Serre-Swan Theorem
for non commutative C*-algebras

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For a Hilbert C*-module X over a C*-algebra $\mathcal{A}$, we construct a vector bundle $\mathcal{E}_X$ associated with X. We represent X as a vector space Γ_X of some class of holomorphic sections on $\mathcal{E}_X$ with the following properties:

(i) Γ_X is a Hilbert $\mathcal{A}$-module and $\Gamma_X \cong X$,

(ii) $\mathcal{E}_X$ has a flat connection which defines the action of $\mathcal{A}$ on Γ_X,

(iii) $\mathcal{E}_X$ has a hermitian metric H which induces the C*-inner product of Γ_X.

This section representation of modules is a kind of generalization of the Serre-Swan theorem to non commutative C*-algebras. We show that $\mathcal{E}_X$ is isomorphic to an associated bundle of an infinite dimensional Hopf bundle with the structure group $U(1)$.

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1 Introduction

The Serre-Swan theorem is described as follows:

Theorem 1.1 *(Serre-Swan) Let Ω be a connected compact Hausdorff space and $C(\Omega)$ the algebra of all complex valued continuous functions on Ω. Assume X is a module over $C(\Omega)$. Then X is finitely generated projective iff there is a complex vector bundle E on Ω such that X is isomorphic onto the module of all continuous sections of E.*

Proof. See [9]. ∎

By this theorem, finitely generated projective modules over $C(\Omega)$ and complex vector bundles on Ω are in one-to-one correspondence up to isomorphisms. In non commutative geometry [5, 13], some class of module over non commutative C*-algebra $\mathcal{A}$ is treated as a vector bundle of "non commutative space" $\mathcal{A}$ by generalizing Serre-Swan theorem for commutative C*-algebra, for example, 2×2 matrix algebra $M_2(\mathbf{C})$ and its free (right)module $M_2(\mathbf{C})^m \equiv M_2(\mathbf{C}) \oplus \cdots \oplus M_2(\mathbf{C})$ of rank m. Specially, it is believed that a finitely generated projective module is regarded as a non commutative vector bundle because the condition of modules is used in Theorem 1.1.

On the other hand, for a unital general non commutative C*-algebra $\mathcal{A}$, there is a uniform Kähler bundle $(\mathcal{P}, p, B)$ [3] unique up to equivalence class of $\mathcal{A}$, such that $\mathcal{A}$ is * isomorphic onto the uniform Kähler function algebra with $*$-product on $(\mathcal{P}, p, B)$ which is a natural generalization of Gel'fand representation.

Example 1.1 We show the example of a functional representation of the simplest, nontrivial and non commutative C*-algebra $\mathcal{A} = M_2(\mathbf{C})$. The uniform Kähler bundle $(\mathcal{P}, p, B)$ of $M_2(\mathbf{C})$ is given as follows: Let $\mathcal{P} = \mathbf{C}P^1$, $B = \{b\}$, where B is a one-point set and p is the trivial surjection from $\mathcal{P}$ to B. The functional representation of $M_2(\mathbf{C})$,

$$f : M_2(\mathbf{C}) \to C^\infty(\mathbf{C}P^1)$$

is defined by $f_A([x]) \equiv {}^t\overline{x} Ax/\|x\|^2$ for $A \in M_2(\mathbf{C})$ and $[x] \in \mathbf{C}P^1$ where $*$-product on $C^\infty(\mathbf{C}P^1)$ is defined by

$$l * m \equiv l \cdot m + \sqrt{-1} X_m l \qquad \left(l, m \in C^\infty(\mathbf{C}P^1) \right).$$

Non-commutativity of this $*$-product comes from the second term of the right hand side on the above equation.

We review the uniform Kähler bundle and the functional representation of non commutative C*-algebras in section 2 intimately. Under the above consideration, we state the following theorem which is a version of the Serre-Swan theorem generalized to non commutative C*-algebras under weaker condition.

Theorem 1.2 *((Main theorem) Serre-Swan theorem for non commutative C*-algebras) Let X be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, $(\mathcal{P}, p, B)$ the uniform Kähler bundle of $\mathcal{A}$, $\mathcal{K}_u(\mathcal{P})$ the C*-algebra of uniform Kähler functions on $\mathcal{P}$ and $f : \mathcal{A} \cong \mathcal{K}_u(\mathcal{P})$ the Gel'fand representation of $\mathcal{A}$.*

(i) *(Construction of vector bundle) There is a complex vector bundle* $\mathcal{E}_X$ *on* $\mathcal{P}$ *such that* $\mathcal{E}_X$ *has a flat connection* D *and a hermitian metric* H*. Furthermore there is a bundle map* P_X *from a trivial bundle* $X \times \mathcal{P}$ *on* $\mathcal{P}$ *to* $\mathcal{E}_X$ *which image is dense in* $\mathcal{E}_X$ *at each fiber.*

(ii) *($*$-action and hermitian metric) Let* $\Gamma_X \equiv P_{X*}(\Gamma_{const}(X \times \mathcal{P})) \subset \Gamma_{hol}(\mathcal{E}_X)$ *where* $\Gamma_{const}(X \times \mathcal{P})$ *is the set of all constant sections on* $X \times \mathcal{P}$ *and* $\Gamma_{hol}(\mathcal{E}_X)$ *is the set of all holomorphic sections of* $\mathcal{E}_X$*. Then* Γ_X *is a Hilbert* $\mathcal{K}_u(\mathcal{P})$*-module with right* $*$*-action*

$$\begin{aligned} \Gamma_X \times \mathcal{K}_u(\mathcal{P}) &\rightarrow \Gamma_X, \\ (s,l) &\mapsto s * l \equiv s \cdot l + \sqrt{-1} D_{X_l} s \qquad (\,(s,l) \in \Gamma_X \times \mathcal{K}_u(\mathcal{P})\,) \end{aligned}$$

and a C-inner product*

$$H|_{\Gamma_X \times \Gamma_X} : \Gamma_X \times \Gamma_X \rightarrow \mathcal{K}_u(\mathcal{P})$$

where X_l *is the holomorphic part of the complex Hamiltonian vector fields of* $l \in \mathcal{K}_u(\mathcal{P}) \subset C^\infty(\mathcal{P})$ *with respect to the Kähler form of* $\mathcal{P}$*.*

(iii) *(Reconstruction) Under an identification* $f : \mathcal{A} \cong \mathcal{K}_u(\mathcal{P})$*,* Γ_X *is isomorphic to* X *as a Hilbert* $\mathcal{A}$*-module.*

Remark 1.1 In spite that the finitely generated projective property (=FP property) is ingredient of modules in the origianl Serre-Swan theorem (Theorem 1.1), reader may note that there is no condition about FP property of Hibert C*-module in Theorem 1.2. We wish to explain for this natural question here.

In the field of non commutative geometry [4, 5, 13], Serre-Swan theorem has been applied as a kind of guiding principle to make theory and give correspondence between modules of non commutative algebra and virtual vector bundles. In this context, the rigid mathematical statement is stretched as a guiding principle of “non commutative-commutative correspondence” along with quantum-classical correspondence in quantum mechanics. In analogy with the case of quantum mechanics, there are many ambiguities and unorganized interpretations about non commutative geometry and there is no exact theory which unifies them because the meaning of this correspondence is not clear and principle itself is not mathematics. Despite of these ambiguity, the non commutative geometry gives us strong interest and desire to study the virtual geometry associated with non commutative algebra. Therefore we regard Serre-Swan theorem as a guiding principle of non commutative-commutative correspondence in the field of non commutative geometry in this paper. We neglect the FP property here because we have no idea to treat the notion of FP property in our theory suitably. In this reason, we decide to use a word "Serre-Swan" in our paper in order to stress the significance of the role of guiding principle in non commutative geometry. FP property may be a problem which may be studied in future. Inversely, we can regard the Hilbert C*-structure is more essential than FP property according to our theorem. Here we would mention that there is a kind of FP property of Hilbert C*-module by Kasparov stability theorem [10].

In subsection 3.2, we introduce $\mathcal{E}_X$ in Theorem 1.2. $\mathcal{E}_X$ is called the atomic bundle of a Hilbert C*-module X which is a Hilbert bundle on the uniform Kähler bundle. We show its geometrical structure in subsection 3.4 (e.g. $\mathcal{A} = M_2(\mathbf{C})$ and $X = M_2(\mathbf{C})^m$, then $\mathcal{E}_X \cong S^3 \times_{U(1)} \mathbf{C}^{2m}$ which is an associated vector bundle of a Hopf bundle $(S^3, \mu, \mathbf{C}P^1, U(1))$). In subsection 4.1, we give a flat connection D in Theorem 1.2. D is called the atomic connection of the atomic bundle. In subsection 4.2, by using a connection of the vector bundle, we define a $*$-action of a function algebra on the vector space of holomorphic sections on a vector bundle under more general situation than the case of Hilbert C*-modules. In section 5, we give a proof of Theorem 1.2.

Here we summarize correspondences between geometry and algebra.

Gel'fand representation

	space	algebra
CG	Ω	$C(\Omega)$ point wise product
NCG	$\mathcal{P} \to B$	$\mathcal{K}_u(\mathcal{P})$ $*$-product

Serre-Swan theorem

	vector bundle	module
CG	$E \to \Omega$	$\Gamma(E)$ point wise action
NCG	$\mathcal{E}_X \to \mathcal{P}$	Γ_X $*$-action

where we call respectively, *CG = commutative geometry* as a geometry associated with commutative C*-algebras, and *NCG = non commutative geometry* as a geometry associated with non commutative C*-algebras by following A.Connes [4].

2 C*-geometry

In a naive sense of correspondence between algebraic objects and geometric ones defined by algebraic equations, we propose a version of the algebraic geometry for non commutative C*-algebras. Or, it may be considered as a generalized Gel'fand representation for non commutative C*-algebraic objects (= the automorphism group, modules, etc.) [3]. As a preparation, we review the basic for an assurance that such correspondence exits by the following [3].

What do we want to do ? It is often stated that usual geometry is commutative geometry in comparison to non commutative geometry ([4, 5, 13]). Commutative geometry means a topological, or geometrical space consisting points or its function algebra. On the other hand, non commutative geometry means the algebra which is non commutative and not function algebra in general, and it is considered as a *virtual* function space on a *virtual* topological, or *virtual* geometrical space. But there exist several kind of non-commutativity outside the region of genuine non commutative geometry, too. For example, the fundamental group of topological spaces, transformation group, the Lie algebra of vector fields of a differential manifold are generally non commutative. Furthermore, for a symplectic manifold M, the set $C^\infty(M)$ of all smooth functions on M has the Poisson bracket determined by the symplectic form of M [6]. It is nothing but which is non commutative Lie bracket. In this sense, the function algebra is not always commutative. As a special case, for a Kähler manifold, the smooth function algebra has Kähler bracket from its Kähler form [11] which is the special case of symplectic form. For example, a complex projective space $\mathbf{C}P^n$ is a Kähler manifold with a metric called Fubini-Study type. Cirelli, Manià and Pizzocchero realize non-commutativity of C*-algebra by Kähler bracket of projective space. We are trying to explain the non-commutativity appearing in non commutative geometry by that inherent in commutative geometry, such as the automorphism group, module, subalgebra, Atiyah-Singer index, K-group and so on in the theory of C*-algebras.

2.1 Uniform Kähler bundle

We start from the geometric characterization of the set of all pure states and the spectrum of a C*-algebra [3].

Assume now that E and M are topological spaces.

Definition 2.1 *(E, μ, M) is called a uniform Kähler bundle if it satisfies the following conditions:*

(i) *μ is an open, continuous surjection between topological spaces E and M,*

(ii) *the topology of E is a uniform topology,*

(iii) *each fiber $E_m \equiv \mu^{-1}(m)$ is a Kähler manifold.*

Remark 2.1 This definition of uniform Kähler bundle is weaker than the original one [3]. We do not use the condition of equivalence of Kähler topology and uniform topology in this paper.

The local triviality of uniform Kähler bundle is not assumed. M is always neither compact nor Hausdorff.

We simply denote (E, μ, M) by E. For a uniform topology, see [2]. Any metric space is a uniform space. Examples and relations with C*-algebra are given later. Roughly speaking, the fiber of uniform Kähler bundle is related to non-commutativity of C*-algebra.

Definition 2.2 *Two uniform Kähler bundles (E, μ, M), $(E^{'}, \mu^{'}, M^{'})$ are isomorphic if there is a pair (β, ϕ) of homeomorphisms $\beta : E \to E^{'}$ and $\phi : M \to M^{'}$, such that $\mu^{'} \circ \beta = \phi \circ \mu$*

$$\begin{array}{ccccc} & & \beta & & \\ & E & \cong & E^{'} & \\ \mu & \downarrow & & \downarrow & \mu^{'} \\ & M & \cong & M^{'} & \\ & & \phi & & \end{array}$$

and any restriction $\beta|_{\mu^{-1}(m)} : \mu^{-1}(m) \to (\mu^{'})^{-1}(\phi(m))$ is a holomorphic Kähler isometry for any $m \in M$. We call (β, ϕ) a uniform Kähler isomorphism between (E, μ, M) and $(E^{'}, \mu^{'} M^{'})$.

Example 2.1 (i) Any Kähler manifold N is a uniform Kähler bundle with a one-point set as the base space. In the same way, the direct sum of Kähler manifolds $\{N_i\}_{i=1}^{n}$ as a metric space is a uniform Kähler bundle with a n-point set as the base space endowed with discrete topology.

(ii) Any compact Hausdorff space X is a uniform space. X is a uniform Kähler bundle with 0-dimensional fiber which itself as the base space [2].

We explain the nontrivial third example of uniform Kähler bundles as follows.
Let $\mathcal{A}$ be a unital C*-algebra. Denote

$\mathcal{P}$: the set of all pure states on $\mathcal{A}$, *and*
B : the set of all equivalence classes of irreducible representations of $\mathcal{A}$.

B is called *the spectrum of* $\mathcal{A}$. The weak* topology on $\mathcal{P}$ is a uniform topology. By the GNS representation of $\mathcal{A}$, there is a natural projection from $\mathcal{P}$ onto B :

$$p : \mathcal{P} \to B.$$

If $\mathcal{A}$ is commutative, then $\mathcal{P} \cong B \cong$ "the set of all maximal ideals of $\mathcal{A}$" as a compact Hausdorff space. We consider $(\mathcal{P}, p, B)$ as a map of topological spaces where $\mathcal{P}$ is endowed with weak* topology and B is endowed with the Jacobson topology [12].

In Ref.[3], the following results are proved.

Theorem 2.1 *(Reduced atomic realization) For any unital C*-algebra $\mathcal{A}$, $(\mathcal{P}, p, B)$ is a uniform Kähler bundle.*

For a fiber $\mathcal{P}_b \equiv p^{-1}(b)$, let $(\mathcal{H}_b, \pi_b)$ be an irreducible representation belonging to $b \in B$. $\rho \in \mathcal{P}_b$ corresponds $[x_\rho] \in \mathcal{P}(\mathcal{H}_b) \equiv (\mathcal{H}_b \setminus \{0\})/\mathbf{C}^\times$ where $\rho = \omega_{x_\rho} \circ \pi_b$ with ω_{x_ρ} denoting a vector state $\omega_{x_\rho} =< x_\rho|(\cdot)x_\rho >$. Then $\mathcal{P}_b$ has a Kähler manifold structure induced by this correspondence from the projective Hilbert space $\mathcal{P}(\mathcal{H}_b)$. We denote this correspondence by τ^b:

$$\tau^b : \mathcal{P}_b \to \mathcal{P}(\mathcal{H}_b); \quad \tau^b(\rho) \equiv [x_\rho]. \tag{2.1}$$

The Kähler distance d_b of a fiber $\mathcal{P}_b$ is given by

$$d_b(\rho, \rho^{'}) \equiv \sqrt{2}\mathrm{arcos}| < x_\rho | x_{\rho^{'}} > | \qquad (\rho, \rho^{'} \in \mathcal{P}_b)$$

which is the length of shortest geodesic between ρ and $\rho^{'}$ in $\mathcal{P}_b$.

Theorem 2.2 *Let $\mathcal{A}_i$ be C*-algebras with associated uniform Kähler bundles $(\mathcal{P}_i, p_i, B_i)$, $i = 1, 2$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are * isomorphic if and only if $(\mathcal{P}_1, p_1, B_1)$ and $(\mathcal{P}_2, p_2, B_2)$ are isomorphic as a uniform Kähler bundle.*

By this theorem $(\mathcal{P}, p, B)$ associated with $\mathcal{A}$ is uniquely determined up to a uniform Kähler isomorphism. From now, we call $(\mathcal{P}, p, B)$ in Theorem 2.1 *the uniform Kähler bundle associated with C*-algebra $\mathcal{A}$.*

2.2 A functional representation of non commutative C*-algebras

We reconstruct $\mathcal{A}$ from the uniform Kähler bundle $(\mathcal{P}, p, B)$ associated with $\mathcal{A}$. Since $\mathcal{P}_b \equiv p^{-1}(b) \subset \mathcal{P}$ is a Kähler manifold for each $b \in B$, consider the fiberwise smooth (= smooth in $\mathcal{P}_b$ for each $b \in B$) function. Let

$C^\infty(\mathcal{P})$: the set of all fiberwise smooth complex valued functions on $\mathcal{P}$.

Define a product $*$ on $C^\infty(\mathcal{P})$ denoted by $l * m$ for $l, m \in C^\infty(\mathcal{P})$ by

$$l * m \equiv l \cdot m + \sqrt{-1} X_m l \tag{2.2}$$

Define an involution * on $C^\infty(\mathcal{P})$ by complex conjugate. Then $(C^\infty(\mathcal{P}), *)$ becomes a * algebra with unit which is not associative in general, where X_l is the holomorphic Hamiltonian vector field of l defined by an equation

$$\omega_\rho\left((X_l)_\rho, \bar{Y}_\rho\right) = \bar{\partial}_\rho l(\bar{Y}_\rho) \qquad (\bar{Y}_\rho \in \overline{T_\rho \mathcal{P}}) \tag{2.3}$$

for each $\rho \in \mathcal{P}$, Kähler form ω on $\mathcal{P}$ which is defined each fiber, $\bar{\partial}$ is the anti-holomorphic differential operator on $C^\infty(\mathcal{P})$, and $\overline{T_\rho \mathcal{P}}$ is the anti-holomorphic tangent space of $\mathcal{P}$ at $\rho \in \mathcal{P}$. By using Eq.2.3, Eq.2.2 can be written as follows:

$$l * m = l \cdot m + \sqrt{-1}\, \omega(\bar{X}_l, X_m).$$

If $\{\cdot, \cdot\}$ is the Kähler bracket with respect to ω, then the following equality holds:

$$l * m - m * l = \sqrt{-1}\{l, m\} \quad (\, l, m \in C^\infty(\mathcal{P})\,). \tag{2.4}$$

Theorem 2.3 *(Gel'fand representation of non commutative C^*-algebras) For a non commutative C^*-algebra $\mathcal{A}$, the Gel'fand representation*

$$f_A(\rho) \equiv \rho(A), \qquad\qquad (A \in \mathcal{A},\ \rho \in \mathcal{P})$$

gives an injective * homomorphism of unital * algebras :

$$f: \ \mathcal{A} \ \to \ C^\infty(\mathcal{P}); \quad A \mapsto f_A$$

where $C^\infty(\mathcal{P})$ is endowed with $*$-product defined by (2.2). For a function l in the image $f(\mathcal{A})$ of a map f,

$$\|l\| \equiv \sup_{\rho \in \mathcal{P}} \left| (\bar{l} * l)\,(\rho) \right|^{\frac{1}{2}} \tag{2.5}$$

defines a C^**-norm of* * *algebra* $f(\mathcal{A})$*. By this norm, an associative* * *subalgebra* $(f(\mathcal{A}),*)$ *is a* C^**-algebra which is* * *isomorphic onto* $\mathcal{A}$*.*

Furthermore $f(\mathcal{A})$ *is equal to a subset* $\mathcal{K}_u(\mathcal{P})(\subset C^\infty(\mathcal{P}))$ *defined by*

$$\mathcal{K}_u(\mathcal{P}) \equiv \left\{ l \in C^\infty(\mathcal{P}) : \begin{array}{c} D^2 l = 0, \quad \bar{D}^2 l = 0, \\ \bar{l} * l, \quad l * \bar{l}, \quad l \text{ are uniformly continuous on } \mathcal{P} \end{array} \right\}, \tag{2.6}$$

where D*,* $\bar{D}$ *are the holomorphic and anti-holomorphic part, respectively, of covariant derivative of Kähler metric defined on each fiber of* $\mathcal{P}$*. Hence, the following equivalence of* C^**-algebras holds:*

$$\mathcal{A} \cong \mathcal{K}_u(\mathcal{P}).$$

We call these objects *C^*-geometry* since any C*-algebra can be reconstructed from the associated uniform Kähler bundle [3] and, therefore, any C*-algebra is determined by such a geometry.

Remark 2.2 In Theorem 2.3, the function f_A of the image of a functional representation on $A \in \mathcal{A}$ is not holomorphic. It is a complex quadratic form on a Hilbert projective space like Example 1.1. Furthermore, f_A satisfies not only continuous on across fiber but also more strong uniform continuity like the definition of $\mathcal{K}_u(\mathcal{P})$.

Remark 2.3 *(Non-commutativity and differential structure)* It is not known yet whether the function space of uniform Kähler functions on a general uniform Kähler bundle satisfying conditions in Definition 2.1 becomes a C*-algebra. In this sense, the class of uniform Kähler bundles may be larger than the class of C*-algebras.

Remark 2.4 In [4], non commutative generalization of differential geometry is claimed by using the theory of C*-algebras. A non commutative C*-algebra is treated as a function algebra $C^\infty(M)$ with respect to a *virtual* smooth manifold M. On the other hand, the smooth structure of the uniform Kähler bundle of a C*-algebra $\mathcal{A}$ appears only when $\mathcal{A}$ is not commutative since statements above Theorem 2.1. Furthermore it is known that a commutative C*-algebra has no continuous derivation except 0. Under these considerations, our opinion is that the differential structure of a geometry makes a C*-algebra non commutative. In commutative case, the differential structure do not arise from algebraic structure of a C*-algebra.

Hence the geometry of non commutative C*-algebras is necessarily differential geometry with the w* topology of the space of states, and that of commutative C*-algebras is only (compact Hausdorff)topology.

By the above results, we obtain a fundamental correspondence between algebra and geometry as follows:

unital commutative C*-algebra	$\Leftrightarrow$	compact Hausdorff space
$\cap$		$\cap$
unital generally non commutative C*-algebra	$\Leftrightarrow$	uniform Kähler bundle. associated with a C*-algebra

The upper correspondence above is just the Gel'fand representation of unital commutative C*-algebra. Since Remark 2.3, we must restrict the class of uniform Kähler bundles to like the above.

Example 2.2 Assume that $\mathcal{H}$ is a separable infinite dimensional Hilbert space.

(i) When $\mathcal{A} \equiv \mathcal{L}(\mathcal{H})$ which is the algebra of all bounded linear operators on $\mathcal{H}$, the uniform Käher bundle of $\mathcal{A}$ is $(\mathcal{P}(\mathcal{H}) \cup \mathcal{P}_-,\, p,\, 2^{[0,1]} \cup \{b_0\})$ where $\mathcal{P}(\mathcal{H})$ is the projective Hilbert space of $\mathcal{H}$, $\mathcal{P}_-$ is the union of projective Hilbert spaces of continuous dimensional Hilbert space indexed by $2^{[0,1]}$, $2^{[0,1]}$ is the power set of closed interval $[0,1]$ and $\{b_0\}$ is the one-point set corresponding to the equivalence class of identity representation $(\mathcal{H}, id_{\mathcal{L}(\mathcal{H})})$ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$. Since the primitive spectrum of $\mathcal{L}(\mathcal{H})$ is a two-point set, the topology of $2^{[0,1]} \cup \{b_0\}$ is equal to $\{\, \emptyset,\, 2^{[0,1]},\, \{b_0\},\, 2^{[0,1]} \cup \{b_0\}\,\}$ [8]. In this way, the base space of the UKB is not always a one-point set when an algebra is type I.

(ii) For a C^*-algebra $\mathcal{A}$ generated by the Weyl form of 1-dimensional canonical commutation relation

$$U(s)V(t) = e^{\sqrt{-1}st}V(t)U(s) \quad (s, t \in \mathbf{R}),$$

its uniform Kähler bundle is $(\mathcal{P}(\mathcal{H}), p, \{1pt\})$. The spectrum is a one-point set $\{1pt\}$ since von Neumann uniqueness theorem [1].

(iii) *CAR-algebra* $\mathcal{A}$ is a UHF algebra with the nest $\{M_{2^n}(\mathbf{C})\}_{n\in\mathbf{N}}$. The uniform Kähler bundle has the base space $2^{\mathbf{N}}$ and each fiber on $2^{\mathbf{N}}$ is a separable infinite dimensional projective Hilbert space where $2^{\mathbf{N}}$ is the power set of the set $\mathbf{N}$ of all natural numbers with trivial topology, that is, the topology of $2^{\mathbf{N}}$ is just $\{\emptyset, 2^{\mathbf{N}}\}$. In general, the Jacobson topology of the spectrum of a simple C*-algebra is trivial [8].

3 The atomic bundle of a Hilbert C*-module

The aim of this section is the construction of a vector bundle by a given Hilbert C*-module for a C*-algebra.

3.1 Hopf bundles, Hilbert modules, deformation and canonical quantization

We start from a naive geometric example related to some Hilbert C*-module in this section.

In the text book of fiber bundles, a Hopf bundle is one of typical examples of $U(1)$-principal bundle where the total space is 3-sphere, the base space 1-dimensional complex projective space ($\cong$ 2-sphere) and the typical fiber $U(1)(\cong$ circle). The Hopf bundle $(S^3, \mu, \mathbf{C}P^1)$ is defined as follows:

Definition 3.1 *(Hopf bundle)* $(S^3, \mu, \mathbf{C}P^1)$ *is called the Hopf bundle if*

$$\begin{aligned} S^3 \equiv\ & \left\{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\}, \\ \mathbf{C}P^1 \equiv\ & (\mathbf{C}^2 \setminus \{0\})/\mathbf{C}^\times = S^3/U(1), \\ \mu :\ & S^3 \to \mathbf{C}P^1, \\ \mu(z_1, z_2) \equiv\ & [(z_1, z_2)] \\ =\ & \left\{e^{i\theta}(z_1, z_2) : e^{i\theta} \in U(1)\right\} \quad (\,(z_1, z_2) \in S^3\,). \end{aligned}$$

The base space $\mathbf{C}P^1$ of a Hopf bundle is the total space of the uniform Kähler bundle of C*-algebra $M_2(\mathbf{C})$ by Theorem 2.1. Some examples of vector bundles of $\mathbf{C}P^1$ can be obtained from associated vector bundles $S^3 \times_{U(1)} \mathbf{C}^m$ of the Hopf bundle defined by the space of all $U(1)$-orbits in a direct product space $S^3 \times \mathbf{C}^m$ associated to a $U(1)$-action on $\mathbf{C}^m$. Such vector bundle has the structure group $U(1)$. By Serre-Swan theorem for vector bundles of a smooth compact manifold, a finitely generated projective $C^\infty(\mathbf{C}P^1)$-module X corresponds to a vector bundle E of $\mathbf{C}P^1$ unique up to isomorphisms [9, 13]. In this situation, $C^\infty(\mathbf{C}P^1)$-action on $\Gamma_\infty(E)$ is an action by pointwise product

$$\Gamma_\infty(E) \times C^\infty(\mathbf{C}P^1) \ni (s,l) \longmapsto s \cdot l \in \Gamma_\infty(E),$$
$$(s \cdot l)(x) \equiv s(x)l(x) \quad (x \in \mathbf{C}P^1).$$

Let

$$\{\cdot,\cdot\} : C^\infty(\mathbf{C}P^1) \times C^\infty(\mathbf{C}P^1) \to C^\infty(\mathbf{C}P^1)$$

be the Kähler bracket of $\mathbf{C}P^1$. To define an action of a Lie algebra $(C^\infty(\mathbf{C}P^1), \{\cdot,\cdot\})$, we define a $*$-action by deforming the above action by a connection D of E (deformation of action):

$$s \cdot l \quad \Longrightarrow \quad s * l \equiv s \cdot l + \sqrt{-1} D_{X_l} s \tag{3.7}$$

where X_l is defined in (2.3). Rewrite a right action of $C^\infty(\mathbf{C}P^1)$

$$M_l : \Gamma_\infty(E) \to \Gamma_\infty(E); \qquad sM_l \equiv s * l.$$

Then we obtain a deformed homomorphism of Lie algebras by curvature.

Lemma 3.1 *For $s \in \Gamma_\infty(E)$ and $l, m \in C^\infty(\mathbf{C}P^1)$, we have the following equation:*

(i) $s[M_l, M_m] = \sqrt{-1} s M_{\{l,m\}} + R_{X_l, X_m} s$
where R is the curvature of D.

(ii) *If M is associated with a flat connection, then*

$$[M_l, M_m] = \sqrt{-1} M_{\{l,m\}} \tag{3.8}$$

on $\Gamma_\infty(E)$. That is, M is a rescaled homomorphism between Lie algebras $C^\infty(\mathbf{C}P^1)$ and $\mathrm{End}(\Gamma_\infty(E))$ with scaled factor $\sqrt{-1}$.

Proof. See Lemma 4.2. ∎

By a flat connection D of E, we obtain a right action M of a Lie algebra $(C^\infty(\mathbf{C}P^1), \{\cdot,\cdot\})$ on $\Gamma_\infty(E)$ rescaled by $\sqrt{-1}$. Define a Lie algebra

$$\mathfrak{h}(C^\infty(\mathbf{C}P^1)) \equiv \left\{ M_l \in \mathrm{End}(\Gamma_\infty(E)) : l \in C^\infty(\mathbf{C}P^1) \right\}.$$

Then $\mathfrak{h}(C^\infty(\mathbf{C}P^1))$ is just the canonical quantization of a classical algebra $(C^\infty(\mathbf{C}P^1), \{\cdot,\cdot\})$ as operators on the space of sections by the quantum-classical correspondence principle.

In this way, we obtain an operator Lie algebra $\mathfrak{h}(C^\infty(\mathbf{C}P^1))$ on $\Gamma_\infty(E)$. By using Example 1.1 and Eq.2.4, $M_2(\mathbf{C})$ is embedded in $C^\infty(\mathbf{C}P^1)$. Hence we obtain an injective homomorphism $M \circ f$ from a Lie algebra $\mathfrak{gl}_2(\mathbf{C}) = M_2(\mathbf{C})$ to $\mathfrak{h}(C^\infty(\mathbf{C}P^1))$ which is the following southeast arrow by composed other two arrows M and f:

$$\begin{array}{ccc} \mathfrak{gl}_2(\mathbf{C}) & \stackrel{f}{\to} & C^\infty(\mathbf{C}P^1) \\ {\scriptstyle M\circ f} & \searrow & \downarrow {\scriptstyle M} \\ & & \mathfrak{h}(C^\infty(\mathbf{C}P^1)). \end{array}$$

In fact, for $A, B \in \mathfrak{gl}_2(\mathbf{C})$,

$$\begin{aligned} [\,(M\circ f)(A),\,(M\circ f)(B)\,] &= [M_{f_A}, M_{f_B}] \\ &= \sqrt{-1}M_{\{f_A,f_B\}} \\ &= M_{\sqrt{-1}\{f_A,f_B\}} \\ &= M_{f_A * f_B - f_B * f_A} \\ &= M_{f_{AB}-f_{BA}} \\ &= M_{f_{AB-BA}} \\ &= (M\circ f)([A,B]). \end{aligned}$$

We obtain an action of $\mathfrak{gl}_2(\mathbf{C})$ on $\Gamma_\infty(E)$ by a flat connection D. We finish to explain a naive example. Note that there is no associativity of $M_2(\mathbf{C})$ in general for E in this example.

In the general case, we construct a vector bundle E associated with a Hilbert C*-module, and a * isomorphism ϕ from a general C*-algebra $\mathcal{A}$ with a uniform Kähler bundle $(\mathcal{P}, p, B)$ into an associative * algebra $\mathcal{B}_E \equiv *Alg < \mathfrak{h}(C^\infty(\mathcal{P})) >$ and an isomorphism Ψ from Hilbert C*-module X into $\Gamma_\infty(E)$ so as for the following diagram to be commutative:

$$\begin{array}{ccccc} X & \times & \mathcal{A} & \to & X \\ {\scriptstyle \Psi\times\phi} \downarrow & & & & \downarrow {\scriptstyle \Psi} \\ \Gamma_\infty(E) & \times & \mathcal{B}_E & \to & \Gamma_\infty(E). \end{array}$$

By this case, we can always construct E which satisfies the associativity of $\mathcal{A}$. On the other hand, we define the atomic bundle as a kind of vector bundle from a Hilbert C*-module and show the bundle structure by using a Hopf bundle.

3.2 The construction of the atomic bundle

Before starting to construct the atomic bundle of a Hilbert C*-module, we state the definition of a Hilbert C*-module.

Definition 3.2 *([7]) X is a Hilbert C*-module over a C*-algebra $\mathcal{A}$ if X is a right $\mathcal{A}$-module and there is an $\mathcal{A}$ valued-sesquilinear form*

$$<\cdot|\cdot>: X \times X \to \mathcal{A}$$

which satisfies the following conditions:

$$\begin{array}{rll} <\eta|\xi a>= & <\eta|\xi>a & (\,\eta,\xi \in X\,), \\ (<\eta|\xi>)^* = & <\xi|\eta> & (\,\xi,\eta \in X, a \in \mathcal{A}\,), \\ <\xi|\xi>\geq & 0 & (\,\xi \in X\,), \\ <\xi|\xi>=0 & \Rightarrow \xi = 0 & (\,\xi \in X\,) \end{array}$$

and X is complete with respect to a norm defined by

$$\|\xi\|_X \equiv \| <\xi|\xi> \|^{1/2} \qquad (\,\xi \in X\,). \tag{3.9}$$

Let X be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ and $(\mathcal{P}, p, B)$ the uniform Kähler bundle associated with $\mathcal{A}$ defined in Theorem 2.1. Defining a closed subspace N_ρ of X with $\rho \in \mathcal{P}$ by

$$N_\rho \equiv \left\{ \xi \in X : \rho(\|\xi\|^2) = 0 \right\} \tag{3.10}$$

(note $\|\xi\|^2 = <\xi|\xi> \in \mathcal{A}$), we consider the quotient vector space

$$\mathcal{E}^o_{X,\rho} \equiv X/N_\rho$$

equipped with a sesquilinear form $<\cdot|\cdot>_\rho$ on $\mathcal{E}^o_{X,\rho}$ defined by

$$<\cdot|\cdot>_\rho : \quad \mathcal{E}^o_{X,\rho} \times \mathcal{E}^o_{X,\rho} \to \mathbf{C},$$

$$<[\xi]_\rho|[\eta]_\rho>_\rho \quad \equiv \rho(<\xi|\eta>) \qquad (\,[\xi]_\rho, [\eta]_\rho \in \mathcal{E}^o_{X,\rho}\,)$$

where

$$[\xi]_\rho \equiv \xi + N_\rho \in \mathcal{E}^o_{X,\rho} \qquad (\,\xi \in X\,). \tag{3.11}$$

Then $<\cdot|\cdot>_\rho$ becomes an inner product on $\mathcal{E}^o_{X,\rho}$. Let $\mathcal{E}_{X,\rho}$ be the completion of $\mathcal{E}^o_{X,\rho}$ by the norm $\|\cdot\|_\rho \equiv (<\cdot|\cdot>_\rho)^{1/2}$. We obtain a Hilbert space $(\,\mathcal{E}_{X,\rho},\, <\cdot|\cdot>_\rho\,)$ from a Hilbert C*-module X for each pure state $\rho \in \mathcal{P}$.

Definition 3.3 *(Atomic bundle) An atomic bundle $\mathcal{E}_X = (\mathcal{E}_X, \Pi_X, \mathcal{P})$ of a Hilbert C*-module X is defined as a fiber bundle $\mathcal{E}_X$ on $\mathcal{P}$:*

$$\mathcal{E}_X \equiv \bigcup_{\rho \in \mathcal{P}} \mathcal{E}_{X,\rho},$$

where the projection map $\Pi_X : \mathcal{E}_X \to \mathcal{P}$ is defined by $\Pi_X(x) = \rho$ for $x \in \mathcal{E}_{X,\rho}$. For $b \in B$, a B-fiber $\mathcal{E}^b_X = (\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$ of X is defined by

$$\mathcal{E}^b_X \equiv \bigcup_{\rho \in \mathcal{P}_b} \mathcal{E}_{X,\rho},$$

$$\Pi^b_X : \mathcal{E}^b_X \to \mathcal{P}_b; \quad \Pi^b_X \equiv \Pi_X|_{\mathcal{E}^b_X}.$$

The name of the *atomic* bundle comes from the (reduced)atomic representation of a C*-algebra. The atomic bundle is a collection of its B-fibers:

$$\mathcal{E}_X = \bigcup_{b \in B} \mathcal{E}^b_X.$$

In this way, $\mathcal{E}_X$ has a two-step fibration $(\mathcal{E}_X, \Pi_X, \mathcal{P})$ and $(\mathcal{E}_X,\, p \circ \Pi_X,\, B)$:

$$\begin{array}{ccccc} \mathcal{E}_X & \stackrel{\Pi_X}{\to} & \mathcal{P} & \\ {\scriptstyle p \circ \Pi_X} & \searrow & \downarrow & {\scriptstyle p} \\ & & B & \end{array}$$

where their fibers are $\mathcal{E}_{X,\rho} = \Pi_X^{-1}(\rho)$ and $\mathcal{E}^b_X = (p \circ \Pi_X)^{-1}(b)$, respectively for $\rho \in \mathcal{P}$ and $b \in B$.

3.3 Unitary group action on the atomic bundle

The aim of this subsection is to give the canonical definition of the typical fiber of the atomic bundle in order to state the structure theorem in subsection 3.4.

Let G be the group of all unitary elements in $\mathcal{A}$. Define an action χ of G on $\mathcal{P}$ by

$$\chi_u(\rho) \equiv \rho \circ \mathrm{Ad}u^* \quad (u \in G,\ \rho \in \mathcal{P}).$$

We note that any pure state is moved to a pure state by any inner automorphism of $\mathcal{A}$. Hence χ_u maps $\mathcal{P}_b$ to $\mathcal{P}_b$ for each $b \in B$ and $u \in G$.

Lemma 3.2 *G acts on $\mathcal{P}_b$ by transitively.*

Proof. Because the GNS representation of any element of $\mathcal{P}_b$ is irreducible, any two unit vectors in $\mathcal{H}_b$ are transformed by $\pi_b(u)$ for some $u \in G$. Assume that $\rho_1, \rho_2 \in \mathcal{P}_b$. Then there are $h, h^{'} \in \mathcal{H}_b$, $\|h\| = 1 = \|h^{'}\|$ such that $\rho_1 =< h|\pi_b(\cdot)h >$ and $\rho_2 =< h^{'}|\pi_b(\cdot)h^{'} >$, there is $u \in G$ such that $\pi_b(u)h = h^{'}$ and

$$\chi_u(\rho_1) = \rho_1 \circ \mathrm{Ad}u^* =< h|\pi_b(u^*)\pi_b(\cdot)\pi_b(u)h >=< h^{'}|\pi_b(\cdot)h^{'} >= \rho_2.$$

Hence the action χ of G is transitive on $\mathcal{P}_b$. ∎

Next, define an action t^b of G on $\mathcal{E}^o_X$ by

$$t^b_u([\xi]_\rho) \equiv [\,\xi u^*\,]_{\chi_u(\rho)} \quad \left(u \in G,\ [\xi]_\rho \in \mathcal{E}^o_{X,\rho}\right).$$

t^b is well defined since a map $\xi \mapsto \xi u^*$ maps N_ρ to $N_{\chi_u(\rho)}$. In fact,

$$\chi_u(\rho)\left(\|\xi u^*\|^2\right) = \rho\left((\mathrm{Ad}u^*)\left(u\|\xi\|^2 u^*\right)\right) = \rho\left(\|\xi\|^2\right).$$

Then t^b_u is a unitary map from $\mathcal{E}^o_{X,\rho}$ to $\mathcal{E}^o_{X,\chi_u(\rho)}$.Hence we can extend t^b_u as a unitary map from $\mathcal{E}_{X,\rho}$ to $\mathcal{E}_{X,\chi_u(\rho)}$. We note that

$$t^b_{cu}(x) = \bar{c}t^b_u(x) \quad (\,u \in G,\ c \in U(1)\,). \tag{3.12}$$

We define an action t of G on $\mathcal{E}_X$ by $t|_{\mathcal{E}^b_X} \equiv t^b$, $b \in B$. Then $T \equiv (t, \chi)$ is an action of G on $(\mathcal{E}_X, \Pi_X, \mathcal{P})$, that is, the following diagram is commutative for each $u \in G$:

$$\begin{array}{ccc} \mathcal{E}_X & \overset{t_u}{\cong} & \mathcal{E}_X \\ \Pi_X \downarrow & & \downarrow \Pi_X \\ \mathcal{P} & \overset{\chi_u}{\cong} & \mathcal{P}. \end{array}$$

In fact,

$$(\Pi_X \circ t_u)([\xi]_\rho) = \Pi_X([\xi u^*]_{\chi_u(\rho)}) = \chi_u(\rho) = (\chi_u \circ \Pi_X)([\xi]_\rho).$$

Hence $\Pi_X \circ t_u = \chi_u \circ \Pi_X$ holds on $\mathcal{E}^o_{X,\rho}$. By continuity, it holds on the whole $\mathcal{E}_{X,\rho}$ for each $\rho \in \mathcal{P}$.

This action preserves B-fiber $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$, $b \in B$, too.

For two fibrations $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$ and $(S(\mathcal{H}_b), \mu_b, \mathcal{P}_b)$, define a fiber product $\mathcal{E}^{b,U(1)}_X \subset \mathcal{E}^b_X \times S(\mathcal{H}_b)$ of them by

$$\begin{aligned}\mathcal{E}^{b,U(1)}_X \equiv\ & \mathcal{E}^b_X \times_{\mathcal{P}_b} S(\mathcal{H}_b)\\ =\ & \Bigl\{(x,h) \in \mathcal{E}^b_X \times S(\mathcal{H}_b) : \Pi^b_X(x) = \mu_b(h)\Bigr\}.\end{aligned}$$

For a Hopf bundle $(S(\mathcal{H}_b), \mu_b, \mathcal{P}_b)$, see a sentence after Theorem 2.1 and Appendix A.

Define an action σ^b of G on $\mathcal{E}^{b,U(1)}_X$ by

$$\sigma^b_u(x,h) \equiv (t_u(x), \pi_b(u)h) \qquad \Bigl((x,h) \in \mathcal{E}^{b,U(1)}_X,\ u \in G\Bigr).$$

In fact

$$\Pi^b_X(t_u(x)) = \chi_u(\Pi^b_X(x)) = \chi_u(\rho) = \rho \circ \mathrm{Ad}u^* = \mu_b(\pi_u h)$$

when $(x,h) \in \mathcal{E}^{b,U(1)}_X$ and $\Pi^b_X(x) = \rho = \mu_b(h)$. Therefore $\sigma^b_u(x,h) \in \mathcal{E}^{b,U(1)}_X$. Hence σ^b_u is well defined.

We note that a representation $(\mathcal{H}_b, \pi_b)$ of $\mathcal{A}$ induces an action of G on $S(\mathcal{H}_b)$.

Lemma 3.3 *For $(x,h) \in \mathcal{E}^{b,U(1)}_X$ and $u \in G$, if $\sigma^b_u(x,h) = (y,h)$, then $x = y$.*

Proof. It is sufficient to show for the case $x \in \mathcal{E}^o_{X,\rho}$. Assume $x = [\xi]_\rho$. By assumption,

$$\begin{aligned}(y,h) =\ & \sigma^b_u(x,h)\\ =\ & (t_u(x),\ \pi_b(u)h)\\ =\ & (\,[\xi u^*]_{\chi_u(\rho)},\ \pi_b(u)h\,).\end{aligned}$$

Hence $h = \pi_b(u)h$. Equivalently we have

$$\pi_b(u^*)h = h. \tag{3.13}$$

By definition of fiber product,

$$\begin{aligned}\chi_u(\rho) = &\ \Pi_X^b\left([\xi u^*]_{\chi_u(\rho)}\right)\\ = &\ \mu_b\left(\pi_b(u)h\right)\\ = &\ \mu_b(h)\\ = &\ \rho.\end{aligned}$$

Therefore we have $\chi_u(\rho) = \rho$ and $y = [\xi u^*]_\rho$. By using the above results, we obtain the following equation:

$$\begin{aligned}\|x-y\|_\rho^2 = &\ \rho(\|\xi - \xi u^*\|^2)\\ = &\ \rho(\|\xi\|^2) + \rho(\|\xi u^*\|^2) - \rho(<\xi|\xi u^*>) - \rho(<\xi u^*|\xi>)\\ = &\ \rho(\|\xi\|^2) + \rho(u\|\xi\|^2 u^*) - \rho(<\xi|\xi> u^*) - \rho(u<\xi|\xi>)\\ = &\ \rho(\|\xi\|^2) + \chi_u(\rho)(\|\xi\|^2)\\ &\ - <h|\pi_b(<\xi|\xi>)\pi_b(u^*)h> - <\pi_b(u^*)h|\pi_b(<\xi|\xi>)h>\\ = &\ 2\rho(\|\xi\|^2) - \rho(<\xi|\xi>) - \rho(<\xi|\xi>) \quad (\text{ by } (3.13)\ )\\ = &\ 0.\end{aligned}$$

We note that $\rho =< h|\pi_b(\cdot)h >$ here. Hence we obtain $x = y$. ∎

Definition 3.4 *F_X^b is the set of all orbits of G in $\mathcal{E}_X^{b,U(1)}$.*

Let $\mathcal{O}(x,h) \in F_X^b$ be an orbit containing $(x,h) \in \mathcal{E}_X^{b,U(1)}$. Since G acts on $S(\mathcal{H}_b)$ transitively,

$$\begin{aligned}\mathcal{O}(x,h) = &\ \{\sigma_u^b(x,h) : u \in G\}\\ = &\ \{(t_u(x), \pi_b(u)h) : u \in G\}\end{aligned}$$

for each $(x,h) \in \mathcal{E}_X^{b,U(1)}$. Hence F_X^b is a family of spheres which is homeomorphic to $S(\mathcal{H}_b)$ in $\mathcal{E}_X^{b,U(1)}$. By Lemma 3.3, any element of $\mathcal{O}(x,h)$ is written as $(y_{h'}, h')$ where $y_{h'}$ is an element of $\mathcal{E}_X^b$ determined by $h' \in S(\mathcal{H}_b)$ uniquely.

Lemma 3.4 *For (y,h') in $\mathcal{O}(x,h)$, if $y = x \neq 0$, then $h = h'$.*

Proof. By choice of (x,h'), there is $u \in G$ such that $\sigma_u^b(x,h) = (x,h')$. $t_u^b(x) = x$ and $\pi_b(u)h = h'$. Since

$$\mu_b(h') = \Pi_X^b(x) = \mu_b(h),$$

there is $c \in U(1)$ such that $h^{'} = ch$. Hence we can choose $u = cI$. If $x \neq 0$, then we have

$$x = t_u^b(x) = t_{cI}^b(x) = \bar{c}t_I^b(x) = \bar{c}x$$

by (3.12). Therefore $c = 1$ and we obtain $h = h^{'}$ when $x \neq 0$. ∎

Corollary 3.1 *For $c \in U(1)$,*

$$\mathcal{O}(x, ch) = \mathcal{O}(cx, h).$$

Proof. If $(y, h) \in \mathcal{O}(x, ch)$, then we can take $u \in G$ as $u = \bar{c}I$. Then

$$y = t_u(x) = \bar{\bar{c}}x = cx.$$

∎

Furthermore $\mathcal{O}(0, h) = \{(0, h^{'}) : h^{'} \in S(\mathcal{H}_b)\}$ because t_u^b is unitary for each $u \in G$. Let $(y, h^{'}) \in \mathcal{O}(x, h) \cap \left(\mathcal{E}_{X,\mu_b(h)} \times S(\mathcal{H}_b)\right)$. Then there is $u \in G$ such that $(y, h^{'}) = \sigma_u(x, h)$. By choice of $(y, h^{'})$, $h^{'} \in \mu_b^{-1}(\mu_b(h))$. Hence there is $c \in U(1)$ such that $h^{'} = ch$.

We note that $\mathcal{E}_{X,\rho}$ and $\mathcal{E}_{X,\rho^{'}}$ are equivalent as a Hilbert space when $\rho, \rho^{'} \in \mathcal{P}_b$.

Proposition 3.1 *F_X^b is a Hilbert space which is isomorphic to $\mathcal{E}_{X,\rho}$ for each $\rho \in \mathcal{P}_b$.*

Proof. Fix $h_0 \in S(\mathcal{H})$ such that $\mu_b(h_0) = \rho$. Define a map

$$R : \mathcal{E}_{X,\rho} \to F_X^b; \quad R(x) \equiv \mathcal{O}(x, h_0).$$

Then R is surjective. If $R(x) = R(y)$ for $x, y \in \mathcal{E}_{X,\rho}$, then $\mathcal{O}(x, h_0) = \mathcal{O}(y, h_0)$. By Lemma 3.4, $x = y$ when $x \neq 0$. If $x = 0$, then $y = 0$. Hence R is bijective. We define a structure of Hilbert space of F_X^b from $\mathcal{E}_{X,\rho}$ by R. Then we have the statement of the proposition. ∎

We denote calculations in F_X^b.

$$\mathcal{O}(x, h) + \mathcal{O}(y, h^{'}) = R(t_{u_h}(x)) + R(t_{u_{h^{'}}}(y))$$

where $\sigma_{u_h}(x,h) = (t_{u_h}(x),\, h_0)$ and $\sigma_{u_{h'}}(y,h^{'}) = (t_{u_{h'}}(y),\, h_0)$. For $k \in \mathbf{C}$,

$$\begin{aligned} k \cdot \mathcal{O}(x,h) = &\ kR(t_{u_h}(x)) \\ = &\ R(k \cdot t_{u_h}(x)) \\ = &\ R(t_{u_h}(kx)) \\ = &\ \mathcal{O}(kx,h). \end{aligned}$$

Specially, $c \in U(1)$,

$$c \cdot \mathcal{O}(x,h) = \mathcal{O}(x,ch)$$

by Corollary 3.1.

3.4 Structure of the atomic bundle

We show that the atomic bundle has a Hilbert bundle structure in order to define sections on it in the next section. Let $(\, S(\mathcal{H}_b) \times_{U(1)} F_X^b,\, \pi_{F_X^b},\, \mathcal{P}(\mathcal{H}_b)\,)$ be the associated bundle of $(S(\mathcal{H}_b), \mu_b, \mathcal{P}(\mathcal{H}_b))$ by F_X^b where a Hilbert space F_X^b is defined in Definition 3.4. By definition of the associated bundle(Appendix A.3), an element of $S(\mathcal{H}_b) \times_{U(1)} F_X^b$ is written as the $U(1)$-orbit $[(h, \mathcal{O}(x,k))]$ which contains $(h, \mathcal{O}(x,k)) \in S(\mathcal{H}_b) \times F_X^b$.

Lemma 3.5 *Any element of* $S(\mathcal{H}_b) \times_{U(1)} F_X^b$ *can be written as* $[(h, \mathcal{O}(x,h))]$ *where* $\mathcal{O}(x,h) \in F_X^b$.

Proof. Take an element $[(h, \mathcal{O}(y,k))] \in S(\mathcal{H}_b) \times_{U(1)} F_X^b$. By definition of $\mathcal{O}(y,k)$ and the transitivity of the action of G on $S(\mathcal{H})$, there is $u \in G$ such that $h = uk$ and $(t_u^b(y), h) \in \mathcal{O}(y,k)$. Denote $x \equiv t_u(y)$ Then $\mathcal{O}(x,h) = \mathcal{O}(y,k)$. Hence

$$[(h, \mathcal{O}(y,k))] = [(h, \mathcal{O}(x,h))].$$

∎

For $h \in S(\mathcal{H})$ and $x \in \mathcal{E}_X^b$, we denote

$$[h,x] \equiv [(h, \mathcal{O}(x,h))] \in S(\mathcal{H}_b) \times_{U(1)} F_X^b$$

from here.

Recall for each $b \in B$, $\mathcal{P}_b$ is a Kähler manifold which is isomorphic to a projective Hilbert space $\mathcal{P}(\mathcal{H}_b)$ by a map τ^b in (2.1).

Theorem 3.1 *For each $b \in B$, the B-fiber $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ at b is a local trivial Hilbert bundle which is isomorphic to $(\, S(\mathcal{H}_b) \times_{U(1)} F_X^b,\, \pi_{F_X^b},\, \mathcal{P}(\mathcal{H}_b)\,)$:*

$$(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b) \cong (\, S(\mathcal{H}_b) \times_{U(1)} F_X^b,\, \pi_{F_X^b},\, \mathcal{P}(\mathcal{H}_b)\,)$$

where F_X^b is a complex Hilbert space defined in Definition 3.4 which is isomorphic to $\mathcal{E}_{X,\rho}$ for each $\rho \in \mathcal{P}_b$.

Proof. Define a map $\Psi^b : \mathcal{E}_X^b \to S(\mathcal{H}_b) \times_{U(1)} F_X^b$ by

$$\Psi^b(x) \equiv [h_x, x] \quad (x \in \mathcal{E}_X^b)$$

where $h_x \in \mu_b^{-1}(\Pi_X^b(x))$. We show this definition is independent in the choice of h_x. If $h^{'} \in \mu_b^{-1}(\Pi_X^b(x))$, then there is $c \in U(1)$ such that $h^{'} = ch_x$. We denote $h = h_x$ for simplicity. Then we have

$$\begin{aligned}(h^{'},\, \mathcal{O}(x, h^{'})) = &\ (ch,\, \mathcal{O}(x, ch)) \\ = &\ (h\gamma_{\bar{c}},\, c\mathcal{O}(x, h)) \\ = &\ (h,\, \mathcal{O}(x, h)) \cdot \bar{c}\end{aligned}$$

where γ is an action of $U(1)$ defined in Appendix A.3. Hence

$$[h, x] = [(h, \mathcal{O}(x, h)] = [(h^{'}, \mathcal{O}(x, h^{'})] = [h^{'} x].$$

In this way, Ψ^b is well defined. If $\Psi^b(x) = \Psi^b(y)$ for $x, y \in \mathcal{E}_X^b$, then $[h, x] = [h^{'}, y]$. Therefore there is $c \in U(1)$ such that $(h, \mathcal{O}(x, h))c = (h^{'}, \mathcal{O}(y, h^{'}))$. By $h^{'} = \bar{c}h$ and Corollary 3.1,

$$\begin{aligned}\mathcal{O}(y, \bar{c}h) = &\ \mathcal{O}(y, h^{'}) \\ = &\ \bar{c}\mathcal{O}(x, h) \\ = &\ \mathcal{O}(x, \bar{c}h).\end{aligned}$$

By Lemma 3.3, $x = y$. Hence Ψ^b is injective. By definition of F_X^b, Ψ^b is surjective. Ψ^b is a bijection. Furthermore the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{E}_X^b & \overset{\Psi^b}{\cong} & S(\mathcal{H}_b) & \times_{U(1)} & F_X^b \\ \Pi_X^b \downarrow & & & \downarrow \pi_{F_X^b} & \\ \mathcal{P}_b & \overset{\tau^b}{\cong} & & \mathcal{P}(\mathcal{H}_b) & . \end{array}$$

We obtain a set-theoretical isomorphism (Ψ^b, τ^b) of fibrations between $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$ and $(\, S(\mathcal{H}_b) \times_{U(1)} F^b_X, \pi_{F^b_X}, \mathcal{P}_b\,)$ such that any restriction $\Psi^b|_{\mathcal{E}_{X,\rho}}$ of Ψ^b at a fiber $\mathcal{E}_{X,\rho}$ is unitary between $\mathcal{E}_{X,\rho}$ and $\pi^{-1}_{F^b_X}(\rho)$ for $\rho \in \mathcal{P}_b$.

We define a Hilbert bundle structure of $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$ from $(\, S(\mathcal{H}_b) \times_{U(1)} F^b_X, \pi_{F^b_X}, \mathcal{P}_b\,)$ by (Ψ^b, τ^b). We have the statement. ∎

We have constructed a Hilbert bundle from a Hilbert C*-module by Definition 3.3 and Theorem 3.1.

Let $(X \times \mathcal{P}, t, \mathcal{P})$ be a rivial complex vector bundle on $\mathcal{P}$. Then there is a map

$$P_X : X \times \mathcal{P} \to \mathcal{E}_X, \tag{3.14}$$

defined by

$$P_X(\xi, \rho) \equiv [\xi]_\rho \qquad (\,(\xi, \rho) \in X \times \mathcal{P}\,).$$

The map P_X has a dense image in $\mathcal{E}_X$ at each B-fiber. For $\rho \in \mathcal{P}$ and $x \in P_X(X \times \mathcal{P}) \cap \mathcal{E}_{X,\rho}$,

$$(P_X)^{-1}(x) = N_\rho$$

where N_ρ is a vector space defined in (3.10).

The following diagram is commutative:

$$\begin{array}{ccccc} & X \times \mathcal{P} & \stackrel{P_X}{\to} & \mathcal{E}_X & \\ t & \downarrow & & \downarrow & \Pi_X \\ & \mathcal{P} & \stackrel{id}{\to} & \mathcal{P} & . \end{array}$$

Hence (P_X, id) is a bundle map from $(X \times \mathcal{P}, t, \mathcal{P})$ to $(\mathcal{E}_X, \Pi_X, \mathcal{P})$.

By Theorem 3.1 and Definition 3.3, the atomic bundle of a Hilbert C*-module is a family of associated bundles of Hopf bundles indexed by spectrum B:

$$\mathcal{E}_X \cong \bigcup_{b \in B} \left(S(\mathcal{H}_b) \times_{U(1)} F^b_X \right).$$

Example 3.1 If $\mathcal{A}$ is commutative, then $\Omega \equiv \mathcal{P} \cong B$ and each B-fiber is

$$\mathcal{E}^b_X \to \mathcal{P}_b \cong \{b\}.$$

Hence the smooth structure of B-fiber collapses at each $b \in B$. Furthermore, if $X = \mathcal{A}^n \equiv \mathcal{A} \oplus \cdots \oplus \mathcal{A} \cong C(\Omega)^n$, then $N_\omega \cong \{\{f_i\}_{i=1}^n \in C(\Omega)^n : f_i(\omega) = 0\}$ for each $\omega \in \Omega$. Since $\mathcal{A}/N_\omega \cong \mathbf{C}$, $\mathcal{E}_{X,\omega} \cong \mathbf{C}^n$. Hence $\mathcal{E}_X \cong \Omega \times \mathbf{C}^n$.

Example 3.2 If $\mathcal{A} \equiv M_n(\mathbf{C})$, then $\mathcal{P} \cong \mathbf{C}P^{n-1}(= n-1$ dimensional complex projective space), B is a one-point set. Hence the atomic bundle of a Hilbert C*-module over $M_n(\mathbf{C})$ is a locally trivial smooth Hilbert bundle on $\mathbf{C}P^{n-1}$. Furthermore the atomic bundle of a Hilbert C*-module over a finite dimensional C*-algebra is a family of locally trivial smooth Hilbert bundles on complex projective spaces.

4 Connection and $*$-action

In this section, we define a flat connection D on the atomic bundle and show a relation between the associativity of $*$-action defined by D and the flatness of D.

4.1 The atomic connection of the atomic bundle

To define the $*$-action of $(C^\infty(\mathcal{P}), *)$ on the smooth sections of the atomic bundle of a Hilbert C*-module X, we define some connection D of $\mathcal{E}_X$ which is called the atomic connection.

Let $\mathcal{E}_X = (\mathcal{E}_X, \Pi_X, \mathcal{P})$ be the atomic bundle of a Hilbert C*-module X over a C*-algebra $\mathcal{A}$ defined in section 3. Let $\Gamma(\mathcal{E}_X)$ be the set of all bounded sections of $\mathcal{E}_X$, that is, $\Gamma(\mathcal{E}_X) \ni s : \mathcal{P} \to \mathcal{E}_X$ is a right inverse of Π_X and satisfies

$$\|s\| \equiv \sup_{\rho \in \mathcal{P}} \|s(\rho)\|_\rho < \infty. \tag{4.15}$$

By the following operations, $\Gamma(\mathcal{E}_X)$ is a complex linear space:

$$(s + s')(\rho) \equiv s(\rho) + s'(\rho) \qquad (\, \rho \in \mathcal{P},\ s, s' \in \Gamma(\mathcal{E}_X)\,),$$

$$(ks)(\rho) \equiv ks(\rho) \qquad (\, \rho \in \mathcal{P},\ s \in \Gamma(\mathcal{E}_X),\ k \in \mathbf{C}\,).$$

Furthermore $\Gamma(\mathcal{E}_X)$ is isometric onto a Banach space $\bigoplus_{\rho \in \mathcal{P}} \mathcal{E}_{X,\rho}$ of the direct sum of $\{\, \mathcal{E}_{X,\rho}\,\}_{\rho \in \mathcal{P}}$. By Theorem 3.1, we can consider the differentiability of $s \in \Gamma(\mathcal{E}_X)$ at each B-fiber

$$s|_{\mathcal{P}_b} : \mathcal{P}_b \to \mathcal{E}_X^b$$

for each $b \in B$ in the sense of Fréchet differential of Hilbert manifolds. Define $\Gamma_\infty(\mathcal{E}_X)$ the set of all B-fiberwise smooth sections in $\Gamma(\mathcal{E}_X)$.

Let $\mathfrak{X}(\mathcal{P})$ be the set of all B-fiberwise smooth vector fields of $\mathcal{P}$. H is a hermitian metric of $\mathcal{E}_X$ defined by

$$H_\rho(s, s^{'}) \equiv \left\langle s(\rho) \,\middle|\, s^{'}(\rho) \right\rangle_\rho \tag{4.16}$$

for $\rho \in \mathcal{P}$, $s, s^{'} \in \Gamma_\infty(\mathcal{E}_X)$ [11].

Definition 4.1 *D is a connection of $\mathcal{E}_X$ if D is a bilinear map of complex vector spaces*

$$D : \mathfrak{X}(\mathcal{P}) \times \Gamma_\infty(\mathcal{E}_X) \to \Gamma_\infty(\mathcal{E}_X)$$

which is $C^\infty(\mathcal{P})$-linear with respect to $\mathfrak{X}(\mathcal{P})$ and satisfies the Leibniz law with respect to $\Gamma_\infty(\mathcal{E}_X)$:

$$D_Y(s \cdot l) = \partial_Y l \cdot s + l \cdot D_Y s$$

for $s \in \Gamma_\infty(\mathcal{E}_X)$, $l \in C^\infty(\mathcal{P})$ and $Y \in \mathfrak{X}(\mathcal{P})$.

For $Y \in \mathfrak{X}(\mathcal{P}_b)$, $h \in S(\mathcal{H}_b)$ and $\rho \in \mathcal{V}_h$, we denote Y^h_ρ a tangent vector at ρ in a local coordinate $\mathcal{H}_h$. We define a linear map

$$A^h_{Y,\rho} : F^b_X \to F^b_X$$

by multiplying a number

$$-\frac{1}{2}\frac{< \beta_h(\rho)|Y^h_\rho >}{1 + \|\beta_h(\rho)\|^2}.$$

That is, for $z \equiv \beta_h(\rho) \in \mathcal{H}_h$,

$$A^h_{Y,\rho} e = -\frac{1}{2}\frac{< z|Y^h_\rho >}{1 + \|z\|^2} e \qquad (\, e \in F^b_X \,). \tag{4.17}$$

For the space of sections of $\mathcal{E}^b_X$, we show that

Proposition 4.1

$$D^h_{Y,\rho} \equiv \partial_{Y^h_\rho} + A^h_{Y,\rho}$$

defines a flat connection D of $\mathcal{E}^b_X$.

Proof. Appendix B. ∎

Definition 4.2 *We call the connection in Proposition 4.1 the atomic connection of the atomic bundle.*

We note that the atomic connection of the atomic bundle $\mathcal{E}_X$ is independent in a Hilbert C*-module X.

We prepare some equations for the main theorem. For $\rho \in \mathcal{V}_h$, define a vector in Ω^h_ρ in $\mathcal{H}_b$ by

$$\Omega^h_\rho \equiv \frac{\beta_h(\rho)+h}{\sqrt{1+\|\beta_h(\rho)\|^2}}.$$

Assume that $\rho = \omega_x \circ \pi_b$ for a $x \in \mathcal{H}_b$. Then

$$\Omega^h_\rho = \frac{x}{<h|x>}\left\|\frac{x}{<h|x>}\right\|^{-1} = \frac{|<h|x>|}{<h|x>}x.$$

Hence $[x] = [\Omega^h_\rho]$ and

$$<h|\Omega^h_\rho>> 0.$$

Let s be a section in $\Gamma(\mathcal{E}_X)$ such that for each $\rho \in \mathcal{P}_b \subset \mathcal{P}$, there is $\xi_\rho \in X$ $s(\rho) = [\xi_\rho]_\rho \in \mathcal{E}_{X,\rho}$. Let $z = \beta_h(\rho)$ for $h \in S(\mathcal{H}_b)$ such that $\rho \in \mathcal{V}_h$.

Lemma 4.1 *The following equations hold:*

$$<e\,|\,\psi_{\alpha,h}(s(\rho))>= \frac{\left\langle \Omega^h_{\rho'}\,\middle|\,\pi_b(<\xi'|\xi_\rho>)(z+h)\right\rangle}{\sqrt{1+\|z\|^2}} \tag{4.18}$$

for $e = \mathcal{O}([\xi']_{\rho'}, h) \in F^b_X$,

$$\partial_Y \phi_h(\rho)(s(\rho)) = \mathcal{O}\left(\left[\partial_Y\hat{\xi}_\rho + \xi_\rho\left(K^h_{Y,\rho} - \frac{<z|Y>}{2(1+\|z\|^2)}\right)\right]_\rho,\quad h\right) \tag{4.19}$$

where an element $K^h_{Y,\rho} \in \mathcal{A}$ *is defined by*

$$\pi_b(K^h_{Y,\rho})(h+z) = Y \tag{4.20}$$

and $[\partial_Y\hat{\xi}_\rho]_\rho \in \mathcal{E}_{X,\rho}$ *is defined by*

$$\left\langle [\eta]_\rho \,\middle|\, [\partial_Y\hat{\xi}_\rho]_\rho \right\rangle_\rho \equiv \rho(\partial_Y <\eta|\xi_\rho>)$$

for $[\eta]_\rho \in \mathcal{E}_{X,\rho}$.

Proof. Appendix C. ∎

4.2 The $*$-action of a function algebra on sections of the atomic bundle

By (2.2), the function space $C^{\infty}(\mathcal{P})$ is a * algebra with $*$-product which is not generally associative. We define the $*$-action of $(C^{\infty}(\mathcal{P}), *)$ on the smooth sections of the atomic bundle of a Hilbert C*-module by using the atomic connection D of $\mathcal{E}_X$. And we characterize algebraic properties, commutativity, associativity, of $*$-action by D and the curvature of $\mathcal{E}_X$ with respect to D.

Let D be any connection of $\mathcal{E}_X$.

Definition 4.3 *We define the (right) $*$-action of $C^{\infty}(\mathcal{P})$ on $\Gamma_{\infty}(\mathcal{E}_X)$ by*

$$s * l \equiv s \cdot l + \sqrt{-1} D_{X_l} s$$

for $l \in C^{\infty}(\mathcal{P})$ and $s \in \Gamma_{\infty}(\mathcal{E}_X)$ where X_l is the holomorphic Hamiltonian vector field of l with respect to the Kähler form of $\mathcal{P}$.

Remark 4.1 (i) By this lemma, the non-commutativity of $*$-action of $C^{\infty}(\mathcal{P})$ on $\Gamma_{\infty}(\mathcal{E}_X)$ depends on the connection D of $\mathcal{E}_X$ and the Kähler form of $\mathcal{P}$.

(ii) We use three different notions, $*$-action, $*$-product and involution by the same symbol "$*$" when they do not make confusions.

We show the geometric characterization of $*$-action.

Lemma 4.2 *For each $s \in \Gamma_{\infty}(\mathcal{E}_X)$ and $l, m \in C^{\infty}(\mathcal{P})$, the following equation holds:*

$$(s * l) * m - (s * m) * l = \left(\sqrt{-1}\{l,\, m\} + [D_{X_l},\, D_{X_m}]\right) s,$$

$$s * (l * m) - s * (m * l) = \left(\sqrt{-1}\{l, m\} + D_{[X_l,\, X_m]}\right) s.$$

Proof. According to Definition 4.3, we have

$(s * l) * m - (s * m) * l$

$$
\begin{aligned}
&= \left(s\cdot l+\sqrt{-1}D_{X_l}s\right)*m-\left(s\cdot m+\sqrt{-1}D_{X_m}s\right)*l \\
&= \sqrt{-1}D_{X_m}(s\cdot l)+\sqrt{-1}\left((D_{X_l}s)\cdot m+\sqrt{-1}D_{X_m}D_{X_l}s\right) \\
&\quad -\sqrt{-1}D_{X_l}(s\cdot m)-\sqrt{-1}\left((D_{X_m}s)\cdot l+\sqrt{-1}D_{X_l}D_{X_m}s\right) \\
&= \sqrt{-1}s\cdot(X_m l-X_l m)-D_{X_m}D_{X_l}s+D_{X_l}D_{X_m}s \\
&= \sqrt{-1}\{l,\,m\}s+[D_{X_l},\,D_{X_m}]s
\end{aligned}
$$

since

$$\partial l(X_m)-\partial m(X_l)=\omega(X_l,\overline{X_m})-\omega(X_m,\overline{X_l})=\{l,\,m\}.$$

Similarly we see

$$
\begin{aligned}
s*(l*m)-s*(m*l) &= s*(l*m-m*l) \\
&= \sqrt{-1}s*\{l,m\} \\
&= \sqrt{-1}\Big(\{l,m\}s+\sqrt{-1}D_{X_{\{l,m\}}}s\Big) \\
&= \sqrt{-1}\{l,m\}s-D_{-[X_l,\,X_m]}s \\
&= \left(\sqrt{-1}\{l,m\}+D_{[X_l,\,X_m]}\right)s
\end{aligned}
$$

since

$$X_{\{l,m\}}=-[X_l,\,X_m]$$

for $l,m\in C^\infty(\mathcal{P})$. ∎

Let the associator $a(l,m)$ of $l,m\in C^\infty(\mathcal{P})$ be an operator

$$a(l,m):\Gamma_\infty(\mathcal{E}_X)\to\Gamma_\infty(\mathcal{E}_X)$$

defined by

$$a(l,m)s\equiv(s*l)*m-s*(l*m)\qquad(s\in\Gamma_\infty(\mathcal{E}_X)).$$

Then we have a relation between associativity and curvature.

Proposition 4.2 *On $\Gamma_\infty(\mathcal{E}_X)$ and for $l,m\in C^\infty(\mathcal{P})$, the following equation holds:*

$$a(l,m)-a(m,l)=R_{X_l,X_m},$$

where R is the curvature of $\mathcal{E}_X$ with respect to D defined by

$$R_{Y,Z}\equiv[D_Y,\,D_Z]-D_{[Y,Z]}\qquad(Y,Z\in\mathfrak{X}(\mathcal{P})).$$

5 A sectional representation of Hilbert C*-modules

Before starting the main part of Theorem 1.2, we summarize notations in this article. Let X be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, $\mathcal{K}_u(\mathcal{P})$ the image of the Gel'fand representation of $\mathcal{A}$ and $\mathcal{E}_X = (\mathcal{E}_X, \Pi_X, \mathcal{P})$ the atomic bundle of X. For the map P_X defined in (3.14), define a linear map

$$P_{X*}: \quad \Gamma(X \times \mathcal{P}) \to \Gamma(\mathcal{E}_X),$$

$$(P_{X*}(s))\,(\rho) \quad \equiv P_X(s(\rho)) \qquad (\, s \in \Gamma(X \times \mathcal{P}),\ \rho \in \mathcal{P}\,).$$

We define a subspace Γ_X of $\Gamma(\mathcal{E}_X)$ as follows:

Definition 5.1

$$\Gamma_X \equiv P_{X*}\,(\Gamma_{const}(X \times \mathcal{P}))$$

where $\Gamma_{const}(X \times \mathcal{P})$ is the subspace of $\Gamma(X \times \mathcal{P})$ consisting of all constant sections.

Remark 5.1 Γ_X is quite smaller that the set of all holomorphic sections of $\mathcal{E}_X$. In fact, by Theorem 5.1, the hermitian form restricted on Γ_X is in $\mathcal{K}_u(\mathcal{P})$. Hence Γ_X is small as same as $\mathcal{K}_u(\mathcal{P})$.

We state a reconstruction theorem of a Hilbert C*-module by the atomic bundle.

Theorem 5.1 (i) *Any element in Γ_X is holomorphic.*

(ii) *Γ_X is a Hilbert C*-module over $\mathcal{K}_u(\mathcal{P})$.*

(iii) *There is an isomorphism between two Banach spaces*

$$\Psi : X \cong \Gamma_X$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \times & \mathcal{A} & \to & X \\ {\scriptstyle \Psi\times f} & \downarrow & & & \downarrow {\scriptstyle \Psi} \\ \Gamma_X & \times & \mathcal{K}_u(\mathcal{P}) & \to & \Gamma_X, \end{array}$$

where two horizontal arrows mean module actions. Hence, under the identification $f : \mathcal{A} \cong \mathcal{K}_u(\mathcal{P})$, Γ_X is isomorphic onto X as a Hilbert $\mathcal{A}$-module.

We prepare some lemmata for the proof of Theorem 5.1 and explain how the structure of Hilbert C*-module is interpreted as the geometrical structure of the atomic bundle.

For $\xi \in X$, we define a section $s_\xi \in \Gamma(\mathcal{E}_X)$ of $\mathcal{E}_X$ by

$$s_\xi(\rho) \equiv [\xi]_\rho \qquad (\rho \in \mathcal{P}).$$

Consider $*$-action defined in Definition 4.3 by the atomic connection D in Proposition 4.1 of $\mathcal{E}_X$. Recall that X and $\Gamma(\mathcal{E}_X)$ have norms defined by (3.9) and (4.15), respectively. Then, the following lemma holds.

Lemma 5.1 *For each $\xi \in X$,*

(i) $\xi \mapsto s_\xi$ *is linear and isometric,*

(ii) $s_\xi \in \Gamma_\infty(\mathcal{E}_X)$ *and it is holomorphic,*

(iii) $s_\xi * f_A = s_{\xi \cdot A}$ *for $A \in \mathcal{A}$.*

Proof. (i) $\xi \mapsto s_\xi$ is linear by definition of linear structure of each fiber of $\mathcal{E}_X$, and we have

$$\begin{aligned} \|s_\xi\| &= \sup_{\rho\in\mathcal{P}} \|s_\xi(\rho)\|_\rho \\ &= \sup_{\rho\in\mathcal{P}} |\rho(<\xi|\xi>)|^{1/2} \\ &= \| <\xi|\xi> \|^{1/2} \\ &= \|\xi\|_X. \end{aligned}$$

Hence s_ξ is bounded on $\mathcal{P}$ and a map s is an isometry.

(ii) Let $\rho \in \mathcal{P}$. Assume $\rho \in \mathcal{P}_b$ for some $b \in B$. Let $(\mathcal{H}, \pi)$ be a representative irreducible representation of b. Take local trivialization $\psi_{\alpha,h}$ at $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ and at ρ with a typical fiber F_X^b defined in Definition 3.4. By (4.19), we obtain

$$\partial_Y \phi_h(\rho)\,(s_\xi(\rho)) = \mathcal{O}\left(\left[\xi\left(K^h_{Y,\rho} - \frac{<z|Y>}{2(1+\|z\|^2)}\right)\right]_\rho,\quad h\right). \tag{5.21}$$

Owing to (4.20), the right-hand side of (5.21) is smooth with respect to $z \equiv \beta_h(\rho) \in \mathcal{H}_h$, and hence, s_ξ is smooth at $\mathcal{P}_b$ for each $b \in B$. For $\rho_0 \in \mathcal{P}_b$, we can choose $h_0 \in S(\mathcal{H}_b)$ such that

$$\rho_0 = < h_0|\pi_b(\cdot)h_0 > .$$

Then $\beta_{h_0}(\rho_0)=0$. By (4.18), we have

$$< e\,|\,\phi_{h_0}(\rho)(s_\xi(\rho)) >= \frac{\left\langle \Omega^{h_0}_{\rho'} \left| \pi_b(<\xi'|\xi>)(z+h_0) \right.\right\rangle}{\sqrt{1+\|z\|^2}}$$

for $z=\beta_{h_0}(\rho)$, $\rho\in\mathcal{V}_{h_0}$. For an anti-holomorphic tangent vector $\bar{Y}$ of $\mathcal{P}_b$, we have

$$\bar{\partial}_{\bar{Y}}\phi_h(\rho)\,(s_\xi(\rho)) = \mathcal{O}\left(\left[-\xi\frac{<Y|z>}{2(1+\|z\|^2)}\right]_\rho, \quad h\right)$$

from which follows

$$\left.\bar{\partial}_{\bar{Y}}\phi_h(\rho)\,(s_\xi(\rho))\right|_{z=0}=0.$$

We see that the anti-holomorphic derivative of s_ξ vanishes at each point in $\mathcal{P}_b$. Hence s_ξ is holomorphic.

(iii) Let $A\in\mathcal{A}$. For $b\in B$ and $\rho_0\in\mathcal{P}_b$, take local coordinate $(\mathcal{V}_h,\beta_h,\mathcal{H}_h)$ at ρ_0 where h is a unit vector in $\mathcal{H}$ and $(\mathcal{H},\pi)$ is a representative irreducible representation of b. Then for $z\in\mathcal{H}_h$, we have

$$\left(f_A\circ\beta_h^{-1}\right)(z)=\frac{<(z+h)|\pi(A)(z+h)>}{1+\|z\|^2}.$$

Then the representation $X^h_{f_A}$ of the Hamiltonian vector filed X_{f_A} at $(\mathcal{V}_h,\beta_h,\mathcal{H}_h)$ is

$$\left(X^h_{f_A}\right)_z = -\sqrt{-1}\Big(\pi(A)(z+h)- <h|\pi(A)(z+h)>(z+h)\Big)$$

for $z\in\mathcal{H}_h$. If we take h such that $\beta_h(\rho_0)=0$, then it holds that

$$\left(X^h_{f_A}\right)_0 = -\sqrt{-1}\Big(\pi(A)h- <h|\pi(A)h>h\Big).$$

D satisfies

$$<v|(D_{X_{f_A}}s)(\rho_0)>_{\rho_0} = \partial_{\rho_0}\Big(<v|s(\cdot)>_{\rho_0}\Big)(X_{f_A})$$

for $v\in E_h$, $s\in\Gamma_\infty(\mathcal{E}_X)$. Hence we have

$$(D_{X_{f_A}}s_\xi)(\rho_0)=\left[\xi a_{X_{f_A},0}\right]_{\rho_0}$$

where $a_{X_{f_A},0}\in\mathcal{A}$ satisfies

$$\begin{aligned}\pi\left(a_{X_{f_A},0}\right)h &= X_{f_A}\\ &= -\sqrt{-1}\Big(\pi(A)- <h|\pi(A)h>\Big)h.\end{aligned}$$

Therefore we have

$$\begin{array}{rcl}\sqrt{-1}(D_{X_{f_A}}s_\xi)(\rho_0) &=& \sqrt{-1}[\,\xi a_{X_{f_A},0}\,]_{\rho_0} \\ &=& \sqrt{-1}\left[\,\xi\cdot\Big(-\sqrt{-1}\Big(A-<h|\pi(A)h>\Big)\Big)\right]_{\rho_0} \\ &=& [\,\xi A]_{\rho_0}-[\xi]_{\rho_0}<h|\pi(A)h> \\ &=& s_{\xi A}(\rho_0)-s_\xi(\rho_0)f_A(\rho_0)\end{array}$$

from which follows

$$\begin{array}{rcl}(s_\xi * f_A)(\rho_0) &=& s_\xi(\rho_0)f_A(\rho_0)+\sqrt{-1}(D_{X_{f_A}}s_\xi)(\rho_0) \\ &=& s_{\xi A}(\rho_0).\end{array}$$

Therefore we arrive at

$$(s_\xi * f_A)(\rho) = s_{\xi A}(\rho) \qquad (\,\rho\in\mathcal{P}\,)$$

which proves lemma (iii). ∎

Proof of Theorem 5.1 (i). Define a map

$$\Psi : X \to \Gamma_\infty(\mathcal{E}_X); \quad \Psi(\xi)\equiv s_\xi \qquad (\,\xi\in X\,).$$

Then Ψ is a linear isometry by Lemma 5.1. For each $\tau\in\Gamma_{const}(X\times\mathcal{P})$, there is $\xi\in X$ such that $\tau(\rho)=(\xi,\rho)$ for $\rho\in\mathcal{P}$. We denote such τ by $\hat{s}_\xi$. Then

$$\begin{array}{rcl}s\in\Gamma_X &\Leftrightarrow& \text{there is } \xi\in X \text{ such that } s=P_{X*}\hat{s}_\xi \\ &\Leftrightarrow& s(\rho)=[\xi]_\rho \qquad (\,\rho\in\mathcal{P}\,) \\ &\Leftrightarrow& s=s_\xi \\ &\Leftrightarrow& s=\Psi(\xi) \\ &\Leftrightarrow& s\in\Psi(X).\end{array}$$

Hence $\Gamma_X=\Psi(X)$. Therefore Theorem 5.1 (i) follows from Lemma 5.1 (ii). ∎

Lemma 5.2 (i) Γ_X *is a right $\mathcal{K}_u(\mathcal{P})$-module by $*$-action defined in Definition 4.3.*

(ii) *For a hermitian metric H of $\mathcal{E}_X$ which is defined by Equation (4.16), let $\mathfrak{h}$ be the restriction $H|_{\Gamma_X}$ of H of $\mathcal{E}_X$ on Γ_X. Then a function-valued sesquilinear form*

$$\mathfrak{h} : \Gamma_X\times\Gamma_X\to C^\infty(\mathcal{P})$$

satisfies

$$\begin{array}{rcll} \mathfrak{h}(s,s^{'}) \in & \mathcal{K}_u(\mathcal{P}) & (\, s,s^{'} \in \Gamma_X\,), & \\ \overline{\mathfrak{h}(s,s^{'})} = & \mathfrak{h}(s^{'},s) & (\, s,s^{'} \in \Gamma_X\,), & \\ \mathfrak{h}(s,s) \geq & 0 & (\, s \in \Gamma_X\,), & (5.22) \\ \mathfrak{h}(\,s,\,s^{'} * f\,) = & \mathfrak{h}(s,s^{'}) * f & (\, s,s^{'} \in \Gamma_X,\ f \in \mathcal{K}_u(\mathcal{P})\,), & \\ \|\mathfrak{h}(s,s)\|^{1/2} = & \|s\| & (\, s \in \Gamma_X\,) & \end{array}$$

where the positivity in (5.22) means $\mathfrak{h}(s,s)$ being a positive-valued function on $\mathcal{P}$ and the norm of $\mathfrak{h}(s,s)$ is the one defined in (4.15).

(iii) *The following equation holds:*

$$\mathfrak{h}_\rho\,(\Psi(\xi),\Psi(\eta)) = \rho(<\xi|\eta>) \qquad (\,\xi,\eta \in X,\ \rho \in \mathcal{P}\,).$$

Proof. (i) From the proof of Theorem 5.1 (i), $\Gamma_X = \Psi(X)$. Since Lemma 5.1 (iii) and $\mathcal{K}_u(\mathcal{P}) = f(\mathcal{A})$, the following map

$$\Gamma_X \times \mathcal{K}_u(\mathcal{P}) = \Psi(X) \times f(\mathcal{A}) \ni (s,l) \longmapsto s * l \in \Psi(X) = \Gamma_X$$

is bilinear. Hence, Γ_X is a right $\mathcal{K}_u(\mathcal{P})$-module. Thus (i) is verified.
(ii) and (iii): Next, we have the following equations

$$\begin{array}{rcl} \mathfrak{h}_\rho(\Psi(\xi),\,\Psi(\xi^{'})) & = & \mathfrak{h}_\rho(s_\xi,s_{\xi^{'}}) \\ & = & H_\rho(s_\xi,s_{\xi^{'}}) \\ & = & <s_\xi(\rho)|s_{\xi^{'}}(\rho)>_\rho \\ & = & \rho(<\xi|\xi^{'}>), \end{array}$$

which proves (iii). Furthermore,

$$\rho(<\xi|\xi^{'}>) = f_{<\xi|\xi^{'}>}(\rho).$$

Therefore $\mathfrak{h}(\Psi(\xi),\,\Psi(\xi^{'})) = f_{<\xi|\xi^{'}>} \in \mathcal{K}_u(\mathcal{P})$. Hence $\mathfrak{h}(s,s^{'}) \in \mathcal{K}_u(\mathcal{P})$ for each $s,s^{'} \in \Gamma_X$. For $\xi,\eta \in X$, $A \in \mathcal{A}$,

$$\begin{array}{rcll} \mathfrak{h}_\rho(\,s_\eta,\ s_\xi * f_A\,) & = & \mathfrak{h}_\rho(s_\eta,\,s_{\xi A}) & \\ & = & \rho(<\eta|\xi A>) & (\text{ by using (iii)}) \\ & = & \rho(<\eta|\xi> A) & \\ & = & (f_{<\eta|\xi>A})(\rho) & \\ & = & \Big(f_{<\eta|\xi>} * f_A\Big)(\rho) & \\ & = & \Big(\mathfrak{h}(s_\eta,\,s_\xi) * f_A\Big)(\rho). & \end{array}$$

Hence it is shown that

$$\mathfrak{h}(s,\, s^{'} * l) = \mathfrak{h}(s,\, s^{'}) * l \qquad (s, s^{'} \in \Gamma_X, l \in \mathcal{K}_u(\mathcal{P})).$$

Remain equations appearing in the statement (ii) follow from the property of C*-valued inner product of X and the proof of Lemma 5.1 (i). This completes the proof of (ii). ∎

Proof of Theorem 5.1 (ii), (iii): (ii) By Lemma 5.2 (i), (ii) and Definition 3.2,

$$\mathfrak{h} : \Gamma_X \times \Gamma_X \to \mathcal{K}_u(\mathcal{P}) \tag{5.23}$$

is a positive definite C*-inner product of Γ_X. Hence Γ_X is a Hilbert C*-module over a C*-algebra $\mathcal{K}_u(\mathcal{P})$.
(iii) By the proof of Lemma 5.1 (i) and Lemma 5.2 (i), Ψ is an isomorphism between X and Γ_X. If we rewrite module actions ϕ and ψ of X and Γ_X, respectively, by

$$\begin{array}{rl} \phi(\xi, A) = & \xi A, \\ \psi(s, l) = & s * l \end{array}$$

for $\xi \in X$, $A \in \mathcal{A}$, $s \in \Gamma_X$ and $l \in \mathcal{K}_u(\mathcal{P})$, then we have

$$\begin{array}{rl} (\psi \circ (\Psi \times f))\,(\xi, A) = & \Psi(\xi) * f_A \\ = & s_\xi * f_A \\ = & s_{\xi A} \\ = & \Psi(\xi A) \\ = & (\Psi \circ \phi)\,(\xi, A) \end{array}$$

by Lemma 5.1 (iii). Hence we obtain the following equation:

$$\psi \circ (\Psi \times f) = \Psi \circ \phi.$$

Therefore the diagram in the statement (iii) is commutative. ∎

We summarize our results. The functional representation f of a non commutative unital C*-algebra $\mathcal{A}$ with the set $\mathcal{P}$ of all pure states on $\mathcal{A}$, and the sectional representation s of a Hilbert $\mathcal{A}$-module X are given as follows:

$$\begin{array}{llll}
\mathcal{A} & \stackrel{f}{\cong} & \mathcal{K}_u(\mathcal{P}) \subset C^\infty(\mathcal{P}), & \\
A & \mapsto & f_A; & f_A(\rho) \equiv \rho(A), \\
 & & & (A \in \mathcal{A},\ \rho \in \mathcal{P}), \\
X & \stackrel{s}{\cong} & \Gamma_X \subset \Gamma_{holo}(\mathcal{E}_X), & \\
\xi & \mapsto & s_\xi; & s_\xi(\rho) \equiv [\xi]_\rho, \\
 & & & (\xi \in X,\ \rho \in \mathcal{P})
\end{array}$$

where $\mathcal{E}_X$ is the atomic bundle of X and $\Gamma_{holo}(\mathcal{E}_X)$ is the set of all holomorphic sections on $\mathcal{E}_X$. The correspondence of module structures of them is the following:

$$\begin{array}{ccccccccc}
(\xi, A) & \in & X \times \mathcal{A} & \cong & \Gamma_X \times \mathcal{K}_u(\mathcal{P}) & \ni & (s_\xi, f_A) \\
 & & \downarrow & & \downarrow & & \\
\xi A & \in & X & \cong & \Gamma_X & \ni & s_\xi * f_A.
\end{array}$$

Remark 5.2 Compairing the characterization of non commutative Gel'fand representation of C*-algebra by [3], it seems that our characterization is not sufficient. Reader may request that another characterization which is defined by uniformity and etc, is necessary. Of course, we have tried to find more suitable characterization of section. Despite of our effort, we have not suceeded yet. The difficulty of characterization is similar to Remark 2.3. This is a problem in our future study.

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Appendix

A Hopf bundle

We summarize about Hopf bundle and its associated bundle.

A.1 Definition

We denote $\mathcal{H}$ a Hilbert space over $\mathbf{C}$ such that $\dim\mathcal{H} \geq 1$. Denote $\mathbf{C}^{\times} \equiv \mathbf{C}\setminus\{0\}$. Define

$$S(\mathcal{H}) \equiv \{z \in \mathcal{H} : \|z\| = 1\},$$

$$\mathcal{P}(\mathcal{H}) \equiv (\mathcal{H}\setminus\{0\})/\mathbf{C}^{\times}.$$

We call $S(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ *a Hilbert sphere* and *a projective Hilbert space* over $\mathcal{H}$, respectively. We denote an element of $\mathcal{P}(\mathcal{H})$ by $[z]$ for $z \in \mathcal{H}\setminus\{0\}$. We define a topology of $S(\mathcal{H})$ the relative topology of $\mathcal{H}$, and that of $\mathcal{P}(\mathcal{H})$ the quotient topology from $\mathcal{H}\setminus\{0\} \subset \mathcal{H}$ by the natural projection. Define a projection μ from $S(\mathcal{H})$ to $\mathcal{P}(\mathcal{H})$ by

$$\mu : \quad S(\mathcal{H}) \to \mathcal{P}(\mathcal{H}),$$

$$\mu(z) \quad \equiv [z] \quad (z \in S(\mathcal{H})).$$

Definition A.1 *We call* $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ *a Hopf (fiber)bundle over* $\mathcal{H}$*.*

Clearly, $\mu^{-1}([z]) \cong S^1$ for each $[z] \in \mathcal{P}(\mathcal{H})$.

Example A.1 When $\mathcal{H} = \mathbf{C}$, then

$$S(\mathcal{H}) = S^1, \quad \mathcal{P}(\mathcal{H}) = \{1pt\}.$$

When $\mathcal{H} = \mathbf{C}^2$,

$$S(\mathcal{H}) = S^3, \quad \mathcal{P}(\mathcal{H}) = \mathbf{C}P^1.$$

We define local trivial neighborhoods of a Hopf bundle ([3]).

Fix $h \in S(\mathcal{H})$ and define

$$\mathcal{W}_h \equiv \{z \in S(\mathcal{H}) :< h|z >> 0\},$$

$$\mathcal{V}_h \equiv \{[z] \in \mathcal{P}(\mathcal{H}) :< h|z >\neq 0\},$$

$$\mathcal{H}_h \equiv \{z \in \mathcal{H} :< h|z >= 0\},$$

$$\beta_h : \quad \mathcal{V}_h \to \mathcal{H}_h; \quad \beta_h([z]) \equiv \frac{z}{< h|z >} - h \quad ([z] \in \mathcal{V}_h).$$

Then $\{(\mathcal{V}_h, \beta_h, \mathcal{H}_h)\}_{h\in S(\mathcal{H})}$ is a system of local coordinates of $\mathcal{P}(\mathcal{H})$. $\mathcal{P}(\mathcal{H})$ is a Kähler manifold by this local coordinate system [3].

Let ψ_h be the local trivialization of $S(\mathcal{H})$ at $\mathcal{V}_h$ defined by

$$\psi_h : \mu^{-1}(\mathcal{V}_h) \cong \mathcal{V}_h \times U(1)$$

$$\psi_h(z) \equiv \ ([z],\, \phi_h(z) \,),$$

$$\phi_h(z) \equiv \ \frac{< z|h >}{|< h|z >|} \qquad (\, z \in \mu^{-1}(\mathcal{V}_h) \,),$$

$$\psi_h^{-1}([z], g) \equiv \ z\frac{< h|z >}{|< h|z >|}g \qquad (\, [z] \in \mathcal{V}_h,\, g \in U(1) \,).$$

Hence $\{\mathcal{V}_h\}_{h\in S(\mathcal{H})}$ is a system of local trivial neighborhoods of $\mathcal{P}(\mathcal{H})$ for $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$. Let R be a right action of $U(1)$ on $S(\mathcal{H})$ defined by

$$S(\mathcal{H}) \times U(1) \to S(\mathcal{H}); \quad (z, c) \mapsto z \cdot c = R_c z \equiv \bar{c}z.$$

Then the following conditions are satisfied:

(i) $\mu(R_c z) = \mu(z)$,

(ii) R is free, that is, if $R_c z = z$, then $c = 1$,

(iii) for each $h \in S(\mathcal{H})$,

$$\phi_h(R_c z) = \frac{< z|h >}{|< h|z >|}c \quad (z \in S(\mathcal{H}),\, c \in U(1)).$$

Hence $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ is a principal $U(1)$-bundle.

Lemma A.1 *For $h, h^{'} \in S(\mathcal{H})$, assume $\mathcal{V}_{h'} \cap \mathcal{V}_h \neq \emptyset$. For $z, X \in \mathcal{H}_h$, we have*

$$(\beta_{h'} \circ \beta_h^{-1})(z) = \ \frac{h + z}{< h'|h + z >} - h^{'},$$

$$\partial_z(\beta_{h'} \circ \beta_h^{-1})(X) = \ \frac{1}{< h'|h + z >}X - \frac{< h^{'}|X >}{< h'|h + z >^2}(h + z).$$

Definition A.2 *For a local trivial neighborhood $\mathcal{V}_h$,*

$$\Omega_h : \mathcal{V}_h \to S(\mathcal{H})$$

is a local section defined by

$$\Omega_h([z]) \equiv \frac{< z|h >}{|< z|h >|}z \quad ([z] \in \mathcal{P}(\mathcal{H}))$$

where $z \in S(\mathcal{H})$.

By definition, $< h|\Omega_h(\rho) >> 0$ for $\rho \in \mathcal{V}_h$.

A.2 Transition function

If $h, h' \in S(\mathcal{H})$ such that $h' \in \mathcal{V}_h$, then the transition function

$$Q_{h'h} : \mathcal{V}_h \cap \mathcal{V}_{h'} \to U(1)$$

is defined by

$$\begin{aligned} Q_{h'h}([z]) \equiv & \frac{< z|h' >}{| < h'|z > |}\left(\frac{< z|h >}{| < h|z > |}\right)^{-1} \\ = & \frac{< z|h' >}{| < h'|z > |}\frac{< h|z >}{| < h|z > |}. \end{aligned}$$

Fact A.1 (i)

$$Q_{hh}([z]) = 1 \quad ([z] \in \mathcal{V}_h).$$

(ii) *If $h, h' \in S(\mathcal{H})$ satisfy $< h'|h > \neq 0$, then*

$$Q_{h'h} = Q_{hh'}^{-1}.$$

(iii) *If $h, h', h'' \in S(\mathcal{H})$ are mutually non orthogonal, then*

$$Q_{h''h'}([z]) \cdot Q_{h'h}([z]) = Q_{h''h}([z]) \quad ([z] \in \mathcal{V}_h \cap \mathcal{V}_{h'} \cap \mathcal{V}_{h''}).$$

Lemma A.2 *Let X be a tangent vector of $\mathcal{P}(\mathcal{H})$ at $\rho \in \mathcal{V}_h \cap \mathcal{V}_{h'}$ which is realized in $\mathcal{H}_{h'}$ and $\beta_{h'}(\rho) = z$. Then we have*

$$\partial_z \left(Q_{h'h}^{-1} \circ \beta_{h'}^{-1}\right)(X) = -\frac{1}{2}\frac{< z + h'|h >^2 < h|X >}{| < h|z + h' > |^3}.$$

Proof. For $w \in \mathcal{H}_{h'}$, we have

$$\begin{aligned} \left(Q_{h'h}^{-1} \circ \beta_{h'}^{-1}\right)(w) = & \left(Q_{hh'} \circ \beta_{h'}^{-1}\right)(w) \\ = & \frac{< w + h'|h >}{| < h|w + h' > |}\frac{< h'|w + h >}{| < h'|w + h > |} \\ = & \frac{< w + h'|h >}{| < h|w + h' > |} \end{aligned}$$

because $< h'|w >= 0$ by definition of $\mathcal{H}_{h'}$.

$$\partial_z \left(Q^{-1}_{h'h} \circ \beta^{-1}_{h'}\right)(X) = \quad -\frac{1}{2}\frac{< z+h'|h >^2 < h|X >}{| < h|z+h' > |^3}.$$

∎

Lemma A.3

$$\left(Q_{h'h} \circ \beta^{-1}_{h'}\right)(w) \cdot \partial_w \left(Q^{-1}_{h'h} \circ \beta^{-1}_{h'}\right)(X) = -\frac{1}{2}\frac{< h|X >}{< h|w+h' >}.$$

Proof. By the previous lemma,

$$\begin{aligned}
&\left(Q_{h'h} \circ \beta^{-1}_{h'}\right)(w) \cdot \partial_w \left(Q^{-1}_{h'h} \circ \beta^{-1}_{h'}\right)(X) \\
&\quad = \quad \frac{< h|w+h' >}{| < w+h'|h > |} \cdot \left(-\frac{1}{2}\frac{< h|X >< w+h'|h >^2}{| < h|w+h' > |^3}\right) \\
&\quad = \quad -\frac{1}{2}\frac{< h|w+h' >< h|X >< w+h'|h >^2}{< w+h'|h >^2 < h|w+h' >^2} \\
&\quad = \quad -\frac{1}{2}\frac{< h|X >}{< h|w+h' >}.
\end{aligned}$$

∎

A.3 Associated bundles of Hopf bundles

Let F be a C^∞-manifold with left $U(1)$-action α and $S(\mathcal{H}) \times F$ the direct product space of $S(\mathcal{H})$ and F. Define a right $U(1)$-action γ on $S(\mathcal{H})$ by

$$z\gamma_c \equiv \bar{c}z \quad (\, c \in U(1),\, z \in S(\mathcal{H})\,).$$

We define $S(\mathcal{H}) \times_{U(1)} F$ by the set of all orbits of $U(1)$ in $S(\mathcal{H}) \times F$ where a $U(1)$-action is defined by

$$(z, f)c \equiv (z\gamma_c,\, \alpha(\bar{c})f) \qquad (\, c \in U(1),\, (z, f) \in S(\mathcal{H}) \times F\,).$$

The topology of $S(\mathcal{H}) \times_{U(1)} F$ is induced from $S(\mathcal{H}) \times F$ by the natural projection $\pi : S(\mathcal{H}) \times F \to S(\mathcal{H}) \times_{U(1)} F$. We denote the element of $S(\mathcal{H}) \times_{U(1)} F$ containing (x, f) by $[(x, f)]$. Define a projection

$$\pi_F : S(\mathcal{H}) \times_{U(1)} F \to \mathcal{P}(\mathcal{H})$$

by

$$\pi_F([(x, f)]) \equiv \mu(x) \qquad \Big([(x, f)] \in S(\mathcal{H}) \times_{U(1)} F \Big).$$

Definition A.3 *A fibration* $\mathbf{F} \equiv \Big(S(\mathcal{H}) \times_{U(1)} F,\, \pi_F,\, \mathcal{P}(\mathcal{H}) \Big)$ *is called the associated bundle of* $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ *by* F*.*

For $h \in S(\mathcal{H})$, define a map

$$\psi_{\alpha,h} : \pi_F^{-1}(\mathcal{V}_h) \to \mathcal{V}_h \times F$$

by

$$\psi_{\alpha,h}([(z, f)]) \equiv \big(\mu(z),\, \phi_{\alpha,h}([(z, f)]) \big),$$

$$\phi_{\alpha,h}([(z, f)]) \equiv \alpha(\phi_h(z)) f \qquad \big([(z, f)] \in \pi_F^{-1}(\mathcal{V}_h) \big).$$

Hence we have

$$\psi_{\alpha,h}([(z, f)]) = \left([z], \quad \alpha\left(\frac{< z|h >}{| < h|z > |} \right) f \right) \quad \big([(z, f)] \in \pi_F^{-1}(\mathcal{V}_h) \big),$$

$$\psi_{\alpha,h}^{-1}([z], f) = \left[\left(z, \quad \alpha\left(\frac{< h|z >}{| < h|z > |} \right) f \right) \right] \quad \big(([z], f) \in \mathcal{V}_h \times F \big).$$

The definition of $\psi_{\alpha,h}$ is independent choice of (z, f). In fact, if $(z', f') = (z, f)c$ for $c \in U(1)$, then $(z', f') = (\bar{c}z, \alpha(\bar{c})f)$ and

$$\begin{aligned}
\psi_{\alpha,h}([(z', f')] &= \left([z'], \quad \alpha\left(\frac{< z'|h >}{| < h|z' > |} \right) f' \right) \\
&= \left([z], \quad \alpha\left(c\frac{< z|h >}{| < h|z > |} \right) \alpha(\bar{c}) f \right) \\
&= \left([z], \quad \alpha\left(\frac{< z|h >}{| < h|z > |} \right) f \right) \\
&= \psi_{\alpha,h}([(z, f)]).
\end{aligned}$$

$\psi_{\alpha,h}$ is a local trivialization of $\mathbf{F}$ at $\mathcal{V}_h$. The transition function is given by

$$\hat{Q}_{\alpha,h',h} \equiv \psi_{\alpha,h'} \circ \psi_{\alpha,h}^{-1} : \; (\mathcal{V}_{h'} \cap \mathcal{V}_h) \times F \;\to\; (\mathcal{V}_{h'} \cap \mathcal{V}_h) \times F,$$

$$([z], f) \;\mapsto\; (\,[z],\, \alpha(Q_{h'h}([z]))f\,)\,.$$

Each vector space V over $\mathbf{C}$ has the scalar multiplication as $U(1)$-action α. If $F = V$, then we have

$$\psi_{\alpha,h}\,([(z,f)]) = \left([z], \quad \frac{< z|h >}{|< h|z >|} f \right) \quad (\,[(z,f)] \in \pi_F^{-1}(\mathcal{V}_h)\,),$$

$$\hat{Q}_{\alpha,h',h}([z],f) = \left([z], \quad \frac{< z|h' >}{|< h'|z >|} \frac{< h|z >}{|< h|z >|} f \right)$$

$$(\,([z],f) \in (\mathcal{V}_{h'} \cap \mathcal{V}_h) \times F\,)\,.$$

A.4 Recovery of the typical fiber

Let $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ be a Hopf bundle and F a complex Hilbert space. We consider $S(\mathcal{H}) \times_{U(1)} F$ by the scalar multiple of $U(1)$.

Proposition A.1 *There is the following equivalence relation of fiber bundles on $\mathcal{P}(\mathcal{H})$:*

$$\Big(S(\mathcal{H}) \times_{U(1)} F\Big) \times_{\mathcal{P}(\mathcal{H})} S(\mathcal{H}) \quad \cong \quad S(\mathcal{H}) \times F$$

where the left hand side is the fiber product of $(S(\mathcal{H}) \times_{U(1)} F, \pi_F, \mathcal{P}(\mathcal{H}))$ and $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$, and $(S(\mathcal{H}) \times F, \mu_F, \mathcal{P}(\mathcal{H}))$ is the trivial bundle.

Proof. Let

$$X_1 \equiv \Big(S(\mathcal{H}) \times_{U(1)} F\Big) \times_{\mathcal{P}(\mathcal{H})} S(\mathcal{H}).$$

We note that any element of X_1 is written as $([(h,v)], h)$ where $[(h,v)] \in S(\mathcal{H}) \times_{U(1)} F$ because

$$\pi_F([(h,v)]) = \mu(h)$$

and we can choose phase factor of (h,v) according to h. Let

$$\hat{\pi_F} : X_1 \to \mathcal{P}(\mathcal{H}); \quad \hat{\pi_F}([(h,v)], h) \equiv h.$$

Define a map

$$\Phi : X_1 \to S(\mathcal{H}) \times F; \quad \Phi\left([(h,v)],\, h\right) \equiv (h,v).$$

If $(h^{'},v^{'}) \in [(h,v)]$, then there is $c \in U(1)$ such that

$$(h^{'},v^{'}) = (h,v)c = (h\gamma_c, \bar{c}v) = (\bar{c}h, \bar{c}v).$$

But our notation restricts $c = 1$ since $h^{'} = h$. Hence Φ is well defined. Φ is bijective. Furthermore

$$\begin{array}{rl}(\mu_F \circ \Phi)([(h,v)],h) = & \mu_F(h,v) \\ = & h \\ = & \hat{\pi}_F([(h,v)],h).\end{array}$$

Therefore $\mu_F \circ \Phi = \hat{\pi}_F$ and (Φ, id) is a bundle map between X_1 and $(S(\mathcal{H}) \times F, \mu_F, \mathcal{P}(\mathcal{H}))$. Hence we have the statement. ∎

Let G be a group such that G acts on $S(\mathcal{H})$ transitively and acts on F trivially. Then we have the following proposition.

Proposition A.2 *There is the following equivalence of linear spaces:*

$$\left(\left(S(\mathcal{H}) \times_{U(1)} F\right) \times_{\mathcal{P}(\mathcal{H})} S(\mathcal{H})\right)\Big/ G \cong F.$$

Proof. We use the symbol in the previous proposition. Let α and β be actions of G on $\mathcal{S}(\mathcal{H})$ and F, respectively. Denote the action

$$\hat{\alpha} \equiv (\alpha \times_{U(1)} 1) \times_{\mathcal{P}(\mathcal{H})} \alpha$$

of G on X_1 and

$$Y_1 \equiv \left(\left(S(\mathcal{H}) \times_{U(1)} F\right) \times_{\mathcal{P}(\mathcal{H})} S(\mathcal{H})\right)\Big/ G.$$

For $[x] = [([(h,v)],h)] \in Y_1$,

$$[x] = \{([(\alpha_g h, v)], \alpha_g h) : g \in G\}.$$

Hence we can extend Φ as

$$\hat{\Phi} : Y_1 \to S(\mathcal{H}) \times F; \quad \hat{\Phi}([x]) \equiv [\Phi(x)].$$

Then

$$\begin{aligned}\hat{\Phi}([([(h,v)],h)]) = & \; [\Phi([([(h,v)],h))] \\ = & \; \{\Phi([(\alpha_g h, v)], \alpha_g h) : g \in G\} \\ = & \; \{(\alpha_g h, v) : g \in G\} \\ = & \; S(\mathcal{H}) \times \{v\}.\end{aligned}$$

Note $\hat{\Phi}$ is bijection, too. Hence we denote $\hat{v} \equiv \hat{\Phi}([([(h,v)],h)])$.] We define a linear structure for Y_1 by

$$a\hat{v} + b\hat{w} \equiv \widehat{(av + bw)} \quad (v, w \in F,\, a, b \in \mathbf{C}).$$

Then

$$\hat{} : Y_1 \to F$$

is a linear isomorphism. ∎

A.5 Connections of an associated bundle of a Hopf bundle

Let $\mathbf{F} \equiv (S(\mathcal{H}) \times_{U(1)} F, \pi_F, \mathcal{P}(\mathcal{H}))$ be an associated vector bundle of a Hopf bundle $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ by a complex Hilbert space F. Let $\Gamma(\mathbf{F})$ be the set of all smooth sections of $\mathbf{F}$, that is the set of right inverses of a projection π_F. By the following operations, $\Gamma(\mathbf{F})$ is a complex linear space:

$$(s + s^{'})(\rho) \equiv \; s(\rho) + s^{'}(\rho) \qquad (\rho \in \mathcal{P}(\mathcal{H}),\, s, s^{'} \in \Gamma(\mathbf{F})),$$

$$(ks)(\rho) \equiv \; ks(\rho) \qquad (\rho \in \mathcal{P}(\mathcal{H}),\, s \in \Gamma(\mathbf{F}),\, k \in \mathbf{C}).$$

Definition A.4 *D is a connection of $\mathbf{F}$ if D is a bilinear map of complex vector spaces*

$$D : \mathfrak{X}(\mathcal{P}(\mathcal{H})) \times \Gamma(\mathbf{F}) \to \Gamma(\mathbf{F})$$

which is $C^\infty(\mathcal{P}(\mathcal{H}))$-linear with respect to $\mathfrak{X}(\mathcal{P}(\mathcal{H}))$ and satisfies the Leibniz law with respect to $\Gamma(\mathbf{F})$:

$$D_Y(s \cdot l) = \partial_Y l \cdot s + l \cdot D_Y s$$

for $s \in \Gamma(\mathbf{F})$, $l \in C^\infty(\mathcal{P}(\mathcal{H})$ and $Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$.

For $Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$, $h \in S(\mathcal{H})$ and $\rho \in \mathcal{V}_h$, we denote Y^h_ρ a tangent vector at ρ in a local coordinate $\mathcal{H}_h$. We consider a linear map

$$A^h_{Y,\rho} : F \to F$$

such that $\partial_Y|^h_\rho + A^h_{Y,\rho}$ is a connection of $\mathbf{F}$.

Fact A.2

$$D \equiv \partial + A$$

is a connection of $\mathbf{F}$ *if and only if a family* $\{A^h\}_{h\in S(\mathcal{H})}$ *satisfies the following equality:*

$$A^{h'}_{Y,\rho} = -\frac{1}{2}\frac{<h|Y>}{<h|z+h'>} + A^h_{Y,\rho} \quad (\,\rho \in \mathcal{V}_{h'} \cap \mathcal{V}_h\,) \tag{A.24}$$

where Y *is a holomorphic tangent vector of* $\mathcal{P}(\mathcal{H})$ *at* ρ *which is realized on* $\mathcal{H}_{h'}$ *and* $z = \beta_{h'}(\rho)$*.*

Proof. By using Leibniz rule and Lemma A.3,

$$\left(A^{h'}_Y \circ \beta^{-1}_{h'}\right)(z) = -\frac{1}{2}\frac{<h|Y>}{<h|z+h'>} + \left(A^h_Y \circ \beta^{-1}_{h'}\right)(z) \quad (\,z \in \beta_{h'}(\mathcal{V}_{h'} \cap \mathcal{V}_h)\,).$$

We have the statement. ∎

B The atomic connection

Proof of Proposition 4.1.

At the beginning, we show the cocycle condition for $A \equiv \{A^h\}_{h\in S(\mathcal{H}_b)}$ defined by (4.17). For $\rho \in \mathcal{P}_b$, choose $h, h' \in S(\mathcal{H}_b)$ such that $\rho \in \mathcal{V}_h \cap \mathcal{V}_{h'}$. The cocycle condition for A is given by (A.24) in Appendix A.2.

Let $z' \equiv \beta_{h'}(\rho)$, $z \equiv \beta_h(\rho)$. Then we have

$$
\begin{aligned}
-2 \cdot A^h_{X,\rho} &= \frac{< z|X^h_\rho >}{1 + \|z\|^2} \\
&= \frac{< (\beta_h \circ \beta_{h'}^{-1})(z') \,|\, \partial_{z'}(\beta_h \circ \beta_{h'}^{-1})(X^{h'}_\rho) >}{1 + \|(\beta_h \circ \beta_{h'}^{-1})(z')\|^2} \\
&= \frac{\left\langle \dfrac{z' + h'}{< h|z' + h' >} \,\middle|\, \dfrac{X^{h'}_\rho}{< h|z' + h' >} - \dfrac{< h|X^{h'}_\rho > (z' + h')}{< h|z' + h' >^2} \right\rangle}{\left\| \dfrac{z' + h'}{< h|z' + h' >} \right\|^2} \\
&= \frac{< z' + h'|X^{h'}_\rho > - \dfrac{\|z' + h'\|^2 < h|X^{h'}_\rho >}{< h|z' + h' >}}{\|z' + h'\|^2} \\
&= \frac{< z'|X^{h'}_\rho >}{1 + \|z'\|^2} - \frac{< h|X^{h'}_\rho >}{< h|z' + h' >} \quad (\text{ since } < h'|X^{h'}_\rho >= 0) \\
&= -2 \cdot A^{h'}_{X,\rho} - \frac{< h|X^{h'}_\rho >}{< h|z' + h' >}.
\end{aligned}
$$

We obtain (A.24). Therefore D which is defined in Proposition 4.1 is a connection.

The curvature R of D is written by using A defined in (4.17) as follows:

$$R_{X,Y} = (dA)(X, Y) + (A \wedge A)(X, Y) \qquad (X, Y \in \mathfrak{X}(\mathcal{P}_b)\,).$$

Since A is scalar,

$$A \wedge A = 0.$$

In a local coordinate $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ of $\rho \in \mathcal{P}_b$ and $z = \beta_h(\rho) \in \mathcal{H}_h$, we have

$$
\begin{aligned}
(d_z A)(X,Y) = \; & X A^h_{Y,z} - Y A^h_{X,z} - A^h_{[X,Y],z} \\
= \; & \frac{X < z|Y > - Y < z|X >}{2(1+\|z\|^2)} \\
& + \frac{< z|X >< z|Y > - < z|X >< z|Y >}{2(1+\|z\|^2)^2} \\
& - \frac{< z|[X,Y] >}{2(1+\|z\|^2)} \\
= \; & \frac{d_z(< z| \cdot >)(X,Y)}{2(1+\|z\|^2)} \\
= \; & \frac{(d_z^2 k)(X,Y)}{2(1+\|z\|^2)} \\
= \; & 0,
\end{aligned}
$$

where a one-form $< z|\cdot >$ is defined by

$$< z|\cdot > (X) \equiv < z|X >$$

and a function $k(z) \equiv \|z\|^2 = < z|z >$ defined on $\mathcal{H}_h$. Hence we arrive at

$$R = 0.$$

Thus D is flat. ∎

C Proof of Lemma 4.1

Proof. Let $\phi_{\alpha,h} : (\Pi^b_X)^{-1}(\mathcal{V}_h) \to F^b_X$ be a map defined by

$$\psi_{\alpha,h}(x) = (\mu_b(h),\, \phi_{\alpha,h}(x))\,.$$

For $e = \mathcal{O}([\xi^{'}]_{\rho^{'}}, h) \in F_X^b$ such that $h \in \mu_b^{-1}(\rho)$, we have

$$
\begin{aligned}
< e \,|\, \phi_{\alpha,h}(s(\rho)) > &= \; < \mathcal{O}([\xi^{'}]_{\rho^{'}}, h^{'}) \,|\mathcal{O}([\xi]_\rho, h) > \\
&= \left\langle [\xi^{'}]_\rho \left| \left[\xi u^h_{\rho,\rho^{'}}\right]_{\rho^{'}} \right\rangle_{\rho^{'}} \right. \\
&= \rho \left(< \xi^{'} | \xi_\rho u^h_{\rho^{'},\rho^{'}} > \right) \\
&= \left\langle \Omega^h_{\rho^{'}} | \pi_b(< \xi^{'} | \xi_\rho u^h_{\rho^{'},\rho} >) \Omega^h_{\rho^{'}} \right\rangle \\
&= \left\langle \Omega^h_{\rho^{'}} | \pi_b \left(< \xi^{'} | \xi_\rho > \right) \pi_b \left(u^h_{\rho^{'},\rho} \right) \Omega^h_{\rho^{'}} \right\rangle \\
&= \left\langle \Omega^h_{\rho^{'}} | \pi_b \left(< \xi^{'} | \xi_\rho > \right) \Omega^h_\rho \right\rangle \\
&= \frac{\left\langle \Omega^h_{\rho^{'}} | \pi_b(< \xi^{'} | \xi_\rho >)(z+h) \right\rangle}{\sqrt{1 + \|z\|^2}}.
\end{aligned}
$$

From this, the following equality holds:

$$
\begin{aligned}
< e \,|\, \partial_Y \phi_h(\rho)(s(\rho)) > &= \; \partial_Y < e \,|\, \phi_h(\rho)(s(\rho)) > \\
&= \partial_Y \left(\frac{\left\langle \Omega^h_{\rho^{'}} | \pi_b(< \xi^{'} | \xi_\rho >)(z+h) \right\rangle}{\sqrt{1 + \|z\|^2}} \right) \\
&= \frac{\left\langle \Omega^h_{\rho^{'}} | \pi_b(\partial_X < \xi^{'} | \xi_\rho >)(z+h) \right\rangle}{\sqrt{1 + \|z\|^2}} \\
&\quad + \frac{\left\langle \Omega^h_{\rho^{'}} | \pi_b(< \xi^{'} | \xi_\rho >) Y \right\rangle}{\sqrt{1 + \|z\|^2}} \\
&\quad - \frac{\left\langle \Omega^h_{\rho^{'}} | \pi_b(< \xi^{'} | \xi_\rho >)(z+h) \right\rangle < z | Y >}{2(\sqrt{1 + \|z\|^2})^3}.
\end{aligned}
$$

Hence we have

$$
\partial_Y \phi_h(\rho) \, (s(\rho)) = \mathcal{O} \left(\left[\partial_Y \hat{\xi}_\rho + \xi_\rho \left(K^b_Y - \frac{< z | Y >}{2(1 + \|z\|^2)} \right) \right]_\rho , \quad h \right). \quad \text{(C.25)}
$$

We finish to prove Lemma. ∎

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