k+1} \wedge \widehat{p}_1^{k+1})(v) \qquad (v \in V_0 \cup \widehat{V}_0,\ k \geq k')$$

for a sufficiently large number $k' \in \mathbf{Z}_{++}$. Furthermore, the L-convexity of g_1 and g_2 yields

$$\begin{aligned} 2g(p) + \frac{2}{k} &\geq g_1(p_1^k) + g_2(p_2^k) + g_1(\widehat{p}_1^k) + g_2(\widehat{p}_2^k) \\ &\geq g_1(p_1^k \wedge \widehat{p}_1^k) + g_2(p_2^k \vee \widehat{p}_2^k) + g_1(p_1^k \vee \widehat{p}_1^k) + g_2(p_2^k \wedge \widehat{p}_2^k) \\ &\geq g(p) + g_1(p_1^k \wedge \widehat{p}_1^k) + g_2(p_2^k \vee \widehat{p}_2^k). \end{aligned}$$

Thus, there exists a subsequence of $\{(p_1^k \wedge \widehat{p}_1^k, p_2^k \vee \widehat{p}_2^k)\}_{k\in\mathbf{Z}_{++}}$ having (13), (14) and (15) for $(V_0 \cup \widehat{V}_0, V_\infty \cap \widehat{V}_\infty)$. ■

For each $p \in \mathrm{dom}\, g$, Proposition 11 guarantees the existence of the maximum V_0 and the minimum V_∞ with respect to set-inclusion such that there is a sequence satisfying (13), (14) and (15) for (V_0, V_∞). Here we denote by $V_0(p)$ and $V_\infty(p)$ the maximum V_0 and the minimum V_∞, respectively, for $p \in \mathrm{dom}\, g$.

Proposition 12 *For any $p, q \in \mathrm{dom}\, g$, $V_0(p) = V_0(q)$ and $V_\infty(p) = V_\infty(q)$ hold.*

Proof. By Proposition 7, without loss of generality, we deal with the case where $q \geq p$ and q is adjacent to p. Let $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ and $\{(q_1^k, q_2^k)\}_{k\in\mathbf{Z}_{++}}$ be sequences satisfying (13), (14) and (15) for $(V_0(p), V_\infty(p))$ and $(V_0(q), V_\infty(q))$, respectively. By Proposition 9 and (13), for any $k \in \mathbf{Z}_{++}$, there exists $\widehat{q}_1^k$ and $\widehat{q}_2^k$ satisfying

$$\begin{aligned} &g(q) + \tfrac{3}{k} \geq g_1(\widehat{q}_1^k) + g_2(\widehat{q}_2^k), \\ &\widehat{q}_1^k + \widehat{q}_2^k = q, \quad \widehat{q}_1^k \in \mathrm{dom}\, g_1, \quad \widehat{q}_2^k \in \mathrm{dom}\, g_2, \\ &\| p_1^k - \widehat{q}_1^k \|_\infty \leq 1, \quad \| p_2^k - \widehat{q}_2^k \|_\infty \leq 1. \end{aligned}$$

This and the hypothesis, that $\{(p_1^k, p_2^k)\}_{k\in\mathbf{Z}_{++}}$ satisfies (14) and (15), guarantee the existence of a subsequence of $\{(\widehat{q}_1^k, \widehat{q}_2^k)\}_{k\in\mathbf{Z}_{++}}$ satisfying (13), (14) and (15). Since $\| p_1^k - \widehat{q}_1^k \|_\infty \leq 1$, we must have $V_0(p) \subseteq V_0(q)$. Since $V_0(p) = V_0(p + \mathbf{1})$ holds, we also obtain $V_0(q) \subseteq V_0(p)$ by the symmetric argument. ■

By Proposition 12, for all $p \in \mathrm{dom}\, g$, we can denote $V_0(p)$ and $V_\infty(p)$ by V_0 and V_∞, respectively, without reference to a particular point p. For the L_2-convex function g in Example 1, we have $(V_0, V_\infty) = (\{2\}, \{1\})$.

Proposition 13 *For $p_1 \in \mathrm{dom}\, g_1$ and $p_2 \in \mathrm{dom}\, g_2$, the following statements hold.*

(a) *For any* $\alpha \in \mathbf{Z}_{++}$, $p_1 + \alpha\chi_{V_\infty} \in \mathrm{dom}\, g_1$ *and* $p_2 - \alpha\chi_{V_\infty} \in \mathrm{dom}\, g_2$.

(b) *Functions* $g_1^{p_1}, g_2^{p_2} : \mathbf{Z}_{++} \to \mathbf{R}$ *defined by*

$$g_1^{p_1}(k) = g_1(p_1 + k\chi_{V_\infty}), \quad g_2^{p_2}(k) = g_2(p_2 - k\chi_{V_\infty}) \quad (k \in \mathbf{Z}_{++})$$

satisfy

$$\begin{aligned} g_1^{p_1}(k_1) + g_1^{p_1}(k_2) &\geq g_1^{p_1}(k_1 + l) + g_1^{p_1}(k_2 - l), \\ g_2^{p_2}(k_1) + g_2^{p_2}(k_2) &\geq g_2^{p_2}(k_1 + l) + g_2^{p_2}(k_2 - l) \end{aligned}$$

for $k_1, k_2 \in \mathbf{Z}_{++}$ *with* $k_1 < k_2$ *and* $l \in \mathbf{Z}_{++}$ *with* $0 \leq l \leq k_2 - k_1$. *(The above inequalities say that there exist piecewise linear convex functions* $\overline{g}_1^{p_1}, \overline{g}_2^{p_2} : \mathbf{R} \to \mathbf{R}$ *such that* $\overline{g}_1^{p_1}(k) = g_1^{p_1}(k)$ *and* $\overline{g}_2^{p_2}(k) = g_2^{p_2}(k)$ *for any* $k \in \mathbf{Z}_{++}$.*)*

(c) $g_1^{p_1} + g_2^{p_2}$ *is a non-increasing function bounded by* $g(p_1 + p_2)$ *from below.*

(d) *There exists a constant* $c \in \mathbf{R}$ *such that* $\lim_{k\to\infty} (g_1^{p_1}(k+1) - g_1^{p_1}(k)) = c$ *and* $\lim_{k\to\infty} (g_2^{p_2}(k+1) - g_2^{p_2}(k)) = -c$. *Furthermore, c is independent of the choice of* p_1 *and* p_2.

(e) *Let* $\tilde{g}_1^{p_1}(k) = g_1^{p_1}(k) - ck$ *and* $\tilde{g}_2^{p_2}(k) = g_2^{p_2}(k) + ck$. *Then,* $\{\tilde{g}_1^{p_1}(k)\}_{k\in\mathbf{Z}_{++}}$ *and* $\{\tilde{g}_2^{p_2}(k)\}_{k\in\mathbf{Z}_{++}}$ *converge to certain reals.*

Proof. Here we assume $V_\infty \neq \emptyset$.

(a): Let $\{(q_1^k, q_2^k)\}_{k\in\mathbf{Z}_{++}}$ be an arbitrary sequence having (13), (14) and (15) for (V_0, V_∞) and a certain point $q \in \mathrm{dom}\, g$. By (3), we can assume that $q_1^k(v) < p_1(v)$ and $q_2^k(v) > p_2(v)$ for any $v \in V_0$ and $k \in \mathbf{Z}_{++}$. For any sufficiently large number $k \in \mathbf{Z}_{++}$ such that $\alpha < \min\{q_1^k(v) - p_1(v) \mid v \in V_\infty\}$ and $\alpha < \min\{p_2(v) - q_2^k(v) \mid v \in V_\infty\}$, the L-convexity of g_1 and g_2 yield

$$\begin{aligned} g_1(p_1) + g_1(q_1^k) &\geq g_1(p_1 \wedge q_1^k) + g_1(p_1 \vee q_1^k), \\ g_1(p_1 \vee q_1^k) + g_1(p_1 + \alpha\mathbf{1}) &\geq g_1(p_1 + \alpha\chi_{V_\infty}) + g_1(q_1^k - \alpha\chi_{V_\infty} + \alpha\mathbf{1}), \\ g_2(p_2) + g_2(q_2^k) &\geq g_2(p_2 \wedge q_2^k) + g_2(p_2 \vee q_2^k), \\ g_2(p_2 \wedge q_2^k) + g_2(p_2 - \alpha\mathbf{1}) &\geq g_2(p_2 - \alpha\chi_{V_\infty}) + g_2(q_2^k + \alpha\chi_{V_\infty} - \alpha\mathbf{1}). \end{aligned} \tag{16}$$

(16) guarantees the assertion.

(b): Here we show the assertion for $g_1^{p_1}$. The assertion is obtained as follows:

$$\begin{aligned} g_1^{p_1}(k_1) + g_1^{p_1}(k_2) &= g_1(p_1 + k_1\chi_{V_\infty}) + g_1(p_1 + k_2\chi_{V_\infty}) \\ &= g_1(p_1 + k_1\chi_{V_\infty} + l\mathbf{1}) + g_1(p_1 + k_2\chi_{V_\infty}) - lr \\ &\geq g_1(p_1 + (k_1{+}l)\chi_{V_\infty}) + g_1(p_1 + (k_2{-}l)\chi_{V_\infty} + l\mathbf{1}) - lr \\ &= g_1^{p_1}(k_1 + l) + g_1^{p_1}(k_2 - l), \end{aligned}$$

where r is the constant defined in (3).

(c): By summing up inequalities in (16), $g_1(p_1 + \alpha\chi_{V_\infty}) + g_2(p_2 - \alpha\chi_{V_\infty})$ is bounded by

$$2g_1(p_1) + 2g_2(p_2) + \frac{1}{k} - g_1(p_1 \wedge q_1^k) - g_2(p_2 \vee q_2^k)$$

from above, where $p_1 \wedge q_1^k$ and $p_2 \vee q_2^k$ are independent of k because k is sufficiently large. Thus, $g_1^{p_1} + g_2^{p_2}$ is bounded from above, and furthermore, it must be non-increasing by (b). Obviously, $g_1^{p_1}(k) + g_2^{p_2}(k) \geq g(p_1 + p_2)$ holds. Since g is L_2-convex, $g(p_1 + p_2)$ must be a finite value. Hence $g_1^{p_1} + g_2^{p_2}$ is bounded from below.

(d): By (c), we can assume $g_1^{p_1}$ is non-increasing without loss of generality. Then, we have

$$0 \geq g_1^{p_1}(k+1) - g_1^{p_1}(k) \geq g_1^{p_1}(k) - g_1^{p_1}(k-1),$$

where the second inequality follows from (b). Since $\{g_1^{p_1}(k+1) - g_1^{p_1}(k)\}_{k \in \mathbf{Z}_{++}}$ is bounded from above and is non-decreasing, it converges to some $c \in \mathbf{R}$. It follows from (c) that

$$\lim_{k\to\infty} ((g_1^{p_1}{+}g_2^{p_2})(k+1) - (g_1^{p_1}{+}g_2^{p_2})(k)) = 0. \tag{17}$$

Hence, $\lim_{k\to\infty} (g_2^{p_2}(k+1) - g_2^{p_2}(k)) = -c$ must hold. Furthermore, (17) holds for any $p_1 \in \mathrm{dom}\, g_1$ and $p_2 \in \mathrm{dom}\, g_2$. Thus, c is independent of the choice of p_1 and p_2.

(e): The assertion (b) says that $g_1^{p_1}(k{+}1) - g_1^{p_1}(k)$ and $g_2^{p_2}(k{+}1) - g_2^{p_2}(k)$ are non-decreasing. Furthermore, by (d), we have

$$g_1^{p_1}(k+1) - g_1^{p_1}(k) \leq c, \quad g_2^{p_2}(k+1) - g_2^{p_2}(k) \leq -c,$$

and hence, $\tilde{g}_1^{p_1}(k{+}1) - \tilde{g}_1^{p_1}(k)$ and $\tilde{g}_2^{p_2}(k{+}1) - \tilde{g}_2^{p_2}(k)$ are non-positive for any $k \in \mathbf{Z}_{++}$. Namely, $\tilde{g}_1^{p_1}$ and $\tilde{g}_2^{p_2}$ are non-increasing. On the other hand, $\tilde{g}_1^{p_1} + \tilde{g}_2^{p_2} = g_1^{p_1} + g_2^{p_2}$ is bounded from below by (c). Hence, the assertion must hold. ∎

Proposition 14 *Let $\hat{g}_1$ and $\hat{g}_2$ be functions defined by*

$$\hat{g}_1(p_1) = \begin{cases} \lim_{k\to\infty} (g_1(p_1 + k\chi_{V_\infty}) - ck) & (p_1 \in \mathrm{dom}\, g_1) \\ +\infty & (p_1 \notin \mathrm{dom}\, g_1) \end{cases} \quad (p_1 \in \mathbf{Z}^V),$$

$$\hat{g}_2(p_2) = \begin{cases} \lim_{k\to\infty} (g_2(p_2 - k\chi_{V_\infty}) + ck) & (p_2 \in \mathrm{dom}\, g_2) \\ +\infty & (p_2 \notin \mathrm{dom}\, g_2) \end{cases} \quad (p_2 \in \mathbf{Z}^V),$$

where c is the constant in (d) *of Proposition 13. Then, $\hat{g}_1$ and $\hat{g}_2$ are L-convex.*

Proof. Here we verify the L-convexity of $\widehat{g}_1$. By (e) of Proposition 13, we have $\widehat{g}_1(p_1) \in \mathbf{R}$ for any $p_1 \in \mathrm{dom}\, g_1$, and hence, $\mathrm{dom}\, \widehat{g}_1 \supseteq \mathrm{dom}\, g_1$. Since $\mathrm{dom}\, \widehat{g}_1 \subseteq \mathrm{dom}\, g_1$ obviously holds, we have $\mathrm{dom}\, \widehat{g}_1 = \mathrm{dom}\, g_1$. For $k \in \mathbf{Z}_{++}$, let us define g_1^k by

$$g_1^k(p) = \begin{cases} g_1(p + k\chi_{V_\infty}) - ck & (p \in \mathrm{dom}\, g_1) \\ +\infty & (p \notin \mathrm{dom}\, g_1) \end{cases} \qquad (p \in \mathbf{Z}^V).$$

Trivially, $\mathrm{dom}\, g_1^k = \mathrm{dom}\, g_1 = \mathrm{dom}\, \widehat{g}_1$ holds. For $p, q \in \mathbf{Z}^V$, we have

$$\begin{aligned} g_1^k(p) + g_1^k(q) &= g_1(p{+}k\chi_{V_\infty}) + g_1(q{+}k\chi_{V_\infty}) - 2ck \\ &\geq g_1((p{+}k\chi_{V_\infty}) \wedge (q{+}k\chi_{V_\infty})) + g_1((p{+}k\chi_{V_\infty}) \vee (q{+}k\chi_{V_\infty})) - 2ck \\ &= g_1((p \wedge q) + k\chi_{V_\infty}) + g_1((p \vee q) + k\chi_{V_\infty}) - 2ck \\ &= g_1^k(p \wedge q) + g_1^k(p \vee q). \end{aligned}$$

The submodularity of g_1^k implies that of $\widehat{g}_1$. Obviously, $\widehat{g}_1$ possesses (TRF). Thus, $\widehat{g}_1$ is L-convex. ∎

Proposition 14 shows Theorem 2.

Proof of Theorem 2. Suppose that $g = g_1 \Box g_2$ but g_1 and g_2 do not satisfy (2). Let $\widehat{g}_1$ and $\widehat{g}_2$ be L-convex functions defined in Proposition 14. First, we show that $g = \widehat{g}_1 \Box \widehat{g}_2$. Let p be any point in $\mathrm{dom}\, g$ and let $\{(p_1^k, p_2^k)\}_{k \in \mathbf{Z}_{++}}$ a sequence possessing (13), (14) and (15) for p, (V_0, V_∞), g_1 and g_2. It is enough to show that $\{(p_1^k, p_2^k)\}_{k \in \mathbf{Z}_{++}}$ also satisfies (13) for $\widehat{g}_1$ and $\widehat{g}_2$. By (c) of Proposition 13, we have

$$g(p) + \frac{1}{k} \geq g_1(p_1^k) + g_2(p_2^k) \geq \widehat{g}_1(p_1^k) + \widehat{g}_2(p_2^k) \geq g(p)$$

for any $k \in \mathbf{Z}_{++}$, and hence, $g = \widehat{g}_1 \Box \widehat{g}_2$.

In the same way as discussions for g_1 and g_2, there exists a partition $(\widehat{V}_0, \widehat{V}_\infty)$ of V for $\widehat{g}_1$ and $\widehat{g}_2$ such that (15) and $\widehat{V}_0(p) = \widehat{V}_0(q)$ are satisfied for any $p, q \in \mathrm{dom}\, g$. Next, we show that V_0 is a proper subset of $\widehat{V}_0$. Let us consider the sequence $\{(p_1^k, p_2^k)\}_{k \in \mathbf{Z}_{++}}$ defined in the previous paragraph. We denote $\min\{p_1^k(v) - p_1^1(v) \mid v \in V_\infty\}$ by β_k for each $k \in \mathbf{Z}_{++}$. Without loss of generality, we assume that there is an element $u \in V_\infty$ with $p_1^k(u) - p_1^1(u) = \beta_k$ for all $k \in \mathbf{Z}_{++}$. By the definition of β_k, we have

$$\begin{aligned} (p_1^k - \beta_k\chi_{V_\infty})(u) &= (p_1^{k+1} - \beta_{k+1}\chi_{V_\infty})(u), \\ (p_2^k + \beta_k\chi_{V_\infty})(u) &= (p_2^{k+1} + \beta_{k+1}\chi_{V_\infty})(u) \end{aligned}$$

for all k. On the other hand,

$$\lim_{k \to \infty} (g_1(p_1 + k\chi_{V_\infty}) + g_2(p_2 - k\chi_{V_\infty})) = \widehat{g}_1(p_1) + \widehat{g}_2(p_2) \qquad (18)$$

holds for any $p_1 \in \mathrm{dom}\, g_1$ and $p_2 \in \mathrm{dom}\, g_2$. Equation (18) guarantees that

$$\widehat{g}_1(p_1 + \chi_{V_\infty}) + \widehat{g}_2(p_2 - \chi_{V_\infty}) = \widehat{g}_1(p_1) + \widehat{g}_2(p_2).$$

Thus, the sequence

$$\{(p_1^k - \beta_k \chi_{V_\infty},\ p_2^k + \beta_k \chi_{V_\infty})\}_{k \in \mathbf{Z}_{++}} \tag{19}$$

also satisfies (13) for $\widehat{g}_1$ and $\widehat{g}_2$. In a manner similar to the proof of Proposition 10, there exists a subsequence of (19) such that (13), (14) and (15) hold for $\widehat{g}_1$ and $\widehat{g}_2$. Furthermore, for any $v \in V_0 \cup \{u\}$, the v-th components of all points in the subsequence are fixed. Hence V_0 is a proper subset of $\widehat{V}_0$.

If $\widehat{V}_\infty = \emptyset$ then we have L-convex functions $\widehat{g}_1$ and $\widehat{g}_2$ satisfying (2); otherwise, we repeat the modifications of L-convex functions defined in Proposition 14 until $\widehat{V}_\infty = \emptyset$. Since V_0 is strictly enlarged, the above process is terminated in at most $|V|$ iterations. Hence, there exist L-convex functions g_1' and g_2' satisfying (2). ∎

3 Applications

This section gives proofs of three theoretical applications of Theorem 2, namely, Theorems 3, 4 and 5, and also gives elementary proofs of two results on the convolution of L-convex functions as applications of Proposition 9.

Proof of Theorem 3. The implication ($\Rightarrow$) is trivial. We prove the opposite direction. By Theorem 2, let us assume that g is defined by two L-convex functions g_1 and g_2 satisfying (2). By the hypothesis and Proposition 6, we have $g_1(q+\mathbf{1}) = g_1(q)$ and $g_2(q+\mathbf{1}) = g_2(q)$ for all $q \in \mathbf{Z}^V$. Let p_1^* and p_2^* be such that $g(p^*) = g_1(p_1^*) + g_2(p_2^*)$, $p_1^* + p_2^* = p^*$, $p_1^* \in \mathrm{dom}\, g_1$ and $p_2^* \in \mathrm{dom}\, g_2$. By the definition of the convolution, we have

$$g(p^* + \chi_S) \le \min\{g_1(p_1^* + \chi_S) + g_2(p_2^*),\quad g_1(p_1^*) + g_2(p_2^* + \chi_S)\}.$$

This inequality and the assumption that $g(p^*) = g_1(p_1^*) + g_2(p_2^*) \le g(p^* + \chi_S)$ yield

$$g_1(p_1^*) \le g_1(p_1^* + \chi_S), \quad g_2(p_2^*) \le g_2(p_2^* + \chi_S)$$

for any $S \subseteq V$. By an optimality criterion for L-convex functions [9], we have $p_1^* \in \arg\min g_1$ and $p_2^* \in \arg\min g_2$. This says that p^* must be a minimizer of g. ∎

Proof of Theorem 4. By Theorem 2, let us assume that g is defined by two L-convex functions g_1 and g_2 satisfying (2). By the hypothesis and Proposition 6, we have $g_1(q+\mathbf{1}) = g_1(q)$ and $g_2(q+\mathbf{1}) = g_2(q)$ for all $q \in \mathbf{Z}^V$. Let p_1 and p_2 be such that $g(p) = g_1(p_1) + g_2(p_2)$, $p_1 + p_2 = p$, $p_1 \in \mathrm{dom}\, g_1$ and $p_2 \in \mathrm{dom}\, g_2$. In the same way as the proof of Theorem 3, we can show that

$$g_1(p_1) \le g_1(p_1 + \alpha\chi_S), \quad g_2(p_2) \le g_2(p_2 + \alpha\chi_S)$$

for any $S \subseteq V$. By the proximity theorem for L-convex functions [5], there exist $p_1^* \in \arg\min g_1$ and $p_2^* \in \arg\min g_2$ such that

$$p_1 \le p_1^* \le p_1 + (n{-}1)(\alpha{-}1)\mathbf{1}, \quad p_2 \le p_2^* \le p_2 + (n{-}1)(\alpha{-}1)\mathbf{1}.$$

The above inequalities guarantee that $p^* = p_1^* + p_2^*$ satisfies $p \le p^* \le p + 2(n{-}1)(\alpha{-}1)\mathbf{1}$. Moreover, p^* must be a minimizer of g because $p_1^* \in \arg\min g_1$ and $p_2^* \in \arg\min g_2$. ∎

Proof of Theorem 5. By Theorem 2, we assume that g is defined by two L-convex functions g_1 and g_2 satisfying (2). Obviously, $\arg\min g_1 + \arg\min g_2 \subseteq \arg\min g$ holds. Let p be an arbitrary element of $\arg\min g$. By (2), there exist $p_1 \in \mathrm{dom}\, g_1$ and $p_2 \in \mathrm{dom}\, g_2$ such that $g(p) = g_1(p_1) + g_2(p_2)$ and $p_1 + p_2 = p$. This says that p_1 and p_2 must belong to $\arg\min g_1$ and $\arg\min g_2$, respectively. Thus, we have

$$\arg\min g = \arg\min g_1 + \arg\min g_2.$$

We can easily show that $\arg\min g_1$ and $\arg\min g_2$ are L-convex sets. Hence, $\arg\min g$ is an L_2-convex set. ∎

Finally, we give elementary proofs of two results on the convolution of L-convex functions. The first result, Theorem 15 below, says that if the convolution g of two L-convex functions g_1 and g_2 has a finite value at some point p, then it has a finite value at any point in $\mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$. The original proof of Theorem 15 in [7] utilizes the conjugacy between L-convexity and M-convexity and the Fenchel-type min-max identity, while our proof utilizes only Propositions 7 and 9.

Theorem 15 ([7]) *Let $g_1, g_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be L-convex and $g = g_1 \Box g_2$. If $g(p) \in \mathbf{R}$ for some $p \in \mathbf{Z}^V$, then*

$$\begin{cases} g(q) \in \mathbf{R} & (q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2), \\ g(q) = +\infty & (\textit{otherwise}). \end{cases}$$

Proof. It is sufficient to show $g(q) > -\infty$ for each $q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$. By (3) and Proposition 7, it is enough to show $g(q) > -\infty$ for each $q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ such that $q \ge p$ and q is adjacent to p. Suppose to the contrary that $g(q) = -\infty$. Since $g(p) \in \mathbf{R}$ holds, for any $\gamma > 0$, there exist p_1 and p_2 such that

$$g(p) + \gamma \ge g_1(p_1) + g_2(p_2) \ge g(p), \quad p_1 + p_2 = p, \quad p_1 \in \mathrm{dom}\, g_1, \quad p_2 \in \mathrm{dom}\, g_2.$$

On the other hand, by the assumption $g(q) = -\infty$, for any positive number $M \in \mathbf{R}$, there exist q_1 and q_2 such that

$$-M \geq g_1(q_1) + g_2(q_2), \quad q_1 + q_2 = q, \quad q_1 \in \mathrm{dom}\, g_1, \quad q_2 \in \mathrm{dom}\, g_2.$$

Proposition 9 guarantees that there exist q_1'' and q_2'' satisfying

$$\begin{aligned} &-M + 2\gamma \geq g_1(q_1'') + g_2(q_2''), \\ &q_1'' + q_2'' = q, \quad q_1'' \in \mathrm{dom}\, g_1, \quad q_2'' \in \mathrm{dom}\, g_2, \\ &\| p_1 - q_1'' \|_\infty \leq 1, \quad \| p_2 - q_2'' \|_\infty \leq 1. \end{aligned}$$

This says that the neighborhood of either p_1 or p_2 must have a point whose function value is $-\infty$, because M is any positive number. However, this contradicts the fact that $g_1 > -\infty$ and $g_2 > -\infty$. Therefore, $g(q) > -\infty$ must hold. ∎

The second result, Theorem 16 below, says that if the infimum in (1) is attained at some point p then the infimum in (1) is attained at any point, i.e., g_1 and g_2 satisfy (2).

Theorem 16 *Let $g_1, g_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be L-convex and $g = g_1 \Box g_2$. If there exist p_1 and p_2 for some $p \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ such that*

$$g(p) = g_1(p_1) + g_2(p_2), \quad p_1 + p_2 = p, \quad p_1 \in \mathrm{dom}\, g_1, \quad p_2 \in \mathrm{dom}\, g_2, \tag{20}$$

then for any $q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ there exist q_1 and q_2 satisfying (20) with $\{p, p_1, p_2\}$ replaced by $\{q, q_1, q_2\}$.

Proof. By (3) and Proposition 7, it is sufficient to show that there exist q_1 and q_2 satisfying (20) for each $q \in \mathrm{dom}\, g_1 + \mathrm{dom}\, g_2$ such that $q \geq p$ and q is adjacent to p. Theorem 15 and Proposition 9 guarantee that for any $\gamma > 0$, there exist q_1^γ and q_2^γ satisfying

$$\begin{aligned} &g(q) + \gamma \geq g_1(q_1^\gamma) + g_2(q_2^\gamma) \geq g(q), \\ &q_1^\gamma + q_2^\gamma = q, \quad q_1^\gamma \in \mathrm{dom}\, g_1, \quad q_2^\gamma \in \mathrm{dom}\, g_2, \\ &\| p_1 - q_1^\gamma \|_\infty \leq 1, \quad \| p_2 - q_2^\gamma \|_\infty \leq 1. \end{aligned}$$

Since γ is an arbitrary positive number, there must exist q_1 and q_2 satisfying

$$\begin{aligned} &g(q) = g_1(q_1) + g_2(q_2), \quad q_1 + q_2 = q, \\ &q_1 \in \mathrm{dom}\, g_1, \quad q_2 \in \mathrm{dom}\, g_2, \\ &\| p_1 - q_1 \|_\infty \leq 1, \quad \| p_2 - q_2 \|_\infty \leq 1. \end{aligned}$$

Hence, the assertion holds. ∎

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