Geometric realization of T-shaped root systems and counterexamples to Hilbert's fourteenth problem

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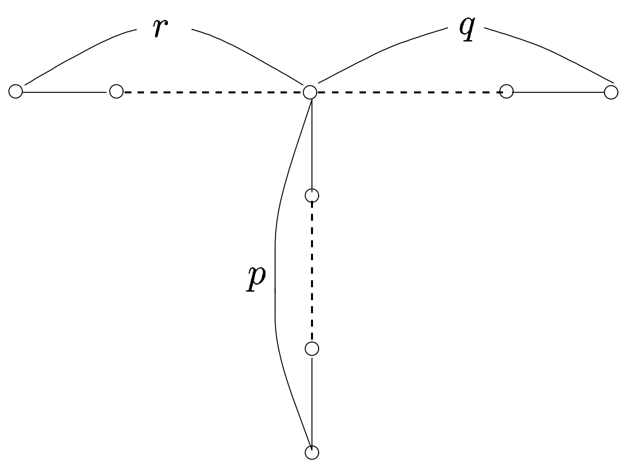
Abstract: Generalizing a result of Dolgachev, we realize the root system $T_{p,q,r}$ in the cohomology group of a certain rational variety of Picard number $p+q+r-1$. As an application we show that the invariant ring of a tensor product of the actions of Nagata type is infinitely generated if the Weyl group of the corresponding root system $T_{p,q,r}$ is infinite. In this sense this article is a continuation of [4].

1

The Dynkin diagram $T_{p,q,r}$



defines a lattice $L_{p,q,r}$ of rank $p+q+r-2$. The set of vertices α_i's is its basis as a free $\mathbf{Z}$-module. The bilinear form is defined to be $(\alpha_i.\alpha_j) = -2, 0$ or 1 according as $i = j$, α_i and α_j are disjoint or joined by an edge. It is known that the root system $T_{p,q,r}$ is of finite type, affine or infinite according as $1/p + 1/q + 1/r$ is $> 1, = 1$ or < 1 ([2] Ex. 4.2).

In the case $p = 2$, Dolgachev[1] realizes this root system $T_{2,q,r}$ in the cohomology group of the blow-up $Bl_{q+r\,\mathrm{pts}}\,\mathbf{P}^{r-1}$ of the $(r-1)$-dimensional

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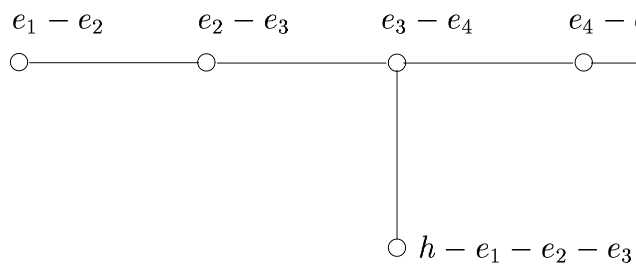
projective space $\mathbf{P}^{r-1}$ at $q+r$ points in general position. This is a generalization of the classical discovery that the configuration of the 27 lines on a nonsingular cubic surface $S \subset \mathbf{P}^3$, which is the blow-up of $\mathbf{P}^2$ at six points, has a symmetry of the Weyl group of $E_6 = T_{2,3,3}$ (cf. [3]). The moduli of cubic surfaces is a quotient of an open set of $\mathbf{P}^2 \times \mathbf{P}^2$ by $W(E_6)$ (cf. [1]) and the cohomology group $H^2(S, \mathbf{Z})$ has a monodromy action of $W(E_6)$.



The reflections by $e_i - e_{i+1}$, $1 \le i \le 5$, generate all permutations of the 6 points which are the centers of blowing up and that by $h - e_1 - e_2 - e_3$ corresponds to the quadratic Cremona transformation

$$\mathbf{P}^2 \cdots \to \mathbf{P}^2, \qquad (x_1 : x_2 : x_3) \mapsto (1/x_1 : 1/x_2 : 1/x_3).$$

It is natural to extend the result of Dolgachev to all diagrams of T-shape. The answer is simple: just generalize $\mathbf{P}^{r-1}$ to the product $(\mathbf{P}^{r-1})^{p-1}$ of its $p-1$ copies. Let X be a blow-up of the product $(\mathbf{P}^{r-1})^{p-1}$ at $q+r$ points in general position. The second cohomology group $H^2(X, \mathbf{Z})$, or equivalently $\operatorname{Pic} X$, is a free $\mathbf{Z}$-module of rank $p+q+r-1$ and has a basis consisting of

$$h_i, \quad 1 \le i \le p-1, \quad \text{and} \quad e_j, \quad 1 \le j \le q+r, \tag{1}$$

where h_i is the pull-back of the hyperplane class on the ith factor of $(\mathbf{P}^{r-1})^{p-1}$ and e_j the class of the exceptional divisor over the jth center of blowing up. We refer (1) as the *tautological basis.*

Theorem 1 *The root system $T_{p,q,r}$ is realized in the orthogonal complement L of the anti-canonical class $c_1(X)$ in the second cohomology group $H^2(X, \mathbf{Z})$ endowed with a certain symmetric bilinear form. (See §3.) Moreover, for each element $w : H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathbf{Z})$ of the Weyl group $W(T_{p,q,r})$, there is a* strong *birational map $\Psi_w : X_w \cdots \to X$ of a a blow-up X_w of $(\mathbf{P}^{r-1})^{p-1}$ at $q+r$ points such that the pull-back of a tautological basis of X_w coincides with the transformation of that of X by w.*

A birational map is called *strong* if it is an isomorphism in codimension one.

In the special case $q = 1$, X has a birational action of $W(T_{p,1,r}) = W(A_{p+r-1})$, which is the symmetric group of degree $p+r$. In fact X is a GIT quotient of the Grassmannian variety $G(p, p+r)$ by the maximal torus $T \simeq (\mathbf{C}^*)^{p+r-1}$ of its automorphism group $G \simeq PGL(p+r)$. Hence the Weyl group of G acts on X birationally. X is a compactification of the configuration space of ordered $p+r$ points on $\mathbf{P}^{r-1}$.

Remark The isomorphism between $G(p, p+r)$ and $G(r, p+r)$ induces a strong birational map between $Bl_{r+1\,\mathrm{pts}}\,(\mathbf{P}^{r-1})^{p-1}$ and $Bl_{p+1\,\mathrm{pts}}\,(\mathbf{P}^{p-1})^{r-1}$ and hence that between $Bl_{q+r\,\mathrm{pts}}\,(\mathbf{P}^{r-1})^{p-1}$ and $Bl_{p+q\,\mathrm{pts}}\,(\mathbf{P}^{p-1})^{r-1}$. For example $Bl_{q+r\,\mathrm{pts}}\,\mathbf{P}^{r-1}$ is strongly birationally equivalent to $Bl_{2+q\,\mathrm{pts}}\,(\mathbf{P}^{1})^{r-1}$.

2

Our interest in Theorem 1 comes from Nagata's counterexample to Hilbert's fourteenth problem also. Let

$$(t_1, \ldots, t_n) \in \mathbf{C}^n \ \downarrow \mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] =: S \tag{2}$$

$$\begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases}, \quad 1 \le i \le n,$$

be the standard unipotent action of $\mathbf{C}^n$, or the additive algebraic group $\mathbf{G}_a^n$ more precisely, on the polynomial ring S of $2n$ variables and $G \subset \mathbf{C}^n$ a general linear subspace. In [5], Nagata studied the invariant ring S^G of the subaction of G. The key fact is that the ring S^G is isomorphic to the total coordinate ring

$$\mathcal{TC}(X) := \bigoplus_{a, b_1, \ldots, b_n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(ah - b_1 e_1 - \cdots - b_n e_n)) \simeq \bigoplus_{L \in \operatorname{Pic} X} H^0(X, L) \tag{3}$$

of the variety $X = Bl_{n\,\mathrm{pts}}\,\mathbf{P}^{r-1}$, where r is the codimension of $G \subset \mathbf{C}^n$.

In [4], we pay attention to the support of this graded ring $\mathcal{TC}(X)$, which is the semi-group $\operatorname{Eff} X \subset \operatorname{Pic} X$ of effective divisor classes on X. A divisor $D \subset X$ is called a (-1)-divisor if there is a strong birational map $X \cdots \to X'$ such that the image of D can be contracted to a smooth point. Obviously the linear equivalence class of a (-1)-divisor is indispensable as generator of $\operatorname{Eff} X$.

Assume that the inequality

$$\frac{1}{2}+\frac{1}{n-r}+\frac{1}{r}\leq 1. \tag{4}$$

holds. Then the Weyl group $W(T_{2,n-r,r})$ of X is infinite and infinitely many (-1)-divisors on X are obtained as its orbit. Hence Eff X and $\mathcal{TC}(X)$ are not finitely generated. This is an outline of the main argument of [4].

In order to obtain more examples, we take $p-1$ actions

$$G_i \ \downarrow \mathbf{C}[x_1,\dots,x_n,y_1,\dots,y_n]=:S, \qquad G_i\subset \mathbf{C}^n,\ 1\leq i\leq p-1$$

of Nagata type on the same polynomial ring S and take their tensor product

$$G=\bigoplus_{i=1}^{p-1} G_i \downarrow S\otimes_{\mathbf{C}[x]}\cdots\otimes_{\mathbf{C}[x]} S=:\tilde{S} \tag{5}$$

over $\mathbf{C}[x_1,\dots,x_n]$. $\tilde{S}$ is a polynomial ring of pn variables.

Theorem 2 *The invariant ring $\tilde{S}^G$ of the above action (5) is isomorphic to the total coordinate ring*

$$\bigoplus_{a_1,\dots,a_{p-1},b_1,\dots,b_n\in\mathbf{Z}} H^0(X,\mathcal{O}_X(a_1h_1+\cdots+a_{p-1}h_{p-1}-b_1e_1-\cdots-b_ne_n))$$

of the blow-up X of the product $\mathbf{P}^{r_1-1}\times\cdots\times\mathbf{P}^{r_{p-1}-1}$ of $p-1$ projective spaces at n points, where h_i is the pull-back of the hyperplane class of $\mathbf{P}^{r_i-1}$.

We can localize the action (5) by $x_1,\dots,x_n$ since they are G-invariant. Then the additive group G acts on

$$\tilde{S}[x_1^{-1},\dots,x_n^{-1}]=S[x_1^{-1},\dots,x_n^{-1}]\otimes_{\mathbf{C}[x,x^{-1}]}\cdots\otimes_{\mathbf{C}[x,x^{-1}]} S[x_1^{-1},\dots,x_n^{-1}]$$

$$=\mathbf{C}[x_1^{\pm1},\dots,x_n^{\pm1},\frac{y_1}{x_1},\dots,\frac{y_n}{x_n}]\otimes_{\mathbf{C}[x,x^{-1}]}\cdots\otimes_{\mathbf{C}[x,x^{-1}]}\mathbf{C}[x_1^{\pm1},\dots,x_n^{\pm1},\frac{y_1}{x_1},\dots,\frac{y_n}{x_n}].$$

Since $(t_1,\dots,t_n)\in G_i$ acts by the translation $y_j/x_j\mapsto y_j/x_j+t_j$, $1\leq j\leq n$, the invariant ring $\tilde{S}[x_1^{-1},\dots,x_n^{-1}]^G$ is a polynomial ring of $r_1+\cdots+r_{p-1}$ variables. The rest of the proof is similar to that of the case $p=2$ in [4] and we omit it.

If r_i's are all the same, then we can apply Theorem 1 and obtain the following by the same reason, that is, X has infinitely many (-1)-divisors.

Theorem 3 *The invariant ring S^G of (5) is not finitely generated if $G_i \subset \mathbf{C}^n$ are general subspaces of codimension r and if the inequality*

$$\frac{1}{p} + \frac{1}{n-r} + \frac{1}{r} \leq 1$$

holds.

In the case $p = 2$ there are three cases where the diagram is of affine type:

r	q	X	diagram	$\dim G = q$
3	6	$Bl_{9\,\mathrm{pts}}\,\mathbf{P}^2$	$T_{2,[6],3}$	6
4	4	$Bl_{8\,\mathrm{pts}}\,\mathbf{P}^3$	$T_{2,[4],4}$	4
6	3	$Bl_{9\,\mathrm{pts}}\,\mathbf{P}^5$	$T_{2,[3],6}$	3

Allowing $p \geq 3$, we obtain three new ones with $p \leq r$. (See Remark at the end of §1.)

p	r	q	X	diagram	$\dim G = (p-1)q$
3	3	3	$Bl_{6\,\mathrm{pts}}\,\mathbf{P}^2 \times \mathbf{P}^2$	$T_{3,[3],3}$	6
3	6	2	$Bl_{8\,\mathrm{pts}}\,\mathbf{P}^5 \times \mathbf{P}^5$	$T_{3,[2],6}$	4
4	4	2	$Bl_{6\,\mathrm{pts}}\,\mathbf{P}^3 \times \mathbf{P}^3 \times \mathbf{P}^3$	$T_{4,[2],4}$	6

$T_{p,[q],r}$ is the diagram $T_{p,q,r}$ plus an extra vertex, which is defined in the next section.

3 Proof of Theorem 1

Let X be as in the theorem. The anti-canonical class $c_1(X)$ is equal to

$$r(h_1 + \cdots + h_{p-1}) - (r-2)(e_1 + \cdots + e_{q+r}).$$

We define an integral symmetric bilinear form on $H^2(X, \mathbf{Z})$ as follows:

1. h_i and e_j are orthogonal for every $1 \leq i \leq p-1$ and $1 \leq j \leq q+r$,
2. $$(e_i.e_j) = \begin{cases} -1 & i = j \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (h_i.h_j) = \begin{cases} r-2 & i = j \\ r-1 & \text{otherwise.} \end{cases}$$

We take a new $\mathbf{Z}$-basis of $H^2(X, \mathbf{Z})$ consisting of

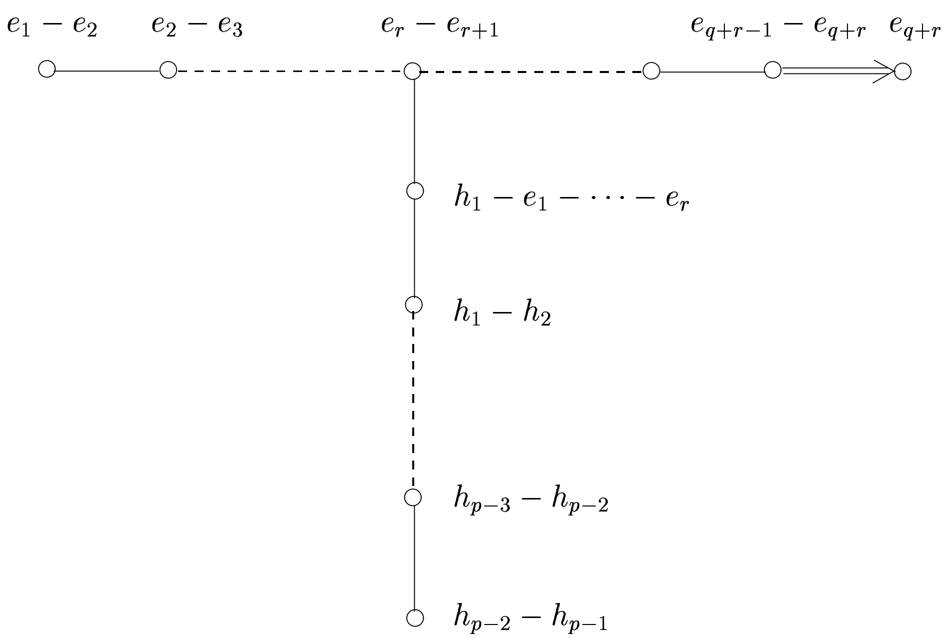
1. $h_1 - e_1 - \cdots - e_r$,

2. $h_i - h_{i+1}, \quad 1 \le i \le p-2$,

3. $e_j - e_{j+1}, \quad 1 \le j \le q+r-1$, and

4. e_{q+r}.

Then $H^2(X, \mathbf{Z})$ becomes a root system with the following diagram, which is denoted by $T_{p,[q],r}$. (See [2] §5.11 also for $T_{2,[q],3}$.)



The $p+q+r-2$ classes except for e_n are of length -2 and form a basis of the orthogonal complement L of $c_1(X)$ in $H^2(X, \mathbf{Z})$. Hence L is isomorphic to the root lattice $L_{p,q,r}$.

In order to show the latter half of Theorem 1, it is enough to check it for the simple reflections. This is obvious for $e_j - e_{j+1}$'s and $h_i - h_{i+1}$'s since they correspond to transpositions of a pair of centers and a pair of factors. The reflection with respect to $h_1 - e_1 - \cdots - e_r$ transforms the tautological basis (1) of $H^2(X, \mathbf{Z})$ as follows:

$$\begin{cases} h_1 \mapsto (r-1)h_1 - (r-2)\sum_{j=1}^r e_j \\ h_i \mapsto (r-1)h_1 + h_i - (r-1)\sum_{j=1}^r e_j, & 2 \le i \le p-1, \\ e_j \mapsto h_1 - e_1 - \cdots - \check{e}_j - \cdots - e_r, & 1 \le j \le r, \\ e_j \mapsto e_j, & r+1 \le j \le q+r. \end{cases} \tag{6}$$

Let $P = \{p_1, \dots, p_r\}$ be a set of r distinct points on $(\mathbf{P}^{r-1})^{p-1}$. P is *non-degenerate* if ith components $p_1^{(i)}, \dots, p_r^{(i)}$ spans $\mathbf{P}^{r-1}$ for every $1 \leq i \leq p-1$. If P is non-degenerate we can choose homogeneous coordinates of $\mathbf{P}^{r-1}$'s such that P is the image of the r coordinate points by the diagonal morphism $\Delta : \mathbf{P}^{r-1} \hookrightarrow (\mathbf{P}^{r-1})^{p-1}$.

Lemma *Let $P = \{p_1, \dots, p_r\}$ and $Q = \{q_1, \dots, q_r\}$ be non-degenerate sets of r points of $(\mathbf{P}^{r-1})^{p-1}$ and X_P and X_Q be the blow-ups with center P and Q, respectively. Then there exists a strong birational map*

$$\Psi = \Psi_{P,Q} : X_P \cdots \longrightarrow X_Q$$

such that

$$\begin{cases} \Psi^* h'_1 = (r-1)h_1 - (r-2)\sum_{j=1}^r e_j \\ \Psi^* h'_i = (r-1)h_1 + h_i - (r-1)\sum_{j=1}^r e_j, \quad 2 \leq i \leq p-1, \\ \Psi^* e'_j = h_1 - e_1 - \cdots - \check{e}_j - \cdots - e_r, \quad 1 \leq j \leq r, \end{cases} \tag{7}$$

where $\{h_i, e_j\}$ and $\{h'_i, e'_j\}$ are tautological bases of $\operatorname{Pic} X_P$ *and* $\operatorname{Pic} X_Q$, *respectively.*

Proof. We may assume that both P and Q are the image of the coordinate points by the diagonal morphism Δ. Consider the (toric) Cremona transformation

$$\bar{\Psi} : \mathbf{P}^{r-1} \times \mathbf{P}^{r-1} \times \cdots \times \mathbf{P}^{r-1} \cdots \rightarrow \mathbf{P}^{r-1} \times \mathbf{P}^{r-1} \times \cdots \times \mathbf{P}^{r-1}$$

$$((x_1 : x_2 : \ldots : x_r), (y_1 : y_2 : \ldots : y_r), \dots, (z_1 : z_2 : \ldots : z_r)) \mapsto$$
$$((\frac{1}{x_1} : \frac{1}{x_2} : \ldots : \frac{1}{x_r}), (\frac{y_1}{x_1} : \frac{y_2}{x_2} : \ldots : \frac{y_r}{x_r}), \dots, (\frac{z_1}{x_1} : \frac{z_1}{x_2} : \ldots : \frac{z_1}{x_r})).$$

Its indeterminacy locus is the union $\cup_{1 \leq i < j \leq r} H_i \cap H_j$ of the intersection of all pairs of H_i's, where $H_1, \dots, H_r$ are the pull-backs of coordinate hyperplanes of the first factor. The map $\bar{\Psi}$ is an isomorphism off the union $\cup_{1 \leq i \leq r} H_i$ and $\bar{\Psi}^2$ is the identity. By blowing-up, we obtain the commutative diagram:

$$\begin{array}{ccc} & \Psi & \\ X_P & \cdots \longrightarrow & X_Q \\ \downarrow & & \downarrow \\ \mathbf{P}^{r-1} & \cdots\cdots \rightarrow & \mathbf{P}^{r-1} \\ & \bar{\Psi} & \end{array}$$

Let $(X_2, \ldots, X_r), (Y_2, \ldots, Y_r), \ldots, (Z_2, \ldots, Z_r)$ be the standard inhomogeneous coordinate of $(\mathbf{P}^{r-1})^{p-1}$ arround $p_1 = \Delta(1 : 0 : \ldots : 0)$. Then the rational map $X_P \cdots \to (\mathbf{P}^{r-1})^{p-1}$ is given by

$$E_1 \ni (X_2 : \ldots : X_r : Y_2 : \ldots : Y_r : \ldots : Z_2 : \ldots : Z_r) \mapsto$$

$$((0 : \frac{1}{X_2} : \ldots : \frac{1}{X_r}), (1 : \frac{Y_2}{X_2} : \ldots : \frac{Y_r}{X_r}), \ldots, (1 : \frac{Z_2}{X_2} : \ldots : \frac{Z_r}{X_r})).$$

over the exceptional divisor E_1 over p_1. Hence Ψ resrtricted to E_1 is a birational map onto (the strict transform of) the divisor H_1. Therefore, Ψ is an isomorphism in codimension one. (7) is obvious. □

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