Polynomial embedding of Cuntz-Krieger algebra into Cuntz algebra

Katsunori Kawamura*
College of Science and Engineering Ritsumeikan University,
1-1-1 Noji Higashi, Kusatsu, Shiga 525-8577, Japan

March 21, 2008

Abstract

For any Cuntz-Krieger algebra $\mathcal{O}_A$, we construct embeddings of $\mathcal{O}_A$ into the Cuntz algebra $\mathcal{O}_2$ such that the canonical generators of $\mathcal{O}_A$ are written as polynomials in those of $\mathcal{O}_2$.

Mathematics Subject Classifications (2000). 47L55, 81T05.

Key words. Cuntz-Krieger algebra, embedding.

1 Introduction

We have studied embeddings of operator algebras [1, 2, 3, 5, 7]. Especially, our interest is how an algebra is embedded into the other but not whether an embedding exists or not. For example, we constructed a non-symmetric tensor product of representations [8] and a C*-bialgebra [9] from a "good" class of embeddings.

It is well-known that there always exists a $*$-embedding of a C*-algebra which satisfies some conditions into the Cuntz algebra $\mathcal{O}_2$ [10, 12]. Although, concrete methods of construction of embedding are not known very well. We construct embeddings of any Cuntz-Krieger algebra into $\mathcal{O}_2$ by using concrete polynomials in the following sense.

Theorem 1.1 *For $N \geq 2$, let $\mathcal{O}_A$ denote the Cuntz-Krieger algebra by a nondegenerate matrix A with entries* 0 *or* 1. *Then there exist elements $t_1, \ldots, t_N$ in $\mathcal{O}_2$ such that*

*e-mail: kawamura@kurims.kyoto-u.ac.jp.

(i) *they satisfy the relations of the canonical generators of* $\mathcal{O}_A$*, and*

(ii) *they are polynomials in the canonical generators of* $\mathcal{O}_2$ *and their conjugates.*

We will show Theorem 1.1 in Section 2 (Theorem 2.4). Examples of these generators and the naturality of our construction are shown in Section 3. In order to construct generators of $\mathcal{O}_A$ in $\mathcal{O}_2$, we prepare several notions in this section.

A matrix A is *nondegenerate* if any column and any row are not zero. For $N \geq 2$, let $\mathbf{M}_N(\{0,1\})$ denote the set of all nondegenerate $N \times N$ matrices with entries 0 or 1. For $A = (a_{ij}) \in \mathbf{M}_N(\{0,1\})$, $\mathcal{O}_A$ is the *Cuntz-Krieger algebra by* A [4, 6] if $\mathcal{O}_A$ is a C*-algebra which is universally generated by partial isometries $s_1, \ldots, s_n$ satisfying

$$s_i^* s_i = \sum_{j=1}^{N} a_{ij} s_j s_j^* \quad (i = 1, \ldots, N), \quad \sum_{i=1}^{N} s_i s_i^* = I. \tag{1.1}$$

Especially, when $a_{ij} = 1$ for each $i, j = 1, \ldots, n$, $\mathcal{O}_A$ is $*$-isomorphic to the Cuntz algebra $\mathcal{O}_N$.

Let R be a nonempty subset of $\mathbf{C}$. For $M \geq 2$, let $s_1, \ldots, s_M$ denote canonical generators of $\mathcal{O}_M$. Define subsets $\mathcal{M}(\mathcal{O}_M)$ and $\mathcal{O}_M^o(R)$ of $\mathcal{O}_M$ by

$$\mathcal{M}(\mathcal{O}_M) \equiv \bigcup_{k+l\geq 1,\, k,l\geq 0} \left\{ s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^* \in \mathcal{O}_M : \begin{array}{c} i_\alpha, j_\beta = 1, \ldots, M, \\ \alpha = 1, \ldots, k, \\ \beta = 1, \ldots, l, \end{array} \right\},$$

$$\mathcal{O}_M^o(R) \equiv \bigcup_{n\geq 1} \left\{ \sum_{\lambda=1}^{n} b_\lambda x_\lambda \in \mathcal{O}_M : x_\lambda \in \mathcal{M}(\mathcal{O}_M),\, b_\lambda \in R,\, \lambda = 1, \ldots, n \right\}.$$

In this paper, any homomorphism and embedding are assumed unital. The generators of $\mathcal{O}_A$ always mean those which satisfy (1.1).

Definition 1.2 (i) *An element in* $\mathcal{O}_M^o(R)$ *(resp.* $\mathcal{M}(\mathcal{O}_M)$*) is called an* R*-polynomial (resp. a monomial) in* $\mathcal{O}_M$*.*

(ii) *A* $*$*-homomorphism* Φ *from* $\mathcal{O}_A$ *to* $\mathcal{O}_M$ *is polynomial type over* R *(resp. monomial type) if* $\Phi(t_1), \ldots, \Phi(t_N)$ *belong to* $\mathcal{O}_M^o(R)$ *(resp.* $\mathcal{M}(\mathcal{O}_M)$*) where* $t_1, \ldots, t_N$ *denote canonical generators of* $\mathcal{O}_A$*.*

(iii) *A* $*$*-embedding* Φ *from* $\mathcal{O}_A$ *into* $\mathcal{O}_M$ *is polynomial type over* R *(resp. monomial type) if* Φ *is polynomial type over* R *(resp. monomial type) as a* $*$*-homomorphism.*

(iv) *The algebra* $\mathcal{O}_A$ *is R-polynomially (resp. monomially) embedded into* $\mathcal{O}_M$ *if there exists a* $*$*-embedding from* $\mathcal{O}_A$ *into* $\mathcal{O}_M$ *which is polynomial type over* R *(resp. monomial type).*

(v) *Elements* $x_1, \ldots, x_N$ *in* $\mathcal{O}_M$ *are R-polynomial (resp. monomial) generators of* $\mathcal{O}_A$ *if* $x_1, \ldots, x_N$ *belong to* $\mathcal{O}^o_M(R)$ *(resp.* $\mathcal{M}(\mathcal{O}_M)$*) and satisfy (1.1).*

Remark 1.3 Strictly speaking, an element in $\mathcal{O}^{(0)}_M(R)$ is not a polynomial in canonical generators of $\mathcal{O}_M$ and its conjugates, but a non-commutative polynomial because generators are not commutative. In this paper, we call a polynomial as a non-commutative polynomial.

Especially, if R is a subring of $\mathbf{C}$, then $\mathcal{O}^o_M(R)$ is a subalgebra of $\mathcal{O}_M$ over R. Furthermore if R is closed under complex conjugation, then $\mathcal{O}^o_M(R)$ is a $*$-subalgebra of $\mathcal{O}_M$ over R. Note that $\mathcal{O}^o_M \equiv \mathcal{O}^o_M(\mathbf{C})$ is dense in $\mathcal{O}_M$ and it is regarded as the (non-commutative) polynomial ring with generators $s_1, \ldots, s_M, s_1^*, \ldots, s_M^*$ over $\mathbf{C}$ under relations of $\mathcal{O}_M$.

In subsection 2.1 in [2], we showed that there are various polynomial embeddings among Cuntz algebras. We review known embeddings associated with our works [2, 3].

Lemma 1.4 (i) *For each* $N \geq 2$, $\mathcal{O}_N$ *is monomially embedded into* $\mathcal{O}_2$.

(ii) *For each* $M \in \{(N-1)k+1 : k \geq 1\}$, $\mathcal{O}_M$ *is monomially embedded into* $\mathcal{O}_N$.

Proof. (i) Let s_1, s_2 denote the canonical generators of $\mathcal{O}_2$. The case $N = 2$ is trivial. Assume $N \geq 3$. Define

$$\begin{cases} t_1 \equiv & s_1, \\ t_i \equiv & (s_2)^{i-1} s_1 \quad (i = 2, \ldots, N-1), \\ t_N \equiv & (s_2)^{N-1}. \end{cases} \tag{1.2}$$

Then $t_1, \ldots, t_N$ satisfy relations of canonical generators of $\mathcal{O}_N$ and they belong to $\mathcal{M}(\mathcal{O}_2)$.

(ii) Let $s_1, \ldots, s_N$ denote canonical generators of $\mathcal{O}_N$. The case $M = N$ is trivial. Assume that $M=(N-1)k+1$ and $k\geq 2$. Define

$$\begin{cases} t_i \equiv & s_i \qquad\qquad (i=1,\ldots,N-1),\\ t_{(N-1)l+i} \equiv & (s_N)^l s_i \qquad \left(\begin{array}{l} l=1,\ldots,k-1,\\ i=1,\ldots,N-1 \end{array}\right),\\ t_M \equiv & (s_N)^k. \end{cases} \tag{1.3}$$

Then $t_1,\ldots,t_M$ satisfy relations of canonical generators of $\mathcal{O}_M$ and they belong to $\mathcal{M}(\mathcal{O}_N)$. ∎

Corollary 1.5 *For each $n\geq 1$, there exists a monomial embedding of $\mathcal{O}_{2n+1}$ into $\mathcal{O}_3$.*

Note that a choice of polynomial embedding of $\mathcal{O}_N$ into $\mathcal{O}_2$ is not unique. For example, the following embedding of $\mathcal{O}_4$ into $\mathcal{O}_2$ exists:

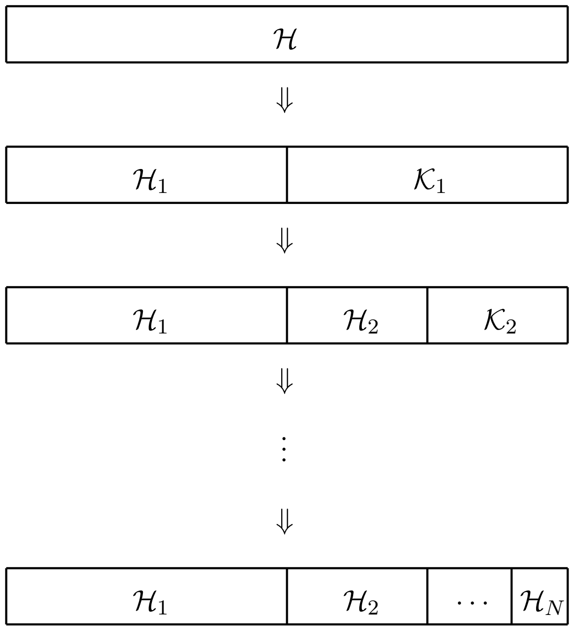
$$t_1\equiv \; s_1,\quad t_2\equiv s_2s_2,\quad t_3\equiv s_2s_1s_2,\quad t_4\equiv s_2s_1s_1. \tag{1.4}$$

Furthermore, the following embedding of $\mathcal{O}_5$ into $\mathcal{O}_2$ exists:

$$t_1\equiv \; s_1s_1,\quad t_2\equiv s_1s_2s_1,\quad t_3\equiv s_1s_2s_2,\quad t_4\equiv s_2s_1,\quad t_5\equiv s_2s_2. \tag{1.5}$$

We illustrate our construction of embeddings among Cuntz algebras in Lemma 1.4 (i). Assume that $\mathcal{O}_2$ is represented on a Hilbert space $\mathcal{H}$. Then we have an orthogonal decomposition $\{\mathcal{H}_i\}_{i=1}^N$ of $\mathcal{H}$ by

$$\mathcal{H}_1\equiv s_1\mathcal{H},\quad \mathcal{H}_2\equiv s_2s_1\mathcal{H},\ldots,\mathcal{H}_{N-1}\equiv s_2^{N-2}s_1\mathcal{H},\quad \mathcal{H}_N\equiv s_2^{N-1}\mathcal{H}.$$



where

$$\mathcal{K}_i \equiv \left[\bigoplus_{j=1}^{i} \mathcal{H}_j\right]^{\perp} \quad (i = 1, \ldots, N-1).$$

2 Construction of polynomial generators of $\mathcal{O}_A$ in $\mathcal{O}_M$

We prepare several tools associated with matrices. Fix $A = (a_{ij}) \in \mathbf{M}_N(\{0,1\})$. Define the set $\{(M_i, q_i, B_i)\}_{i=1}^N$ by

$$B_i \equiv \{\, j \in \{1, \ldots, N\} : a_{ij} = 1 \,\}, \quad M_i \equiv \sum_{j=1}^{N} a_{ij},$$

$$q_i : \ B_i \to \{1, \ldots, M_i\}; \quad q_i(j) \equiv \#\{k \in B_i : k \le j\}$$

for $i = 1, \ldots, N$. Note that q_i is bijective for each $i = 1, \ldots, N$.

Definition 2.1 *The set $\{(M_i, q_i, B_i)\}_{i=1}^N$ is called the (canonical) A-coordinate. The set $\{M_i\}_{i=1}^N$ is called the set of row sums of A.*

Lemma 2.2 *Let* $A = (a_{ij}) \in \mathbf{M}_N(\{0,1\})$ *with the* A*-coordinate* $\{(M_i, q_i, B_i)\}_{i=1}^N$. *Assume that a unital* C^**-algebra* $\mathcal{B}$ *satisfies the following condition:*

$$\mathcal{B} \text{ contains } \mathcal{O}_N \text{ and } \mathcal{O}_{M_i} \text{ for each } i = 1, \ldots, N \text{ when } M_i \geq 2 \text{ as } C^*\text{-subalgebras with common unit.} \tag{2.1}$$

Let $s_1, \ldots, s_N$ *and* $t_{i,1}, \ldots, t_{i,M_i}$ *denote canonical generators of* $\mathcal{O}_N$ *and those of* $\mathcal{O}_{M_i}$ *for* $i = 1, \ldots, N$ *as elements in* $\mathcal{B}$*, respectively. Especially, we define* $\mathcal{O}_1 = \mathbf{C}I$ *and* $t_{i,1} = I$ *when* $M_i = 1$*. Under these assumptions, define*

$$x_i \equiv \sum_{j=1}^N a_{ij} s_i t_{i,q_i(j)} s_j^* \quad (i = 1, \ldots, N). \tag{2.2}$$

Then $x_1, \ldots, x_N$ *satisfy (1.1) with respect to* A*.*

Proof. Define

$$F_i \equiv \sum_{j=1}^N a_{ij} t_{i,q_i(j)} s_j^* \quad (i = 1, \ldots, N).$$

Then $x_i = s_i F_i$ and the following holds:

$$F_i^* F_i = \sum_{j=1}^N a_{ij} s_j s_j^*, \quad F_i F_i^* = \sum_{j=1}^N a_{ij} t_{i,q_i(j)} t_{i,q_i(j)}^* = I \quad (i = 1, \ldots, N).$$

We show the condition (1.1) by direct computation.

$$x_i^* x_i = \quad F_i^* s_i^* s_i F_i = \sum_{j=1}^N a_{ij} s_j s_j^*, \quad x_i x_i^* = s_i F_i F_i^* s_i^* = s_i s_i^*$$

for each $i = 1, \ldots, N$. Hence we have the condition (1.1):

$$x_i^* x_i = \sum_{j=1}^N a_{ij} x_j x_j^*, \quad \sum_{i=1}^N x_i x_i^* = I.$$

∎

Note that Lemma 2.2 holds when the choice of q_i are replaced as any bijection from B_i to $\{1, \ldots, M_i\}$ for each $i = 1, \ldots, N$.

Corollary 2.3 *Let* $N \geq 2$*. For* $A \in \mathbf{M}_N(\{0,1\})$ *and the set* $\{M_i\}_{i=1}^N$ *of row sums of* A*, there exists a* $*$*-embedding of* $\mathcal{O}_A$ *into* $\mathcal{B}$ *if* $\mathcal{B}$ *is a unital* C^**-algebra which satisfies (2.1).*

Proof. From Lemma 2.2, it holds immediately. ∎

Let $\mathbf{Z}_{n\geq 0}$ denote the set of all non-negative integers. Recall the definition of properties of embeddings in Definition 1.2.

Theorem 2.4 *For any $A \in \mathbf{M}_N(\{0,1\})$, there exists a $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_A$ into $\mathcal{O}_2$.*

Proof. For any $M \geq 2$, there exists a $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_M$ into $\mathcal{O}_2$ by Lemma 1.4 (i). Furthermore $\mathcal{O}_2$ satisfies (2.1) in Lemma 2.2 such that $s_i, t_{i,j}$ in (2.2) are written as monomials in $\mathcal{O}_2$. From the form of x_i in (2.2), $x_1, \ldots, x_N$ are written as $\mathbf{Z}_{n\geq 0}$-polynomials in $\mathcal{O}_2$. Therefore the statement holds. ∎

Theorem 1.1 is shown by Theorem 2.4. The embedding in Theorem 2.4 depends on the choice of embeddings of $\mathcal{O}_M$ into $\mathcal{O}_2$.

Corollary 2.5 *Let $A \in \mathbf{M}_N(\{0,1\})$, the set $\{M_i\}_{i=1}^N$ of row sums of A and $M \geq 2$.*

(i) *If $\{N, M_i : i = 1, \ldots, N\}$ is a subset of $\{(M-1)k + 1 : k \geq 0\}$, then there exists a $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_A$ into $\mathcal{O}_M$.*

(ii) *If M_i and N are odd for each $i = 1, \ldots, N$, then there exists a $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_A$ into $\mathcal{O}_3$.*

Proof. (i) It follows from Corollary 2.3, the form of generators in (2.2) and Lemma 1.4 (ii). (ii) By Corollary 1.5, $\mathcal{O}_3$ satisfies the condition in (i) with respect to all odd number N, M_i, $i = 1, \ldots, N$. Hence there are $\mathbf{Z}_{n\geq 0}$-polynomial generators of $\mathcal{O}_A$ in $\mathcal{O}_3$. ∎

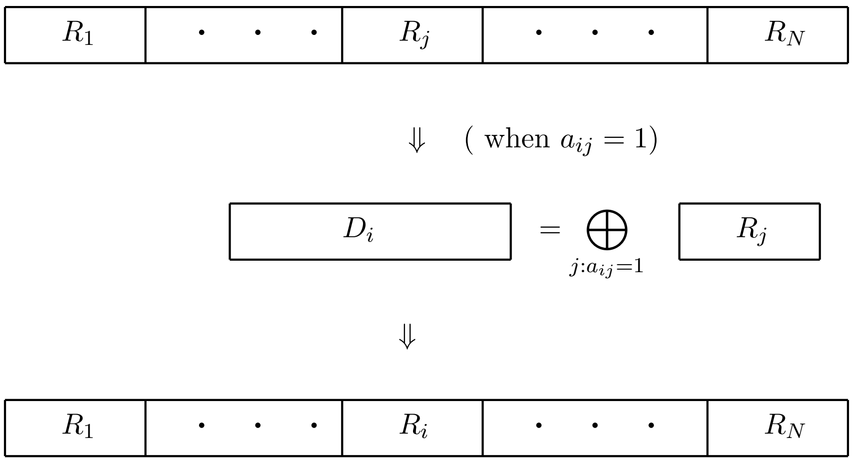
We illustrate our construction of embeddings as a decomposition of a Hilbert space by partial isometries, where we assume that $\mathcal{B}$ in Lemma 2.2 is represented on an infinite dimensional Hilbert space $\mathcal{H}$. Fix $A \in \mathbf{M}_N(\{0,1\})$ and $\{M_i\}_{i=1}^N$ is the set of row sums of A.

(i) At first, decompose a Hilbert space $\mathcal{H}$ into N parts $R_1, \ldots, R_N$ as infinite dimensional Hilbert subspaces of $\mathcal{H}$. This is the role of $s_1^*, \ldots, s_N^*$ in (2.2).

(ii) Next, choose M_i number of components from $R_1, \ldots, R_N$ by the rule associated with a matrix A and make a new subspace D_i of $\mathcal{H}$ for each $i = 1, \ldots, N$, respectively. This process is executed by $t_{i,q_i(j)}$ and the sum in (2.2).

(iii) At the end, we maps D_i into R_i by s_i for $i = 1, \ldots, N$ in (2.2), respectively.

By these procedure, we have a partial isometry x_i in (2.2) with the initial projection D_i and the final projection R_i for $i = 1, \ldots, N$.



3 Examples

We show examples in this section.

Example 3.1 Assume that $A = (a_{ij}) \in \mathbf{M}_N(\{0,1\})$ satisfies $a_{ij} = 1$ for each $i, j = 1, \ldots, N$. In this case, $\mathcal{O}_A \cong \mathcal{O}_N$. Then the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^N$ is given by $(M_i, q_i, B_i) = (N, id_{\{1,\ldots,N\}}, \{1, \ldots, N\})$ for each $i = 1, \ldots, N$. By Corollary 2.5 (i), we obtain an embedding of $\mathcal{O}_N$ into $\mathcal{O}_N$. That is, this is an endomorphism of $\mathcal{O}_N$. Let $s_1, \ldots, s_N$ denote canonical generators of $\mathcal{O}_N$. Then $u_j = t_{i,j} = s_j$ for $i, j = 1, \ldots, N$. Hence the $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_N \cong \mathcal{O}_A$ into $\mathcal{O}_N$ is given by

$$x_i = \sum_{j=1}^{N} a_{ij} u_i t_{i,q_i(j)} u_j^* = \sum_{j=1}^{N} s_i s_j s_j^* = s_i \quad (i = 1, \ldots, N).$$

Therefore this embedding is the identity map on $\mathcal{O}_N$. In this sense, the method of construction of embeddings by Corollary 2.5 is natural.

Example 3.2 If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $M_1 = 2$, $M_2 = 1$, $B_1 = \{1, 2\}$, $B_2 = \{1\}$, $q_1 = id_{\{1,2\}}$ and $q_2 = id_{\{1\}}$. Let s_1, s_2 denote canonical generators of $\mathcal{O}_2$. Define

$$u_i = s_i, \quad t_{1,i} = s_i \quad (i = 1, 2), \quad t_{2,1} = I.$$

Then we have the following well-known embedding of $\mathcal{O}_A$ into $\mathcal{O}_2$:

$$x_1 = s_1, \quad x_2 = s_2 s_1^*.$$

Since this correspondence is invertible, it is a $*$-isomorphism from $\mathcal{O}_A$ to $\mathcal{O}_2$.

Example 3.3 We show cases of matrices in p 268, [4]. For the matrix

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

consider the embedding of $\mathcal{O}_{A_1}$ into $\mathcal{O}_2$. Let s_1, s_2 denote canonical generators of $\mathcal{O}_2$. Then $(M_i)_{i=1}^3 = (1, 2, 3)$, $(B_i)_{i=1}^3 = (\{3\}, \{1, 3\}, \{1, 2, 3\})$, $q_1(3) = 1$, $q_2(1) = 1$, $q_2(3) = 2$, $q_3 = id$. $u_1 = s_1, u_2 = s_2 s_1, u_3 = s_2^2$. From these preparations,

$$\begin{cases} x_1 = & u_1 u_3^* = s_1 s_2^* s_2^*, \\ x_2 = & u_2(s_1 u_1^* + s_2 u_3^*) = s_2 s_1 (s_1 s_1^* + s_2 s_2^* s_2^*), \\ x_3 = & u_3 = s_2^2. \end{cases} \tag{3.1}$$

Note $\mathcal{O}_{A_1} \cong \mathcal{O}_4$. In fact,

$$v_1 \equiv x_1 x_3, \quad v_2 \equiv x_3, \quad v_3 \equiv x_2 x_3, \quad v_4 \equiv x_2 x_1 x_3 \tag{3.2}$$

satisfy the relations of canonical generators of $\mathcal{O}_4$. On the other hand,

$$x_1 = v_1 v_2^*, \quad x_2 = v_4 v_1^* + v_3 v_2^*, \quad x_3 = v_2.$$

This shows that (3.2) is a $*$-isomorphism from $\mathcal{O}_{A_1}$ to $\mathcal{O}_4$. If we write ψ, φ_c, ϕ as embeddings in (1.4), (3.1), (3.2), respectively, then $\psi \circ \phi = \varphi_c$.

In the same way, the following embeddings of $\mathcal{O}_A$ are obtained for

$A = A_2, A_3, A_4$:

$$A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad \begin{cases} x_1 = & s_1(s_1s_1^*s_2^* + s_2s_2^*s_2^*) = s_1s_2^*, \\ x_2 = & s_2s_1(s_1s_1^* + s_2s_2^*s_2^*), \\ x_3 = & s_2^2, \end{cases} \tag{3.3}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \quad \begin{cases} x_1 = & s_1s_2^*, \\ x_2 = & s_2s_1(s_1s_1^* + s_2s_2^*s_2^*), \\ x_3 = & s_2^2(s_1s_1^* + s_2s_1^*s_2^*), \end{cases} \tag{3.4}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad \begin{cases} x_1 = & s_1(s_1s_1^* + s_2s_2^*s_2^*), \\ x_2 = & s_2s_1(s_1s_1^*s_2^* + s_2s_2^*s_2^*) = s_2s_1s_2^*, \\ x_3 = & s_2^2. \end{cases} \tag{3.5}$$

Note that $\mathcal{O}_{A_2} \cong \mathcal{O}_5 \otimes M_2(\mathbf{C})$. In fact, for x_1, x_2, x_3 in (3.3), define $t_1, \ldots, t_5$ by

$$\begin{cases} t_1 = & x_1x_2x_1x_1^* + x_2x_1, \\ t_2 = & x_1x_2x_3x_1x_1^* + x_2x_3x_1, \\ t_3 = & x_1x_2x_3x_1^* + x_2x_3x_1^*x_1, \\ t_4 = & x_1x_3x_1x_1^* + x_3x_1, \\ t_5 = & x_1x_3x_1^* + x_3x_1^*x_1. \end{cases} \tag{3.6}$$

Then $t_1, \ldots, t_5$ satisfy the relations of canonical generators of $\mathcal{O}_5$. Furthermore $[t_i, x_1] = 0 = [t_i^*, x_1]$ for each $i = 1, \ldots, 5$. Hence $C^*\langle\{t_1, \ldots, t_5, x_1\}\rangle \cong \mathcal{O}_5 \otimes M_2(\mathbf{C})$. On the other hand,

$$x_2 = x_1^*x_1(t_1x_1^* + (t_2x_1^* + t_3)x_3^*), \quad x_3 = x_1^*x_1t_4.$$

Hence $C^*\langle\{t_1, \ldots, t_5, x_1\}\rangle = C^*\langle\{x_1, x_2, x_3\}\rangle = \varphi_c^{'}(\mathcal{O}_{A_2})$ where $\varphi_c^{'}$ is the embedding which is defined in (3.3). Hence we obtain a $*$-isomorphism from $\mathcal{O}_{A_2}$ to $\mathcal{O}_5 \otimes M_2(\mathbf{C})$.

Define the map ϕ' from $\mathcal{O}_5$ to $\varphi'_c(\mathcal{O}_{A_2})$ by (3.6). If ρ and ψ' are the canonical endomorphism of $\mathcal{O}_2$ and the embedding in (1.5), respectively, then $\rho \circ \psi' = \phi'$.

Example 3.4 Define $A = (a_{ij}) \in \mathbf{M}_N(\{0,1\})$ by $a_{ij} = 0$ $(i < j)$, $a_{ij} = 1$ $(i \geq j)$. Then the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^N$ is given by $M_i = i$, $B_i = \{1, \ldots, i\}$, $q_i = id_{B_i}$ for each $i = 1, \ldots, N$. Then

$$t_{1,1} = I, \quad t_{j,j} = s_2^{j-1}, \quad t_{j,i} = s_2^{i-1} s_1 \quad (2 \leq j \leq N,\ i = 1, \ldots, j-1),$$

$$x_j = t_{N,j} \sum_{i=1}^{j} t_{j,i} t_{N,i}^*.$$

Hence

$$\begin{cases} x_1 = & s_1 s_1^*, \\ x_2 = & s_2 s_1 (s_1 s_1^* + s_2 s_1^* s_2^*), \\ x_3 = & s_2^2 s_1 \{s_1 s_1^* + s_2 s_1 s_1^* s_2^* + s_2^2 s_1^* (s_2^*)^2\}, \\ \vdots & \vdots \\ x_{N-1} = & s_2^{N-2} s_1 \{s_1 s_1^* + \cdots + s_2^{N-2} s_1^* (s_2^*)^{N-2}\}, \\ x_N = & s_2^{N-1}. \end{cases}$$

For example, the case $N = 4$ is given as follows:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}; \quad \begin{cases} x_1 = & s_1 s_1^*, \\ x_2 = & s_2 s_1 (s_1 s_1^* + s_2 s_1^* s_2^*), \\ x_3 = & s_2^2 s_1 \{s_1 s_1^* + s_2 s_1 s_1^* s_2^* + s_2^2 s_1^* (s_2^*)^2\}, \\ x_4 = & s_2^3. \end{cases}$$

Example 3.5 Assume that $N \geq 3$ and define $A = (a_{ij}) \in \mathbf{M}_N(\{0,1\})$ by

$$a_{NN} = 0 \quad \text{and} \quad a_{ij} = 1 \quad \text{when } (i,j) \neq (N,N).$$

Then $M_i = N$, $B_i = B \equiv \{1, \ldots, N\}$, $q_i = id_B$ for $i = 1, \ldots, N-1$, $M_N = N-1$, $B_N = \{1, \ldots, N-1\}$, $q_N = id_{B_N}$. Let s_1, s_2 denote canonical generators of $\mathcal{O}_2$. Define

$$u_1 \equiv s_1, u_2 \equiv s_2 s_1, u_3 \equiv s_2 s_2 s_1, \ldots, u_{N-1} \equiv s_2^{N-2} s_1, u_N \equiv s_2^{N-1},$$

$$t_{i,j} \equiv \; u_j \quad (i = 1, \ldots, N-1,\, j = 1, \ldots, N),$$

$$t_{N,j} \equiv \; u_j \quad (j = 1, \ldots, N-2), \quad t_{N,N-1} \equiv s_2^{N-2}.$$

Remark that $u_1, \ldots, u_N$ and $t_{N,1}, \ldots, t_{N,N-1}$ satisfy relations of canonical generators of $\mathcal{O}_N$ and those of $\mathcal{O}_{N-1}$, respectively. Then

$$x_i = \; u_i = s_2^{i-1} s_1 \qquad (i = 1, \ldots, N-1),$$

$$x_N = \; u_N \left(\sum_{j=1}^{N-2} t_{N,j} t_{N,j}^* + t_{N,N-1} u_{N-1}^* \right)$$

$$= \; s_2^{N-1} \left(\sum_{j=1}^{N-2} s_2^{j-1} s_1 s_1^* (s_2^*)^{j-1} + s_2^{N-2} s_1^* (s_2^*)^{N-2} \right)$$

where we use 0-th power $s_i^0 \equiv I$ for $i = 1, \ldots, N$. Hence

$$x_1 = s_1, \quad x_2 = s_2 s_1, \quad \ldots, \quad x_{N-1} = s_2^{N-2} s_1, \quad x_N = s_2^{N-1} F_N$$

where

$$F_N \equiv \sum_{j=1}^{N-2} s_2^{j-1} s_1 s_1^* (s_2^*)^{j-1} + s_2^{N-2} s_1^* (s_2^*)^{N-2}.$$

For example, if $N = 3$, then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \qquad \begin{cases} x_1 = & s_1, \\ x_2 = & s_2 s_1, \\ x_3 = & s_1 s_1^* + s_2^3 s_1^* s_2^*. \end{cases}$$

Example 3.6 We show an example of Corollary 2.5 (ii) when $N = 5$. Define

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^5$ is given as follows:

$$(M_i)_{i=1}^5 = \quad (3,3,5,3,3),$$

$$(B_i)_{i=1}^5 = \quad (\{1,2,3\},\{1,2,3\},\{1,2,3,4,5\},\{1,3,5\},\{1,3,5\}),$$

$q_1 = q_2 = id_{\{1,2,3\}}, \quad q_3 = id_{\{1,2,3,4,5\}}, \quad q_4(2n-1) = n \quad (n = 1,2,3), \quad q_5 = q_4.$
Let s_1, s_2, s_3 denote canonical generators of $\mathcal{O}_3$. Define

$$t_{i,1} \equiv s_1, \quad t_{i,1} \equiv s_2, \quad t_{i,1} \equiv s_3 \quad (i = 1,2,4,5),$$

$$t_{3,1} \equiv s_1, \quad t_{3,2} \equiv s_2, \quad t_{3,3} \equiv s_3 s_1, \quad t_{3,4} \equiv s_3 s_2, \quad t_{3,5} \equiv s_3 s_3,$$

$$u_i \equiv t_{3,i} \quad (i = 1,\ldots,5).$$

Under these preparations, define generators of $\mathcal{O}_A$ by

$$x_i = \sum_{j=1}^{5} a_{ij} u_i t_{i,q_i(j)} u_j^* \quad (i = 1,2,3,4,5).$$

Then we have

$$\begin{cases} x_1 = & s_1 (s_1 s_1^* + s_2 s_2^* + s_3 s_1 s_3^*), \\ x_2 = & s_2 (s_1 s_1^* + s_2 s_2^* + s_3 s_1 s_3^*), \\ x_3 = & s_3 s_1, \\ x_4 = & s_3 s_2 (s_1 s_1^* + s_2 s_1 s_3^* + s_3 s_3 s_3^*), \\ x_5 = & s_3 s_3 (s_1 s_1^* + s_2 s_1 s_3^* + s_3 s_3 s_3^*). \end{cases}$$

In this case, we have a polynomial $*$-embedding of $\mathcal{O}_A$ into $\mathcal{O}_3$ with coefficient 1.

Example 3.7 Define $A \in \mathbf{M}_7(\{0,1\})$ by

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^7$ is given as follows:

$$(M_i)_{i=1}^7 = (4,4,7,4,7,4,1),$$

$$(B_i)_{i=1}^7 = \left(\begin{array}{l} \{2,4,6,7\},\{1,3,5,6\},\{1,\ldots,7\},\{1,2,3,4\}, \\ \{1,\ldots,7\},\{4,5,6,7\},\{1\} \end{array}\right)$$

and $\{q_i\}_{i=1}^7$ is taken as Definition 2.1. Since $\{M_i\}_{i=1}^7 = \{1,4,7\} \subset \{3k+1 : k \geq 0\}$, there is a $*$-embedding of $\mathcal{O}_A$ into $\mathcal{O}_4$. Let $s_1,\ldots,s_4$ denote canonical generators of $\mathcal{O}_4$. Define

$$u_i \equiv s_i \quad (i=1,2,3), \quad u_{3+i} \equiv s_4 s_i \quad (i=1,2,3,4).$$

Then polynomial generators of $\mathcal{O}_A$ in $\mathcal{O}_4$ are given as follows:

$$\begin{cases} x_1 = & s_1(s_1 s_2^* + s_2 s_1^* s_4^* + s_3 s_3^* s_4^* + s_4 (s_4^*)^2), \\ x_2 = & s_2(s_1 s_1^* + s_2 s_3^* + s_3 s_2 s_4^* + s_4 (s_4^*)^2), \\ x_3 = & s_3, \\ x_4 = & s_4 s_1 (s_1 s_1^* + s_2 s_2^* + s_3 s_3^* + s_4 s_1^* s_4^*), \\ x_5 = & s_4 s_2, \\ x_6 = & s_4 s_3 (s_1 s_1^* s_4^* + s_2 s_2^* s_4^* + s_3 s_3^* s_4^* + s_4 (s_4^*)^2) = s_4 s_3 s_4^*, \\ x_7 = & s_4^2 s_1 s_1^*. \end{cases}$$

Acknowledgement: The author would like to thank Kengo Matsumoto for his nice explanation of Cuntz-Krieger algebra [11].

References

[1] M. Abe and K. Kawamura, *Recursive fermion system in Cuntz algebra. I — Embeddings of fermion algebra into Cuntz algebra —*, Commun. Math. Phys. **228** (2002) 85-101.

[2] M. Abe and K. Kawamura, *Recursive fermion system in Cuntz algebra. II — Endomorphism, automorphism and branching of representation —*, preprint RIMS-1362 (2002).

[3] M. Abe and K. Kawamura, *Branching laws for endomorphisms of fermions and the Cuntz algebra $\mathcal{O}_2$*, J. Math. Phys., to appear.

[4] J. Cuntz and W. Krieger, *A class of C^*-algebra and topological Markov chains*, Invent. Math. **56** (1980) 251–268.

[5] K. Kawamura, *Extensions of representations of the CAR algebra to the Cuntz algebra $\mathcal{O}_2$ —the Fock and the infinite wedge—*, J. Math. Phys. **46**(7) (2005) 073509-1–073509-12.

[6] K. Kawamura, *The Perron-Frobenius operators, invariant measures and representations of the Cuntz-Krieger algebras*, J. Math. Phys. **46**(8) (2005) 083514-1–083514-6.

[7] K. Kawamura, *Recursive boson system in the Cuntz algebra $\mathcal{O}_\infty$*, J. Math. Phys. **48**(9) (2007) 093510-1–093510-16.

[8] K. Kawamura, *A tensor product of representations of Cuntz algebras*, Lett. Math. Phys. **82** (2007) 91–104.

[9] K. Kawamura, *C^*-bialgebra defined by the direct sum of Cuntz algebras*, J. Algebra, to appear.

[10] E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebra in the Cuntz algebra $\mathcal{O}_2$*, J. reine angew. Math. **525** (2000) 17–53.

[11] K. Matsumoto, *The Cuntz algebra and the Cuntz-Krieger algebra from the viewpoint of symbolic dynamical system*, RIMS workshop: Representations of Cuntz algebras on fractal sets and their application for mathematical physics, 26-29, Nov. 2002.

[12] M. Rørdam and E.Størmer, *Classification of nuclear C^*-algebras. Entropy in operator algebras,* Springer-Verlag Berlin Heidelberg (2002).