Polynomial embedding of the Cuntz-Krieger algebra into the Cuntz algebra

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Abstract

For any Cuntz-Krieger algebra $\mathcal{O}_A$, we construct embeddings of $\mathcal{O}_A$ into the Cuntz algebra $\mathcal{O}_2$ such that the generators of $\mathcal{O}_A$ are written as polynomials of those of $\mathcal{O}_2$.

1 Introduction

It is well known that there always exists a *-embedding of a C*-algebra which satisfies some conditions into the Cuntz algebra $\mathcal{O}_2$ by [3]. Although, concrete method of construction of embedding is not known very well. We construct embeddings of any Cuntz-Krieger algebra into $\mathcal{O}_2$ by concrete polynomials in the following sense.

Let $\mathcal{O}_A$ be the Cuntz-Krieger algebra by a matrix A.

Theorem 1.1 *(Main theorem) Let $N \geq 2$. For any $N \times N$-matrix A which consists only 0 or 1, there exists a family $\{t_1, \ldots, t_N\}$ of elements in $\mathcal{O}_2$ such that*

(i) *they satisfy the relations of generators of $\mathcal{O}_A$, and*

(ii) *they are polynomials of generators s_1, s_2 of $\mathcal{O}_2$ and their conjugations s_1^*, s_2^*.*

We show this theorem in section 2(Theorem 2.4). Examples of these generators and the naturality of our construction are shown in section 3. In order to construct generators of $\mathcal{O}_A$ in $\mathcal{O}_2$, we prepare several notions in this section.

For $N \geq 2$, let $M_N(\{0,1\})$ be the set of all $N \times N$ matrices such that each element is 0 or 1. For $A = (a_{ij}) \in M_N(\{0,1\})$, $\mathcal{O}_A$ is *the Cuntz-Krieger algebra by A* if $\mathcal{O}_A$ is a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ and they satisfy the following conditions ([2]):

$$s_i^* s_i = \sum_{j=1}^{N} a_{ij} s_j s_j^* \quad (i = 1, \ldots, N), \quad \sum_{i=1}^{N} s_i s_i^* = I. \tag{1.1}$$

Specially, when $a_{ij} = 1$ for each $i, j = 1, \ldots, N$, $\mathcal{O}_A$ is the Cuntz algebra $\mathcal{O}_N$.

Let $M \geq 2$, a subset $R \subset \mathbf{C}$ and generators $s_1, \ldots, s_M$ of $\mathcal{O}_M$. Denote subsets of $\mathcal{O}_M$

$$\mathcal{M}(\mathcal{O}_M) \equiv \bigcup_{k+l\geq 1,\, k,l\geq 0} \left\{ s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^* \in \mathcal{O}_M : \begin{array}{c} i_\alpha, j_\beta = 1, \ldots, M, \\ \alpha = 1, \ldots, k, \\ \beta = 1, \ldots, l, \end{array} \right\},$$

$$\mathcal{O}_M^o(R) \equiv \bigcup_{n\geq 1} \left\{ \sum_{\lambda=1}^{n} b_\lambda x_\lambda \in \mathcal{O}_M : x_\lambda \in \mathcal{M}(\mathcal{O}_M),\, b_\lambda \in R,\, \lambda = 1, \ldots, n \right\}.$$

In this paper, any homomorphism and embedding are assumed unital. Generators of $\mathcal{O}_A$ means always those which satisfies (1.1).

Definition 1.2 (i) *An element in $\mathcal{O}_M^o(R)$ ($\mathcal{M}(\mathcal{O}_M)$) is called a R-polynomial (a monomial) in $\mathcal{O}_M$.*

(ii) *A $*$-homomorphism $\Phi : \mathcal{O}_A \to \mathcal{O}_M$ is polynomial type over R (monomial type) if $\Phi(t_1), \ldots, \Phi(t_N)$ are in $\mathcal{O}_M^o(R)$ ($\mathcal{M}(\mathcal{O}_M)$) where $t_1, \ldots, t_N$ are generators of $\mathcal{O}_A$.*

(iii) *A $*$-embedding $\Phi : \mathcal{O}_A \hookrightarrow \mathcal{O}_M$ is polynomial type over R (monomial type) if Φ is polynomial type over R(monomial type) as $*$-homomorphism.*

(iv) *$\mathcal{O}_A$ is R-polynomially (monomially) embedded into $\mathcal{O}_M$ if there exists $*$-embedding from $\mathcal{O}_A$ into $\mathcal{O}_M$ which is polynomial type over R(monomial type).*

(v) *$x_1, \ldots, x_N$ are R-polynomial (monomial) generators of $\mathcal{O}_A$ in $\mathcal{O}_M$ if $x_1, \ldots, x_N$ are in $\mathcal{O}_M^o(R)$ ($\mathcal{M}(\mathcal{O}_M)$) and satisfy (1.1)*

Remark 1.3 For a non commutative polynomial $f \in \mathbf{C}[x_1, \ldots, x_M, y_1, \ldots, y_M]$, it is natural to regard $f(s_1, \ldots, s_M, s_1^*, \ldots, s_M^*)$ as a polynomial in $\mathcal{O}_M$ with respect to generators $s_1, \ldots, s_M$. But it is reasonable to regard an element in $\mathcal{O}_M^o(R)$ as a polynomial in $\mathcal{O}_M$ because such $f(s_1, \ldots, s_M, s_1^*, \ldots, s_M^*)$ is always in $\mathcal{O}_M^0(R)$ by the relations (1.1).

Specially, if R is a subring of $\mathbf{C}$, then $\mathcal{O}_M^o(R)$ is a subalgebra of $\mathcal{O}_M$ over R. Furthermore if R is closed under complex conjugation, then $\mathcal{O}_M^o(R)$ is a $*$-subalgebra of $\mathcal{O}_M$ over R. Note $\mathcal{O}_M^o \equiv \mathcal{O}_M^o(\mathbf{C})$ is dense in $\mathcal{O}_M$ and it is regarded as the (non commutative)polynomial ring of generators $s_1, \ldots, s_M, s_1^*, \ldots, s_M^*$ over $\mathbf{C}$ under relations of $\mathcal{O}_M$.

In subsection 2.1 in [1], there are many polynomial embeddings among Cuntz algebras. We review known embeddings associated our article from [1].

Lemma 1.4 (i) *For each $N \geq 2$, $\mathcal{O}_N$ can be monomially embedded into $\mathcal{O}_2$.*

(ii) *For each $M \in \{(N-1)k+1 : k \geq 1\}$, $\mathcal{O}_M$ can be monomially embedded into $\mathcal{O}_N$.*

Proof. (i) Let s_1, s_2 be generators of $\mathcal{O}_2$. The case $N = 2$ is trivial. Assume $N \geq 3$. Put

$$\begin{cases} t_1 \equiv & s_1, \\ t_i \equiv & (s_2)^{i-1} s_1 \quad (i = 2, \ldots, N-1), \\ t_N \equiv & (s_2)^{N-1}. \end{cases} \tag{1.2}$$

Then $t_1, \ldots, t_N$ satisfy relations of generators of $\mathcal{O}_N$ and they belong to $\mathcal{M}(\mathcal{O}_2)$.

(ii) Let $s_1, \ldots, s_N$ be generators of $\mathcal{O}_N$. The case $M = N$ is trivial. Assume that $M = (N-1)k + 1$, $k \geq 2$. Put

$$\begin{cases} t_i \equiv & s_i \qquad (i = 1, \ldots, N-1), \\ t_{(N-1)l+i} \equiv & (s_N)^l s_i \qquad \begin{pmatrix} l = 1, \ldots, k-1, \\ i = 1, \ldots, N-1 \end{pmatrix}, \\ t_M \equiv & (s_N)^k. \end{cases} \tag{1.3}$$

Then $t_1, \ldots, t_M$ satisfy relations of generators of $\mathcal{O}_M$ and they belong to $\mathcal{M}(\mathcal{O}_N)$. ∎

Corollary 1.5 *For each $n \geq 1$, there exists a monomial embedding of $\mathcal{O}_{2n+1}$ into $\mathcal{O}_3$.*

Note that the choice of polynomial embedding of $\mathcal{O}_N$ into $\mathcal{O}_2$ is not unique. For example, we have the followings: An embedding of $\mathcal{O}_4$ into $\mathcal{O}_2$:

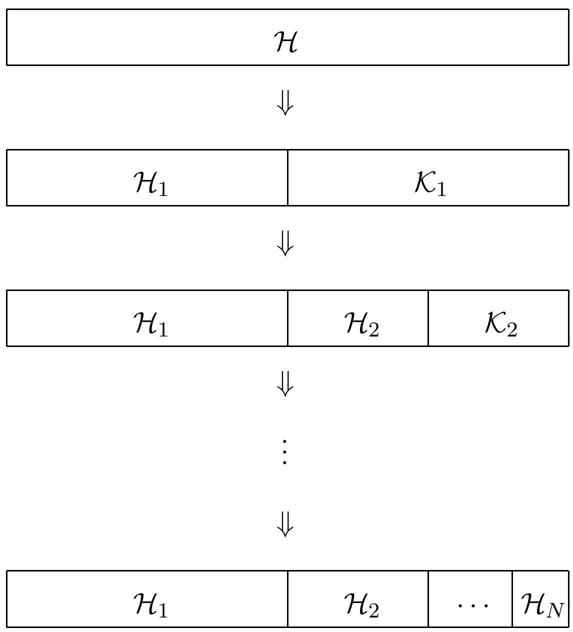
$$t_1 \equiv s_1, \quad t_2 \equiv s_2 s_2, \quad t_3 \equiv s_2 s_1 s_2, \quad t_4 \equiv s_2 s_1 s_1. \tag{1.4}$$

An embedding of $\mathcal{O}_5$ into $\mathcal{O}_2$:

$$t_1 \equiv \quad s_1 s_1, \quad t_2 \equiv s_1 s_2 s_1, \quad t_3 \equiv s_1 s_2 s_2, \quad t_4 \equiv s_2 s_1, \quad t_5 \equiv s_2 s_2. \tag{1.5}$$

We illustrate our construction of embeddings among Cuntz algebras in Lemma 1.4 (i). Assume that $\mathcal{O}_2$ is represented on a Hilbert space $\mathcal{H}$. Then we have an orthogonal decomposition $\{\mathcal{H}_i\}_{i=1}^N$ of $\mathcal{H}$ by

$$\mathcal{H}_1 \equiv s_1\mathcal{H}, \quad \mathcal{H}_2 \equiv s_2 s_1 \mathcal{H}, \ldots, \mathcal{H}_{N-1} \equiv s_2^{N-2} s_1 \mathcal{H}, \quad \mathcal{H}_N \equiv s_2^{N-1}\mathcal{H}.$$



where

$$\mathcal{K}_i \equiv \left(\bigoplus_{j=1}^{i} \mathcal{H}_j \right)^{\perp} \quad (i = 1, \ldots, N-1).$$

2 Construction of polynomial generators of $\mathcal{O}_A$ in $\mathcal{O}_M$

We prepare several tools associated with a matrix A.

Fix $A=(a_{ij})\in M_N(\{0,1\})$. Put

$$B_i \equiv \ \{\, j\in\{1,\ldots,N\} : a_{ij}=1\,\}, \quad M_i\equiv \sum_{j=1}^{N} a_{ij},$$

$$q_i : \ \ B_i \to \{1,\ldots,M_i\}; \quad q_i(j)\equiv \#\{k\in B_i : k\leq j\}$$

for $i=1,\ldots,N$. Note that q_i is bijective for each $i=1,\ldots,N$.

Definition 2.1 *$\{(M_i,q_i,B_i)\}_{i=1}^{N}$ is called the (canonical)A-coordinate. $\{M_i\}_{i=1}^{N}$ is called the set of row sums of A.*

Lemma 2.2 *Let $A=(a_{ij})\in M_N(\{0,1\})$ and $\{(M_i,q_i,B_i)\}_{i=1}^{N}$ the A-coordinate. Assume that a unital C^*-algebra $\mathcal{B}$ satisfies the following condition:*

$\mathcal{B}$ contains $\mathcal{O}_N$ and $\mathcal{O}_{M_i}$ for each $i=1,\ldots,N$ when $M_i\geq 2$ as C^*-subalgebras with common unit. (2.1)

Let $\{s_1,\ldots,s_N\}$ be generators of $\mathcal{O}_N$ and $\{t_{i,j} : j=1,\ldots,M_i\}$ those of $\mathcal{O}_{M_i}$ for $i=1,\ldots,N$ as elements in $\mathcal{B}$, respectively. Specially, we put $\mathcal{O}_1=\mathbf{C}I$ and $t_{i,1}=I$ when $M_i=1$. Under these assumptions, put

$$x_i\equiv \sum_{j=1}^{N} a_{ij}s_i t_{i,q_i(j)}s_j^*. \tag{2.2}$$

Then $\{x_i\}_{i=1}^{N}$ satisfies the condition (1.1) with respect to A.

Proof. Denote

$$F_i\equiv \sum_{j=1}^{N} a_{ij}t_{i,q_i(j)}s_j^* \quad (i=1,\ldots,N).$$

Then $x_i=s_iF_i$ and the followings hold:

$$F_i^*F_i=\sum_{j=1}^{N} a_{ij}s_js_j^*, \quad F_iF_i^*=\sum_{j=1}^{N} a_{ij}t_{i,q_i(j)}t_{i,q_i(j)}^*=I \quad (i=1,\ldots,N).$$

We show the condition (1.1) by direct computation.

$$x_i^*x_i = \ \ F_i^*s_i^*s_iF_i=\sum_{j=1}^{N} a_{ij}s_js_j^*, \quad x_ix_i^*=s_iF_iF_i^*s_i^*=s_is_i^*$$

for each $i = 1, \ldots, N$. Hence we have the condition (1.1):

$$x_i^* x_i = \sum_{j=1}^{N} a_{ij} x_j x_j^*, \quad \sum_{i=1}^{N} x_i x_i^* = I.$$

∎

Note that Lemma 2.2 holds when the choice of q_i are replaced as any bijection from B_i to $\{1, \ldots, M_i\}$ for each $i = 1, \ldots, N$, too.

Corollary 2.3 *Let $N \geq 2$. For $A \in M_N(\{0,1\})$ and the set $\{M_i\}_{i=1}^N$ of row sums of A, there exists a $*$-homomorphism from $\mathcal{O}_A$ to $\mathcal{B}$ if $\mathcal{B}$ is a unital C^*-algebra which satisfies (2.1).*

Proof. By Lemma 2.2, it holds immediately. ∎

Let $\mathbf{Z}_{n\geq 0} \equiv \{n \in \mathbf{Z} : n \geq 0\}$ be the set of all non-negative integers. Recall the definition of properties of embeddings in Definition 1.2.

Theorem 2.4 *For any $A \in M_N(\{0,1\})$, there exists a $\mathbf{Z}_{n\geq 0}$-polynomial homomorphism from $\mathcal{O}_A$ to $\mathcal{O}_2$. Specially if $\mathcal{O}_A$ is simple, then there exists a $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_A$ into $\mathcal{O}_2$.*

Proof. For any $M \geq 2$, there exists $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_M$ into $\mathcal{O}_2$ by Lemma 1.4 (i). Furthermore $\mathcal{O}_2$ satisfies (2.1) in Lemma 2.2 such that $s_i, t_{i,j}$ in (2.2) are written as monomials of $\mathcal{O}_2$. Since the form of x_i in (2.2), $x_1, \ldots, x_N$ are written by $\mathbf{Z}_{n\geq 0}$-polynomials in $\mathcal{O}_2$. Therefore the first statement holds. Specially, if $\mathcal{O}_A$ is simple, this homomorphism is injective automatically. Hence the second statement follows. ∎

Theorem 1.1 is shown by the above theorem. The embedding in Theorem 2.4 depends on the choice of embeddings of $\mathcal{O}_M$ into $\mathcal{O}_2$.

Corollary 2.5 *Let $A \in M_N(\{0,1\})$, the set $\{M_i\}_{i=1}^N$ of row sums of A and $M \geq 2$.*

(i) *If there is the following inclusion $\{N, M_i : i = 1, \ldots, N\} \subset \{(M-1)k + 1 : k \geq 0\}$, then there exists a $\mathbf{Z}_{n\geq 0}$-polynomial homomorphism from $\mathcal{O}_A$ to $\mathcal{O}_M$.*

(ii) *Assume that M_i and N are odd for each $i = 1, \ldots, N$. Then there exists a $\mathbf{Z}_{n\geq 0}$-polynomial homomorphism from $\mathcal{O}_A$ to $\mathcal{O}_3$.*

Proof. (i) It follows from Corollary 2.3, the form of generators in (2.2) and Lemma 1.4 (ii). (ii) By Corollary 1.5, $\mathcal{O}_3$ satisfies the condition in (i) with respect to all odd number N, M_i, $i = 1, \ldots, N$. Hence there are $\mathbf{Z}_{n\geq 0}$-polynomial generators of $\mathcal{O}_A$ in $\mathcal{O}_3$. ∎

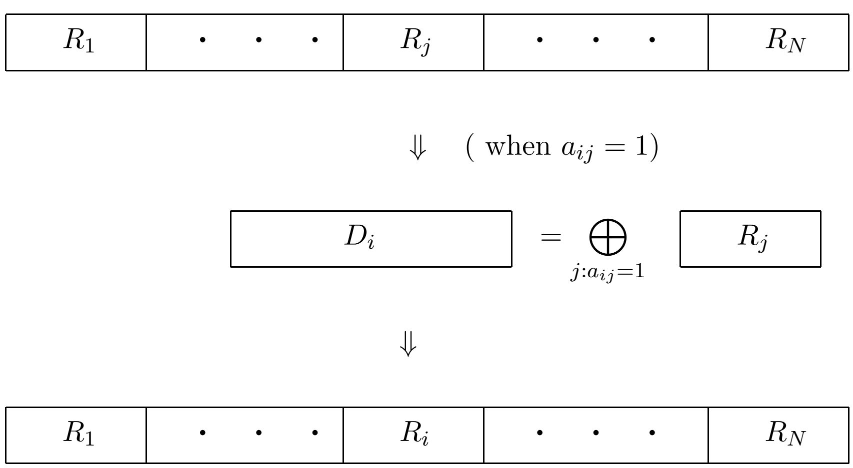
We illustrate our construction of embeddings as a decomposition of Hilbert space by partial isometries, where we assume that $\mathcal{B}$ in Lemma 2.2 is represented on an infinite dimensional Hilbert space $\mathcal{H}$. Fix $A \in M_N(\{0,1\})$ and $\{M_i\}_{i=1}^N$ is the set of row sums of A.

(i) At first, decompose a Hilbert space $\mathcal{H}$ into N-parts $R_1, \ldots, R_N$ as infinite dimensional Hilbert subspaces of $\mathcal{H}$. This is the role of $s_1^*, \ldots, s_N^*$ in (2.2).

(ii) Next, choose M_i-number of components from $R_1, \ldots, R_N$ by the rule associated with a matrix A and make a new subspace D_i of $\mathcal{H}$ for each $i = 1, \ldots, N$, respectively. This process is executed by $t_{i,q_i(j)}$ and the sum in (2.2).

(iii) At the end, we maps D_i into R_i by s_i for $i = 1, \ldots, N$ in (2.2), respectively.

By these procedure, we have a partial isometry $x_i : D_i \to R_i$ in (2.2) for $i = 1, \ldots, N$.



3 Examples

Example 3.1 Assume that $A = (a_{ij}) \in M_N(\{0,1\})$ satisfies $a_{ij} = 1$ for each $i, j = 1, \ldots, N$. In this case, $\mathcal{O}_A \cong \mathcal{O}_N$. Then the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^N$ is given by $(M_i, q_i, B_i) = (N, id_{\{1,\ldots,N\}}, \{1, \ldots, N\})$ for each $i = 1, \ldots, N$. By Corollary 2.5 (i), we obtain an embedding of $\mathcal{O}_N$ into $\mathcal{O}_N$. That is, this is an endomorphism of $\mathcal{O}_N$.

Let $s_1, \ldots, s_N$ be generators of $\mathcal{O}_N$. Hence $u_j \equiv t_{i,j} = s_j$ for $i, j = 1, \ldots, N$. Hence $\mathbf{Z}_{n\geq 0}$-polynomial embedding of $\mathcal{O}_N \cong \mathcal{O}_A$ into $\mathcal{O}_N$ is given by

$$x_i = \sum_{j=1}^{N} a_{ij} u_i t_{i,q_i(j)} u_j^* = \sum_{j=1}^{N} s_i s_j s_j^* = s_i \quad (i = 1, \ldots, N).$$

Therefore this embedding is the identity map on $\mathcal{O}_N$. In this sense, the method of construction of embeddings by Corollary 2.5 is natural.

Example 3.2 If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $M_1 = 2$, $M_2 = 1$, $B_1 = \{1, 2\}$, $B_2 = \{1\}$, $q_1 = id_{\{1,2\}}$ and $q_2 = id_{\{1\}}$. Let s_1, s_2 be generators of $\mathcal{O}_2$. Put

$$u_i = s_i, \quad t_{1,i} = s_i \quad (i = 1, 2), \quad t_{2,1} = I.$$

Then we have the well known following embedding of $\mathcal{O}_A$ into $\mathcal{O}_2$:

$$x_1 = s_1, \quad x_2 = s_2 s_1^*.$$

This correspondence is invertible. Hence $\mathcal{O}_A \cong \mathcal{O}_2$.

Example 3.3 We show cases of matrices in p 268, [2]. For a matrix

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

consider the embedding of $\mathcal{O}_{A_1}$ into $\mathcal{O}_2$. Let s_1, s_2 be generators of $\mathcal{O}_2$. $(M_i)_{i=1}^3 = (1, 2, 3)$, $(B_i)_{i=1}^3 = (\{3\}, \{1, 3\}, \{1, 2, 3\})$, $q_1(3) = 1$, $q_2(1) = 1$, $q_2(3) = 2$, $q_3 = id$. $u_1 = s_1, u_2 = s_2 s_1, u_3 = s_2^2$. From these preparations,

$$\begin{cases} x_1 = & u_1 u_3^* = s_1 s_2^* s_2^*, \\ x_2 = & u_2(s_1 u_1^* + s_2 u_3^*) = s_2 s_1 (s_1 s_1^* + s_2 s_2^* s_2^*), \\ x_3 = & u_3 = s_2^2. \end{cases} \tag{3.1}$$

Note $\mathcal{O}_{A_1} \cong \mathcal{O}_4$. In fact,

$$v_1 \equiv x_1x_3, \quad v_2 \equiv x_3, \quad v_3 \equiv x_2x_3, \quad v_4 \equiv x_2x_1x_3 \tag{3.2}$$

satisfy the relations of generators of $\mathcal{O}_4$. On the contrary

$$x_1 = v_1v_2^*, \quad x_2 = v_4v_1^* + v_3v_2^*, \quad x_3 = v_2.$$

This shows (3.2) is an isomorphism from $\mathcal{O}_{A_1}$ to $\mathcal{O}_4$. If we denote ψ, φ_c, ϕ as homomorphisms in (1.4), (3.1), (3.2), respectively, then $\psi \circ \phi = \varphi_c$.

In the same way, we have the followings:

$$A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad \begin{cases} x_1 = & s_1(s_1s_1^*s_2^* + s_2s_2^*s_2^*) = s_1s_2^*, \\ x_2 = & s_2s_1(s_1s_1^* + s_2s_2^*s_2^*), \\ x_3 = & s_2^2, \end{cases} \tag{3.3}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \quad \begin{cases} x_1 = & s_1s_2^*, \\ x_2 = & s_2s_1(s_1s_1^* + s_2s_2^*s_2^*), \\ x_3 = & s_2^2(s_1s_1^* + s_2s_1^*s_2^*), \end{cases} \tag{3.4}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad \begin{cases} x_1 = & s_1(s_1s_1^* + s_2s_2^*s_2^*), \\ x_2 = & s_2s_1(s_1s_1^*s_2^* + s_2s_2^*s_2^*) = s_2s_1s_2^*, \\ x_3 = & s_2^2. \end{cases} \tag{3.5}$$

Note that $\mathcal{O}_{A_2} \cong \mathcal{O}_5 \otimes M_2(\mathbf{C})$. In fact, for x_1, x_2, x_3 in (3.3), put $t_1, \ldots, t_5$ by

$$\begin{cases} t_1 = & x_1x_2x_1x_1^* + x_2x_1, \\ t_2 = & x_1x_2x_3x_1x_1^* + x_2x_3x_1, \\ t_3 = & x_1x_2x_3x_1^* + x_2x_3x_1^*x_1, \\ t_4 = & x_1x_3x_1x_1^* + x_3x_1, \\ t_5 = & x_1x_3x_1^* + x_3x_1^*x_1. \end{cases} \tag{3.6}$$

Then $t_1, \ldots, t_5$ satisfy the relations of $\mathcal{O}_5$. Furthermore $[t_i, x_1] = 0 = [t_i^*, x_1]$ for each $i = 1, \ldots, 5$. Hence $C^* < \{t_1, \ldots, t_5, x_1\} >\cong \mathcal{O}_5 \otimes M_2(\mathbf{C})$. On the contrary,

$$x_2 = x_1^* x_1 (t_1 x_1^* + (t_2 x_1^* + t_3) x_3^*), \quad x_3 = x_1^* x_1 t_4.$$

Hence $C^* < \{t_1, \ldots, t_5, x_1\} >= C^* < \{x_1, x_2, x_3\} >= \varphi_c^{'}(\mathcal{O}_{A_2})$ where $\varphi_c^{'}$ is the embedding which is defined in (3.3). Since $\mathcal{O}_{A_2}$ is simple, we have the isomorphism from $\mathcal{O}_{A_2}$ to $\mathcal{O}_5 \otimes M_2(\mathbf{C})$.

Define a map $\phi^{'} : \mathcal{O}_5 \to \varphi_c^{'}(\mathcal{O}_{A_2}) \subset \mathcal{O}_2$ by (3.6). If $\rho, \psi^{'}$ are the canonical endomorphism of $\mathcal{O}_2$ and the embedding in (1.5) respectively, then $\rho \circ \psi^{'} = \phi^{'}$.

Example 3.4 Put $A = (a_{ij}) \in M_N(\{0,1\})$ by $a_{ij} = 0$ $(i < j)$, $a_{ij} = 1$ $(i \geq j)$. The A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^N$ is given by $M_i = i$, $B_i = \{1, \ldots, i\}$, $q_i = id_{B_i}$ for each $i = 1, \ldots, N$. Then

$$t_{1,1} = I, \quad t_{j,j} = s_2^{j-1} \quad (2 \leq j \leq N),$$

$$t_{j,i} = s_2^{i-1} s_1 \quad (2 \leq j \leq N,\, i = 1, \ldots, j-1),$$

$$x_j = t_{N,j} \left(\sum_{i=1}^{j} t_{j,i} t_{N,i}^* \right).$$

Hence

$$\begin{cases} x_1 = & s_1 s_1^*, \\ x_2 = & s_2 s_1 (s_1 s_1^* + s_2 s_1^* s_2^*), \\ x_3 = & s_2^2 s_1 (s_1 s_1^* + s_2 s_1 s_1^* s_2^* + s_2^2 s_1^* (s_2^*)^2), \\ \vdots & \vdots \\ x_{N-1} = & s_2^{N-2} s_1 (s_1 s_1^* + \cdots + s_2^{N-2} s_1^* (s_2^*)^{N-2}), \\ x_N = & s_2^{N-1}. \end{cases}$$

For example, the case $N=4$,

$$A=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix};\quad \begin{cases} x_1 = & s_1s_1^*,\\ x_2 = & s_2s_1(s_1s_1^*+s_2s_1^*s_2^*),\\ x_3 = & s_2^2s_1(s_1s_1^*+s_2s_1s_1^*s_2^*+s_2^2s_1^*(s_2^*)^2),\\ x_4 = & s_2^3. \end{cases}$$

Example 3.5 Assume that $N\geq 3$ and put $A=(a_{ij})\in M_N(\{0,1\})$ by

$$a_{NN}=0 \quad \text{and} \quad a_{ij}=1 \quad \text{when } (i,j)\neq(N,N).$$

Then $M_i=N$, $B_i=B\equiv\{1,\dots,N\}$, $q_i=id_B$ for $i=1,\dots,N-1$, $M_N=N-1$, $B_N=\{1,\dots,N-1\}$, $q_N=id_{B_N}$. Let s_1,s_2 be generators of $\mathcal{O}_2$. Put

$$u_1\equiv s_1, u_2\equiv s_2s_1, u_3\equiv s_2s_2s_1,\dots,u_{N-1}\equiv s_2^{N-2}s_1, u_N\equiv s_2^{N-1},$$

$$t_{i,j}\equiv u_j \quad (i=1,\dots,N-1,\ j=1,\dots,N),$$
$$t_{N,j}\equiv u_j \quad (j=1,\dots,N-2), \quad t_{N,N-1}\equiv s_2^{N-2}.$$

Note $u_1,\dots,u_N$ are generators of $\mathcal{O}_N$ and $t_{N,1},\dots,t_{N,N-1}$ are those of $\mathcal{O}_{N-1}$. Then

$$\begin{aligned} x_i = & \quad u_i=s_2^{i-1}s_1 \quad (i=1,\dots,N-1),\\ x_N = & \quad u_N\left(\sum_{j=1}^{N-2}t_{N,j}t_{N,j}^*+t_{N,N-1}u_{N-1}^*\right)\\ = & \quad s_2^{N-1}\left(\sum_{j=1}^{N-2}s_2^{j-1}s_1s_1^*(s_2^*)^{j-1}+s_2^{N-2}s_1^*(s_2^*)^{N-2}\right)\end{aligned}$$

where we use 0-th power $s_i^0\equiv I$ for $i=1,\dots,N$. Hence

$$x_1=s_1,\quad x_2=s_2s_1,\quad \dots,\quad x_{N-1}=s_2^{N-2}s_1,\quad x_N=s_2^{N-1}F_N$$

where

$$F_N\equiv\sum_{j=1}^{N-2}s_2^{j-1}s_1s_1^*(s_2^*)^{j-1}+s_2^{N-2}s_1^*(s_2^*)^{N-2}.$$

For example, if $N = 3$, then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \qquad \begin{cases} x_1 = & s_1, \\ \\ x_2 = & s_2 s_1, \\ \\ x_3 = & s_1 s_1^* + s_2^3 s_1^* s_2^*. \end{cases}$$

Example 3.6 We show an example of Corollary 2.5 (ii) when $N = 5$. Put

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^5$ becomes as follows:

$$(M_i)_{i=1}^5 = \quad (3, 3, 5, 3, 3),$$

$$(B_i)_{i=1}^5 = \quad (\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4, 5\}, \{1, 3, 5\}, \{1, 3, 5\}),$$

$q_1 = q_2 = id_{\{1,2,3\}}$, $\quad q_3 = id_{\{1,2,3,4,5\}}$, $\quad q_4(2n-1) = n \quad (n = 1, 2, 3)$, $\quad q_5 = q_4$.
Let s_1, s_2, s_3 be generators of $\mathcal{O}_3$. Define

$$t_{i,1} \equiv s_1, \quad t_{i,1} \equiv s_2, \quad t_{i,1} \equiv s_3 \quad (i = 1, 2, 4, 5),$$

$$t_{3,1} \equiv s_1, \quad t_{3,2} \equiv s_2, \quad t_{3,3} \equiv s_3 s_1, \quad t_{3,4} \equiv s_3 s_2, \quad t_{3,5} \equiv s_3 s_3,$$

$$u_i \equiv t_{3,i} \quad (i = 1, \ldots, 5).$$

Under these preparations, define generators of $\mathcal{O}_A$ by

$$x_i = \sum_{j=1}^{5} a_{ij} u_i t_{i,q_i(j)} u_j^* \quad (i = 1, 2, 3, 4, 5).$$

Then we have

$$\begin{cases} x_1 = & s_1 \left(s_1 s_1^* + s_2 s_2^* + s_3 s_1 s_3^* \right), \\ \\ x_2 = & s_2 \left(s_1 s_1^* + s_2 s_2^* + s_3 s_1 s_3^* \right), \\ \\ x_3 = & s_3 s_1, \\ \\ x_4 = & s_3 s_2 \left(s_1 s_1^* + s_2 s_1 s_3^* + s_3 s_3 s_3^* \right), \\ \\ x_5 = & s_3 s_3 \left(s_1 s_1^* + s_2 s_1 s_3^* + s_3 s_3 s_3^* \right). \end{cases}$$

In this case, we have a polynomial $*$-homomorphism from $\mathcal{O}_A$ to $\mathcal{O}_3$ with coefficient 1.

Example 3.7 Let $A \in M_7(\{0,1\})$ be

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^7$ becomes as follows:

$$(M_i)_{i=1}^7 = \quad (4,4,7,4,7,4,1),$$

$$(B_i)_{i=1}^7 = \quad \begin{pmatrix} \{2,4,6,7\}, \{1,3,5,6\}, \{1,\ldots,7\}, \{1,2,3,4\}, \\ \{1,\ldots,7\}, \{4,5,6,7\}, \{1\} \end{pmatrix}$$

and $\{q_i\}_{i=1}^7$ is taken as Definition 2.1. Since $\{M_i\}_{i=1}^7 = \{1,4,7\} \subset \{3k+1 : k \geq 0\}$, there is a homomorphism from $\mathcal{O}_A$ to $\mathcal{O}_4$. Let $s_1, \ldots, s_4$ be generators of $\mathcal{O}_4$. Put

$$u_i \equiv s_i \quad (i = 1,2,3), \quad u_{3+i} \equiv s_4 s_i \quad (i = 1,2,3,4).$$

Then polynomial generators of $\mathcal{O}_A$ in $\mathcal{O}_4$ are given as follows:

$$\begin{cases} x_1 = & s_1(s_1 s_2^* + s_2 s_1^* s_4^* + s_3 s_3^* s_4^* + s_4 (s_4^*)^2), \\ x_2 = & s_2(s_1 s_1^* + s_2 s_3^* + s_3 s_2 s_4^* + s_4 (s_4^*)^2), \\ x_3 = & s_3, \\ x_4 = & s_4 s_1 (s_1 s_1^* + s_2 s_2^* + s_3 s_3^* + s_4 s_1^* s_4^*), \\ x_5 = & s_4 s_2, \\ x_6 = & s_4 s_3 (s_1 s_1^* s_4^* + s_2 s_2^* s_4^* + s_3 s_3^* s_4^* + s_4 (s_4^*)^2) = s_4 s_3 s_4^*, \\ x_7 = & s_4^2 s_1 s_1^*. \end{cases}$$

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