Substitutes and Complements in Network Flows Viewed as Discrete Convexity [1]

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Abstract

A new light is shed on "substitutes and complements" in the maximum weight circulation problem with reference to the concepts of L-convexity and M-convexity in the theory of discrete convex analysis. This provides us with a deeper understanding of the relationship between convexity and submodularity in combinatorial optimization.

Keywords: network flow, submodularity, convexity, combinatorial optimization

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1 Introduction

The relationship between convexity and submodularity has been discussed in the literature of combinatorial optimization (see [1, 2, 3, 10]). In this paper, we address this issue with reference to "substitutes and complements in network flows" discussed by Gale–Politof [5], and show that the concepts of L-convexity and M-convexity due to Murota [11, 12] help us better understand the relationship between convexity and submodularity.

We consider a network flow problem. Let $G = (V, A)$ be a directed graph with vertex set V and arc set A. For each arc $a \in A$, we are given a nonnegative capacity $c(a)$ for flow and a weight $w(a)$ per unit flow. The *maximum weight circulation problem* is to find a flow $\xi = (\xi(a) \mid a \in A)$ that maximizes the total weight $\sum_{a \in A} w(a)\xi(a)$ subject to the capacity (feasibility) constraint:

$$0 \leq \xi(a) \leq c(a) \qquad (a \in A)$$

and the conservation constraint:

$$\sum\{\xi(a) \mid a \text{ leaves } v\} - \sum\{\xi(a) \mid a \text{ enters } v\} = 0 \qquad (v \in V). \tag{1.1}$$

We denote by F the maximum weight of a feasible circulation, i.e.,

$$F = \max\{w^{\mathrm{T}}\xi \mid N\xi = \mathbf{0},\ \mathbf{0} \leq \xi \leq c\}, \tag{1.2}$$

where $N\xi = \mathbf{0}$ represents the conservation constraint (1.1).

Our concern here is how the weight F depends on the problem parameters (w, c). Namely, we are interested in the function $F = F(w, c)$ in $w \in \mathbf{R}^A$ and $c \in \mathbf{R}^A_+$. We first look at convexity and concavity.

Proposition 1. *F is convex in w and concave in c.*

Proof. The function $F = F(w, c)$ given by (1.2) is the maximum of linear functions in w and hence convex in w. By linear programming duality, we obtain an alternative expression $F = \min\{c^{\mathrm{T}}\eta \mid N^{\mathrm{T}}p + \eta \geq w,\ \eta \geq \mathbf{0}\}$, which shows the concavity of F in c. □

We next consider submodularity and supermodularity. A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *submodular* if

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (x, y \in \mathbf{R}^n),$$

and *supermodular* if

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y) \qquad (x, y \in \mathbf{R}^n),$$

where $x \vee y$ and $x \wedge y$ are defined by

$$(x \vee y)(i) = \max\{x(i), y(i)\}, \qquad (x \wedge y)(i) = \min\{x(i), y(i)\} \qquad (i = 1, 2, \ldots, n).$$

With economic terms of *substitutes* and *complements* we have the following correspondences:

$$\begin{array}{lcl} f \text{ is submodular} & \Longleftrightarrow & \text{goods are substitutes,} \\ f \text{ is supermodular} & \Longleftrightarrow & \text{goods are complements,} \end{array}$$

where f is interpreted as representing a utility function.

Two arcs are said to be *"parallel"* if every (undirected) simple cycle containing both of them orients them in the opposite direction, and *"series"* if every (undirected) simple cycle containing both of them orients them in the same direction. A set of arcs is said to be *"parallel"* if it consists of pairwise "parallel" arcs, and *"series"* if it consists of pairwise "series" arcs. With notations $w_P = (w(a) \mid a \in P)$, $c_P = (c(a) \mid a \in P)$, $w_S = (w(a) \mid a \in S)$, and $c_S = (c(a) \mid a \in S)$, the following statements hold true.

Theorem 2 (Gale–Politof [5]). *Let P be a "parallel" arc set and S a "series" arc set.*
(i) *F is submodular in w_P and in c_P.*
(ii) *F is supermodular in w_S and in c_S.*

See [6, 7, 8] for some extensions and generalizations of this result.

Combination of Proposition 1 and Theorem 2 yields that

$$\left. \begin{array}{lcccl} F \text{ is} & \text{convex} & \text{and} & \text{submodular} & \text{in } w_P, \\ F \text{ is} & \text{concave} & \text{and} & \text{submodular} & \text{in } c_P, \\ F \text{ is} & \text{convex} & \text{and} & \text{supermodular} & \text{in } w_S, \\ F \text{ is} & \text{concave} & \text{and} & \text{supermodular} & \text{in } c_S. \end{array} \right\} \qquad (1.3)$$

Thus all combinations of convexity/concavity and submodularity/supermodularity arise in our network flow problem. This demonstrates that convexity and submodularity are mutually independent properties.

Although convexity and submodularity are mutually independent, the combinations of convexity/concavity and submodularity/supermodularity in (1.3) are not accidental phenomena but logical consequences that can be explained in terms of L-convexity and M-convexity.

The concepts of M-convex and L-convex functions are introduced by Murota [11, 12] (see also [13, 14]), aiming to identify the well-behaved structure in (nonlinear) combinatorial optimization. These concepts were originally introduced for functions over the integer lattice; subsequently, their variants called M$^\natural$-convexity and L$^\natural$-convexity were introduced by Murota–Shioura [15] and by Fujishige–Murota [4], respectively. Recently, Murota–Shioura [16] extended these concepts to polyhedral convex functions defined over the real space.

A polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]),$$

where

$$\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\},$$
$$\mathrm{supp}^+(x) = \{i \mid x(i) > 0\}, \quad \mathrm{supp}^-(x) = \{i \mid x(i) < 0\} \qquad (x \in \mathbf{R}^n),$$
$$\chi_i \in \{0,1\}^n\text{: the } i\text{-th unit vector } (i = 1, 2, \ldots, n),$$
$$[0, \alpha_0] = \{\alpha \in \mathbf{R} \mid 0 \leq \alpha \leq \alpha_0\}.$$

A polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M$^\natural$-convex if the function $\widehat{f} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x, x_0) = \begin{cases} f(x) & ((x, x_0) \in \mathbf{R}^n \times \mathbf{R},\ x_0 = -\sum_{i=1}^n x(i)), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex. On the other hand, a polyhedral convex function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2):

(LF1) g is submodular,

(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathbf{R}^n,\ \lambda \in \mathbf{R})$;

it is called L$^\natural$-convex if the function $\widehat{g} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p, p_0) = g(p - p_0 \mathbf{1}) \qquad ((p, p_0) \in \mathbf{R}^n \times \mathbf{R})$$

is L-convex.

The main aim of this paper is to show that the function F defined by (1.2) is endowed with L$^\natural$-convexity and M$^\natural$-convexity, as follows. The proof is given in Section 2.

Theorem 3. *Let P be a "parallel" arc set and S a "series" arc set.*
(i) *F is L$^\natural$-convex in w_P and M$^\natural$-concave in c_P.*
(ii) *F is M$^\natural$-convex in w_S and L$^\natural$-concave in c_S.*

In general, L$^\natural$-convexity implies submodularity by definition, whereas M$^\natural$-convexity implies supermodularity [16, Theorem 4.24].

Theorem 4.
(i) *An L$^\natural$-convex function is submodular.*
(ii) *An M$^\natural$-convex function is supermodular.*

Accordingly, L$^\natural$-concavity implies supermodularity and M$^\natural$-concavity submodularity. With the aid of these general results on L$^\natural$-convex and M$^\natural$-convex functions, Theorem 3 above provides us with a somewhat deeper understanding of (1.3). Namely, it is understood that

F is L$^\natural$-convex,	hence	convex and submodular,	in	w_P,
F is M$^\natural$-concave,	hence	concave and submodular,	in	c_P,
F is M$^\natural$-convex,	hence	convex and supermodular,	in	w_S,
F is L$^\natural$-concave,	hence	concave and supermodular,	in	c_S.

It is left for future research to consider the results of [6, 7, 8] from the viewpoint of discrete convexity.

2 Proofs

This section gives the proof of Theorem 3. We start with basic properties of "parallel" and "series" arc sets that we use in the proof. Let us call $\pi : A \to \{0, \pm 1\}$ a *circuit* if $\partial \pi = \mathbf{0}$ and the set $\mathrm{supp}^{+}(\pi) \cup \mathrm{supp}^{-}(\pi)$ forms a simple cycle.

Proposition 5. *Let π be a circuit.*
(i) $|\mathrm{supp}^{+}(\pi) \cap P| \leq 1$ *and* $|\mathrm{supp}^{-}(\pi) \cap P| \leq 1$ *for a "parallel" arc set P.*
(ii) $|\mathrm{supp}^{+}(\pi) \cap S| = 0$ *or* $|\mathrm{supp}^{-}(\pi) \cap S| = 0$ *for a "series" arc set S.*

Proposition 6. *Let S be a "series" arc set, and π_1 and π_2 be circuits. If* $\mathrm{supp}^{+}(\pi_1) \cap \mathrm{supp}^{+}(\pi_2) \cap S \neq \emptyset$, *there exists a circuit π such that* $\mathrm{supp}^{+}(\pi) \subseteq \mathrm{supp}^{+}(\pi_1) \cup \mathrm{supp}^{+}(\pi_2)$, $\mathrm{supp}^{-}(\pi) \subseteq \mathrm{supp}^{-}(\pi_1) \cup \mathrm{supp}^{-}(\pi_2)$, *and* $\mathrm{supp}^{+}(\pi) \cap S = (\mathrm{supp}^{+}(\pi_1) \cup \mathrm{supp}^{+}(\pi_2)) \cap S$.

Proof. Suppose $a \in (\mathrm{supp}^{+}(\pi_2) \setminus \mathrm{supp}^{+}(\pi_1)) \cap S$. By elementary graph argument we can find a circuit π' such that $\mathrm{supp}^{+}(\pi') \subseteq \mathrm{supp}^{+}(\pi_1) \cup \mathrm{supp}^{+}(\pi_2)$, $\mathrm{supp}^{-}(\pi') \subseteq \mathrm{supp}^{-}(\pi_1) \cup \mathrm{supp}^{-}(\pi_2)$, and $\mathrm{supp}^{+}(\pi') \cap S \supseteq (\mathrm{supp}^{+}(\pi_1) \cap S) \cup \{a\}$. Repeating this we can find π. □

The main technical tool in the proof is the *conformal decomposition* (see, e.g., [9, 17]) of a circulation ξ, which is a representation of ξ as a positive sum of circuits conformal to ξ, i.e.,

$$\xi = \sum_{i=1}^{m} \beta_i \pi_i,$$

where $\beta_i > 0$ and $\pi_i : A \to \{0, \pm 1\}$ is a circuit with $\mathrm{supp}^{+}(\pi_i) \subseteq \mathrm{supp}^{+}(\xi)$ and $\mathrm{supp}^{-}(\pi_i) \subseteq \mathrm{supp}^{-}(\xi)$ for $i = 1, 2, \ldots, m$.

2.1 Proof of L$^\natural$-convexity in w_P

L$^\natural$-convexity of F in w_P is equivalent to submodularity of $F(w - w_0 \chi_P, c)$ in (w_P, w_0), which in turn is equivalent to

$$F(w + \lambda\chi_a, c) + F(w + \mu\chi_b, c) \geq F(w, c) + F(w + \lambda\chi_a + \mu\chi_b, c), \tag{2.1}$$
$$F(w + \lambda\chi_a, c) + F(w - \mu\chi_P, c) \geq F(w, c) + F(w + \lambda\chi_a - \mu\chi_P, c) \tag{2.2}$$

for $a, b \in P$ with $a \neq b$ and $\lambda, \mu \in \mathbf{R}_+$, where $\chi_P \in \{0,1\}^A$ denotes the characteristic vector of $P \subseteq A$.

To show (2.1) let ξ and $\overline{\xi}$ be optimal circulations for w and $w + \lambda\chi_a + \mu\chi_b$. We can establish (2.1) by constructing feasible circulations ξ_a and ξ_b such that

$$\xi_a + \xi_b = \xi + \overline{\xi}, \qquad \lambda[\xi_a(a) - \overline{\xi}(a)] + \mu[\xi_b(b) - \overline{\xi}(b)] \geq 0, \tag{2.3}$$

since this implies

$$(w + \lambda\chi_a)^{\mathrm{T}}\xi_a + (w + \mu\chi_b)^{\mathrm{T}}\xi_b \geq w^{\mathrm{T}}\xi + (w + \lambda\chi_a + \mu\chi_b)^{\mathrm{T}}\overline{\xi},$$

of which the left-hand side is bounded by $F(w + \lambda\chi_a, c) + F(w + \mu\chi_b, c)$ and the right-hand side is equal to $F(w, c) + F(w + \lambda\chi_a + \mu\chi_b, c)$. If $\overline{\xi}(a) \leq \xi(a)$, we can take $\xi_a = \xi$ and $\xi_b = \overline{\xi}$ to meet (2.3).

If $\overline{\xi}(b) \le \xi(b)$, we can take $\xi_a = \overline{\xi}$ and $\xi_b = \xi$ to meet (2.3). Otherwise, we make use of the conformal decomposition $\overline{\xi} - \xi = \sum_{i=1}^{m} \beta_i \pi_i$. Since $a \in \mathrm{supp}^+(\overline{\xi} - \xi)$ we may assume $\pi_i(a) = 1$ for $i = 1, 2, \ldots, \ell$ and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$. We have $\pi_i(b) = 0$ for $i = 1, 2, \ldots, \ell$ by Proposition 5 (i), since P is "parallel" and $\{a, b\} \subseteq \mathrm{supp}^+(\overline{\xi} - \xi)$. Then $\xi_a = \xi + \sum_{i=1}^{\ell} \beta_i \pi_i$ and $\xi_b = \xi + \sum_{i=\ell+1}^{m} \beta_i \pi_i$ are feasible circulations that satisfy (2.3).

To show (2.2) let ξ and $\overline{\xi}$ be optimal circulations for w and $w + \lambda \chi_a - \mu \chi_P$. We can establish (2.2) by constructing feasible circulations ξ_a and ξ_P such that

$$\xi_a + \xi_P = \xi + \overline{\xi}, \qquad \lambda[\xi_a(a) - \overline{\xi}(a)] + \mu[\sum_{a' \in P} \overline{\xi}(a') - \sum_{a' \in P} \xi_P(a')] \ge 0 \tag{2.4}$$

since this implies

$$(w + \lambda \chi_a)^{\mathrm{T}} \xi_a + (w - \mu \chi_P)^{\mathrm{T}} \xi_P \ge w^{\mathrm{T}} \xi + (w + \lambda \chi_a - \mu \chi_P)^{\mathrm{T}} \overline{\xi}.$$

If $\overline{\xi}(a) \le \xi(a)$, we can take $\xi_a = \xi$ and $\xi_P = \overline{\xi}$ to meet (2.4). Otherwise we use the conformal decomposition $\overline{\xi} - \xi = \sum_{i=1}^{m} \beta_i \pi_i$, in which we assume $\pi_i(a) = 1$ for $i = 1, 2, \ldots, \ell$ and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$. Since P is "parallel" we have $|\mathrm{supp}^-(\pi_i) \cap P| \le 1$ by Proposition 5 (i), and hence $\sum_{a' \in P} \pi_i(a') \ge 0$ for $i = 1, 2, \ldots, \ell$. Therefore, $\xi_a = \xi + \sum_{i=1}^{\ell} \beta_i \pi_i$ and $\xi_P = \xi + \sum_{i=\ell+1}^{m} \beta_i \pi_i$ are feasible circulations with the properties in (2.4).

2.2 Proof of M$^\natural$-concavity in c_P

M$^\natural$-convexity of a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is characterized by the following property [16, Theorem 4.21]:

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \ge f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]),$$

where $\chi_0 = \mathbf{0}$ by convention. We prove the M$^\natural$-concavity of F in c_P by establishing (M$^\natural$-EXC) for $-F$ as a function in c_P. In our notation this reads as follows:

> Let $c_1, c_2 \in \mathbf{R}_+^A$ be capacities with $c_1(a') = c_2(a')$ for all $a' \in A \setminus P$. For each $a \in \mathrm{supp}^+(c_1 - c_2)$, there exist $b \in \mathrm{supp}^-(c_1 - c_2) \cup \{0\}$ and a positive number α_0 such that
>
> $$F(w, c_1) + F(w, c_2) \le F(w, c_1 - \alpha(\chi_a - \chi_b)) + F(w, c_2 + \alpha(\chi_a - \chi_b)) \qquad (\forall \alpha \in [0, \alpha_0]).$$

Let ξ_1 and ξ_2 be optimal circulations for c_1 and c_2, respectively. We shall find $\alpha_0 > 0$ and $b \in \mathrm{supp}^-(c_1 - c_2) \cup \{0\}$ such that, for any $\alpha \in [0, \alpha_0]$, there exist circulations ξ_1' and ξ_2' such that

$$\xi_1' + \xi_2' = \xi_1 + \xi_2, \quad \mathbf{0} \le \xi_1' \le c_1 - \alpha(\chi_a - \chi_b), \quad \mathbf{0} \le \xi_2' \le c_2 + \alpha(\chi_a - \chi_b). \tag{2.5}$$

If $\xi_1(a) < c_1(a)$, we can take $\alpha_0 = c_1(a) - \xi_1(a)$, $b = 0$, $\xi_1' = \xi_1$ and $\xi_2' = \xi_2$ to meet (2.5). Suppose $\xi_1(a) = c_1(a)$. We have $\xi_1(a) = c_1(a) > c_2(a) \ge \xi_2(a)$. Let π be a circuit such that $a \in \mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi_1 - \xi_2)$ and $\mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi_1 - \xi_2)$. Since P is "parallel" and $a \in \mathrm{supp}^+(\pi)$, we have $\mathrm{supp}^+(\pi) \cap P = \{a\}$ and $|\mathrm{supp}^-(\pi) \cap P| \le 1$ by Proposition 5 (i). If $|\mathrm{supp}^-(\pi) \cap P| = 1$, define b by $\{b\} = \mathrm{supp}^-(\pi) \cap P$; otherwise put $b = 0$. We can take $\alpha_0 > 0$ such that $\alpha_0 \le |\xi_1(a') - \xi_2(a')|$ for all $a' \in \mathrm{supp}^+(\pi) \cup \mathrm{supp}^-(\pi)$. Then $\xi_1' = \xi_1 - \alpha\pi$ and $\xi_2' = \xi_2 + \alpha\pi$ satisfy (2.5) if $\alpha \in [0, \alpha_0]$.

2.3 Proof of M$^\natural$-convexity in w_S

We prove the M$^\natural$-convexity of F in w_S by establishing (M$^\natural$-EXC). In our notation this reads as follows:

> Let $w_1, w_2 \in \mathbf{R}^A$ be weight vectors with $w_1(a') = w_2(a')$ for all $a' \in A \setminus S$. For each $a \in \mathrm{supp}^+(w_1 - w_2)$, there exist $b \in \mathrm{supp}^-(w_1 - w_2) \cup \{0\}$ and a positive number α_0 such that
>
> $$F(w_1, c) + F(w_2, c) \geq F(w_1 - \alpha(\chi_a - \chi_b), c) + F(w_2 + \alpha(\chi_a - \chi_b), c) \qquad (\forall \alpha \in [0, \alpha_0]).$$

Let ξ_1 and ξ_2 be optimal circulations for w_1 and w_2, respectively, with $\xi_1(a)$ minimum and $\xi_2(a)$ maximum.

Proposition 7. *There exists $\alpha_0 > 0$ such that ξ_1 is optimal for $w_1 - \alpha\chi_a$ and ξ_2 is optimal for $w_2 + \alpha\chi_a$ for all $\alpha \in [0, \alpha_0]$.*

Proof. For any circuit π such that $\pi(a) = -1$ and $\mathbf{0} \leq \xi_1 + \beta\pi \leq c$ for some $\beta > 0$, we have $w_1^{\mathrm{T}}(\xi_1 + \beta\pi) < w_1^{\mathrm{T}}\xi_1$ by the choice of ξ_1. Let $\alpha_1 > 0$ be the minimum of $-w_1^{\mathrm{T}}\pi$ over all such circuits π. Then ξ_1 is optimal for $w_1 - \alpha\chi_a$ for all $\alpha \in [0, \alpha_1]$, since $(w_1 - \alpha\chi_a)^{\mathrm{T}}(\xi_1 + \beta\pi) \leq (w_1 - \alpha\chi_a)^{\mathrm{T}}\xi_1$ for any $\beta > 0$ and circuit π such that $\mathbf{0} \leq \xi_1 + \beta\pi \leq c$. Similarly, let $\alpha_2 > 0$ be the minimum of $-w_2^{\mathrm{T}}\pi$ over all circuits π such that $\pi(a) = 1$ and $\mathbf{0} \leq \xi_2 + \beta\pi \leq c$ for some $\beta > 0$. Then ξ_2 is optimal for $w_2 + \alpha\chi_a$ for all $\alpha \in [0, \alpha_2]$. Put $\alpha_0 = \min(\alpha_1, \alpha_2)$. □

If $\xi_1(a) \geq \xi_2(a)$, we can take $b = 0$ in (M$^\natural$-EXC), since

$$\begin{aligned} &F(w_1, c) + F(w_2, c) = w_1^{\mathrm{T}}\xi_1 + w_2^{\mathrm{T}}\xi_2 \\ &\geq (w_1 - \alpha\chi_a)^{\mathrm{T}}\xi_1 + (w_2 + \alpha\chi_a)^{\mathrm{T}}\xi_2 = F(w_1 - \alpha\chi_a, c) + F(w_2 + \alpha\chi_a, c), \end{aligned}$$

where the last equality is by Proposition 7. In what follows we assume $\xi_1(a) < \xi_2(a)$.

By Proposition 5 (ii), we can impose further conditions on ξ_1 and ξ_2 that, for each $b \in S \setminus \{a\}$, $\xi_1(b)$ is maximum among all optimal ξ_1 for w_1 with $\xi_1(a)$ minimum, and $\xi_2(b)$ is minimum among all optimal ξ_2 for w_2 with $\xi_2(a)$ maximum.

Proposition 8. *There exists $\alpha_0 > 0$ such that ξ_1 is optimal for $w_1 - \alpha(\chi_a - \chi_b)$ and ξ_2 is optimal for $w_2 + \alpha(\chi_a - \chi_b)$ for all $b \in S \setminus \{a\}$ and for all $\alpha \in [0, \alpha_0]$.*

Proof. For any circuit π such that $\pi(a) - \pi(b) = -1$ for some $b \in S \setminus \{a\}$ and $\mathbf{0} \leq \xi_1 + \beta\pi \leq c$ for some $\beta > 0$, we have $w_1^{\mathrm{T}}(\xi_1 + \beta\pi) < w_1^{\mathrm{T}}\xi_1$ by the choice of ξ_1. Let $\alpha_1 > 0$ be the minimum of $-w_1^{\mathrm{T}}\pi$ over all such circuits π. Then ξ_1 is optimal for $w_1 - \alpha(\chi_a - \chi_b)$ for all $\alpha \in [0, \alpha_1]$. Similarly, let $\alpha_2 > 0$ be the minimum of $-w_2^{\mathrm{T}}\pi$ over all circuits π such that $\pi(a) - \pi(b) = 1$ for some $b \in S \setminus \{a\}$ and $\mathbf{0} \leq \xi_2 + \beta\pi \leq c$ for some $\beta > 0$. Then ξ_2 is optimal for $w_2 + \alpha(\chi_a - \chi_b)$ for all $\alpha \in [0, \alpha_2]$. Put $\alpha_0 = \min(\alpha_1, \alpha_2)$. □

Proposition 8 implies that for all $b \in S \setminus \{a\}$ we have

$$\begin{aligned}
&F(w_1, c) + F(w_2, c) - F(w_1 - \alpha(\chi_a - \chi_b), c) - F(w_2 + \alpha(\chi_a - \chi_b), c) \\
&= w_1^{\mathrm{T}}\xi_1 + w_2^{\mathrm{T}}\xi_2 - (w_1 - \alpha(\chi_a - \chi_b))^{\mathrm{T}}\xi_1 - (w_2 + \alpha(\chi_a - \chi_b))^{\mathrm{T}}\xi_2 \\
&= \alpha[(\xi_2(b) - \xi_1(b)) - (\xi_2(a) - \xi_1(a))]. \qquad (2.6)
\end{aligned}$$

We want to find $b \in \mathrm{supp}^-(w_1 - w_2)$ for which (2.6) is nonnegative.

We make use of the conformal decomposition $\xi_2 - \xi_1 = \sum_{i=1}^m \beta_i \pi_i$. Since S is "series" we may assume, by Proposition 6, that

$$a \in \mathrm{supp}^+(\pi_1) \cap S \subseteq \mathrm{supp}^+(\pi_2) \cap S \subseteq \cdots \subseteq \mathrm{supp}^+(\pi_\ell) \cap S$$

and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$; then $\mathrm{supp}^-(\pi_i) \cap S = \emptyset$ for $i = 1, 2, \ldots, \ell$.

Proposition 9. *There exists* $b \in (\mathrm{supp}^+(\pi_1) \cap S) \cap \mathrm{supp}^-(w_1 - w_2)$.

Proof. We have $w_1^{\mathrm{T}}\pi_1 \leq 0$, since ξ_1 is optimal for w_1 and $\mathbf{0} \leq \xi_1 + \beta_1\pi_1 \leq c$. Similarly, we have $-w_2^{\mathrm{T}}\pi_1 \leq 0$. Hence

$$0 \geq (w_1 - w_2)^{\mathrm{T}}\pi_1 = \sum_{b \in S}(w_1(b) - w_2(b))\pi_1(b) = \sum_{b \in \mathrm{supp}^+(\pi_1) \cap S}(w_1(b) - w_2(b)).$$

Since $w_1(a) - w_2(a) > 0$ in this summation, we must have $w_1(b) - w_2(b) < 0$ for some $b \in \mathrm{supp}^+(\pi_1) \cap S$. □

For $b \in (\mathrm{supp}^+(\pi_1) \cap S) \cap \mathrm{supp}^-(w_1 - w_2)$ in Proposition 9, we have

$$\xi_2(b) - \xi_1(b) = \sum_{i=1}^{\ell}\beta_i + \sum_{i=\ell+1}^{m}\beta_i\pi_i(b) \geq \sum_{i=1}^{\ell}\beta_i = \xi_2(a) - \xi_1(a),$$

which shows the nonnegativity of (2.6).

2.4 Proof of L$^\natural$-concavity in c_S

L$^\natural$-concavity of F in c_S is equivalent to supermodularity of $F(w, c - c_0\chi_S)$ in (c_S, c_0), which in turn is equivalent to

$$F(w, c + \lambda\chi_a) + F(w, c + \mu\chi_b) \leq F(w, c) + F(w, c + \lambda\chi_a + \mu\chi_b), \qquad (2.7)$$

$$F(w, c + \lambda\chi_a) + F(w, c - \mu\chi_S) \leq F(w, c) + F(w, c + \lambda\chi_a - \mu\chi_S) \qquad (2.8)$$

for $a, b \in S$ with $a \neq b$ and $\lambda, \mu \in \mathbf{R}_+$, where $\chi_S \in \{0,1\}^A$ denotes the characteristic vector of $S \subseteq A$.

To show (2.7) let ξ_a and ξ_b be optimal circulations for $c + \lambda\chi_a$ and $c + \mu\chi_b$. We can establish (2.7) by constructing circulations ξ and $\overline{\xi}$ such that

$$\xi + \overline{\xi} = \xi_a + \xi_b, \quad \mathbf{0} \leq \xi \leq c, \quad \mathbf{0} \leq \overline{\xi} \leq c + \lambda\chi_a + \mu\chi_b. \qquad (2.9)$$

If $\xi_a(a) \le c(a)$, we can take $\xi = \xi_a$ and $\overline{\xi} = \xi_b$ to meet (2.9). If $\xi_b(b) \le c(b)$, we can take $\xi = \xi_b$ and $\overline{\xi} = \xi_a$ to meet (2.9). Otherwise, we have $\xi_a(a) > c(a) \ge \xi_b(a)$ and $\xi_a(b) \le c(b) < \xi_b(b)$, and therefore $a \in \mathrm{supp}^+(\xi_a - \xi_b)$ and $b \in \mathrm{supp}^-(\xi_a - \xi_b)$. We make use of the conformal decomposition $\xi_a - \xi_b = \sum_{i=1}^m \beta_i \pi_i$, where we assume $\pi_i(a) = 1$ for $i = 1, 2, \ldots, \ell$ and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$. We have $\pi_i(b) = 0$ for $i = 1, 2, \ldots, \ell$ by Proposition 5 (ii), since S is "series" and $a \in \mathrm{supp}^+(\xi_a - \xi_b)$ and $b \in \mathrm{supp}^-(\xi_a - \xi_b)$. Then $\xi = \xi_a - \sum_{i=1}^{\ell} \beta_i \pi_i$ and $\overline{\xi} = \xi_b + \sum_{i=1}^{\ell} \beta_i \pi_i$ satisfy (2.9).

To show (2.8) let ξ_a and ξ_S be optimal circulations for $c + \lambda \chi_a$ and $c - \mu \chi_S$. We can establish (2.8) by constructing circulations ξ and $\overline{\xi}$ such that

$$\xi + \overline{\xi} = \xi_a + \xi_S, \quad \mathbf{0} \le \xi \le c, \quad \mathbf{0} \le \overline{\xi} \le c + \lambda \chi_a - \mu \chi_S. \tag{2.10}$$

If $\xi_a(a) \le c(a)$, we can take $\xi = \xi_a$ and $\overline{\xi} = \xi_S$ to meet (2.10). Otherwise, we have $\xi_a(a) > c(a) \ge \xi_S(a)$, and therefore $a \in \mathrm{supp}^+(\xi_a - \xi_S)$. We use the conformal decomposition $\xi_a - \xi_S = \sum_{i=1}^m \beta_i \pi_i$. Since S is "series" we may assume by Proposition 6 that

$$a \in \mathrm{supp}^+(\pi_1) \cap S \subseteq \mathrm{supp}^+(\pi_2) \cap S \subseteq \cdots \subseteq \mathrm{supp}^+(\pi_\ell) \cap S$$

and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$; then $\mathrm{supp}^-(\pi_i) \cap S = \emptyset$ for $i = 1, 2, \ldots, \ell$. Noting $\sum_{i=1}^{\ell} \beta_i = \xi_a(a) - \xi_S(a) \ge \xi_a(a) - c(a)$, let k be the smallest integer with $\sum_{i=1}^{k} \beta_i \ge \xi_a(a) - c(a)$ and define $\beta' = [\xi_a(a) - c(a)] - \sum_{i=1}^{k-1} \beta_i$. Then $\xi = \xi_a - \sum_{i=1}^{k-1} \beta_i \pi_i - \beta' \pi_k$ and $\overline{\xi} = \xi_S + \sum_{i=1}^{k-1} \beta_i \pi_i + \beta' \pi_k$ satisfy (2.10), since $\xi(a) = \xi_a(a) - \sum_{i=1}^{k-1} \beta_i - \beta' = c(a)$, $\overline{\xi}(a) = \xi_S(a) + \sum_{i=1}^{k-1} \beta_i + \beta' = \xi_S(a) + \xi_a(a) - c(a) \le c(a) + \lambda - \mu$, and, for any $b \in \mathrm{supp}^+(\pi_k) \cap S$, we have

$$\begin{aligned}
\overline{\xi}(b) &= \xi_S(b) + \sum_{i=1}^{k-1} \beta_i \pi_i(b) + \beta' = \xi_S(b) + \sum_{i=1}^{k-1} \beta_i \pi_i(b) + [\sum_{i=k}^{\ell} \beta_i + \xi_S(a) - c(a)] \\
&= [\xi_S(b) + \sum_{i=1}^{\ell} \beta_i \pi_i(b)] + \xi_S(a) - c(a) \le \xi_a(b) + \xi_S(a) - c(a) \le c(b) - \mu.
\end{aligned}$$

This completes the proof of Theorem 3.

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