The Singularity of Kontsevich's Solution for $QH^*(CP^2)$

Davide Guzzetti

Research Institute for Mathematical Sciences (RIMS)
Kyoto University
Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan

Phone +81 75-753-7214, Fax +81 75-753-7266
E-mail: guzzetti@kurims.kyoto-u.ac.jp

Abstract.

In this paper we study the nature of the singularity of the Kontsevich's solution of the WDVV equations of associativity. We prove that it corresponds to a singularity in the change of two coordinates systems of the Frobenius manifold given by the quantum cohomology of CP^2.

1 Introduction

In this paper we study the nature of the singularity of the solution of the WDVV equations of associativity for the quantum cohomology of the complex projective space CP^2. As we will explain in detail below, the *quantum cohomology* of a projective space CP^d (d integer) is a Frobenius Manifold which has a structure specified by a solution to a WDVV equation. In the case of CP^2 such a solution was found by Kontsevich [20] in the form of a convergent series in the flat coordinates (t^1, t^2, t^3) of the corresponding Frobenius manifold:

$$F(t) := \frac{1}{2}[(t^1)^2 t^3 + t^1 (t^2)^2)] + \frac{1}{t^3} \sum_{k=1}^{\infty} A_k \left[(t^3)^3 \exp(t^2)\right]^k, \quad A_k \in \mathbf{R} \tag{1}$$

The series converges in a neighborhood of $(t^3)^3 \exp(t^2) = 0$ with a certain radius of convergence extimated by Di Francesco and Itzykson [5]. The coefficients A_k are real and are the Gromov-Witten invariants of genus zero. We will explain this point later. As for the Gromov-Witten invariants of genus one of CP^2, we refer to [13], where B. Dubrovin and Y. Zhang proved that their G-function has the same radius of convergence of (1).

As we will explain below, the nature of the boundary points of the ball of convergence of (1) is important to study of the global structure of the manifold.

In the following, we first state rigorously the problem of the global structure of a Frobenius manifold, then we introduce the quantum cohomology of CP^d as a Frobenius manifold and we explain its importance in enumerative geometry. Finally, we study the boundary points of the ball of convergence of Kontsevich's solution. We prove that they correspond to a singularity in the change of two coordinates systems.

Our paper is part of a project to study of the global structure of Frobenius manifolds that we started in [15].

1.1 Frobenius Manifolds and their Global Structure

The subject of this subsection can be found in [9], [10] or, in a more synthetic way, in [15].

The WDVV equations of associativity were introduced by Witten [28], Dijkgraaf, Verlinde E., Verlinde H. [6]. They are differential equations satisfied by the *primary free energy* $F(t)$ in two-dimensional topological field theory. $F(t)$ is a function of the coupling constants $t := (t^1, t^2, ..., t^n)$ $t^i \in \mathbf{C}$. Let $\partial_\alpha := \frac{\partial}{\partial t_\alpha}$. Given a non-degenerate symmetric matrix $\eta^{\alpha\beta}$, $\alpha, \beta = 1, ..., n$, and numbers $q_1, q_2, ..., q_n$, $r_1, r_2, ..., r_n$, d, ($r_\alpha = 0$ if $q_\alpha \neq 1$, $\alpha = 1, ..., n$), the WDVV equations are

$$\partial_\alpha \partial_\beta \partial_\lambda F \; \eta^{\lambda\mu} \; \partial_\mu \partial_\gamma \partial_\delta F \;=\; \text{the same with } \alpha,\, \delta \text{ exchanged}, \tag{2}$$

$$\partial_1\partial_\alpha\partial_\beta F = \eta_{\alpha\beta}, \tag{3}$$

$$E(F) = (3-d)F + \text{ (at most) quadratic terms}, \tag{4}$$

where the matrix $(\eta_{\alpha\beta})$ is the inverse of the matrix $(\eta^{\alpha\beta})$ and the differential operator E is $E := \sum_{\alpha=1}^{n} E^\alpha\partial_\alpha$, $E^\alpha := (1-q_\alpha)t^\alpha + r_\alpha$, $\alpha = 1, ..., n$, and will be called *Euler vector field.*

Frobenius structures first appeared in the works of K. Saito [25] [26] with the name of *flat scructures*. The complete theory of Frobenius manifolds was then developed by B. Dubrovin as a geometrical setting for topological field theory and WDVV equations in [7]. Such a theory has links to many branches of mathematics like singularity theory and reflection groups [25] [26] [12] [9], algebraic and enumerative geometry [20] [22], isomonodromic deformations theory, boundary value problems and Painlevé equations [10].

If we define $c_{\alpha\beta\gamma}(t) := \partial_\alpha\partial_\beta\partial_\gamma F(t)$, $c_{\alpha\beta}^\gamma(t) := \eta^{\gamma\mu}c_{\alpha\beta\mu}(t)$ (sum over repeated indices is always omitted in the paper), and we consider a vector space A=span$(e_1, ..., e_n)$, then we obtain a family of commutative algebras A_t with the multiplication $e_\alpha \cdot e_\beta := c_{\alpha\beta}^\gamma(t)e_\gamma$. Equation (2) is equivalent to associativity and (3) implies that e_1 is the unity.

Definition: A *Frobenius manifold* is a smooth/analytic manifold M over $\mathbf{C}$ whose tangent space T_tM at any $t \in M$ is an *associative, commutative algebra* with *unity* e. Moreover, there exists a non-degenerate bilinear form $< \quad , \quad >$ defining a *flat metric* (flat means that the curvature associated to the Levi-Civita connection is zero).

We denote by $\cdot$ the product and by ∇ the covariant derivative of $< \cdot, \cdot >$. We require that the tensors $c(u, v, w) :=< u \cdot v, w >$, and $\nabla_y c(u, v, w)$, $u, v, w, y \in T_tM$, be symmetric. Let $t^1, .., t^n$ be (local) flat coordinates for $t \in M$. Let $e_\alpha := \partial_\alpha$ be the canonical basis in T_tM, $\eta_{\alpha\beta} :=< \partial_\alpha, \partial_\beta >$, $c_{\alpha\beta\gamma}(t) :=< \partial_\alpha \cdot \partial_\beta, \partial_\gamma >$. The symmetry of c corresponds to the complete symmetry of $\partial_\delta c_{\alpha\beta\gamma}(t)$ in the indices. This implies the existence of a function $F(t)$ such that $\partial_\alpha\partial_\beta\partial_\gamma F(t) = c_{\alpha\beta\gamma}(t)$ satisfying the WDVV (2). The equation (3) follows from the axiom $\nabla e = 0$ which yields $e = \partial_1$. . Some more axioms are needed to formulate the quasi-homogeneity condition (4) and we refer the reader to [9] [10] [11]. In this way the WDVV equations are reformulated in a geometrical terms.

We first consider the problem of the local structure of Frobenius manifolds. A Frobenius manifold is characterized by a family of *flat* connections $\tilde{\nabla}(z)$ parameterized by a complex number z, such that for $z = 0$ the connection is associated to $< \quad , \quad >$. For this reason $\tilde{\nabla}(z)$ are called *deformed connections.* Let $u, v \in T_tM$, $\frac{d}{dz} \in T_z\mathbf{C}$; the family is defined on $M \times \mathbf{C}$ as follows:

$$\tilde{\nabla}_u v := \nabla_u v + zu \cdot v,$$

$$\tilde{\nabla}_{\frac{d}{dz}} v := \frac{\partial}{\partial z}v + E \cdot v - \frac{1}{z}\hat{\mu}v,$$

$$\tilde{\nabla}_{\frac{d}{dz}}\frac{d}{dz} = 0, \qquad \tilde{\nabla}_u\frac{d}{dz} = 0$$

where E is the Euler vector field and

$$\hat{\mu} := I - \frac{d}{2} - \nabla E$$

is an operator acting on v. In flat coordinates $t = (t^1, ..., t^n)$, $\hat{\mu}$ becomes:

$$\hat{\mu} = \mathrm{diag}(\mu_1, ..., \mu_n), \qquad \mu_\alpha = q_\alpha - \frac{d}{2},$$

provided that ∇E is diagonalizable. This will be assumed in the paper. A flat coordinate $\tilde{t}(t, z)$ is a solution of $\tilde{\nabla} d\tilde{t} = 0$, which is a linear system

$$\partial_\alpha\xi = zC_\alpha(t)\xi, \tag{5}$$

$$\partial_z\xi = \left[\mathcal{U}(t) + \frac{\hat{\mu}}{z}\right]\xi, \tag{6}$$

where ξ is a column vector of components $\xi^\alpha = \eta^{\alpha\mu}\frac{\partial\tilde{t}}{\partial t^\mu}$, $\alpha = 1, ..., n$ and $C_\alpha(t) := \left(c_{\alpha\gamma}^\beta(t)\right)$, $\mathcal{U} := \left(E^\mu c_{\mu\gamma}^\beta(t)\right)$ are $n \times n$ matrices.

The quantum cohomology of projective spaces, to be introduce below, belongs to the class of *semi-simple* Frobenius manifolds, namely analytic Frobenius manifolds such that the matrix $\mathcal{U}$ can be diagonalized with *distinct eigenvalues* on an open dense subset $\mathcal{M}$ of M. Then, there exists an invertible matrix $\phi_0 = \phi_0(t)$ such that $\phi_0 \mathcal{U} \phi_0^{-1} = \mathrm{diag}(u_1, ..., u_n) =: U$, $u_i \neq u_j$ for $i \neq j$ on $\mathcal{M}$. The systems (5) and (6) become:

$$\frac{\partial y}{\partial u_i} = [zE_i + V_i]\ y \tag{7}$$

$$\frac{\partial y}{\partial z} = \left[U + \frac{V}{z}\right] y, \tag{8}$$

where the row-vector y is $y := \phi_0\ \xi$, E_i is a diagonal matrix such that $(E_i)_{ii} = 1$ and $(E_i)_{jk} = 0$ otherwise, and

$$V_i := \frac{\partial \phi_0}{\partial u_i}\ \phi_0^{-1} \qquad V := \phi_0\ \hat{\mu}\ \phi_0^{-1},$$

As it is proved in [9] [10], u_1,...,u_n are local coordinates on $\mathcal{M}$. The two bases $\frac{\partial}{\partial t^\nu}$, $\nu = 1, ..., n$ and $\frac{\partial}{\partial u_i}$, $i = 1, ..., n$ are related by ϕ_0 according to the linear combination $\frac{\partial}{\partial t^\nu} = \sum_{i=1}^n \frac{(\phi_0)_{i\nu}}{(\phi_0)_{i1}}\ \frac{\partial}{\partial u_i}$. Locally we obtain a change of coordinates, $t^\alpha = t^\alpha(u)$, then $\phi_0 = \phi_0(u)$, $V = V(u)$. The local Frobenius structure of $\mathcal{M}$ is given by parametric formulae:

$$t^\alpha = t^\alpha(u), \qquad F = F(u) \tag{9}$$

where $t^\alpha(u)$, $F(u)$ are certain meromorphic functions of $(u_1, ..., u_n)$, $u_i \neq u_j$, which can be obtained from $\phi_0(u)$ and $V(u)$. Their explicit construction was the object of [15]. We stress here that the condition $u_i \neq u_j$ is crucial. We will further comment on this when we face the problem of the global structure.

The dependence of the system on u is *isomonodromic*. This means that the monodromy data of the system (8), to be introduced below, do not change for a small deformation of u. Therefore, the coefficients of the system in every local chart of $\mathcal{M}$ are naturally labeled by the monodromy data. To calculate the functions (9) in every local chart one has to reconstruct the system (8) from its *monodromy data*. This is the *inverse problem.*

We briefly explain what are the monodromy data of the system (8) and why they do not depend on u (locally). For details the reader is referred to [10]. At $z = 0$ the system (8) has a fundamental matrix solution (i.e. an invertible $n \times n$ matrix solution) of the form

$$Y_0(z, u) = \left[\sum_{p=0}^{\infty} \phi_p(u)\ z^p\right] z^{\hat{\mu}} z^R, \tag{10}$$

where $R_{\alpha\beta} = 0$ if $\mu_\alpha - \mu_\beta \neq k > 0$, $k \in \mathbf{N}$. At $z = \infty$ there is a formal $n \times n$ matrix solution of (8) given by $Y_F = \left[I + \frac{F_1(u)}{z} + \frac{F_2(u)}{z^2} + ...\right]\ e^{z\ U}$ where $F_j(u)$'s are $n \times n$ matrices. It is a well known result that there exist fundamental matrix solutions with asymptotic expansion Y_F as $z \to \infty$ [2]. Let l be a generic oriented line passing through the origin. Let l_+ be the positive half-line and l_- the negative one. Let Π_L and Π_R be two sectors in the complex plane to the left and to the right of l respectively. There exist unique fundamental matrix solutions Y_L and Y_R having the asymptotic expansion Y_F for $x \to \infty$ in Π_L and Π_R respectively [2]. They are related by an invertible connection matrix S, called *Stokes matrix*, such that $Y_L(z) = Y_R(z)S$ for $z \in l_+$. As it is proved in [10] we also have $Y_L(z) = Y_R(z)S^T$ on l_-.

Finally, there exists a $n \times n$ invertible *connection matrix* C such that $Y_0 = Y_R C$ on Π_R.

Definition: The matrices R, C, $\hat{\mu}$ and the Stokes matrix S of the system (8) are the *monodromy data* of the Frobenius manifold in a neighborhood of the point $u = (u_1, ..., u_n)$. It is also necessary to specify which is the first eigenvalue of $\hat{\mu}$, because the dimension of the manifold is $d = -2\mu_1$ (a more precise definition of monodromy data is in [10]).

The definition makes sense because the data do not change if u undergoes a small deformation. This problem is discussed in [10]. We also refer the reader to [17] for a general discussion of isomonodromic deformations. Here we just observe that since a fundamental matrix solution $Y(z, u)$ of (8) also satisfies (7), then the monodromy data can not depend on u (locally). In fact, $\frac{\partial Y}{\partial u_i} Y^{-1} = zE_i + V_i$ is single-valued in z.

The inverse problem can be formulated as a *boundary value problem* (b.v.p.). Let's fix $u = u^{(0)} = (u_1^{(0)}, ..., u_n^{(0)})$ such that $u_i^{(0)} \neq u_j^{(0)}$ for $i \neq j$. Suppose we give μ, μ_1, R, an admissible line l, S and C. Some more technical conditions must be added, but we refer to [10]. Let D be a disk specified by $|z| < \rho$ for some small ρ. Let P_L and P_R be the intersection of the complement of the disk with Π_L and Π_R respectively. We denote by ∂D_R and ∂D_L the lines on the boundary of D on the side of P_R and P_L respectively; we denote by $\tilde{l}_+$ and $\tilde{l}_-$ the portion of l_+ and l_- on the common boundary of P_R and P_L. Let's consider the following discontinuous b.v.p.: we want to construct a piecewise holomorphic $n \times n$ matrix function

$$\Phi(z) = \begin{cases} \Phi_R(z), & z \in P_R \\ \Phi_L(z), & z \in P_L \\ \Phi_0(z), & z \in D \end{cases},$$

continuous on the boundary of P_R, P_L, D respectively, such that

$$\Phi_L(\zeta) = \Phi_R(\zeta)\ e^{\zeta U} S e^{-\zeta U}, \quad \zeta \in \tilde{l}_+$$

$$\Phi_L(\zeta) = \Phi_R(\zeta)\ e^{\zeta U} S^T e^{-\zeta U}, \quad \zeta \in \tilde{l}_-$$

$$\Phi_0(\zeta) = \Phi_R(\zeta)\ e^{\zeta U} C \zeta^{-R} \zeta^{-\hat{\mu}}, \quad \zeta \in \partial D_R$$

$$\Phi_0(\zeta) = \Phi_L(\zeta)\ e^{\zeta U} S^{-1} C \zeta^{-R} \zeta^{-\hat{\mu}}, \quad \zeta \in \partial D_L$$

$$\Phi_{L/R}(z) \to I \text{ if } z \to \infty \text{ in } P_{L/R}\ .$$

The reader may observe that $\tilde{Y}_{L/R}(z) := \Phi_{L/R}(z) e^{zU}$, $\tilde{Y}^{(0)}(z) := \Phi_0(z, u) z^{\hat{\mu}} z^R$ have precisely the monodromy properties of the solutions of (8).

Theorem [23][21][10]: *If the above boundary value problem has solution for a given* $u^{(0)} = (u_1^{(0)}, ..., u_n^{(0)})$ *such that* $u_i^{(0)} \neq u_j^{(0)}$ *for* $i \neq j$*, then:*

i) it is unique.

ii) The solution exists and it is analytic for u *in a neighborhood of* $u^{(0)}$*.*

iii) The solution has analytic continuation as a meromorphic function on the universal covering of $\mathbf{C}^n \backslash \{diagonals\}$*, where "diagonals" stands for the union of all the sets* $\{u \in \mathbf{C}^n \mid u_i = u_j, \quad i \neq j\}$*.*

A solution $\tilde{Y}_{L/R}$, $\tilde{Y}^{(0)}$ of the b.v.p. solves the system (7), (8). This means that we can locally reconstruct $V(u)$, $\phi_0(u)$ and (9) from the local solution of the b.v.p.. It follows that every local chart of the atlas covering the manifold is labeled by monodromy data. Moreover, $V(u)$, $\phi_0(u)$ and (9) can be continued analytically as meromorphic functions on the universal covering of $\mathbf{C}^n\backslash$diagonals.

Let $\mathcal{S}_n$ be the symmetric group of n elements. Local coordinates $(u_1, ..., u_n)$ are defined up to permutation. Thus, the analytic continuation of the local structure of $\mathcal{M}$ is described by the *braid group* $\mathcal{B}_n$, namely the fundamental group of $(\mathbf{C}^n\backslash\text{diagonals})/\mathcal{S}_n$. There exists an action of the braid group itself on the monodromy data, corresponding to the change of coordinate chart. The group is generated by $n-1$ elements β_1,...,β_{n-1} such that β_i is represented as a deformation consisting of a permutation of u_i, u_{i+1} moving counter-clockwise (clockwise or counter-clockwise is a matter of convention).

If u_1, ..., u_n are in lexicographical order w.r.t. l, so that S is upper triangular, the braid β_i acts on S as follows [10]:

$$S \mapsto S^{\beta_i} = A_i(S)\ S\ A_i(S)$$

where

$$(A_i(S))_{kk} = 1 \qquad k = 1, ..., n \quad n \neq\ i,\ i+1$$

$$(A_i(S))_{i+1,i+1} = -s_{i,i+1}$$

$$(A_i(S))_{i,i+1} = (A_i(S))_{i+1,i} = 1$$

and all the other entries are zero. For a generic braid β the action $S \to S^\beta$ is decomposed into a sequence of elementary transformations as above. In this way, we are able to describe the analytic continuation of the local structure in terms of monodromy data.

Not all the braids are actually to be considered. Suppose we do the following gauge $y \mapsto Jy$, J =diag$(\pm1, ..., \pm1)$, on the system (8). Therefore $JUJ^{-1} \equiv U$ but S is transformed to JSJ^{-1}, where some entries change sign. The formulae which define a local chart of the manifold in terms of monodromy data, which are given in [10], [15], are not affected by this transformation. The analytic continuation of the local structure on the universal covering of $(\mathbf{C}^n\backslash\text{diagonals})/\mathcal{S}_n$ is therefore described by the elements of the quotient group

$$\mathcal{B}_n/\{\beta \in \mathcal{B}_n \mid S^\beta = JSJ\} \tag{11}$$

From these considerations it is proved in [10] that:

Theorem [10]: *Given monodromy data (μ_1, $\hat{\mu}$, R, S, C), the local Frobenius structure obtained from the solution of the b.v.p. extends to an open dense subset of the covering of $(\mathbf{C}^n\backslash diagonals)/\mathcal{S}_n$ w.r.t. the covering transformations (11).*

Let's start from a Frobenius manifold M of dimension d. Let $\mathcal{M}$ be the open sub-manifold where $\mathcal{U}(t)$ has distinct eigenvalues. If we compute its monodromy data ($\mu_1 = -\frac{d}{2}$, $\hat{\mu}$, R, S, C) at a point $u^{(0)} \in \mathcal{M}$ and we construct the Frobenius structure from the analytic continuation of the corresponding b.v.p. on the covering of $(\mathbf{C}^n\backslash diagonals)/\mathcal{S}_n$ w.r.t. the quotient (11), then there is an equivalence of Frobenius structures between this last manifold and $\mathcal{M}$.

To understanding the *global structure* of a Frobenius manifold we have to study (9) when two or more distinct coordinates u_i, u_j, etc, merge. $\phi_0(u)$, $V(u)$ and (9) are multi-valued meromorphic functions of $u = (u_1, ..., u_n)$ and the branching occurs when u goes around a loop around the set of diagonals $\bigcup_{ij}\{u \in \mathbf{C}^n \mid u_i = u_j, \quad i \neq j\}$. $\phi_0(u)$, $V(u)$ and (9) have singular behavior if $u_i \to u_j$ $(i \neq j)$. We call such behavior *critical behavior*.

The Kontsevich's solution introduced at the beginning has a radius of convergence which might be due to the fact that some coordinates u_i, u_j merge at the boundary of the ball of convergence. We will prove that this is not the case. Rather, there is a singularity in the change of coordinates $u \mapsto t$.

1.2 Intersection Form of a Frobenius Manifold

The deformed flat connection was introduced as a natural structure on a Frobenius manifold and allows to transform the problem of solving the WDVV equations to a problem of isomonodromic deformations. There is a further natural structure on a Frobenius manifold which makes it possible to do the same. It is the intersection form. We need it as a tool to calculate the canonical coordinates later.

There is a natural isomorphism $\varphi : T_tM \to T_t^*M$ induced by $< .,. >$. Namely, let $v \in T_tM$ and define $\varphi(v) :=< v,. >$. This allow us to define the product in T_t^*M as follows: for $v, w \in T_tM$ we define $\varphi(v) \cdot \varphi(w) :=< v \cdot w,. >$. In flat coordinates $t^1, ..., t^n$ the product is

$$dt^\alpha \cdot dt^\beta = c_\gamma^{\alpha\beta}(t)\ dt^\gamma, \qquad c_\gamma^{\alpha\beta}(t) = \eta^{\beta\delta}c_{\delta\gamma}^\alpha(t),$$

(sums over repeated indices are omitted).

Definition: The *intersection form* at $t \in M$ is a bilinear form on T_t^*M defined by

$$(\omega_1, \omega_2) := (\omega_1 \cdot \omega_2)(E(t))$$

where $E(t)$ is the Euler vector field. In coordinates

$$g^{\alpha\beta}(t) := (dt^\alpha, dt^\beta) = E^\gamma(t)c_\gamma^{\alpha\beta}.$$

In the semi-simple case, let $u_1, ..., u_n$ be local canonical coordinates, equal to the distinct eigenvalues of $\mathcal{U}(t)$. From the definitions we have

$$du_i \cdot du_j = \frac{1}{\eta_{ii}}\delta_{ij}du_i, \quad g^{ij}(u) = (du_i, du_j) = \frac{u_i}{\eta_{ii}}\delta_{ij}, \qquad \eta_{ii} = (\phi_0)_{i1}^2$$

Then $g^{ij} - \lambda\eta^{ij} = \frac{u_i - \lambda}{\eta_{ii}}\delta_{ij}$ and

$$\det((g^{ij} - \lambda\eta^{ij})) = \frac{1}{\det((\eta_{ij}))}\ (u_1 - \lambda)(u_2 - \lambda)...(u_n - \lambda).$$

Namely, the roots λ of the above polynomial are the canonical coordinates.

In order to compute $g^{\alpha\beta}$, in the paper we are going to use the following formula. We differentiate twice the expression

$$E^\gamma\partial_\gamma F = (2 - d)F + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C$$

which is the quasi-homogeneity of F up to quadratic terms. By recalling that $E^\gamma = (1-q_\gamma)t^\gamma + r_\gamma$ and that $\partial_\alpha\partial_\beta\partial_\gamma F = c_{\alpha\beta\gamma}$ we obtain

$$g^{\alpha\beta}(t) = (1+d-q_\alpha-q_\beta)\partial^\alpha\partial^\beta F(t) + A^{\alpha\beta} \tag{12}$$

where $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$, $A^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}A_{\gamma\delta}$.

2 Quantum Cohomology of Projective spaces

In this section we introduce the Frobenius manifold called quantum cohomology of the projective space CP^d and we describe its connections to enumerative geometry.

It is possible to introduce a structure of Frobenius algebra on the cohomology $H^*(X, \mathbf{C})$ of a closed oriented manifold X of dimension d such that

$$H^i(X,\mathbf{C}) = 0 \quad \text{for } i \text{ odd}.$$

Then

$$H^*(X,\mathbf{C}) = \otimes_{i=0}^d \, H^{2i}(X,\mathbf{C}).$$

For brevity we omit $\mathbf{C}$ in H. $H^*(X)$ can be realized by classes of closed differential forms. The unit element is a 0-form $e_1 \in H^0(X)$. Let us denote by ω_α a form in $H^{2q_\alpha}(X)$, where $q_1 = 0$, $q_2 = 1$, ..., $q_{d+1} = d$. The product of two forms ω_α, ω_β is defined by the wedge product $\omega_\alpha \wedge \omega_\beta \in H^{2(q_\alpha+q_\beta)}(X)$ and the bilinear form is

$$<\omega_\alpha, \omega_\beta> := \int_X \omega_\alpha \wedge \omega_\beta \neq 0 \iff q_\alpha + q_\beta = d$$

It is not degenerate by Poincaré duality and $q_\alpha + q_{d-\alpha+1} = d$.

Let $X = CP^d$. Let $e_1 = 1 \in H^0(CP^d)$, $e_2 \in H^2(CP^d)$, ..., $e_{d+1} \in H^{2d}(CP^d)$ be a basis in $H^*(CP^d)$. For a suitable normalization we have

$$(\eta_{\alpha\beta}) := (<e_\alpha, e_\beta>) = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

The multiplication is

$$e_\alpha \wedge e_\beta = e_{\alpha+\beta-1}.$$

We observe that it can also be written as

$$e_\alpha \wedge e_\beta = c_{\alpha\beta}^\gamma e_\gamma, \quad \text{sums on } \gamma$$

where

$$\eta_{\alpha\delta} c_{\beta\gamma}^\delta := \frac{\partial^3 F_0(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

$$F_0(t) := \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1}$$

F_0 is the trivial solution of WDVV equations. We can construct a trivial Frobenius manifold whose points are $t := \sum_{\alpha=1}^{d+1} t^\alpha e_\alpha$. It has tangent space $H^*(CP^d)$ at any t. By *quantum cohomology* of CP^d (denoted by $QH^*(CP^d)$) we mean a Frobenius manifold whose structure is specified by

$$F(t) = F_0(t) + \text{ analytic perturbation}$$

This manifold has therefore tangent spaces $T_t QH^*(CP^d) = H^*(CP^d)$, with the same $<.,.>$ as above, but the multiplication is a deformation, depending on t, of the wedge product (this is the origin of the adjective "quantum").

3 The case of CP^2

To start with, we restrict to CP^2. In this case

$$F_0(t) = \frac{1}{2}\left[(t^1)^2 t^3 + t^1 (t^2)^2\right]$$

which generates the product for the basis $e_1 = 1 \in H^0$, $e_2 \in H^2$, $e_3 \in H^4$. The deformation was introduced by Kontsevich [20].

3.1 Kontsevich's solution

The WDVV equations for $n = 3$ variables have solutions

$$F(t_1, t_2, t_3) = F_0(t_1, t_2, t_3) + f(t_2, t_3).$$

$f(t_2, t_3)$ satisfies a differential equation obtained by substituting $F(t)$ into the WDVV equations. Namely:

$$f_{222} f_{233} + f_{333} = (f_{223})^2 \tag{13}$$

with the notation $f_{ijk} := \frac{\partial^3 f}{\partial t_i \partial t_j \partial t_k}$. As for notations, the variables t_j are flat coordinates in the Frobenius manifold associate to F. They should be written with upper indices, but we use the lower for convenience of notation.

Let N_k be the number of rational curves $CP^1 \to CP^2$ of degree k through $3k - 1$ generic points. Kontsevich [20] constructed the solution

$$f(t_2, t_3) = \frac{1}{t_3}\varphi(\tau), \qquad \varphi(\tau) = \sum_{k=1}^{\infty} A_k \tau^k, \qquad \tau = t_3^3\, e^{t_2} \tag{14}$$

where

$$A_k = \frac{N_k}{(3k-1)!}$$

The A_k (or N_k) are called Gromov-Witten invariants of genus zero. We note that this solution has precisely the form of the general solution of the WDVV eqs. for $n = 3$, $d = 2$ and $r_2 = 3$ [9]. If we put $\tau = e^X$ and we define

$$\Phi(X) := \varphi(e^X) = \sum_{k=1}^{\infty} A_k\, e^{kX},$$

we rewrite (13) as follows:

$$-6\Phi + 33\Phi' - 54\Phi'' - (\Phi'')^2 + \Phi'''\,(27 + 2\Phi' - 3\Phi'') = 0 \tag{15}$$

The prime stands for the derivative w.r.t X. If we fix A_1, the above (15) determines the A_k uniquely. Since $N_1 = 1$, we fix

$$A_1 = \frac{1}{2}.$$

Then (15) yields the recurrence relation

$$A_k = \sum_{i=1}^{k-1}\left[\frac{A_i A_{k-i}\; i(k-i)\big((3i-2)(3k-3i-2)(k+2) + 8k - 8\big)}{6(3k-1)(3k-2)(3k-3)}\right] \tag{16}$$

The convergence of (14) was studied by Di Francesco and Itzykson [5]. They proved that

$$A_k = b\; a^k\; k^{-\frac{7}{2}}\;\left(1 + O\left(\frac{1}{k}\right)\right), \qquad k \to \infty$$

and numerically extimated

$$a = 0.138, \qquad b = 6.1 \quad .$$

We remark that the problem of the exact computation of a and b is open. The result implies that $\varphi(\tau)$ converges in a neighborhood of $\tau = 0$ with radius of convergence $\frac{1}{a}$.

We remark that as far as the Gromov-Witten invariants of genus one are concerned, B. Dubrovin and Y. Zhang proved in [13] that their G-function has the same radius of convergence of (1). Moreover, they proved the asymptotic formula for such invariants as conjectuder by Di Francesco–Itzykson. As far as I know, such a result was explained in lectures, but not published.

The proof of [5] is divided in two steps. The first is based on the relation (16), to prove that

$$A_k^{\frac{1}{k}} \to a \text{ for } k \to \infty, \qquad \frac{1}{108} < a < \frac{2}{3}$$

a is real positive because the A_k's are such. It follows that we can rewrite

$$A_k = ba^k \ k^{\omega} \ \left(1 + O\left(\frac{1}{k}\right)\right), \qquad \omega \in \mathbf{R}$$

The above estimate implies that $\varphi(\tau)$ has the radius of convergence $\frac{1}{a}$. The second step is the determination of ω making use of the differential equation (15). Let's write

$$A_k := C_k \ a^k$$

$$\Phi(X) = \sum_{k=1}^{\infty} \ A_k \ e^{kX} = \sum_{k=1}^{\infty} \ C_k \ e^{k(X-X_0)}, \qquad X_0 := \ln \frac{1}{a}$$

The inequality $\frac{1}{108} < a < \frac{2}{3}$ implies that $X_0 > 0$. The series converges at least for $\Re X < X_0$. To determine ω we divide $\Phi(X)$ into a regular part at X_0 and a singular one. Namely

$$\Phi(X) = \sum_{k=0}^{\infty} d_k (X - X_0)^k + (X - X_0)^{\gamma} \ \sum_{k=0}^{\infty} e_k (X - X_0)^k, \qquad \gamma > 0, \quad \gamma \notin \mathbf{N},$$

d_k and e_k are coefficients. By substituting into (15) we see that the equation is satisfied only if $\gamma = \frac{5}{2}$. Namely:

$$\Phi(X) = d_0 + d_1(X - X_0) + d_2(X - X_0)^2 + e_0(X - X0)^{\frac{5}{2}} + ...$$

This implies that $\Phi(X)$, $\Phi'(X)$ and $\Phi''(X)$ exist at X_0 but $\Phi'''(X)$ diverges like

$$\Phi'''(X) \asymp \frac{1}{\sqrt{X - X_0}}, \qquad X \to X_0 \tag{17}$$

On the other hand $\Phi'''(X)$ behaves like the series

$$\sum_{k=1}^{\infty} \ b \ k^{\omega+3} \ e^{k(X-X_0)}, \qquad \Re(X - X_0) < 0$$

Suppose $X \in \mathbf{R}$, $X < X_0$. Let us put $\Delta := X - X_0 < 0$. The above series is

$$\frac{b}{|\Delta|^{3+\omega}} \ \sum_{k=1}^{\infty} (|\Delta| k)^{3+\omega} e^{-|\Delta| k} \sim \frac{b}{|\Delta|^{3+\omega}} \ \int_0^{\infty} dx \ x^{3+\omega} e^{-x}$$

It follows from (17) that this must diverge like $\Delta^{-\frac{1}{2}}$, and thus $\omega = -\frac{7}{2}$ (the integral remains finite).

As a consequence of (15) and of the divergence of $\Phi'''(X)$

$$27 + 2\Phi'(X_0) - 3\Phi''(X_0) = 0$$

4 The case of CP^d

The case $d = 1$ is trivial, the deformation being:

$$F(t) = \frac{1}{2} t_1^2 t_2 + e^{t_2}$$

For any $d \geq 2$, the deformation is given by the following solution of the WDVV equations [20] [22]:

$$F(t) = F_0(t) + \sum_{k=1}^{\infty} \left[\sum_{n=2}^{\infty} \tilde{\sum_{\alpha_1,...,\alpha_n}} \frac{N_k(\alpha_1,...,\alpha_n)}{n!} \; t_{\alpha_1}...t_{\alpha_n} \right] e^{kt_2}$$

where

$$\tilde{\sum_{\alpha_1,\dots,\alpha_n}} := \sum_{\alpha_1+...+\alpha_n=2n+d(k+1)+k-3}$$

Here $N_k(\alpha_1,...,\alpha_n)$ is the number of rational curves $CP^1 \to CP^d$ of degree k through n projective subspaces of codimensions $\alpha_1 - 1,...,\alpha_n - 1 \geq 2$ in general position. In particular, there is one line through two points, then

$$N_1(d+1, d+1) = 1$$

Note that in Kontsevich solution $N_k = N_k(d+1, d+1)$.

In flat coordinates the *Euler vector field* is

$$E = \sum_{\alpha \neq 2} (1 - q_\alpha) t^\alpha \frac{\partial}{\partial t^\alpha} + \; k \frac{\partial}{\partial t^2}$$

$$q_1 = 0, \; q_2 = 1, \; q_3 = 2, \; ..., \; q_k = k - 1$$

and

$$\hat{\mu} = \mathrm{diag}(\mu_1, ..., \mu_k) = \mathrm{diag}(-\frac{d}{2}, -\frac{d-2}{2}, ..., \frac{d-2}{2}, \frac{d}{2}), \qquad \mu_\alpha = q_\alpha - \frac{d}{2}$$

5 Nature of the singular point X_0

We are now ready to formulate the problem of the paper. We need to investigate the nature of the singularity X_0, namely whether it corresponds to the fact that two canonical coordinates u_1, u_2, u_3 merge. Actually, we pointed out that the structure of the semi-simple manifold may become singular in such points because the solutions of the boundary value problem are meromorphic on the universal covering of $\mathbf{C}^n \backslash$diagonals and are multi valued if $u_i - u_j$ $(i \neq j)$ goes around a loop around zero. We will verify that actually u_i, u_j do not merge, but the change of coordinates $u \mapsto t$ is singular at X_0. In this section we restore the upper indices for the flat coordinates t^α.

The canonical coordinates can be computed from the intersection form. We recall that the flat metric is

$$\eta = (\eta^{\alpha\beta}) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The intersection form is given by the formula (12):

$$g^{\alpha\beta} = (d + 1 - q_\alpha - q_\beta) \; \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\mu\partial_\nu F + A^{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3,$$

where $d = 2$ and the *charges* are $q_1 = 0$, $q_2 = 1$, $q_3 = 2$. The matrix $A^{\alpha\beta}$ appears in the action of the Euler vector field

$$E := t^1\partial_1 + 3\partial_2 - t^3\partial_3$$

on $F(t^1, t^2, t^3)$:

$$E(F)(t^1, t^2, t^3) = (3 - d)F(t^1, t^2, t^3) + A_{\mu\nu}t^\mu t^\nu \equiv F(t^1, t^2, t^3) + 3t^1t^2$$

Thus

$$(A^{\alpha\beta}) = (\eta^{\alpha\mu}\eta^{\beta\nu}A_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

After the above preliminaries, we are able to compute the intersection form:

$$(g^{\alpha\beta}) = \begin{pmatrix} \frac{3}{[t^3]^3}[2\Phi - 9\Phi' + 9\Phi''] & \frac{2}{[t^3]^2}[3\Phi'' - \Phi'] & t^1 \\ \frac{2}{[t^3]^2}[3\Phi'' - \Phi'] & t^1 + \frac{1}{t^3}\Phi'' & 3 \\ t^1 & 3 & -t^3 \end{pmatrix}$$

The canonical coordinates are roots of

$$\det((g^{\alpha\beta} - u\eta) = 0$$

This is the polynomial

$$u^3 - \left(3t^1 + \frac{1}{t^3}\Phi''\right)\ u^2 - \left(-3[t^1]^2 - 2\frac{t^1}{t^3}\Phi'' + \frac{1}{[t^3]^2}(9\Phi'' + 15\Phi' - 6\Phi)\right)\ u + P(t,\Phi)$$

where

$$P(t,\Phi) = \frac{1}{[t^3]^3}\left(-9t^1t^3\Phi'' + 243\Phi'' - 243\Phi' + 6\Phi\Phi' + \right.$$

$$\left. -9(\Phi'')^2 + 6t^1t^3\Phi + [t^1]^2[t^3]^2\Phi'' - 3\Phi'\Phi'' + [t^1]^3[t^3]^3 - 4(\Phi')^2 + 54\Phi - 15t^1t^3\Phi'\right)$$

It follows that

$$u_i(t^1,t^3,X) = t^1 + \frac{1}{t^3}\mathcal{V}_i(X)$$

$\mathcal{V}_i(X)$ depends on X through $\Phi(X)$ and derivatives. We also observe that

$$u_1 + u_2 + u_3 = 3t^1 + \frac{1}{t^3}\Phi''(X)$$

As a first step, we verify numerically that $u_i \neq u_j$ for $i \neq j$ at $X = X_0$. In order to do this we need to compute $\Phi(X_0)$, $\Phi'(X_0)$, $\Phi''(X_0)$ in the following approximation

$$\Phi(X_0) \cong \sum_{k=1}^{N} A_k\ \frac{1}{a^k}, \quad \Phi'(X_0) \cong \sum_{k=1}^{N}\ k\ A_k\ \frac{1}{a^k}, \qquad \Phi''(X_0) \cong \sum_{k=1}^{N}\ k^2\ A_k\ \frac{1}{a^k},$$

We fixed $N = 1000$ and we computed the A_k, $k = 1, 2, ..., 1000$ exactly using the relation (16). Then we computed a and b by the least squares method. For large k, say for $k \geq N_0$, we assumed that

$$A_k \cong b a^k k^{-\frac{7}{2}} \tag{18}$$

which implies

$$\ln(A_k\ k^{\frac{7}{2}}) \cong (\ln a)\ k + \ln b$$

The corrections to this law are $O\left(\frac{1}{k}\right)$. This is the line to fit the data $k^{\frac{7}{2}}A_k$. Let

$$\bar{y} := \frac{1}{N - N_0 + 1}\sum_{k=N_0}^{N}\ \ln(A_k\ k^{\frac{7}{2}}), \qquad \bar{k} := \frac{1}{N - N_0 + 1}\sum_{N_0}^{N}\ k.$$

By the least squares method

$$\ln a = \frac{\sum_{k=N_0}^{N}\ (k - \bar{k})(\ln(A_k\ k^{\frac{7}{2}}) - \bar{y})}{\sum_{k=N_0}^{N}\ (k - \bar{k})^2}, \quad \text{with error } \left(\frac{1}{\bar{k}^2}\right)$$

$$\ln b = \bar{y} - (\ln a)\ \bar{k}, \ \text{ with error } \left(\frac{1}{\bar{k}}\right)$$

For $N = 1000$, A_{1000} is of the order 10^{-840}. In our computation we set the accuracy to 890 digits. Here is the results, for three choices of N_0. The result should improve as N_0 increases, since the approximation (18) becomes better.

$$N_0 = 500, \quad a = 0.138009444..., \quad b = 6.02651...$$

$$N_0 = 700, \quad a = 0.138009418..., \quad b = 6.03047...$$

$$N_0 = 900, \quad a = 0.138009415..., \quad b = 6.03062...$$

It follows that (for $N_0 = 900$)

$$\Phi(X_0) = 4.268908...\ , \qquad \Phi'(X_0) = 5.408...\ , \qquad \Phi''(X_0) = 12.25...$$

With these values we find

$$27 + 2\Phi'(X_0) - 3\Phi''(X_0) = 1.07...,$$

But the above should vanish! The reason why this does not happen is that $\Phi''(X_0) = \sum_{k=1}^{N} k^2 A_k \frac{1}{a^k}$ converges slowly. To obtain a better approximation we compute it numerically as

$$\Phi''(X_0) = \frac{1}{3}(27 + 2\Phi'(X_0)) = \frac{1}{3}(27 + 2\sum_{k=1}^{N} k A_k \frac{1}{a^k}) = 12.60...$$

Substituting into $g^{\alpha\beta}$ and setting $t^1 = t^3 = 1$ we find

$$u_1 \approx 22.25... , \quad u_2 \approx -(3.5...) - (2.29...)i , \quad u_3 = \bar{u}_2,$$

where $i = \sqrt{-1}$ and the bar means complex conjugation. Thus, with a sufficient accuracy, we have verified that $u_i \neq u_j$ for $i \neq j$.

We now prove that the singularity is a singularity for the change of coordinates

$$(u_1, u_2, u_3) \mapsto (t^1, t^2, t^3)$$

We recall that

$$\frac{\partial u_1}{\partial t^\alpha} = \frac{(\phi_0)_{i\alpha}}{(\phi_0)_{i1}}$$

This may become infinite if $(\phi_0)_{i1} = 0$ for some i. In our case

$$u_1 + u_2 + u_3 = 3t^1 + \frac{1}{t^3}\Phi(X)'', \quad \frac{\partial X}{\partial t^1} = 0, \quad \frac{\partial X}{\partial t^2} = 1, \quad \frac{\partial X}{\partial t^3} = \frac{3}{t^3}$$

and

$$\frac{\partial}{\partial t^1}(u_1 + u_2 + u_3) = 3,$$

$$\frac{\partial}{\partial t^2}(u_1 + u_2 + u_3) = \frac{1}{t^3}\Phi(X)''',$$

$$\frac{\partial}{\partial t^3}(u_1 + u_2 + u_3) = -\frac{1}{[t^3]^2}\Phi(X)'' + \frac{3}{[t^3]^2}\Phi(X)'''.$$

The above proves that the change of coordinates is singular because both $\frac{\partial}{\partial t^2}(u_1+u_2+u_3)$ and $\frac{\partial}{\partial t^3}(u_1 + u_2 + u_3)$ behave like $\Phi(X)''' \asymp \frac{1}{\sqrt{X-X_0}}$ for $X \to X_0$.

Acknowledgments.

I thank A. Its and P. Bleher for suggesting me to try the computations of this paper and for discussions. I thank B. Dubrovin for introducing me to the theory of Frobenius manifolds and for discussing together the problem of this paper. The author is supported by a fellowship of the Japan Society for the Promotion of Science (JSPS).

References

[1] D.V.Anosov - A.A.Bolibruch: *The Riemann-Hilbert Problem* , Publication from the Steklov Institute of Mathematics, (1994).

[2] W.Balser, W.B.Jurkat, D.A.Lutz: *Birkhoff Invariants and Stokes' Multipliers for Meromorphic Linear Differential Equations*, Journal Math. Analysis and Applications, **71**, (1979), 48-94.

[3] W.Balser, W.B.Jurkat, D.A.Lutz: *On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities*, SIAM J. Math. Anal., **12**, (1981), 691-721.

[4] J.S.Birman: *Braids, Links, and Mapping Class Groups*, Annals of Mathematics Studies, **82**, Princeton Univ. Press (1975).

[5] Di Francesco P., Itzykson C.: *Quantum Intersection Rings*, in The Moduli Space of Curves edited by R.Dijkgraaf, C.Faber, G.B.M. van der Geer, (1995).

[6] Dijkgraaf R.,Verlinde E., Verlinde H. Nucl. Phys B, **352**, (1991), 59.

[7] B.Dubrovin: *Integrable Systems in Topological Field Theory*, Nucl. Phys B, **379**, (1992), 627-689.

[8] B.Dubrovin: *Geometry and Itegrability of Topological-Antitopological Fusion*, Comm.Math.Phys, **152**, (1993), 539-564.

[9] B.Dubrovin: *Geometry of 2D topological field theories*, Lecture Notes in Math, **1620**, (1996), 120-348.

[10] B.Dubrovin: *Painlevé trascendents in two-dimensional topological field theory*, in "The Painlevé Property, One Century later" edited by R.Conte, Springer (1999).

[11] B.Dubrovin: *Geometry and Analytic Theory of Frobenius Manifolds*, math.AG/9807034, (1998).

[12] B.Dubrovin: *Differential geometry on the space of orbits of a Coxeter group*, math.AG/9807034, (1998).

[13] B.Dubrovin, Y.Zhang: *Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation*, Comm. Math. Phys. 198 (1998) 311-361.

[14] Guzzetti D: *Stokes matrices and monodromy for the quantum cohomology of projective spaces* Comm.Math.Phys **207**, (1999), 341-383. Also see the preprint math/9904099.

[15] Guzzetti D: *Inverse Problem and Monodromy Data for 3-dimensi onal Frobenius Manifolds* , Journal of Mathematical Physics, Analysis and Geometry, **4**, (2001), 254-291.

[16] A.R.Its - V.Y.Novokshenov: *The isomonodromic deformation method in the theory of Painleve equations*, Lecture Notes in Math, **1191**, (1986).

[17] M.Jimbo, T.Miwa, K.Ueno: *Monodromy Preserving Deformations of Linear Ordinary Differential Equations with Rational Coefficients (I)*, Physica , **D 2** , (1981), 306

[18] M.Jimbo, T.Miwa : *Monodromy Preserving Deformations of Linear Ordinary Differential Equations with Rational Coefficients (II)*, Physica , **D 2** , (1981), 407-448

[19] M.Jimbo, T.Miwa : *Monodromy Preserving Deformations of Linear Ordinary Differential Equations with Rational Coefficients (III)*, Physica , **D 4** , (1981), 26

[20] Kontsevich M., Manin Y.I. : *Gromov-Witten classes, Quantum Cohomology and Enumerative Geometry*, Comm.Math.Phys, **164**, (1994), 525-562

[21] Malgrange B.: *Équations différentielles à coefficientes polynomiaux*, Birkhauser (1991).

[22] Manin V.I. : *Frobenius Manifolds, Quantum Cohomology and Moduli Spaces*, Max Planck Institut fur Mathematik. Bonn. Germany, (1998).

[23] Miwa T.: *Painlevé property of monodromy preserving equations and the analyticity of τ-functions*, Publ. RIMS **17** (1981), 703-721

[24] Ruan Y., Tian G. : *A Mathematical Theory of Quantum Cohomology*, Math.Res.Lett., **1**, (1994), 269-278

[25] Saito K.: Preprint RIMS-288 (1979) and Publ. RIMS **19** (1983), 1231-1264.

[26] Saito K., Yano T., Sekeguchi J.: Comm. in Algebra, **8(4)**, (1980), 373-408.

[27] M.Sato, T.Miwa, M.Jimbo: *Holonomic Quantum Fields. II –The Riemann-Hilbert Problem –*, Publ. RIMS. Kyioto. Univ, **15**, (1979), 201-278.

[28] Witten E.: Nucl. Phys B, **340**, (1990), 281-332.