On global aspects of exact WKB analysis of operators admitting infinitely many phases

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Abstract. A case-study of the Stokes geometry in the large is given for some concrete examples of WKB type operators that admit infinitely many phases. In all the examples discussed in this article virtual turning points and new Stokes curves emanating from them neatly resolve the trouble we encounter. Some computer-assisted study of the configuration of the steepest descent paths is employed to show that the introduction of new Stokes curves is reasonable and appropriate to obtain the complete Stokes geometry.

1. Introduction

In [**AKKT1**] we introduced the notion of operators of WKB type and developed their exact WKB analysis near their turning points. One of our purposes there was to enlarge the scope of applicability of the exact WKB analysis by defining a suitable class of operators whose WKB solutions admit infinitely many phases, so that we might provide fusion physicists with convenient tools. (Cf. [**BB**], [**BP**], [**BNR**], [**WSW**] and references cited therein.) Concerning this point, we feel some further generalization of the class of operators might be needed ([**AKKT2**]). Purely mathematically speaking, however, the class of

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WKB type operators is a subject which is interesting in its own right. In particular, the study of its global aspects raises several interesting and intriguing questions. The purpose of this article is to discuss them by studying some concrete examples numerically (i.e., with the help of a computer). We have no rigorous analytic results so far, but the examples discussed below will convince the reader that the introduction of virtual turning points and new Stokes curves is a reasonable and appropriate approach to the problem. (Cf. [**BNR**], [**AKT1**], [**AKT3**]. See Section 3.2 for the definition of virtual turning points and new Stokes curves for our operators.)

The plan of this paper is as follows: for the convenience of the reader we recall in Section 2 some basic notions and notations in exact WKB analysis which we need in this paper. In Section 3 we study some particular local operator with a large parameter to see what kind of pathologies we encounter in its Stokes geometry in the large and how we can resolve them. Our proposal is to use virtual turning points and new Stokes curves to resolve the trouble, just as in the case of finite order ordinary differential operators with a large parameter. Section 3.3 is devoted to validate this proposal, or at least convince the reader of its validity, through a computer-assisted study of the configuration of the steepest descent paths in an integral that solves the local equation in question. In Section 4, by studying a non-local operator with a large parameter that bears some faint resemblance to the operator discussed in Section 3, we examine whether the locality of the operator in question plays an important role or not; our conclusion is that the locality of the operator, which is important in the "ordinary" analysis, does not play any important role here at the exact WKB analysis but that an important thing in the exact WKB analysis is the balance between the differentiation and the large parameter, thus showing the naturality of the notion of operators of WKB type introduced in [**AKKT1**]. The example discussed in Section 5 is important in that it provides us with an example that is free of the pathologies observed in Section 3.1. We also show why it is free of the pathologies and present a class of operators of WKB type which share this nice property with the particular example in this section. Section 6 is devoted to the study of a particular local operator that originates in some physical problem (∞-level crossing problem in a non-adiabatic transition). (Cf. [**BE**], [**AKT3**].) Appendix provides an analytic proof of an interesting assertion that each new Stokes curve emanating from a virtual turning point passes through the crossing points of Stokes curves in the example discussed in Section 3.

2. Preliminaries

Let us briefly recall the notions and notations which we need in this paper. They are consistent with those used in [**AKKT1**]. In what follows, U stands for an open subset of $\mathbb{C}$ and $U \times \mathbb{C}$ is denoted by X. Let (x, y) be a coordinate of X and let $(x, y; \xi, \eta)$ denote a coordinate of the cotangent bundle T^*X of X. An operator of WKB type defined on U is, by definition, a microdifferential operator of order 0 defined on

$$\Omega = \{(x, y; \xi, \eta) \in T^*X; \eta \neq 0\} \tag{2.1}$$

which commutes with the differentiation with respect to y. Thus the total symbol of a WKB type operator P has the form

$$\sum_{j \geq 0} \eta^{-j} P_j(x, \xi/\eta), \tag{2.2}$$

where $P_j(x, \zeta)$ is holomorphic on $U \times \mathbb{C}$. We note that in the examples discussed in this paper the series (2.2) consists of at most two terms; $P = P_0 + \eta^{-1}P_1$, and actually P_1 is also 0 except in Section 6. A WKB solution ψ of the equation

$$P\psi = P(x, \eta^{-1}\partial_x, \eta)\psi = 0 \tag{2.3}$$

is, by definition, a solution of the form

$$\psi = \exp(\int^x S(x, \eta)dx), \tag{2.4}$$

where

$$S(x, \eta) = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x). \tag{2.5}$$

(A formal series of the form (2.4) is called a WKB symbol in [**AKKT1**]; it reflects the microlocal flavor of [**AKKT1**], and actually a quantized contact transformation determined in terms of the WKB symbol is effectively used to give a mathematically rigorous meaning to the equation (2.3).)

If there exists a (multi-valued) analytic function $S_{-1}(x)$ that satisfies

$$P_0(x, S_{-1}(x)) \text{ vanishes identically in } U \tag{2.6}$$

and

$$\frac{\partial P_0}{\partial \zeta}(x, S_{-1}(x)) \text{ does not vanish identically,} \tag{2.7}$$

we can construct a formal series S recursively and uniquely (once the branch of S_{-1} is fixed) by the following relation:

$$(2.8)\quad S_{l-1}(x) = -\frac{1}{(\partial P_0/\partial\zeta)(x, S_{-1}(x))} \sum \frac{1}{j!} \frac{\partial^{j+k} P_n}{\partial\zeta^{j+k}}(x, S_{-1}(x)) \times \frac{1}{(k_1+1)!} \cdots \frac{1}{(k_j+1)!} \frac{d^{k_1} S_{m_1-1}}{dx^{k_1}}(x) \cdots \frac{d^{k_j} S_{m_j-1}}{dx^{k_j}}(x)$$

where the indices in the summation range over all possible $(j, k, n, k_1, \ldots, k_j, m_1, \ldots, m_j)$ that satisfy

$$(2.9a)\qquad \sum_{i=1}^{j} k_i = k,$$

$$(2.9b)\qquad k_i + m_i \geq 1, \quad i = 1, 2, \ldots, j,$$

$$(2.9c)\qquad k + n + \sum_{i=1}^{j} m_i = l$$

except for one index

$$(2.10)\qquad (j, k, n) = (1, 0, 0), \quad k_1 = 0, \quad m_1 = l.$$

(In (2.9) we regard $\sum k_i$ and $\sum m_i$ as 0 when $j = 0$.) In particular, we have

(2.11)

$$S_0(x) = -\frac{1}{(\partial P_0/\partial\zeta)(x, S_{-1}(x))} \left\{ \frac{1}{2} \frac{\partial^2 P_0}{\partial\zeta^2}(x, S_{-1}(x)) \frac{dS_{-1}}{dx}(x) + P_1(x, S_{-1}(x)) \right\}.$$

The series (2.5) is a divergent one in general, and we want to use the Borel resummation to give an analytically well-defined meaning to (2.4). Concerning the Borel transformability of ψ etc. we refer the reader to [**AKKT1**]. The resurgence of the Borel transformed WKB solutions is, however, still an open problem even for finite order differential operators. Hence we assume the resurgence as an Ansats, and consider the resulting geometric issues. For that purpose we recall the notion of turning points and Stokes curves emanating from them. If

$$(2.12)\qquad P_0(x, \zeta) = \frac{\partial P_0}{\partial\zeta}(x, \zeta) = 0$$

has a solution $(x_*, \zeta_*) \in U \times \mathbb{C}$ and if $P_0(x_*, \zeta)$ does not vanish identically as a function of ζ, then x_* is called a turning point of P with the characteristic value ζ_*.

REMARK 2.1. Except for the second or the third order differential operators, a turning point may, in general, have several (possibly infinitely many) characteristic values. (See Sections 3 and 4 for concrete examples.)

If we further assume

$$\frac{\partial^2 P_0}{\partial \zeta^2}(x_*, \zeta_*) \neq 0, \tag{2.13}$$

we find two (possibly multi-valued) analytic functions $\zeta_\pm(x)$ that satisfy $\zeta_\pm(x_*) = \zeta_*$ and $P_0(x, \zeta_\pm(x)) = 0$. We call $\zeta_\pm(x)$ characteristic roots of P passing through (x_*, ζ_*). All operators discussed in this paper satisfy the condition (2.13). A Stokes curve emanating from a turning point x_* is, by definition, the curve defined by the following:

$$\operatorname{Im} \int_{x_*}^{x} (\zeta_+(s) - \zeta_-(s)) ds = 0. \tag{2.14}$$

In our discussion we sometimes say that a turning point x_* is simple (resp., double) when

$$\frac{\partial P_0}{\partial x}(x_*, \zeta_*) \neq 0 \tag{2.15}$$

(2.16)

$$\text{(resp.,}\quad \frac{\partial P_0}{\partial x}(x_*, \zeta_*) = 0 \quad \text{and} \quad \left.\left[\left(\frac{\partial^2 P_0}{\partial x \partial \zeta}\right)^2 - \frac{\partial^2 P_0}{\partial x^2}\frac{\partial^2 P_0}{\partial \zeta^2}\right]\right|_{(x,\zeta)=(x_*,\zeta_*)} \neq 0)$$

in addition to the condition (2.13). As is well-known, (2.14) determines three (resp., four) curvilinear half-lines near a simple (resp., double) turning point.

3. Study of the Stokes geometry for some local operator with a large parameter

In this section we study the Stokes geometry of a local operator P with a large parameter η given by

$$P = \cosh\left(\sqrt{\frac{1}{i\eta}\frac{d}{dx}}\right) - x = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{1}{i\eta}\frac{d}{dx}\right)^n - x. \tag{3.1}$$

As is noted in [**AKKT1**], the operator P is a local operator, that is, an operator acting on the sheaf of holomorphic functions of x as a sheaf homomorphism, even when we fix the large parameter η at a finite value η_0 $(\neq 0)$. Hence we may expect that among differential operators of WKB type whose WKB solutions admit infinitely many phases P is one of the most natural generalizations of linear differential operators of finite order and with a large parameter.

3.1. Observation of the difficulties in constructing the Stokes geometry for the operator P. Using the terminologies given in Section 2, we can readily confirm that the point $x = 1$ is a simple turning point of P with a characteristic value $-4in^2\pi^2$ $(n = 1, 2, \ldots)$, that is,

$$P(1, \zeta) = \frac{\partial P}{\partial \zeta}(1, \zeta) = 0 \quad \text{for} \quad \zeta = -4in^2\pi^2 \ (n = 1, 2, \ldots). \tag{3.2}$$

We also find that $x = -1$ is a simple turning point of P with a characteristic value $-i(2n-1)^2\pi^2$ $(n = 1, 2, \ldots)$. To describe the Stokes curves emanating from either one of these two turning points we label the characteristic roots $\zeta = \zeta(x)$ of P as follows:

$$\zeta(x) = f_m(x) := i\left(\log(x + \sqrt{x^2 - 1}) + 2im\pi\right)^2 \quad (m \in \mathbb{Z}), \tag{3.3}$$

where we choose and fix the branch of f_m in the following way; by introducing the cuts

$$\{x \in \mathbb{C}\,;\, \arg(x-1) = -\pi/4\} \text{ and } \{x \in \mathbb{C}\,;\, \arg(x+1) = 3\pi/4\}, \tag{3.4}$$

we choose the branch of $\sqrt{x^2 - 1}$ and that of $\log(x + \sqrt{x^2 - 1})$ so that

$$\sqrt{x^2 - 1} > 0 \quad \text{for } x > 1 \tag{3.5a}$$

and

$$\log(x + \sqrt{x^2 - 1})\Big|_{x=0} = \frac{i\pi}{2}. \tag{3.5b}$$

With this choice of the branch of $f_m(x)$, we find $f_m(1) = -4im^2\pi^2$. Hence the characteristic roots of P that pass through $(x, \zeta) = (1, -4in^2\pi^2)$ are given by $f_n(x)$ and $f_{-n}(x)$. Therefore the Stokes curves that emanate from $x = 1$ are given by

$$\operatorname{Im} \int_1^x (f_n(x) - f_{-n}(x))dx = 0 \quad (n = 1, 2, \ldots). \tag{3.6}$$

Note that, for each n, (3.6) is equivalent to

$$\begin{aligned} \operatorname{Im} &\int_1^x \log(x + \sqrt{x^2 - 1})dx \\ &= \operatorname{Im}\left(x\log(x + \sqrt{x^2 - 1}) - \sqrt{x^2 - 1}\right) = 0. \end{aligned} \tag{3.7}$$

Otherwise stated, the curves defined by (3.6) are, set-theoretically speaking, identical for all n. However, as we will see later, the dominance relations among WKB solutions with the leading term

$$\exp(\eta \int^x f_m(x)dx) \quad (m \in \mathbb{Z}) \tag{3.8}$$

play important roles in our discussion. Hence we use the symbol $\sigma^+_{n,-n}$ to designate the curve defined by (3.6) described by a pair of characteristic roots $f_n(x)$ and $f_{-n}(x)$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$.

Similarly, with the choice (3.5) of the branch of $\log(x + \sqrt{x^2-1})$ we find

$$\log(x + \sqrt{x^2-1})\Big|_{x=-1} = i\pi \tag{3.9}$$

and hence $f_m(-1) = -i(2m+1)^2\pi^2$. Therefore the characteristic roots of P that pass through $(x, \zeta) = (-1, -i(2n-1)^2\pi^2)$ $(n \in \mathbb{N})$ are given by $f_{n-1}(x)$ and $f_{-n}(x)$. Thus we find that the Stokes curves emanating from $x = -1$ are given by

$$\operatorname{Im} \int_{-1}^x (f_{n-1}(x) - f_{-n}(x))dx = 0 \quad (n \in \mathbb{N}). \tag{3.10}$$

The curves defined by (3.10) are, again, all identical, that is, they are given by the following equation which is independent of n:

$$\begin{aligned} \operatorname{Im} &\int_{-1}^x (\log(x + \sqrt{x^2-1}) - i\pi)dx \\ = \ \operatorname{Im} &\left(x \log(x + \sqrt{x^2-1}) - \sqrt{x^2-1} - i\pi x\right) = 0. \end{aligned} \tag{3.11}$$

In parallel with the case of Stokes curves emanating from $x = 1$, we use the symbol $\sigma^-_{n-1,-n}$ to designate the curve with the defining equation (3.10).

Figure 3.1 describes the Stokes curves thus obtained. (The wiggly lines designate the cuts given by (3.4).) Checking the dominance relations among the associated WKB solutions, we can readily confirm that the crossing points x_0 and x_1 of the Stokes curves in Figure 3.1 are both ordered ones in the sense of [**BNR**]. (See also [**AKT1**] and [**AKT3**].) To be more precise we find the following:

(i) The crossing points of $\sigma^-_{n-1,-n}$ and $\sigma^+_{n,-n}$ are ordered ones for each $n \in \mathbb{N}$.

(ii) The crossing points of $\sigma^-_{n,-n-1}$ and $\sigma^+_{n,-n}$ are ordered ones for each $n \in \mathbb{N}$.

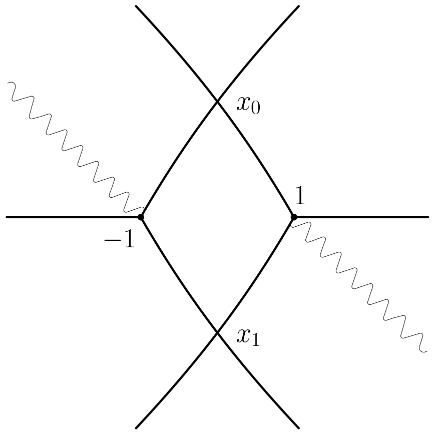


FIGURE 3.1

Note that in the case (i), on a neighborhood of x_0, the type of $\sigma^-_{n-1,-n}$ is "$n-1 < -n$", i.e.,

$$
\begin{aligned}
(3.12)\quad & \operatorname{Re}\int_{-1}^{x}(f_{n-1}(x) - f_{-n}(x))dx \\
&= 2(1-4n)\pi \operatorname{Re}\int_{-1}^{x}(\log(x+\sqrt{x^2-1}) - i\pi)dx \\
&= 2(1-4n)\pi \\
&\qquad \times \operatorname{Re}\,(x\log(x+\sqrt{x^2-1}) - \sqrt{x^2-1} - i\pi x) < 0
\end{aligned}
$$

and the type of $\sigma^+_{n,-n}$ is "$-n < n$", i.e.,

$$
\begin{aligned}
(3.13)\quad & \operatorname{Re}\int_{1}^{x}(f_{n}(x) - f_{-n}(x))dx \\
&= -8n\pi \operatorname{Re}\int_{1}^{x}\log(x+\sqrt{x^2-1})dx \\
&= -8n\pi \operatorname{Re}\,(x\log(x+\sqrt{x^2-1}) - \sqrt{x^2-1}) > 0,
\end{aligned}
$$

and that, on a neighborhood of x_1, the type of $\sigma^-_{n-1,-n}$ is "$-n < n-1$" and the type of $\sigma^+_{n,-n}$ is "$n < -n$". Similar observations can be made in the case (ii). Figure 3.2 (a) and Figure 3.2 (b) respectively summarize these observations. As Berk, Nevins and Roberts ([**BNR**]) first discovered, we need to include a new Stokes curve emanating from an ordered crossing point, i.e., a crossing point of two Stokes curves whose types are "coherent and hinged by a common label" like "$n-1 < -n$" and "$-n < n$", if we want to find a correct connection formula for WKB solutions.

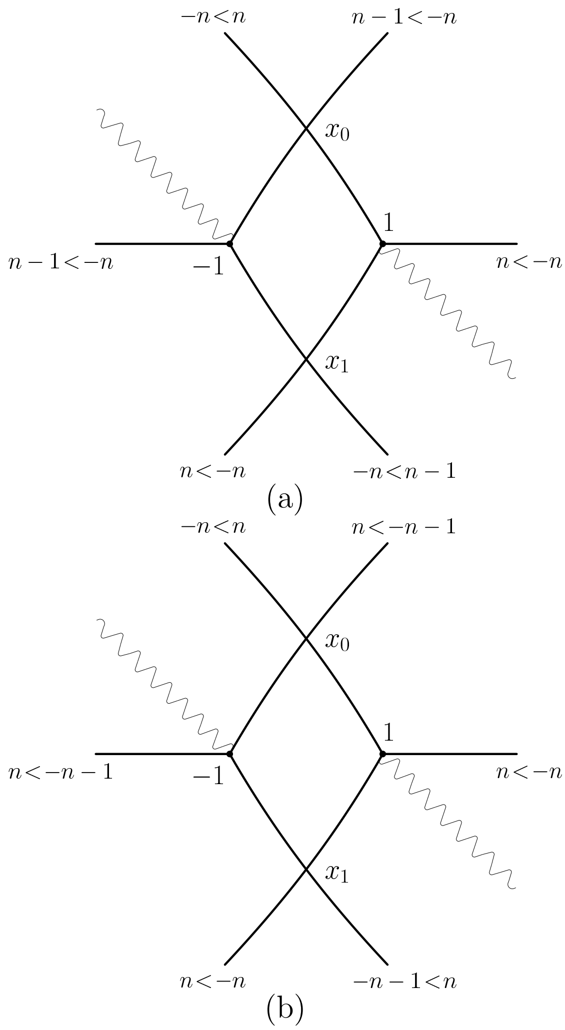


FIGURE 3.2

To find the required new Stokes curves in a systematic manner we introduce, in the next subsection, the notion of virtual turning points by using the bicharacteristic curves of the Borel transform P_B of the operator P.

3.2. Introduction of virtual turning points. Having in mind the general result that singularities of solutions of linear (micro)differential equations propagate along bicharacteristic strips ([**SKK**, Chap. II]), [**AKT1**] proposed to re-interpret a new Stokes curve as a Stokes curve emanating from a new turning point, which is defined as the x-component of a self-intersection point of a bicharacteristic curve of the Borel transform of the operator in question. The arguments of [**AKT1**] are applicable to our current problem without any modifications. In this subsection we first briefly recall the definition of a virtual turning point (= "a new turning point" in the terminology of [**AKT1**]) of the operator P and then concretely discuss how the new Stokes curves emanating from virtual turning points resolve the troubles encountered in Section 3.1. The validity of the resulting Stokes geometry will be endorsed in Section 3.3 by examining the topological changes of the configuration of the steepest descent paths of an integral that solves the equation $P\psi = 0$.

Let P_B denote the Borel transform of $P = \cosh(\sqrt{(i\eta)^{-1}(d/dx)}) - x$;

$$P_B = \cosh(\sqrt{(i\partial_y)^{-1}\partial_x}) - x. \tag{3.14}$$

In this case we may identify P_B with its principal symbol, by replacing ∂_x and ∂_y respectively with ξ and η. Then a bicharacteristic strip is, by definition, a curve $\{(x(t), y(t); \xi(t), \eta(t))\}_{t\in\mathbb{C}}$ in the cotangent bundle $T^*\mathbb{C}^2_{(x,y)}$ that satisfies the following Hamilton-Jacobi equation

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial P_B}{\partial \xi}, & \dfrac{dy}{dt} = \dfrac{\partial P_B}{\partial \eta}, \\[2mm] \dfrac{d\xi}{dt} = -\dfrac{\partial P_B}{\partial x}, & \dfrac{d\eta}{dt} = -\dfrac{\partial P_B}{\partial y} \end{cases} \tag{3.15}$$

with the initial data $(x(0), y(0); \xi(0), \eta(0))$ satisfying

$$P_B(x(0), y(0); \xi(0), \eta(0)) = 0. \tag{3.16}$$

As the initial condition we consider

$$(x(0), y(0); \xi(0), \eta(0)) = (1, 0; 0, 1). \tag{3.17}$$

As we will show later (Remark 3.1), the projection of the bicharacteristic strip to (x,ξ)-space passes through all points $(1, f_n(1))$ and $(-1, f_n(-1))$ $(n \in \mathbb{Z})$. Thus the preferred choice of the initial condition as above does not effect the generality of our discussion.

Now, by definition, a point x_* is called a **virtual turning point** if there exist complex numbers y_*, t and t' $(t \neq t')$ for which

$$(x_*, y_*) = (x(t), y(t)) = (x(t'), y(t')), \tag{3.18}$$

where $\{(x(t), y(t))\}_{t\in\mathbb{C}}$ denotes the bicharacteristic curve, i.e., the projection of the bicharacteristic strip to the base manifold $\mathbb{C}^2_{(x,y)}$. Furthermore, as P_B vanishes identically on the bicharacteristic strip and η remains to be 1 on it, there exists a pair $(f_n(x), f_m(x))$ for which

$$f_n(x_*) = \xi(t) \ \text{ and } \ f_m(x_*) = \xi(t') \tag{3.19}$$

hold. As we will see below (cf. (3.29)), virtual turning points are not turning points $x = \pm 1$ in our current problem. Hence the pair $(f_n(x), f_m(x))$ is uniquely determined. Using the pair $(f_n(x), f_m(x))$, we define a **(new) Stokes curve** $\sigma^0_{n,m}$ by

$$\mathrm{Im} \int_{x_*}^{x} (f_n(x) - f_m(x))dx = 0. \tag{3.20}$$

In the description of a new Stokes curve, we use, as usual (cf. [**AKT1**], [**AKT3**]), a dotted line to denote its portion containing the virtual turning point and no crossing points of Stokes curves; a dotted line segment indicates that no Stokes phenomena occur among WKB solutions when they cross the portion. For the curve $\sigma^0_{n,m}$, we say that it has the type "$n > m$" (resp., "$m > n$") near x if

$$\mathrm{Re} \int_{x_*}^{x} (f_n(x) - f_m(x))dx > 0 \tag{3.21}$$

(resp., if $\mathrm{Re} \int_{x_0}^{x} (f_n(x) - f_m(x))dx < 0$).

Let us now implement the above procedure to find the required Stokes geometry. Substituting (3.14) into (3.15) we obtain

$$\begin{cases} \dfrac{dx}{dt} &= \dfrac{(i\eta)^{-1}}{2\sqrt{(i\eta)^{-1}\xi}} \sinh(\sqrt{(i\eta)^{-1}\xi}) \\ \dfrac{dy}{dt} &= \dfrac{1}{2\eta}\sqrt{(i\eta)^{-1}\xi} \sinh(\sqrt{(i\eta)^{-1}\xi}) \\ \dfrac{d\xi}{dt} &= 1 \\ \dfrac{d\eta}{dt} &= 0. \end{cases} \tag{3.22}$$

With the initial condition (3.17) the solution is given as

$$\begin{cases} x(t) &= \cosh\sqrt{t/i} \\ y(t) &= (t+2i)\cosh\sqrt{t/i} - 2i\sqrt{t/i}\sinh\sqrt{t/i} - 2i \\ \xi(t) &= t \\ \eta(t) &= 1. \end{cases} \tag{3.23}$$

By a straightforward computation we find the pairs (t, t') $(t \neq t')$ for which $x(t) = x(t')$ and $y(t) = y(t')$ hold are given by $(t_{k,l}, t_{k,-l})$ $(k, l \in \mathbb{N})$, where

$$t_{k,l} = -i(w_k + l\pi)^2 \quad (k \in \mathbb{N}, l \in \mathbb{Z}) \tag{3.24}$$

with w_k ($k \in \mathbb{N}$) being positive solutions of the equation

$$w = \tan w \tag{3.25}$$

located so that

$$0 < w_1 < w_2 < w_3 < \cdots . \tag{3.26}$$

Note that all solutions of (3.25) are real and that negative solutions are given by $\{-w_k\}_{k\in\mathbb{N}}$. We also note

$$k\pi < w_k < (k+1/2)\pi. \tag{3.27}$$

It follows from (3.23) and (3.24) that the virtual turning points are

$$x(t_{k,l}) = x(t_{k,-l}) = (-1)^l \cos w_k. \tag{3.28}$$

It also follows from (3.27) that these virtual turning points are located on the real axis and that they are ordered as follows:

$$\begin{aligned} -1 < \cos w_1 < -\cos w_2 < \cos w_3 < \cdots \\ \cdots < -\cos w_3 < \cos w_2 < -\cos w_1 < 1. \end{aligned} \tag{3.29}$$

Remark 3.1. It follows from (3.23) that

$$(x(-4im^2\pi^2), \xi(-4im^2\pi^2)) = (1, f_m(1)) \tag{3.30}$$

and

$$(x(-i(2m+1)^2\pi^2), \xi(-i(2m+1)^2\pi^2)) = (-1, f_m(-1)). \tag{3.31}$$

Furthermore we can confirm that

$$\xi(t_{k,l}) = -i(w_k + l\pi)^2, \tag{3.32}$$

$$\xi(t_{k,-l}) = -i(w_k - l\pi)^2 \tag{3.33}$$

for $k, l \in \mathbb{N}$ and that

$$f_m(|\cos w_k|) = -i(w_k + (2m-k)\pi)^2, \tag{3.34}$$

$$f_m(-|\cos w_k|) = -i(w_k + (2m+k+1)\pi)^2. \tag{3.35}$$

Then, for a positive virtual turning point $x(t_{k,l}) = x(t_{k,-l})$ the pair (f_n, f_m) in (3.19) should satisfy

$$2n - k = l, \quad 2m - k = -l \tag{3.36}$$

because of their uniqueness. Hence we find

$$n + m = k \in \mathbb{N}, \quad n - m = l \in \mathbb{N}. \tag{3.37}$$

Thus the (new) Stokes curve $\sigma^{0+}_{n,m}$ emanating from a positive virtual turning point $|\cos w_k|$ is given by

$$\operatorname{Im} \int_{|\cos w_k|}^{x} (f_n(x) - f_m(x))dx = 0, \tag{3.38}$$

where $n, m \in \mathbb{Z}$, $n+m = k \in \mathbb{N}$ and $n-m \geq 1$. Note that the equation (3.38) is equivalent to the following:

$$\mathrm{Im} \int_{|\cos w_k|}^{x} (\log(x + \sqrt{x^2-1}) + ik\pi)dx = 0 \tag{3.39}$$

(on the condition that we ignore the additional assumption $n - m \geq 1$). Similarly the (new) Stokes curve $\sigma^{0-}_{n,m}$ emanating from a negative virtual turning point $-|\cos w_k|$ is given by

$$\mathrm{Im} \int_{-|\cos w_k|}^{x} (f_n(x) - f_m(x))dx = 0, \tag{3.40}$$

where $n, m \in \mathbb{Z}$, $n + m = -k - 1$ with $k \in \mathbb{N}$ and $n - m \geq 1$. Again this curve is described by one and the same equation

$$\mathrm{Im} \int_{-|\cos w_k|}^{x} (\log(x + \sqrt{x^2-1}) - i(k+1)\pi)dx = 0 \tag{3.41}$$

for each k in $\mathbb{N}$. An important and interesting feature of these curves is that both the curve defined by (3.39) and that defined by (3.41) pass through the points x_0 and x_1 in Figure 3.2. The analytic proof of this fact is given in Appendix. Admitting this fact for the moment, we assert that, in a neighborhood of x_0, the type of the new Stokes curve $\sigma^{0+}_{n,m}$ emanating from a positive virtual turning point $|\cos w_k|$ ($k = m + n$) is "$n > m$" and the new Stokes curve $\sigma^{0-}_{n,m}$ emanating from a negative virtual turning point $-|\cos w_k|$ is "$n < m$"; all these assertions can be readily confirmed analytically using (3.5), but the easiest way to find these types is to note

$$\begin{aligned}
&\int^{x} (f_n(x) - f_m(x))dx \\
&= -4(n-m)\pi \int^{x} (\log(x + \sqrt{x^2-1}) + i(n+m)\pi)dx \\
&= -4(n-m)\pi(x \log(x + \sqrt{x^2-1}) - \sqrt{x^2-1} \\
&\qquad + i(n+m)\pi x + \text{const.})
\end{aligned} \tag{3.42}$$

and the concrete shape of $\sigma^{0\pm}_{n,m}$, which may be readily found with the help of a computer. Note that we made the additional assumption $n - m \geq 1$ in defining $\sigma^{0\pm}_{n,m}$.

Summing up, we present the dominance relations of (ordinary or new) Stokes curves near x_0 (resp., near x_1) in Figure 3.3 (resp., Figure 3.4). Note that we use the equation

$$\mathrm{Im} \int_{x_*}^{x} (f_a(x) - f_b(x))dx = 0 \tag{3.43}$$

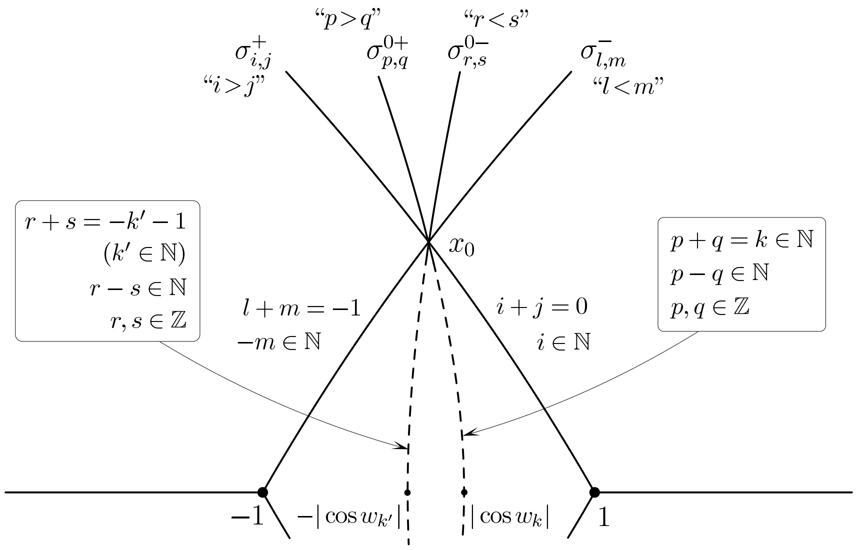


FIGURE 3.3

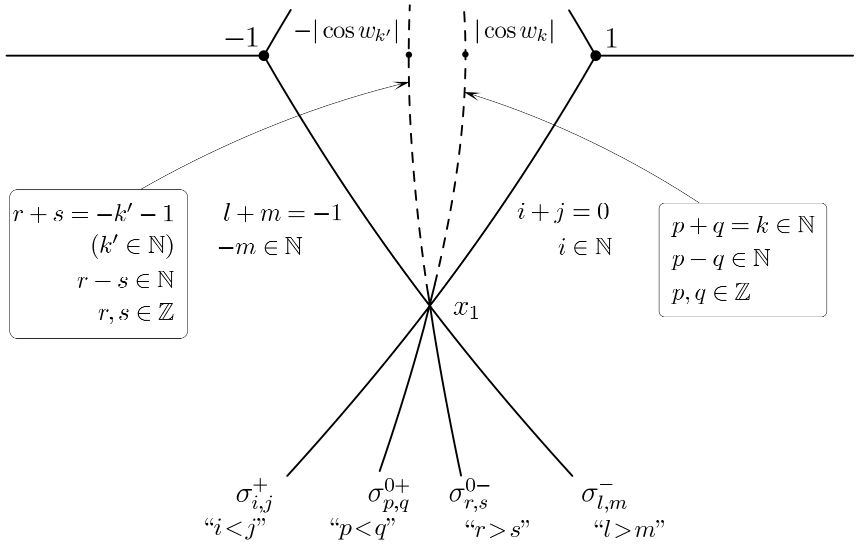


FIGURE 3.4

(x_*: a (virtual or ordinary) turning point) to define the curve $\sigma^{\sharp}_{a,b}$ ($\sharp$: $0+$, $0-$, $+$ or $-$), that is, the order of suffix (a, b) is conventionally determined by fixing the way of writing down the defining equation as above (although

$$\operatorname{Im} \int_{x_*}^{x} (f_b(x) - f_a(x))dx = 0 \tag{3.44}$$

determines the same curve).

It is interesting and instructive to see that any ordered crossing formed by a pair of Stokes curves is resolved by some (ordinary or new) Stokes curves, that is, if we consider all Stokes curves at once, then every ordered crossing is resolved. However, such a clean situation would not be attained if we would consider a restricted family of Stokes curves. Here, by saying "an ordered crossing is resolved", we mean the following:

If Stokes curves with respective types "$j > k$" and "$k > l$" (resp., "$j < k$" and "$k < l$") cross at a point x, then another Stokes curve with its type "$j > l$" (resp., "$j < l$") passes through the point x.

Before discussing the general case, let us see how we are forced to add new Stokes curves one after another if we begin our discussion with resolving the seemingly most basic ordered crossing formed by the pair of ordinary Stokes curves $\sigma^-_{0,-1}$ and $\sigma^+_{1,-1}$. On a neighborhood of x_0 we can read off dominance relations "$-1 > 0$" and "$1 > -1$" from Figure 3.3. Thus they form an ordered crossing, and, to resolve it, we have to seek for a (new) Stokes curve whose type is "$1 > 0$" near x_0. The required one can be readily found; $\sigma^{0+}_{1,0}$ is it. However, the story is not ended. We have to further consider the intersection of $\sigma^-_{1,-2}$ with $\sigma^+_{1,-1}$ and the added new Stokes curve $\sigma^{0+}_{1,0}$. The type of $\sigma^-_{1,-2}$ near x_0 is "$-2 > 1$". Hence $\sigma^-_{1,-2}$ and $\sigma^+_{1,-1}$ form an ordered crossing, which is to be resolved by another new Stokes curve $\sigma^{0-}_{-1,-2}$. But, the added new Stokes curve $\sigma^{0-}_{-1,-2}$ again forms an ordered crossing with $\sigma^-_{0,-1}$, and to resolve it we further need $\sigma^{0-}_{0,-2}$. A slightly comfortable feature of the current situation is that $\sigma^{0-}_{0,-2}$ at once resolves the ordered crossing of $\sigma^-_{1,-2}$ and $\sigma^{0+}_{1,0}$. The Stokes curves so far discussed are shown in Figure 3.5. As the reader easily imagines, the reasoning does not stop here; for example, we have to discuss the ordered crossing formed by $\sigma^+_{2,-2}$ and $\sigma^{0-}_{0,-2}$, and so on. Thus we would eventually need all the Stokes curves to resolve all ordered crossings that sit at x_0, and if we would consider all possible (ordinary or new) Stokes curves at once, the resulting Stokes geometry should be, hopefully, complete in the sense that no ordered crossings remain unresolved. As a matter of fact, we can readily confirm that the Stokes geometry containing all Stokes curves is complete near x_0 as follows; let us consider a pair of a Stokes curve with type "$j > k$" and that with "$k > l$" which cross at x_0. Then, as is clear from Figure 3.3, we can find another Stokes curve with type "$j > l$", which resolves the above ordered crossing. In fact, if $j + l \in \mathbb{N}$ (resp., $j + l = -1 - n$ $(n \in \mathbb{N})$, $j + l = 0$ or $j + l = -1$) it suffices to choose $\sigma^{0+}_{j,l}$ (resp., $\sigma^{0-}_{l,j}$, $\sigma^+_{j,l}$, $\sigma^-_{l,j}$) as the required Stokes

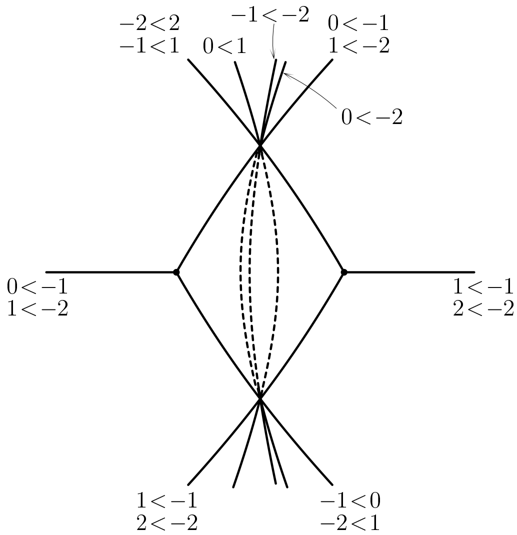


FIGURE 3.5

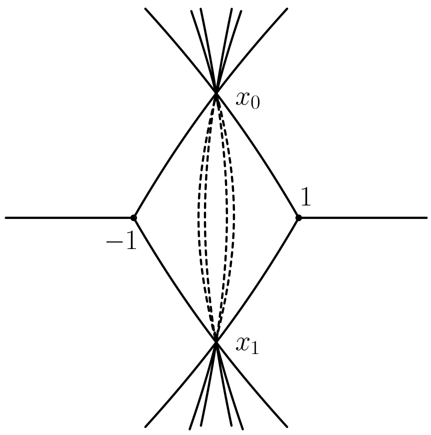


FIGURE 3.6

curve. Thus the Stokes geometry containing all possible Stokes curves is complete; Figure 3.6 shows it, although only finitely many Stokes curves are printed there explicitly for the sake of the clarity of the figure. Furthermore we can show another somewhat more delicate issue: we really need all Stokes curves to obtain a complete Stokes geometry. To confirm this fact in a neat manner let us introduce the following graph-theoretic wording: let us consider the totality of "vertices" v_m which symbolically represent the characteristic roots f_m ($m \in \mathbb{Z}$) considered near the crossing point x_0 in Figure 3.2. When there exists a Stokes curve of type "$j > k$" passing through x_0 we connect two vertices v_j

and v_k by an oriented segment (i.e., an arrow) like "$v_j \to v_k$". Then a graph G is, by definition, a collection of (possibly infinitely many) vertices and arrows in which each arrow connects two different vertices. Note that, in this terminology, a chain $v_j \to v_k \to v_l$ formed by a triplet (v_j, v_k, v_l) of vertices means that two Stokes curves form an ordered crossing at x_0. Then a "reduced" graph $v_j \to v_l$ corresponds to a new Stokes curve resolving this ordered crossing. More generally, if a graph G contains a chain $v_j \to v_k \to v_l$ as a part of it (on the condition that v_k is not connected to any vertices in G other than v_j or v_l) and if we replace $v_j \to v_k \to v_l$ by $v_j \to v_l$, then the resulting graph G' is called a reduction of G at v_k.

For the sake of convenience of the reader, let us summarize the discussion that led us to Figure 3.5. Let us consider the following graph (actually a chain) G_0:

$$G_0 : v_{-2} \to v_1 \to v_{-1} \to v_0. \tag{3.45}$$

Then the reduction of G_0 at v_{-1} reads as

$$G_1 : v_{-2} \to v_1 \to v_0, \tag{3.46}$$

and the reduction at v_1 reads as

$$G_2 : v_{-2} \to v_{-1} \to v_0. \tag{3.47}$$

The introduction of a new Stokes curve $\sigma^{0-}_{0,2}$ makes G_1 reduced at v_1 and, at the same time, G_2 reduced at v_{-1}.

As is easily surmised, the above argument continues indefinitely. To see it clearly let us consider all ordinary Stokes curves and express their dominance relations at x_0 by using the above wording. We then obtain the following chain $G(x_0)$ of infinite length:

$$G(x_0) : \cdots \to v_{-3} \to v_2 \to v_{-2} \to v_1 \to v_{-1} \to v_0. \tag{3.48}$$

Now, for each pair (v_j, v_k) of vertices for which the relation

$$v_j \to \cdots \to v_k \tag{3.49}$$

is observed in $G(x_0)$, we can readily verify that $v_j \to v_k$ should be found in some reduction of $G(x_0)$ through an induction on the length of (3.49). This means that a Stokes curve of type "$j > k$", which corresponds to the relation $v_j \to v_k$, is needed to obtain a complete Stokes geometry. To be more specific we need each Stokes curve corresponding to some graph given below; here $j \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$.

$$\begin{array}{llll} \text{(i)} & v_j \to v_k & \text{with} & j \geq k+1, \\ \text{(ii)} & v_j \to v_{-k} & \text{with} & j \geq k, \\ \text{(iii)} & v_{-j} \to v_{-k} & \text{with} & j > k \geq 1, \\ \text{(iv)} & v_{-j} \to v_k & \text{with} & j \geq k+1. \end{array}$$

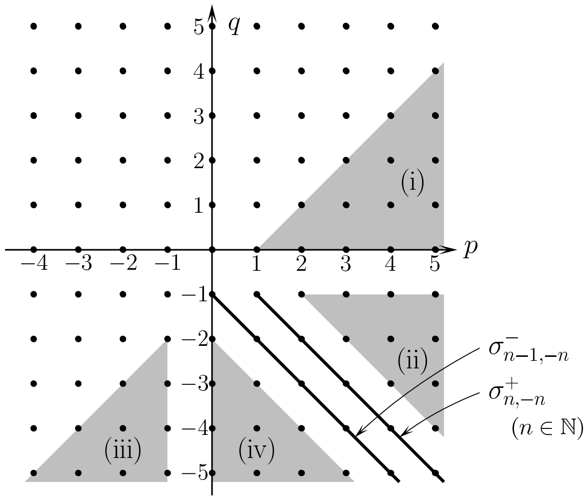


FIGURE 3.7

In order to find an appropriate Stokes curve we thus need all the following new Stokes curves; in case (i) $\sigma^{0+}_{j,k}$ is needed, in case (ii) $\sigma^{0+}_{j,-k}$, in case (iii) $\sigma^{0-}_{-k,-j}$ and in case (iv) $\sigma^{0-}_{k,-j}$. If we plot the indices (p, q) of the required new Stokes curve $\sigma^{0\pm}_{p,q}$ we obtain Figure 3.7. (In Figure 3.7 each lattice point in the region (j) $(j = \mathrm{i}, \mathrm{ii}, \mathrm{iii}, \mathrm{iv})$ corresponds to some new Stokes curve required in case (j).) Each integral lattice point in the shaded regions of Figure 3.7 is in a one-to-one correspondence to a pair of indices of a new Stokes curve $\sigma^{0\pm}_{p,q}$ in Figure 3.3. In other words all new Stokes curves are needed to find a complete Stokes geometry. A similar reasoning applies to the geometry near x_1; in that case we have to consider the following chain $G(x_1)$ instead of $G(x_0)$:

$$G(x_1) : v_0 \to v_{-1} \to v_1 \to v_{-2} \to v_2 \to v_{-3} \to \cdots . \tag{3.50}$$

Very rigorously speaking, we have not yet been able to prove that the resulting Stokes geometry legitimately describes the Stokes phenomena for WKB solutions for the operator P. Instead of trying to prove it, we present in the next subsection a strong supporting evidence for the validity of the Stokes geometry by examining the topological changes of steepest descent paths in the integral representation of a solution of the equation $P\psi = 0$.

3.3. Confirmation of the Stokes geometry through the study of steepest descent paths. So far we have been unable to prove the validity of the Stokes geometry, which probably requires much deeper understanding of the sheet structure of the Borel-transformed WKB solutions. At the same time we are completely confident about the validity of the Stokes geometry given by Figure 3.6. Our belief is endorsed by the incredibly accurate correspondence between the Stokes geometry and the change of topological patterns of the steepest descent paths in the integral

$$\psi(x) = \int \exp(\eta h(x,\zeta))d\zeta, \tag{3.51}$$

where

$$h(x,\zeta) = x\zeta - 2i\sqrt{\zeta/i}\,\sinh(\sqrt{\zeta/i}) + 2i\cosh(\sqrt{\zeta/i}). \tag{3.52}$$

Note that ψ solves the equation $P\psi = 0$ and that h is an entire function of (x,ζ). Since

$$\frac{\partial h}{\partial \zeta} = x - \cosh\sqrt{\zeta/i}, \tag{3.53}$$

we find that saddle points of $h(x,\zeta)$ (or of the integral (3.51)) are given by $\{f_n(x)\}_{n\in\mathbb{Z}}$, the functions defined by (3.3). Let s_n denote the steepest descent path of $\operatorname{Re} h(x,\zeta)$ that passes through the saddle point $\zeta = f_n(x)$. If the Stokes geometry given by Figure 3.6 is a correct one, we will observe topological changes in the configuration of the steepest descent paths $\{s_n\}$ when and only when x crosses the solid lines in Figure 3.6 (cf. e.g. [**T**], [**AKT2**]); this is what we shall confirm numerically (i.e., with the help of a computer) in what follows.

3.3.1. *Effects of new Stokes curves designated by solid lines.* Let us first confirm the effectiveness of ordinary Stokes curves $\sigma^-_{n-1,-n}$ and the portion of new Stokes curves which are designated by solid lines. For this purpose we choose points P_j $(j = 1,2,3,4)$ as in Figure 3.8 and draw the steepest descent paths at $x = P_j$ in Figure 3.9.j $(j = 1,2,3,4)$. (The configuration of the steepest descent paths at $x = P_5$ does not give us any peculiar lessons in this subsection, but it will be compared later (Figure 3.14 in §3.3.3) with the configuration of the steepest descent paths when x is closed to x_1.) In these figures a tiny dot stands for a saddle point, and the number near the dot designates the suffix of the function f_n attached to the saddle point. The right-hand figure in each of Figure 3.9.j is an enlarged figure that focuses on the configuration near the saddle point $f_0(x)$.

(i) When x moves from P_1 to P_2, it crosses ordinary Stokes curves $\sigma^-_{n-1,-n}$ $(n \in \mathbb{N})$, all of which lie on the same curve, and correspondingly we observe in Figure 3.9.1 and Figure 3.9.2 a simultaneous abrupt change of the shape of each s_{-n} $(n \in \mathbb{N})$; the change indicates that, for every $n \in \mathbb{N}$, $s_{-n}(x)$ connects $f_{-n}(x)$ and $f_{n-1}(x)$ when x is on $\sigma^-_{n-1,-n}$.

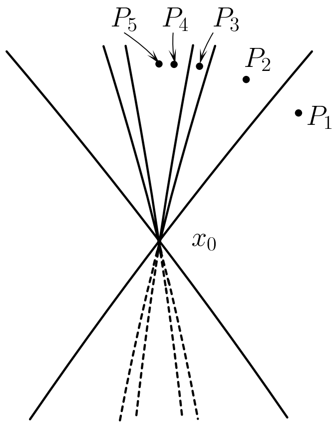


FIGURE 3.8

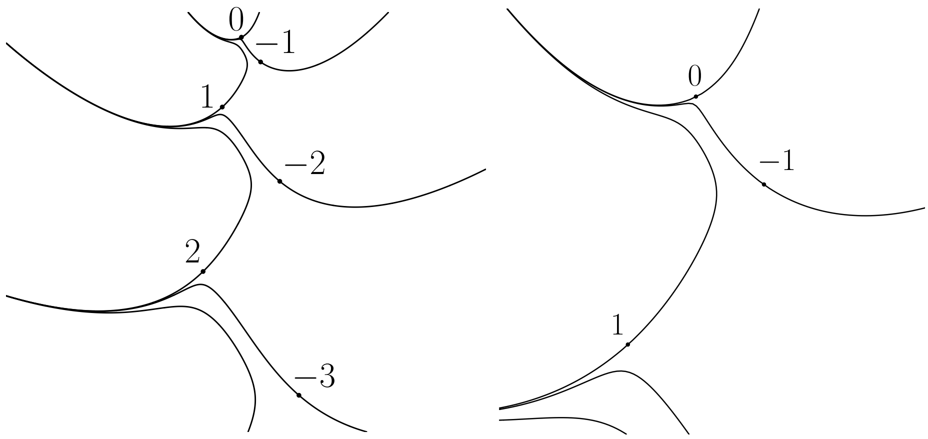


FIGURE 3.9.1

(ii) When x moves from P_2 to P_3, it crosses a solid-line part of a new Stokes curve $\sigma^{0-}_{n-2,-n}$ ($n \geq 2$). Again we observe in Figure 3.9.2 and Figure 3.9.3 a simultaneous abrupt change of the shape of every s_{-n} ($n = 2, 3, \ldots$). The figures indicate that $s_{-n}(x)$ connects $f_{-n}(x)$ and $f_{n-2}(x)$ all at once when x is on $\sigma^{0-}_{n-2,-n}$, in exactly the same manner as in the case of the crossing of an ordinary Stokes curve $\sigma^{-}_{n-1,-n}$. Note that $s_{-1}(x)$ is kept intact topologically when x moves from P_2 to P_3. The same observation can be made when x moves from P_3 to P_4 with the replacement of $f_{n-2}(x)$ by $f_{n-3}(x)$.

It is now convincing to imagine (although the exact visualization becomes technically too difficult to reproduce here) that $s_{-k}(x)$ hits $f_0(x)$ when x lies on $\sigma^{0-}_{n-k,-n}$.

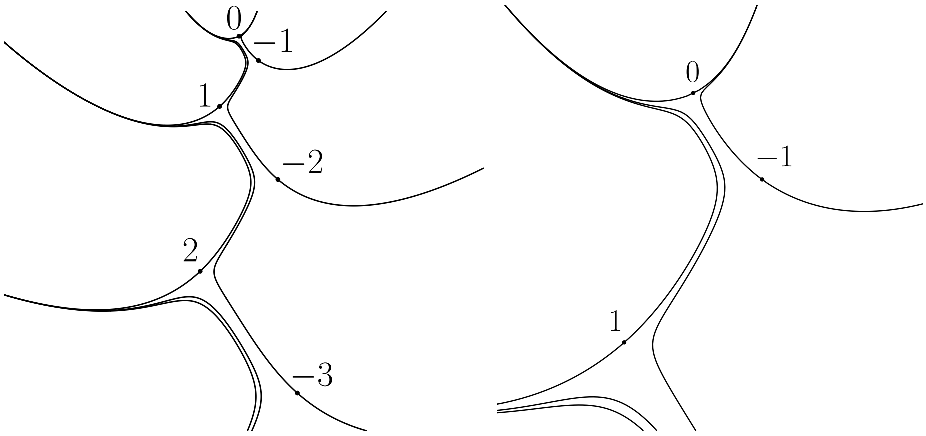


FIGURE 3.9.2

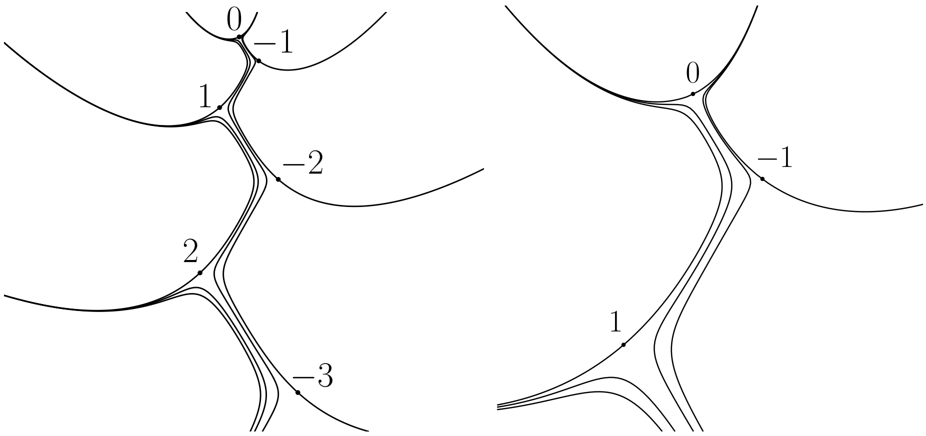


FIGURE 3.9.3

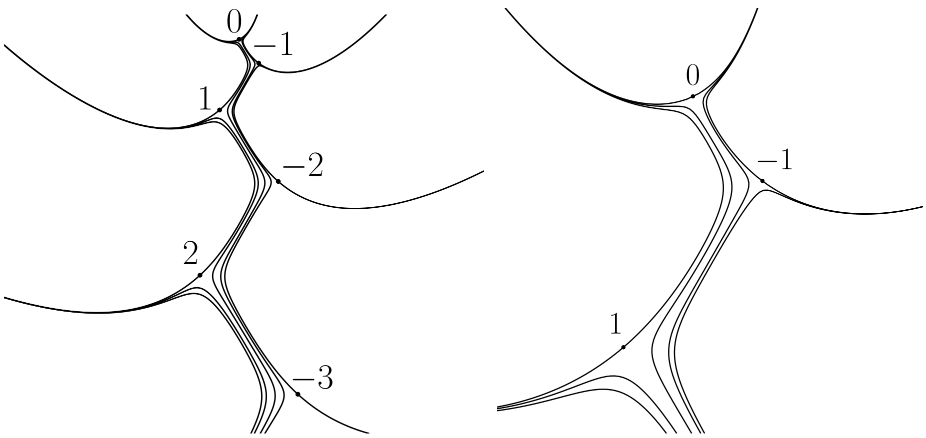


FIGURE 3.9.4

3.3.2. *Differences of the effects of the dotted-line part and the solid-line part of a new Stokes curve.* Next let us study the most interesting part of our discussion: how do the effects of the dotted-line part and the solid-line part differ? As we have seen in §3.3.1, the effect of the solid-line part of a new Stokes curve on the configuration of s_n's was essentially the same as that of an ordinary Stokes curve. Now, what

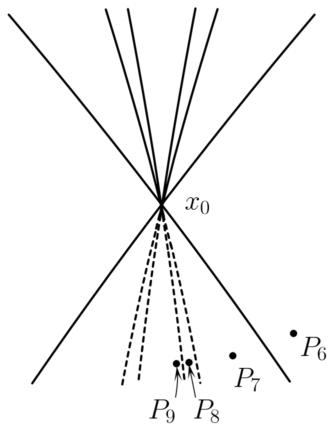


FIGURE 3.10

about the effect of the dotted-line part of a new Stokes curve on the configuration of s_n's? To understand this problem let us consider points P_j ($j = 6, 7, 8, 9$) as shown in Figure 3.10 and visualize the configuration of $s_n(x)$'s when $x = P_j$ in Figure 3.11.j. When x moves from P_6 to P_7 it crosses an ordinary Stokes curve $\sigma^+_{n,-n}$, and correspondingly we observe an abrupt change of the shape of $s_n(x)$ that indicates that $s_n(x)$ hits $f_{-n}(x)$ when x is on $\sigma^+_{n,-n}$. But, surprisingly enough, we do not observe

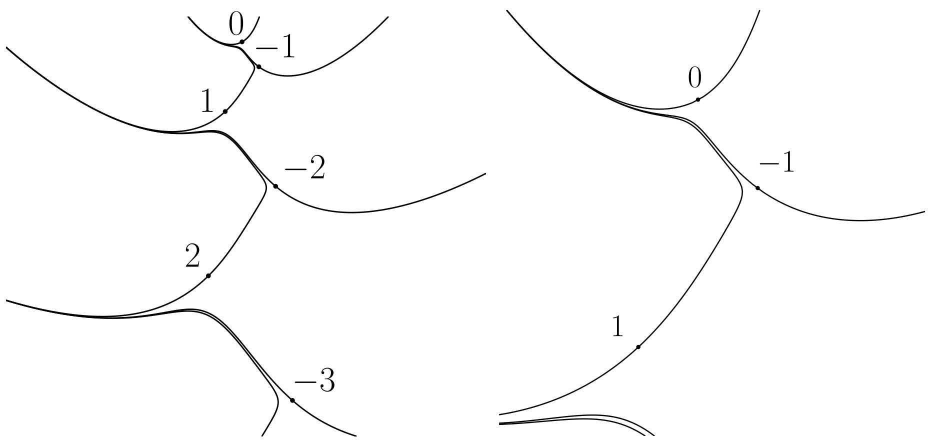


FIGURE 3.11.6

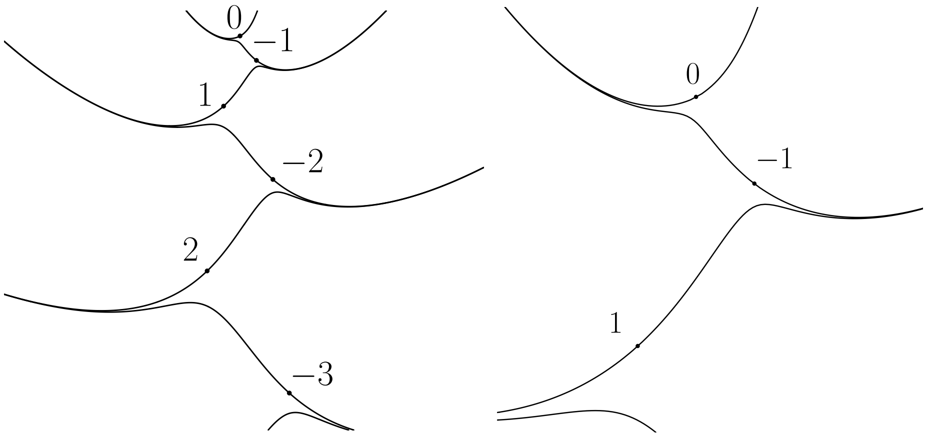


FIGURE 3.11.7

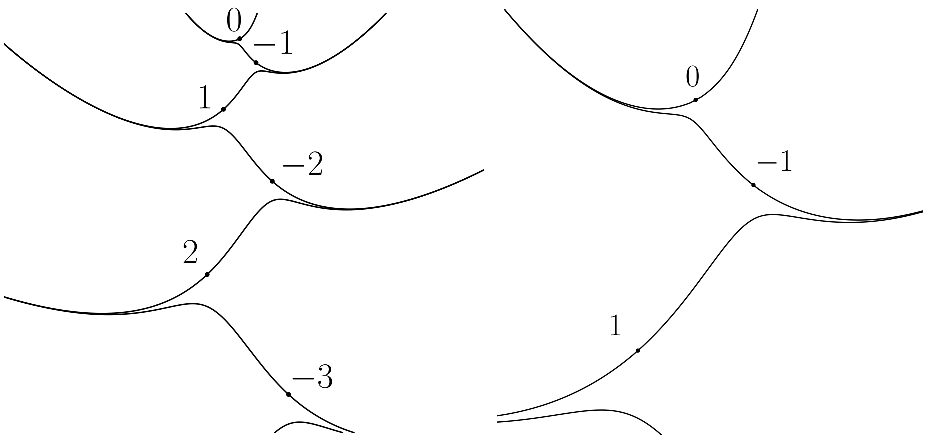


FIGURE 3.11.8

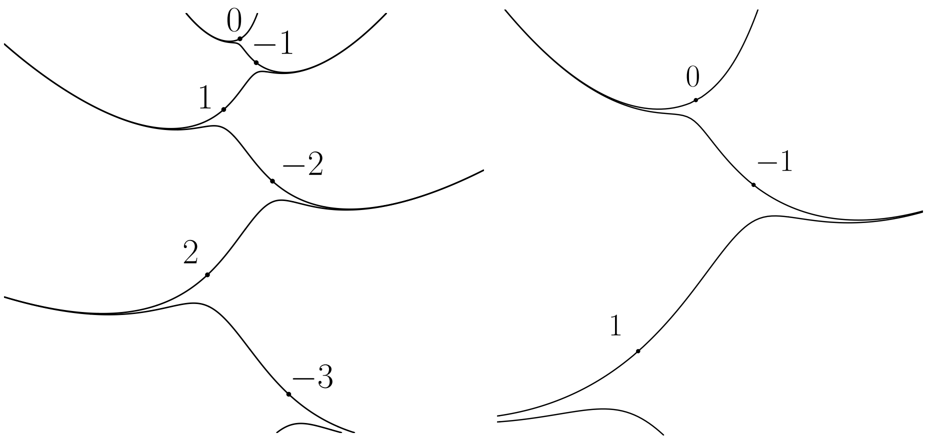


FIGURE 3.11.9

any topological changes of $s_n(x)$'s even when x crosses the dotted-line part of new Stokes curves. This fact is in complete agreement with our subtle but important assertion that no Stokes phenomena should be observed across the dotted-line part of new Stokes curves, and thus it substantially enhances our belief in the validity of our Stokes geometry given in Figure 3.6.

3.3.3. *Effects of new Stokes curves near x_1.* A similar observation can be made if x moves from Q_1 to Q_4 in Figure 3.12. The configuration of $s_n(x)$'s for x at Q_j is shown in Figure 3.13.j. A difference from the case discussed in §3.3.1 is that $s_n(x)$ hits $f_{-n-k}(x)$ when x is on $\sigma^{0-}_{n,-n-k}$, reflecting the dominance relation "$n > -n-k$" at the point in question. To understand the differences between the situation discussed in §3.3.1 and that discussed here, the best way is to concentrate our attention to the role played by $f_0(x)$; the role of $f_0(x)$ in §3.3.1 was rather "passive" in that $s_{-k}(x)$ hits it when x is on an appropriate new Stokes curve, while the role of $f_0(x)$ here is rather "active" in that $s_0(x)$ hits $f_{-k}(x)$ when x is on $\sigma^{0-}_{n,-n-k}$. This observation is consistent with the difference of the structure of chains $G(x_0)$ and $G(x_1)$ used in §3.2. This indicates that the topological patterns of $s_0(x)$ change infinitely many times as x approaches to the imaginary axis along, say, a circle of radius which is larger than $|x_1|$. To demonstrate this fact we show two figures, whose differences are quite impressive; Figure 3.14 shows the configuration of $s_n(x)$'s when x is P_5 in Figure 3.8, and Figure 3.15 shows the configuration of $s_n(x)$'s when x is Q_5 in Figure 3.12. (The points P_5 and Q_5 are chosen to lie in the imaginary axis.)

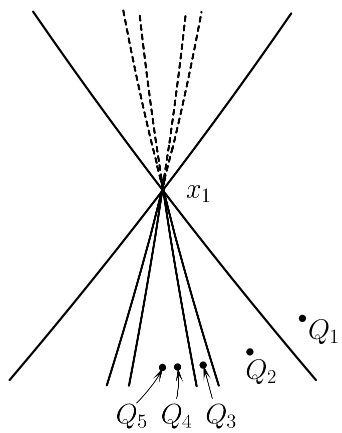


FIGURE 3.12

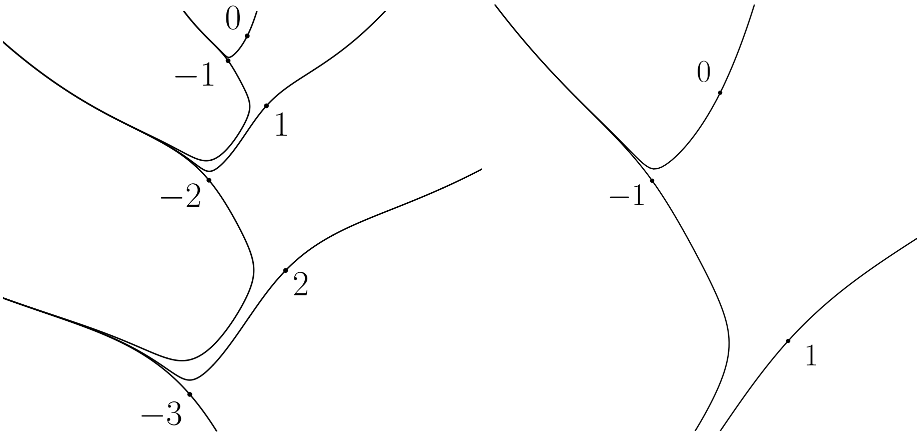


Figure 3.13.1

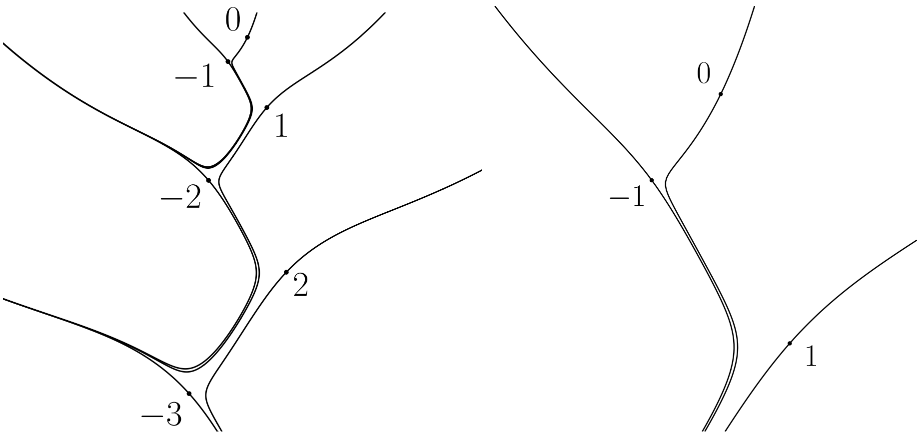


Figure 3.13.2

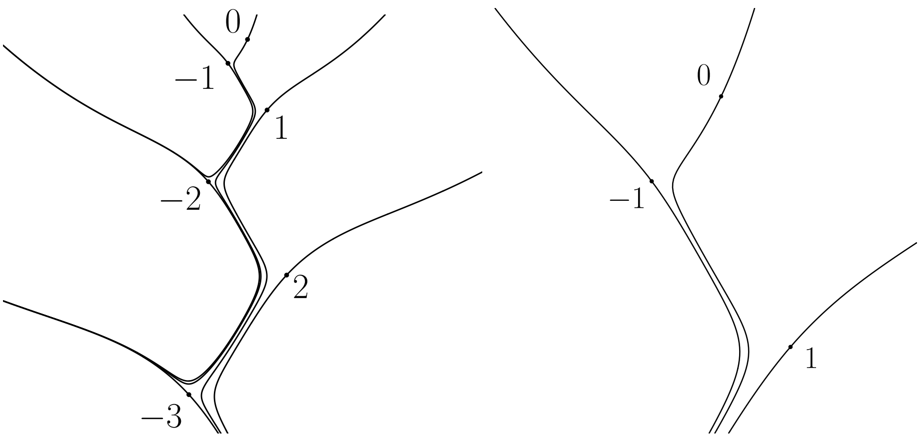


Figure 3.13.3

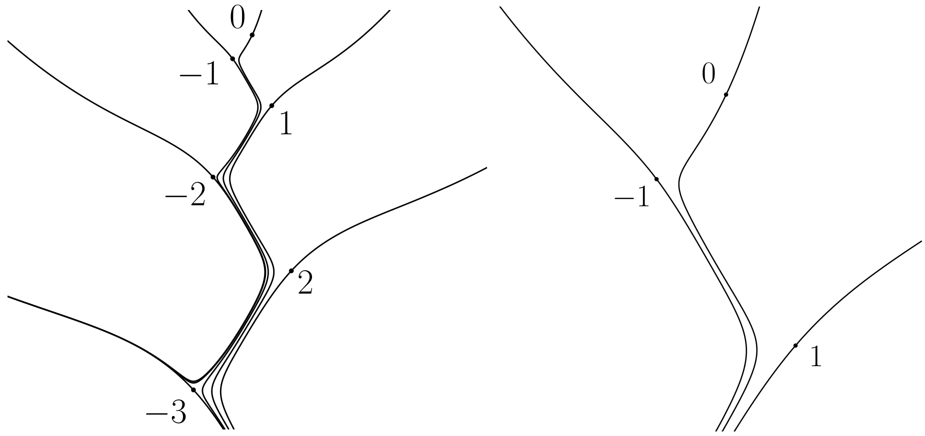


FIGURE 3.13.4

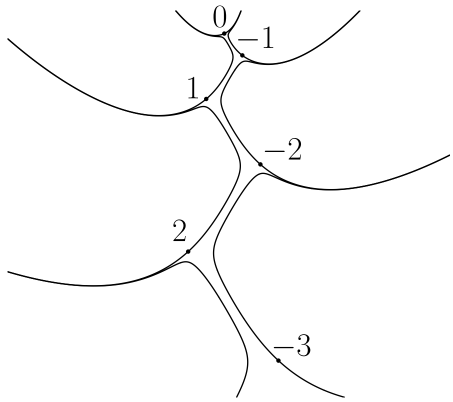


FIGURE 3.14

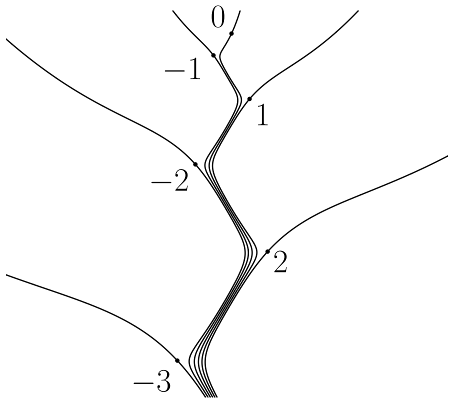


FIGURE 3.15

4. Study of the Stokes geometry for some non-local operator with a large parameter

In the preceding section we discussed some global aspects of WKB solutions of some local operator given by (3.1), expecting that a local operator is the first candidate of the reasonable generalization of a linear differential operator of finite order whose WKB solutions admit infinitely many phases. The results nicely came up to our expectations; they are quite subtle and interesting enough, but more important is the fact that we can understand the global aspect of WKB solutions for the operator in question with exactly the same machinery as that used for finite order operators. Our success seems to be closely linked with the properties of the Borel transform of the operator considered there, which all WKB type operators share. This leads us to a tempting idea that all WKB type operators lie in the scope of applicability of our method (cf. [**AKT1**], [**AKT3**]) as far as their Stokes geometry is concerned, regardless of their particular analytic characters observed when the large parameter is fixed at a finite value. It is consistent with the fact that the role of a large "parameter" in exact WKB analysis is an independent variable and not a fixed constant; we really exemplify the idea for some concrete operators in this section and the next section.

The operator we considered in this section is the following:

$$P = \cosh(\eta^{-1}\frac{d}{dx}) - x. \tag{4.1}$$

In view of the relation

$$\cosh(\eta^{-1}\frac{d}{dx}) = \frac{1}{2}\left\{\exp(\eta^{-1}\frac{d}{dx}) + \exp(-\eta^{-1}\frac{d}{dx})\right\} \tag{4.2}$$

one immediately sees that the operator P is not a local operator if we fix η at a fixed value η_0. Still, as we see below, the WKB-theoretic analysis of this operator proceeds almost in parallel with the reasoning in Section 3.

By solving the simultaneous equations

$$P(x,\zeta) = \frac{\partial P}{\partial \zeta}(x,\zeta) = 0 \tag{4.3}$$

we find that for each n in $\mathbb{Z}$

(4.4) $x = 1$ is a turning point for the operator P with a characteristic value $2in\pi$,

and

(4.5) $x = -1$ is a turning point for the operator P with a characteristic value $i(2n+1)\pi$.

Using the same branch of the function $\log(x + \sqrt{x^2-1})$ as that used in Section 3 (cf. (3.4) and (3.5)), we label the characteristic roots of the equation $P(x,\zeta) = 0$ as

$$f_{+,n}(x) = \log(x + \sqrt{x^2-1}) + 2in\pi, \tag{4.6}$$

$$f_{-,n}(x) = -\log(x + \sqrt{x^2-1}) + 2in\pi. \tag{4.7}$$

It follows from the choice of the branch of $\log(x + \sqrt{x^2-1})$, the characteristic roots passing through $(x,\zeta) = (1, 2in\pi)$ are given by the pair $(f_{+,n}, f_{-,n})$ for each n in $\mathbb{Z}$. Hence the Stokes curves emanating from $x = 1$ are described by the equation

$$\operatorname{Im} \int_1^x (f_{+,n}(x) - f_{-,n}(x))dx = 0, \tag{4.8}$$

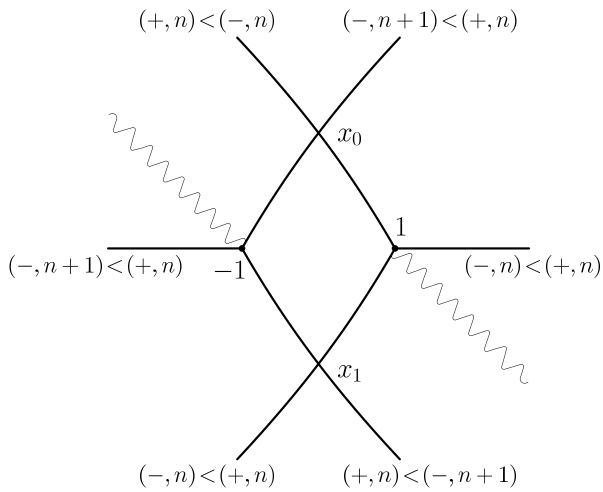


FIGURE 4.1

which is designated as $\sigma^{+}_{(+,n),(-,n)}$. Similarly the Stokes curves emanating from $x=-1$ are described by

$$\mathrm{Im}\int_{-1}^{x}(f_{+,n}(x)-f_{-,n+1}(x))dx=0, \tag{4.9}$$

which is designated as $\sigma^{-}_{(+,n),(-,n+1)}$. Note that (4.8) (resp., (4.9)) is equivalent to (3.7) (resp., (3.11)). Hence, exactly in the same manner as in Section 3, all the Stokes curves $\sigma^{+}_{(+,n),(-,n)}$ and $\sigma^{-}_{(+,n),(-,n+1)}$ respectively sit on one and the same curve. Thus we obtain Figure 4.1 and Figure 4.2, which include the types of relevant curves. One can readily read out that the crossing point x_0 (resp., x_1) is ordered with respect to the pair $\{\sigma^{-}_{(+,n),(-,n+1)},\sigma^{+}_{(+,n),(-,n)}\}$ and the pair $\{\sigma^{-}_{(+,n-1),(-,n)},\sigma^{+}_{(+,n),(-,n)}\}$ (resp., the pair $\{\sigma^{-}_{(+,n),(-,n+1)},\sigma^{+}_{(+,n),(-,n)}\}$ and the pair $\{\sigma^{-}_{(+,n-1),(-,n)},\sigma^{+}_{(+,n),(-,n)}\}$) for each n. To resolve these ordered crossings, we now try to locate virtual turning points. Solving the Hamilton-Jacobi equation for the bicharacteristic strip of $P_B=\cosh(\eta^{-1}\xi)-x$ with the initial condition

$$(x(0),y(0);\xi(0),\eta(0))=(1,0;0,1), \tag{4.10}$$

we obtain

$$(x(t),y(t);\xi(t),\eta(t))=(\cosh t,-t\cosh t+\sinh t;t,1). \tag{4.11}$$

Hence we find the following two classes of virtual turning points.

(i) For each $n\in\mathbb{Z}$ and $k\in\mathbb{N}$, the pair

$$(t,t')=(i\pi(n+\frac{1}{2}),i\pi(2k+n+\frac{1}{2})) \tag{4.12}$$

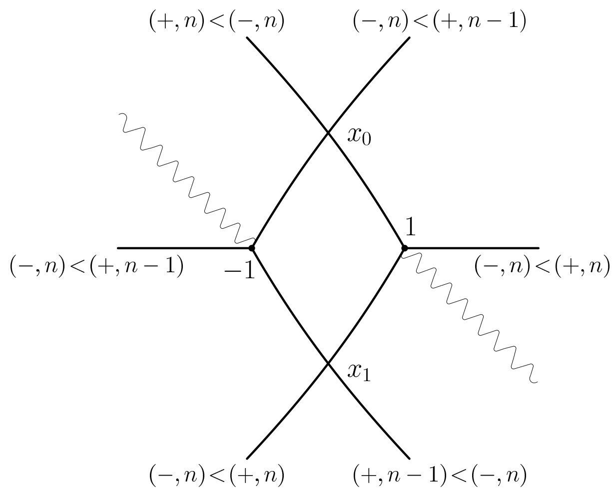


FIGURE 4.2

gives a virtual turning point $x(t) = x(t') = 0$.

(ii) For each $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, the pair

$$(t, t') = (i(w_k + n\pi), -i(w_k - n\pi)), \tag{4.13}$$

where w_k denotes the k-th positive solution of $w = \tan w$ (cf. (3.26)), gives a virtual turning point $x(t) = x(t') = (-1)^n \cos w_k$.

The new Stokes curves emanating from these virtual turning points are as follows:

(i) As $\xi(t) = t$ on the bicharacteristic strip, the new Stokes curves emanating from the origin are given by

$$\operatorname{Im} \int_0^x (f_{+,p}(x) - f_{+,q}(x))dx = 0 \quad (p = \frac{n}{2} + k,\ q = \frac{n}{2}) \tag{4.14}$$

for an even integer n and a positive integer k, and

$$\operatorname{Im} \int_0^x (f_{-,p}(x) - f_{-,q}(x))dx = 0 \quad (p = \frac{n+1}{2} + k,\ q = \frac{n+1}{2}) \tag{4.15}$$

for an odd integer n and a positive integer k.

(ii) As we can verify

$$f_{+,n}(|\cos w_k|) = i(w_k + (2n-k)\pi), \tag{4.16}$$
$$f_{-,n}(|\cos w_k|) = i(-w_k + (2n+k)\pi), \tag{4.17}$$

and

$$f_{+,n}(-|\cos w_k|) = i(-w_k + (2n+k+1)\pi), \tag{4.18}$$
$$f_{-,n}(-|\cos w_k|) = i(w_k + (2n-k-1)\pi), \tag{4.19}$$

the new Stokes curves emanating from $\pm|\cos w_k|$ are given by

$$\operatorname{Im}\int_{|\cos w_k|}^{x}(f_{+,m}(x)-f_{-,n}(x))dx=0 \tag{4.20}$$

for $m-n=k$, and

$$\operatorname{Im}\int_{-|\cos w_k|}^{x}(f_{+,m}(x)-f_{-,n}(x))dx=0 \tag{4.21}$$

for $m-n=-k-1$. The type of the new Stokes curve defined by (4.20) is "$(-,n)>(+,m)$" near the point $x=x_0$, and that of (4.21) is "$(+,m)>(-,n)$" near x_0.

As one can see immediately, (4.20) is the same as (3.39) and (4.21) is the same as (3.41). On the other hand, the curve determined by (4.14) or (4.15) is the imaginary axis, which is not a new Stokes curve for the operator discussed in Section 3.

Let us now employ the same arguments as those given at the end of Section 3.2. Corresponding to the chain $G(x_0)$ and $G(x_1)$ (cf. (3.48) and (3.50)), this time we obtain the following chain of infinite length:

$$\cdots\to(+,1)\to(-,1)\to(+,0)\to(-,0)\to(+,-1)\to(-,-1)\to\cdots \tag{4.22}$$

by considering all ordinary Stokes curves described by (4.8) and (4.9). Then by a similar reason given in §3.2 we see that we can resolve all ordered crossings by adding all new Stokes curves given by (4.14), (4.15), (4.20) and (4.21). Note that the imaginary axis described by (4.14) or (4.15) plays an essential role in the resolution. We will later observe a phenomenon corresponding to this fact in the configuration of the steepest descent paths. Adding all new Stokes curves to Figure 4.1 and 4.2, we finally obtain Figure 4.3. As usual, the portion of a new Stokes curve containing a virtual turning point but neither x_0 nor x_1 is designated by a dotted line.

In parallel with Section 3.3, we study the following integral (4.23) to convince the reader of the validity of the Stokes geometry given by Figure 4.3. Our starting point is an integral representation of a solution ψ of the equation $P\psi=0$ given by

$$\psi=\int\exp(\eta h(x,\zeta))d\zeta, \tag{4.23}$$

where

$$h(x,\zeta)=x\zeta-\sinh\zeta. \tag{4.24}$$

First we note that the saddle points of the integral (4.23) are given by $f_{+,n}(x)$ and $f_{-,m}(x)$ $(n,m\in\mathbb{Z})$. Next we show in Figure 4.5.j

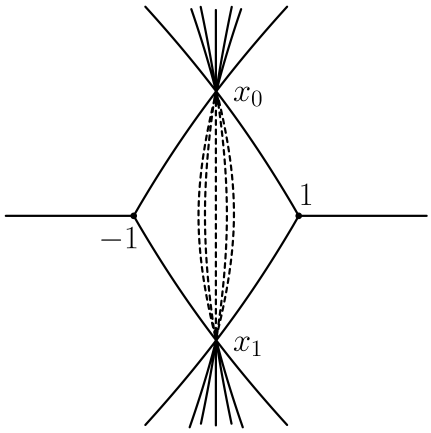


FIGURE 4.3

the configuration of the steepest descent paths when $x = P_j$ ($j = 1, 2, \ldots, 9$) given in Figure 4.4; the point P_5 is chosen to lie in the imaginary axis.

We can read out the following facts from these figures, all of which are consistent with the Stokes geometry given by Figure 4.3.

(i) The configurations of the steepest descent paths topologically change when x crosses an ordinary Stokes curve or the solid-line part of new Stokes curves.

(ii) No topological changes occur in the configuration of steepest descent paths when x crosses the dotted-line part of new Stokes curves.

(iii) Saddle points $\{f_{+,n}(x)\}_{n\in\mathbb{Z}}$ are connected by steepest descent paths and the same holds for $\{f_{-,m}(x)\}_{m\in\mathbb{Z}}$ when x lies in the imaginary axis.

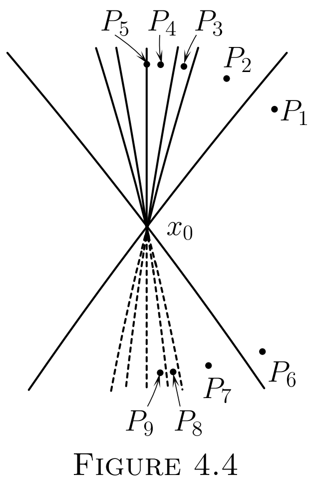


FIGURE 4.4

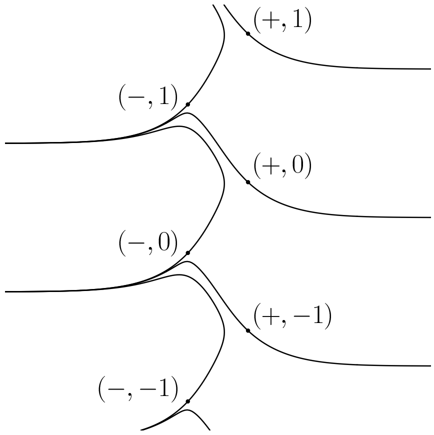


FIGURE 4.5.1

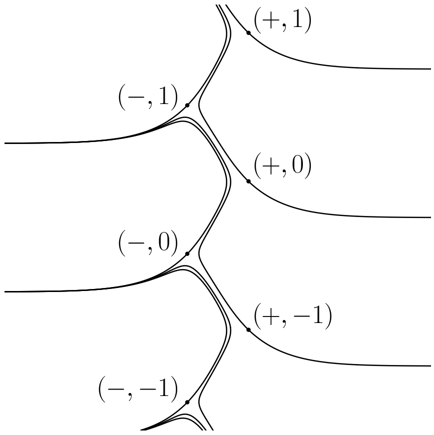


FIGURE 4.5.2

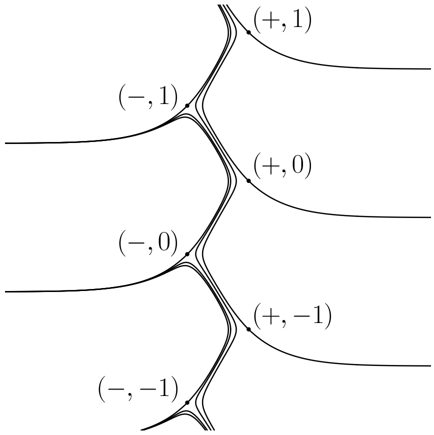


FIGURE 4.5.3

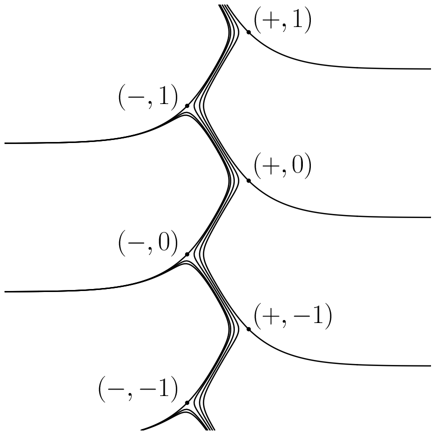


FIGURE 4.5.4

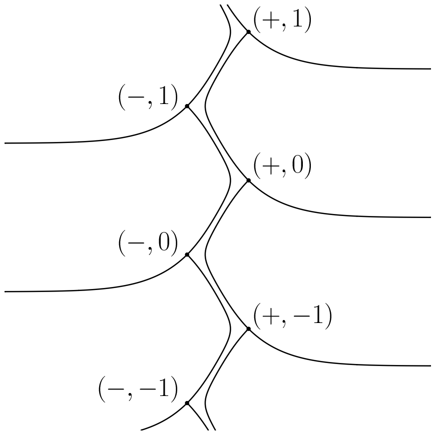


FIGURE 4.5.5

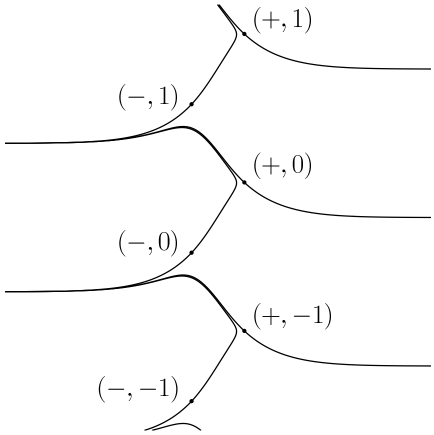


FIGURE 4.5.6

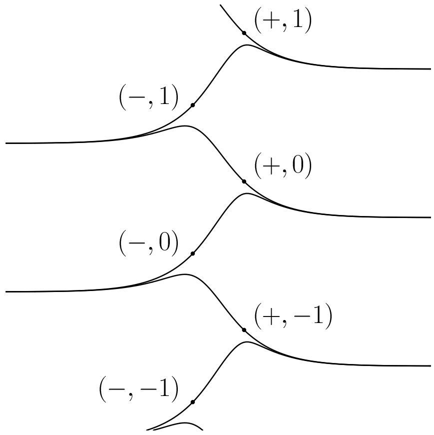


FIGURE 4.5.7

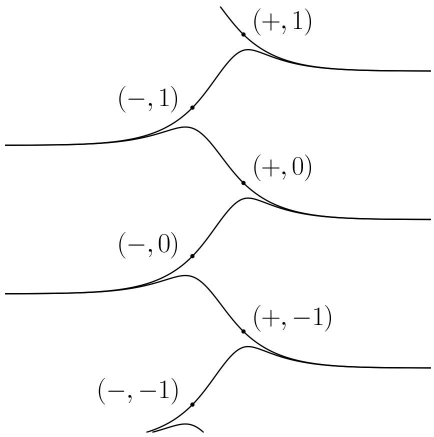


FIGURE 4.5.8

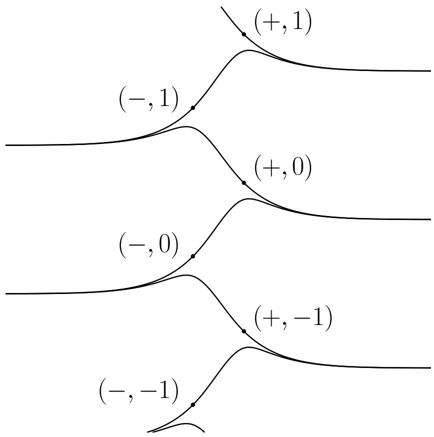


FIGURE 4.5.9

5. Study of the Stokes geometry for some operators without ordered crossing points

Encouraged by the success in the preceding section in the WKB analysis of a non-local operator with a large parameter, we dared to study the following operator:

$$P = \exp\left(\left(\eta^{-1}\frac{d}{dx}\right)^2\right) - \exp(-x^2). \tag{5.1}$$

We can readily find that $x = 0$ is its double turning point and that $x = \pm a_n, \pm b_n$ ($n \in \mathbb{N}$) are its simple turning points with the characteristic value 0, where

$$a_n = e^{i\pi/4}\sqrt{2n\pi}, \tag{5.2}$$

$$b_n = ie^{i\pi/4}\sqrt{2n\pi}. \tag{5.3}$$

The characteristic roots passing through $(x, \zeta) = (0, 0)$ are

$$g_{+,0}(x) = ix \quad \text{and} \quad g_{-,0}(x) = -ix. \tag{5.4}$$

Introducing the cuts to uniformalize

$$g_{+,n}(x) = \sqrt{-x^2 + 2in\pi} \quad (n \in \mathbb{Z}) \tag{5.5}$$

as shown in Figure 5.1, we choose the branch of $g_{+,n}(x)$ by fixing

$$\arg g_{+,n}(0) = i\pi/4 \ (n > 0) \quad \text{and} \quad \arg g_{+,n}(0) = 3i\pi/4 \ (n < 0). \tag{5.6}$$

We then find that the characteristic roots passing through $(x, \zeta) =$

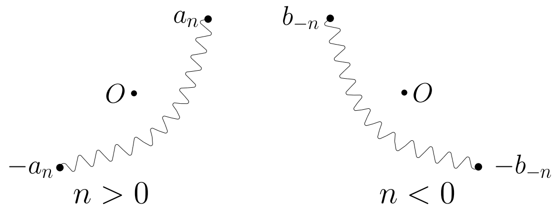


FIGURE 5.1

$(\pm a_n, 0)$ are $g_{+,n}(x)$ and $g_{-,n}(x)$ $(\stackrel{\text{def}}{=} -g_{+,n}(x))$, and that those passing through $(\pm b_n, 0)$ are $g_{+,-n}(x)$ and $g_{-,-n}(x)$. Thus the Stokes curves emanating from $x = a_n$ or $-a_n$ are respectively given by

$$\operatorname{Im} \int_{\pm a_n}^{x} (g_{+,n}(x) - g_{-,n}(x))dx = 0, \tag{5.7}$$

and those emanating from $x = b_n$ or $-b_n$ are respectively given by

$$\operatorname{Im} \int_{\pm b_n}^{x} (g_{+,-n}(x) - g_{-,-n}(x))dx = 0. \tag{5.8}$$

These Stokes curves are shown in Figure 5.2.

At first sight the reader might wonder about the validity of this Stokes geometry, as the Stokes curves emanating from the origin pass through other turning points $\pm a_n$ and $\pm b_n$, and a Stokes curve emanating from a_n (resp., $-a_n$, b_n, $-b_n$) passes through a_m (resp., $-a_m$, b_m, $-b_m$) for $m > n$. However the pair of characteristic roots associated with each turning point is mutually different. Hence we do not anticipate any anomalies caused by these facts. Another interesting fact is observed at each crossing point in the above Stokes geometry; due to the mutual irrelevance of the associated characteristic roots the types of Stokes curves cannot be "hinged" at their crossing points, thus eliminating the possible occurrence of ordered crossings. Thus we are confident of the validity of the Stokes geometry given by Figure 5.2, although its rigorous proof remains to be an open problem. Note that

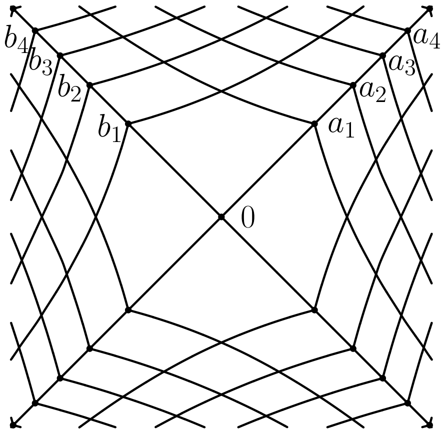


FIGURE 5.2

this interesting feature of the Stokes geometry originates from the following peculiar property of the operator P: Since the symbol of the operator P is equal to

$$\exp(-x^2)\left\{\exp(x^2 + (\xi/\eta)^2) - 1\right\}, \tag{5.9}$$

we can determine the Stokes geometry of P by using

$$\exp(x^2 + (\xi/\eta)^2) - 1 \tag{5.10}$$

instead of P, i.e., by discarding the non-vanishing factor $\exp(-x^2)$. This observation leads us to the following useful conclusion: For any WKB type operator P whose principal symbol has the form $F((\xi/\eta)^2 + Q(x))$ for an entire function $F(\lambda)$, no crossing point of Stokes curves is ordered.

6. Study of the Stokes geometry for a WKB type operator originating in some physical problem

Let us consider a WKB type operator whose symbol P is given by the following:

$$P = P_0(x, \zeta) + \eta^{-1} P_1(x, \zeta), \tag{6.1}$$

where

$$P_0(x, \zeta) = (i\zeta - x) \prod_{n=1}^{\infty} \left(1 - \frac{i\zeta}{n^2}\right), \tag{6.2}$$

$$P_1(x, \zeta) = \sum_{n=1}^{\infty} \frac{|c_n|^2}{n^2} \prod_{p \geq 1, p \neq n} \left(1 - \frac{i\zeta}{p^2}\right) \quad (c_j \in \mathbb{C}) \tag{6.3}$$

with

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty. \tag{6.4}$$

Because of its particular form we may use the symbol P to denote the corresponding operator.

REMARK 6.1. This operator P arises in our trial to generalize the simplest n-level crossing problem in the non-adiabatic transition problem ([**BE**]) to an ∞-level crossing problem. Actually if we consider the following differential equation (6.5) with infinite components, then ψ_0, the first component of ψ, satisfies the equation $P\psi_0 = 0$:

$$i\eta^{-1}\frac{d\psi}{dx} = \left(H_0 + \eta^{-1/2}H_{1/2}\right)\psi, \tag{6.5}$$

where

$$\psi = {}^t(\psi_0, \psi_1, \ldots), \tag{6.6}$$

$$H_0 = \begin{bmatrix} x & & & & \\ & 1 & & 0 & \\ & & 4 & & \\ & 0 & & 9 & \\ & & & & \ddots \end{bmatrix}, \quad H_{1/2} = \begin{bmatrix} 0 & c_1 & c_2 & \cdots \\ \overline{c_1} & & & \\ \overline{c_2} & & 0 & \\ \vdots & & & \end{bmatrix}. \tag{6.7}$$

The turning points of P are given by $\alpha_n \overset{\text{def}}{=} n^2$, and they are double. Defining f_n $(n \geq 0)$ by

$$f_0(x) = -ix, \quad f_n(x) = -in^2 \ \ (n \in \mathbb{N}), \tag{6.8}$$

we find that the Stokes curve emanating from α_n is given by

$$\operatorname{Im} \int_{\alpha_n}^{x} (f_n(x) - f_0(x))dx = 0, \tag{6.9}$$

or, equivalently,

$$\operatorname{Re} \int_{\alpha_n}^{x} (x - \alpha_n)dx = 0. \tag{6.10}$$

These Stokes curves are shown with their types in Figure 6.1. As we immediately read out from Figure 6.1, all crossing points are ordered. Hence we next seek for appropriate virtual turning points to resolve the ordered crossings by adding new Stokes curves emanating from those virtual turning points. For the actual computation we use the method of bicharacteristic diagrams ([**AKT3**]); the method is discussed only for the 3-level problem in [**AKT3**], but the generalization to our problem is straightforward. As a matter of fact, the necessary

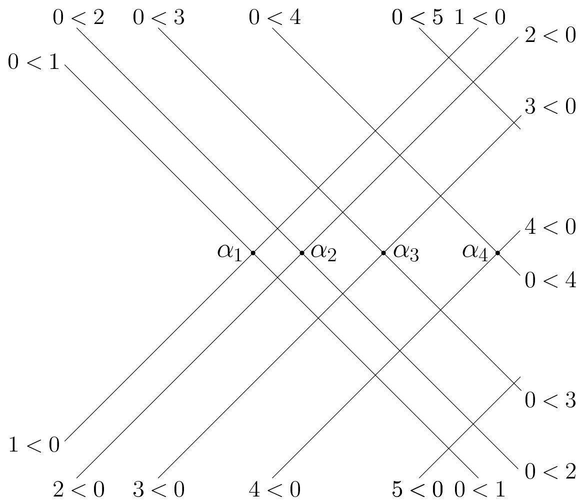


FIGURE 6.1

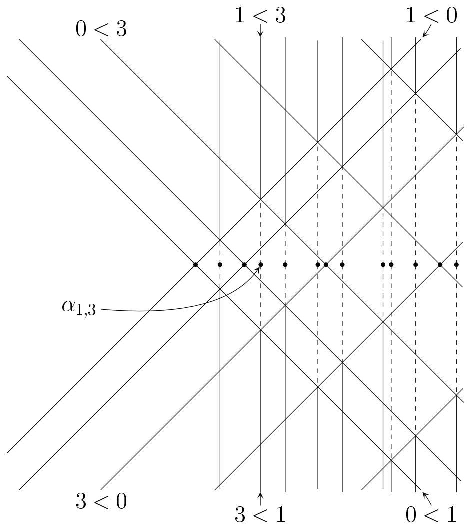
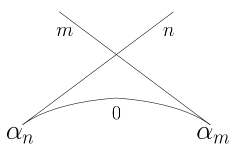


FIGURE 6.2

bicharacteristic diagrams are very limited in our current problem; they are of the following form:



$$(m, n \in \mathbb{N})$$

Correspondingly we obtain the following equation (6.11) to locate the virtual turning point $\alpha_{n,m}$ needed for the resolution of the ordered crossing formed by the Stokes curve emanating from α_n and that emanating from α_m.

$$\int_{\alpha_n}^{\alpha_{n,m}} \alpha_n dx = \int_{\alpha_n}^{\alpha_m} x dx + \int_{\alpha_m}^{\alpha_{n,m}} \alpha_m dx. \tag{6.11}$$

Thus we find

$$\alpha_{n,m} = \frac{1}{2}(\alpha_n + \alpha_m), \tag{6.12}$$

and the new Stokes curve emanating from the virtual turning point is given by

$$\mathrm{Re} \int_{\alpha_{n,m}}^{x} (\alpha_n - \alpha_m) dx = 0. \tag{6.13}$$

Adding the new Stokes curves to Figure 6.1, we obtain Figure 6.2; as usual the portion that contains a virtual turning point but no crossing points is designated by a dotted line. As one can readily confirm, the added new Stokes curves do not produce any new ordered crossings with other (ordinary or new) Stokes curves. (For a quartet (p, q, r, s) of natural numbers satisfying $p^2 + q^2 = r^2 + s^2$, $p < r$ and $p \neq s < q$, the virtual turning points $\alpha_{p,q}$ and $\alpha_{r,s}$ coincide, and the dotted-line parts of new Stokes curves are different. But, since their types are not "hinged", no anomalies are anticipated.) Thus we believe that Figure 6.2 gives a complete Stokes geometry.

In the rest of this section we confirm the above belief by applying the steepest descent method to the following integral (6.14) that solves $P\psi = 0$:

$$\psi = \int e^{\eta(x\zeta - i\zeta^2/2)} \prod_{n=1}^{\infty} (1 - i\zeta/n^2)^{-1-i|c_n|^2} d\zeta. \tag{6.14}$$

The application of the steepest descent method to ψ requires some special care, as besides the ordinary saddle point $\zeta = -ix$ the singular points $\zeta = -in^2$ play the role of saddle points. (Cf. [**KT**].) Note that the saddle point and the singular points are given by $\{f_n(x)\}_{n=0}^{\infty}$, the functions defined by (6.8). Having this point in mind, we show the configuration of the steepest descent paths for $x = P_j$ $(j = 1, 2, \ldots, 6)$ in Figures 6.4.j, where P_j are chosen as in Figure 6.3. From these figures one can clearly see that the topological configuration of the steepest descent paths changes as x crosses a Stokes curve or the solid-line part of a new Stokes curve but that no changes are observed when x crosses the dotted-line part of new Stokes curves.

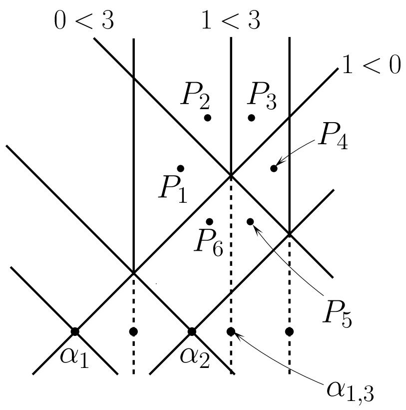


FIGURE 6.3

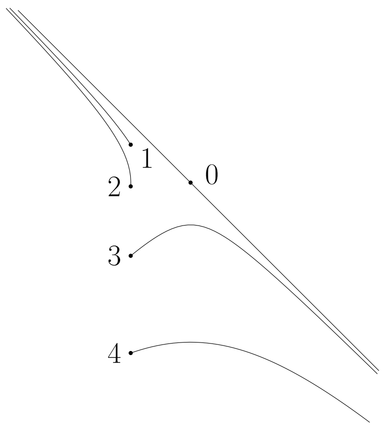


FIGURE 6.4.1

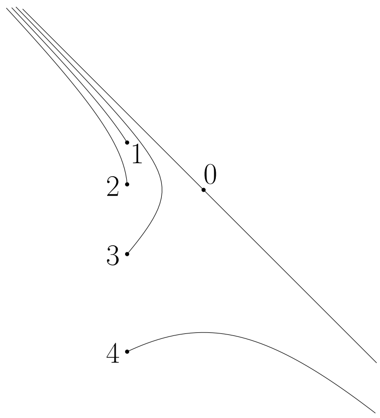


FIGURE 6.4.2

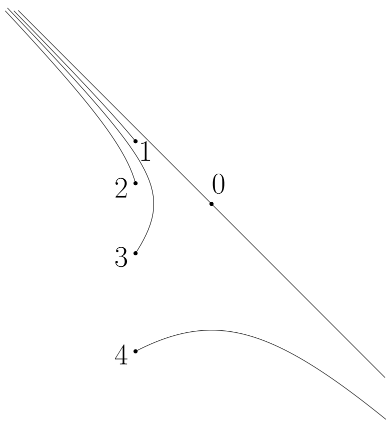


FIGURE 6.4.3

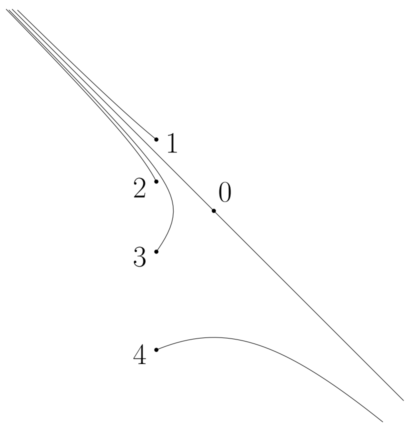


FIGURE 6.4.4

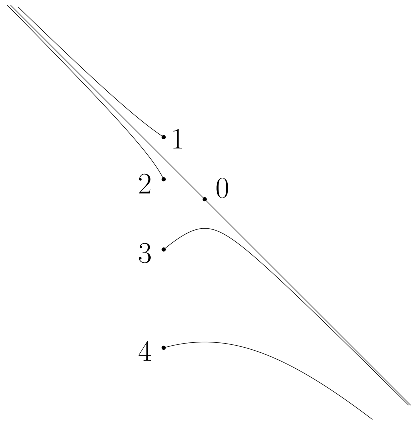


FIGURE 6.4.5

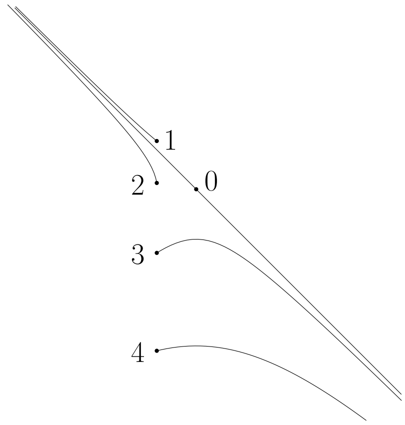


FIGURE 6.4.6

Appendix

In this appendix we show that all new Stokes curves defined by either (3.38) or (3.40) pass through the intersection point x_0 of the ordinary Stokes curves in the example discussed in Section 3 (see Figure 3.1). Exactly the same reasoning shows that all new Stokes curves pass through another intersection point x_1 also.

Because (3.38) (resp., (3.40)) is equivalent to (3.39) (resp., (3.41)), what we have to prove is $\operatorname{Im} I_k^+ = \operatorname{Im} I_k^- = 0$ for each k in $\mathbb{N}$, where

$$I_k^+ = \int_{|\cos w_k|}^{x_0} (\log(x + \sqrt{x^2 - 1}) + ik\pi)dx, \tag{A.1}$$

$$I_k^- = \int_{-|\cos w_k|}^{x_0} (\log(x + \sqrt{x^2 - 1}) - i(k+1)\pi)dx. \tag{A.2}$$

We first observe that

$$I_k^+ = \Big[x \log(x + \sqrt{x^2 - 1}) - \sqrt{x^2 - 1} + ik\pi x\Big]_{x=|\cos w_k|}^{x_0}. \tag{A.3}$$

Because of the choice (3.5) of the branches of $\sqrt{x^2 - 1}$ and $\log(x + \sqrt{x^2 - 1})$, we have

$$\sqrt{x^2 - 1}\Big|_{x=|\cos w_k|} = i|\sin w_k|, \tag{A.4}$$

$$\log(x + \sqrt{x^2 - 1})\Big|_{x=|\cos w_k|} = i(w_k - k\pi). \tag{A.5}$$

(Note that $0 < w_k - k\pi < \pi/2$ in view of (3.27).) Hence we obtain

$$
\begin{aligned}
\text{(A.6)} I_k^+ \;&=\; x_0 \log(x_0 + \sqrt{x_0^2 - 1}) - \sqrt{x_0^2 - 1} + ik\pi x_0 \\
&\quad - \{i(w_k - k\pi)|\cos w_k| - i|\sin w_k| + ik\pi|\cos w_k|\} \\
&=\; x_0 \log(x_0 + \sqrt{x_0^2 - 1}) - \sqrt{x_0^2 - 1} + ik\pi x_0 \\
&\quad - i\left(w_k|\cos w_k| - |\sin w_k|\right).
\end{aligned}
$$

Because w_k is a positive solution of $w = \tan w$, we conclude that

$$
I_k^+ = x_0 \log(x_0 + \sqrt{x_0^2 - 1}) - \sqrt{x_0^2 - 1} + ik\pi x_0 \tag{A.7}
$$

holds for each k. In a similar manner, by using the relation

$$
\sqrt{x^2 - 1}\Big|_{x = -|\cos w_k|} \;=\; i|\sin w_k|, \tag{A.8}
$$

$$
\log(x + \sqrt{x^2 - 1})\Big|_{x = -|\cos w_k|} \;=\; i(-w_k + (k+1)\pi), \tag{A.9}
$$

we obtain

$$
I_k^- = x_0 \log(x_0 + \sqrt{x_0^2 - 1}) - \sqrt{x_0^2 - 1} - i(k+1)\pi x_0. \tag{A.10}
$$

On the other hand, as x_0 in an intersection point of the curves defined by (3.7) and (3.11), we find $\operatorname{Im} J^+ = \operatorname{Im} J^- = 0$, where

$$
\begin{aligned}
\text{(A.11)} \qquad J^+ &= \int_1^{x_0} (\log(x + \sqrt{x^2 - 1}))dx \\
&= x_0 \log(x_0 + \sqrt{x_0^2 - 1}) - \sqrt{x_0^2 - 1}, \\
\text{(A.12)} \qquad J^- &= \int_{-1}^{x_0} (\log(x + \sqrt{x^2 - 1}) - i\pi)dx \\
&= x_0 \log(x_0 + \sqrt{x_0^2 - 1}) - \sqrt{x_0^2 - 1} - i\pi x_0.
\end{aligned}
$$

The relations (A.11) and (A.12) entail that

$$
I_k^+ = (k+1)J^+ - kJ^- \quad \text{and} \quad I_k^- = -kJ^+ + (k+1)J^-. \tag{A.13}
$$

Hence $\operatorname{Im} I_k^+ = \operatorname{Im} I_k^- = 0$ follows from $\operatorname{Im} J^+ = \operatorname{Im} J^- = 0$. This proves our assertion.

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