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A General Two-Sided Matching Market with Discrete Concave Utility Functions †

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Abstract

In the theory of two-sided matching markets there are two standard models called the stable marriage model, due to Gale and Shapley, and the assignment model, due to Shapley and Shubik. Recently, Eriksson and Karlander have introduced a hybrid model of these two and Sotomayor also considered the hybrid model with full generality. In this paper, we propose a common generalization of these models by utilizing a framework of discrete convex analysis introduced by Murota, and verify the existence of a stable solution in our general model.

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1. Introduction

In the theory of two-sided matching markets, there are two standard models called the stable marriage model, due to Gale and Shapley [8], and the assignment model, due to Shapley and Shubik [21]. Great difference between the stable marriage model and the assignment model is that the former does not include money or transferable utilities and the latter permits payments (see [20] for details of these models).

In the original stable marriage model, there are two sets of n men and n women, and each person arbitrarily gives a strict preference order on persons of the opposite gender. A matching is a set of n disjoint pairs of men and women, and is called stable if there is no pair whose members prefer each other to their partners in the matching. Gale and Shapley [8] gave a constructive proof of existence of a stable matching in 1962. Since the advent of their paper a lot of variations and extensions have been proposed in the literature. Recently, a remarkable extension has been made by Fleiner [4] (also see [5]). Fleiner [4] extended the stable marriage model to the framework of matroids, showed existence of a stable solution, and examined a lattice structure and a polyhedral characterization of stable solutions in his matroidal model. Fleiner [5] also gave a strong framework to show existence of a stable solution and a lattice structure of stable solutions by utilizing the Knaster-Tarski fixed point theorem. While in the model of Fleiner [4] preference of each person is described by a linear utility function on a matroidal domain, Eguchi and Fujishige [2] extended the matroidal model [4] to the framework of discrete convex analysis which was recently developed by Murota [12, 13, 15, 16] as a unified framework of discrete optimization. In their model, each agent can express his/her preference by a discrete concave function, called an M$^\natural$-concave function (see Section 2 for M$^\natural$-concavity).

In the assignment model, if man i and woman j form a partnership, then they gain an income c_{ij} and divide it into q_i and r_j such that $q_i + r_j = c_{ij}$ and $q_i, r_j \geq 0$. An outcome $(q, r; X)$ consisting of payoff vectors q and r and a matching X formed by partnerships is called stable if $q_i + r_j \geq c_{ij}$ for all man-woman pairs (i, j). Shapley and Shubik [21] showed existence of a stable outcome by using linear programming duality. Many extensions of the assignment model have also been proposed. Sotomayor [22] showed existence of stable outcomes in a model in which each person is permitted to form several partnerships with persons of the opposite set without repetition of the same pair. Sotomayor [24] also verified nonemptiness of the core in an extended model in which each firm can employ several units of labor-time, each worker can supply several units of time, each pair can earn a specified amount of money per unit time, and each pair is permitted to form partnerships with multiple units. Kelso and Crawford [11] introduced a many-to-one labor market model in which a utility function of each firm has the gross substitutability and a utility function of each worker is strictly increasing (not necessarily linear) in a salary. Danilov, Koshevoy, and Murota [1] provided the first model based on discrete convex analysis.

On the other hand, there has been made research toward unifying the stable marriage model and the assignment model. Kaneko [10] gave a general model including two kinds of models by means of characteristic function and proved nonemptiness of the core. Eriksson and Karlander [3] proposed a hybrid model of the stable marriage model and the assignment model, and verified the existence of a stable matching. In this model, the agents are partitioned into two categories, called flexible agents and rigid agents. The rigid agents do not get side payments, that is, they behave like ones in the stable marriage model, while the flexible agents behave like ones in the assignment model. Thus, it is assumed that there is no side payment between rigid agents and between a flexible agent and a rigid agent. Sotomayor [23] also made further investigation of the hybrid model of Eriksson and Karlander with full generality.

In this paper, we provide a general two-sided model including many of the above models as special cases (see Section 3) by following the ideas of Eguchi and Fujishige and of Eriksson and Karlander. Our model has the following features (also see a model in Remark 1 in Section 2):

- the preference of agents on each side over the agents on the other side is expressed by an $\mathrm{M}^\natural$-concave function,
- each agent is permitted to form partnerships with many agents on the opposite side,
- each pair is permitted to form multiple partnerships,
- the set of pairs is partitioned into a set of flexible pairs and that of rigid pairs.

Our main theorem claims that there always exists a stable solution in this general model.

This paper is organized as follows. Section 2 explains $\mathrm{M}^\natural$-concavity together with its properties and describes our model based on discrete convex analysis. Section 3 gives several existing models that are special cases of our model. In Section 4 we propose an algorithm for finding a stable solution and prove its correctness, which shows our main theorem on existence of a stable solution in our general model.

2. A General Model

2.1. Preliminaries

We first introduce an M$^\natural$-concave function. Let V be a nonempty finite set, and $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integers and reals, respectively. We define the positive support and negative support of $x = (x(v) : v \in V) \in \mathbf{Z}^V$ by

$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\}, \quad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\}. \tag{2.1}$$

For any $x, y \in \mathbf{Z}^V$, the vectors $x \wedge y$ and $x \vee y$ in $\mathbf{Z}^V$ are defined by

$$x \wedge y(v) = \min\{x(v), y(v)\}, \quad x \vee y(v) = \max\{x(v), y(v)\} \quad (v \in V). \tag{2.2}$$

For each $S \subseteq V$, we denote by χ_S the characteristic vector of S defined by

$$\chi_S(v) = \begin{cases} 1 & (v \in S) \\ 0 & (v \in V \setminus S) \end{cases} \tag{2.3}$$

and write simply χ_u instead of $\chi_{\{u\}}$ for each $u \in V$. For a vector $p \in \mathbf{R}^V$ and a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, we define functions $\langle p, x\rangle$ and $f[p](x)$ in $x \in \mathbf{Z}^V$ by

$$\langle p, x\rangle = \sum_{v \in V} p(v)x(v) \quad \text{and} \quad f[p](x) = f(x) + \langle p, x\rangle \qquad (x \in \mathbf{Z}^V), \tag{2.4}$$

where $p(v)$ and $x(v)$ denote the v-th components of p and x, respectively. Also define arg max, the set of maximizers, of f on $U \subseteq \mathbf{Z}^V$ and the *effective domain* of f by

$$\arg\max\{f \mid y \in U\} = \{x \in U \mid \forall y \in U : \ f(x) \geq f(y)\}, \tag{2.5}$$
$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid \ f(x) > -\infty\}. \tag{2.6}$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-concave* [17] if it satisfies

(−M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v), \tag{2.7}$$

where χ_0 is a zero vector.

Two simple examples of M$^\natural$-concave functions are given as follows.

Example 1: Let $\mathcal{I}$ be the family of independent sets of a matroid on V and $w \in \mathbf{R}^V$. Then, a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ defined by

$$f(x) = \begin{cases} \sum_{v \in I} w(v) & (\text{if } x = \chi_I \text{ for } I \in \mathcal{I}) \\ -\infty & (\text{otherwise}) \end{cases} \qquad (x \in \mathbf{Z}^V) \tag{2.8}$$

is M$^\natural$-concave.

Example 2: Let $z \in \mathbf{Z}^V$ be a positive vector, and for each $v \in V$ let $f_v : \mathbf{R} \to \mathbf{R}$ be a concave function. Then, a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ defined by

$$f(x) = \begin{cases} \sum_{v \in V} f_v(x(v)) & (\text{if } \mathbf{0} \leq x \leq z) \\ -\infty & (\text{otherwise}) \end{cases} \qquad (x \in \mathbf{Z}^V) \tag{2.9}$$

is M$^\natural$-concave.

An M$^\natural$-concave function has nice features as a utility function from the point of view of mathematical economics. A utility function is usually assumed to be concave in mathematical economics. For any M$^\natural$-concave function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, there exists a concave function $\bar{f} : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ with $\bar{f}(x) = f(x)$ for any $x \in \mathbf{Z}^V$ [12], that is, any M$^\natural$-concave function on $\mathbf{Z}^V$ has a concave extension on $\mathbf{R}^V$. A utility function usually has decreasing marginal returns, which is equivalent to submodularity in the discrete case. This is also the case for M$^\natural$-concave functions [18], i.e., any M$^\natural$-concave function f on $\mathbf{Z}^V$ satisfies

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (x, y \in \mathrm{dom}\, f). \tag{2.10}$$

We next consider natural generalizations of the gross substitutes condition and the single improvement condition which were originally proposed for set functions by Kelso and Crawford [11], and Gul and Stacchetti [9], respectively.

(−M$^\natural$-GS) For (p_0, p), $(q_0, q) \in \mathbf{R}^{\{0\} \cup V}$ and $x \in \mathrm{dom}\, f$ such that $(p_0, p) \leq (q_0, q)$, $x \in \arg\max f[-p + p_0\mathbf{1}]$, and $\arg\max f[-q + q_0\mathbf{1}] \neq \emptyset$, there exists $y \in \arg\max f[-q + q_0\mathbf{1}]$ such that

$$y(v) \geq x(v) \qquad \text{if} \quad p(v) = q(v), \tag{2.11}$$

$$\sum_{v \in V} y(v) \leq \sum_{v \in V} x(v) \qquad \text{if} \quad p_0 = q_0, \tag{2.12}$$

(−M$^\natural$-SI) For $p \in \mathbf{R}^V$ and $x, y \in \mathrm{dom}\, f$ with $f[-p](x) < f[-p](y)$,

$$f[-p](x) < \max_{u \in \mathrm{supp}^+(x-y) \cup \{0\}} \ \max_{v \in \mathrm{supp}^-(x-y) \cup \{0\}} f[-p](x - \chi_u + \chi_v). \tag{2.13}$$

Here we assume that V and p denote the set of indivisible commodities and prices of units of commodities and that $f(x)$ represents a utility of a consumer for consumption x of commodities. Then the above conditions are interpreted as below. (−M$^\natural$-GS) says that when prices increase ($p \leq q$ and $p_0 = q_0$), the consumer wants a consumption such that the numbers of the commodities whose prices remain the same do not decrease and the total number of commodities does not increase. (−M$^\natural$-GS) also says that when all prices decrease by the same amount ($p = q$ and $p_0 < q_0$), the consumer wants at least the same number of each commodity.

($-\mathrm{M}^\natural$-SI) guarantees that the consumer can bring consumption x nearer to any better consumption y by changing the consumption for at most two commodities. The equivalence between the gross substitutes condition and the single improvement condition for set functions was first pointed out by Gul and Stacchetti [9], and the equivalence between the single improvement condition and $\mathrm{M}^\natural$-concavity for set functions was by Fujishige and Yang [7]. Murota and Tamura [19] showed that an $\mathrm{M}^\natural$-concave function satisfies ($-\mathrm{M}^\natural$-GS) and ($-\mathrm{M}^\natural$-SI), and conversely, $\mathrm{M}^\natural$-concavity is characterized by these conditions under a certain natural assumption. $\mathrm{M}^\natural$-concavity also implies substitutability (see Lemma 4.2).

2.2. Model Description and the Main Theorem

Now we introduce our model that generalizes two-sided matching markets. Let M and W denote two disjoint sets of agents and V be a finite set. In our model, utilities of M and W over V are described by $\mathrm{M}^\natural$-concave functions $f_M, f_W : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, respectively. In the exemplary models described in Section 3 M and W denote disjoint sets of agents, and we have $V = M \times W$, where f_M and f_W can be regarded as aggregations of utilities of M-agents and W-agents in these models, respectively (see Remark 1 given below). We also assume that V is partitioned into two subsets F (the set of *flexible elements*) and R (the set of *rigid elements*). (Recall that in the hybrid model due to Eriksson and Karlander [3] M and W are, respectively, partitioned into $\{M_F, M_R\}$ and $\{W_F, W_R\}$, and we have $F = M_F \times W_F$ and $R = V \setminus F$, where $V = M \times W$.) Furthermore, we assume that f_M and f_W satisfy the following condition:

(A) Effective domains $\mathrm{dom}\, f_M$ and $\mathrm{dom}\, f_W$ are bounded and hereditary, and have a common minimum point. Without loss of generality, we have $\mathbf{0} \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ and $\mathbf{0} \leq x_1 \leq x_2 \in \mathrm{dom}\, f_M$ (respectively $\mathrm{dom}\, f_W$) implies $x_1 \in \mathrm{dom}\, f_M$ (respectively $\mathrm{dom}\, f_W$).

We say that $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is an *$f_M f_W$-stable solution with respect to (F, R)* or simply *$f_M f_W$-stable solution* if there exist $p \in \mathbf{R}^V$, disjoint subsets R_M and R_W of R, $z_M \in \mathbf{Z}^{R_M}$, and $z_W \in \mathbf{Z}^{R_W}$ such that

$$p|_R = \mathbf{0}, \tag{2.14}$$

$$x \in \arg\max\{f_M[+p](y) \mid y \in \mathbf{Z}^V,\ y|_{R_M} \leq z_M\}, \tag{2.15}$$

$$x \in \arg\max\{f_W[-p](y) \mid y \in \mathbf{Z}^V,\ y|_{R_W} \leq z_W\}, \tag{2.16}$$

where $p|_R$ denotes the restriction of p on R. In the present model, p means side payments, and hence, (2.14) is equivalent to that there is no side payment for each rigid element. Since $\mathrm{dom}\, f_M$ and $\mathrm{dom}\, f_W$ are bounded due to Assumption (A), we see that $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is an $f_M f_W$-stable solution with respect to (F, R) if and only if there exist $p \in \mathbf{R}^V$ and $z_M, z_W \in \mathbf{Z}^R$ satisfying (2.14) and the following (2.17)$\sim$(2.19) for a sufficiently large vector $z \in \mathbf{Z}^V$:

$$z|_R = z_M \vee z_W, \tag{2.17}$$

$$x \in \arg\max\{f_M[+p](y) \mid y \in \mathbf{Z}^V,\ y|_R \le z_M\}, \tag{2.18}$$

$$x \in \arg\max\{f_W[-p](y) \mid y \in \mathbf{Z}^V,\ y|_R \le z_W\}. \tag{2.19}$$

In the sequel we will use (2.17)$\sim$(2.19) instead of (2.15) and (2.16).

Our main result claims nonemptiness of the set of $f_M f_W$-stable solutions of our model.

Theorem 2.1 (Main Theorem): *For any* M$^\natural$*-concave functions* $f_M, f_W : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *satisfying* (A) *and for any partition* (F, R) *of* V*, there always exists an* $f_M f_W$*-stable solution with respect to* (F, R).

A proof of the main theorem will be given in Section 4.

Remark 1: In our model given above each of M and W is regarded as a single aggregate agent but it can be interpreted as a set of agents as follows. Let $M = \{1, \cdots, m\}$, $W = \{1, \cdots, w\}$, and $V = M \times W$. Also define $V_i = \{i\} \times W$ $(i \in M)$ and $V_j = M \times \{j\}$ $(j \in W)$. Suppose that each agent $i \in M$ has an M$^\natural$-concave utility function $f_i : \mathbf{Z}^{V_i} \to \mathbf{R} \cup \{-\infty\}$ on V_i and that each agent $j \in W$ has an M$^\natural$-concave utility function $f_j : \mathbf{Z}^{V_j} \to \mathbf{R} \cup \{-\infty\}$ on V_j. Aggregations $f_M(x) = \sum_{i \in M} f_i(x|_{V_i})$ and $f_W(x) = \sum_{j \in W} f_j(x|_{V_j})$ $(x \in \mathbf{Z}^V)$ are also M$^\natural$-concave. Moreover, V can be arbitrarily partitioned into two sets of flexible pairs and rigid pairs of M-agents and W-agents. It should be noted that this modified model is equivalent to our original model. □

Remark 2: When M and W are, respectively, a set of workers and a set of firms, p expresses salaries from firms to workers, and hence, it should be nonnegative. Our model, however, does not impose such a condition, since the nonnegativity of p is a property that should be derived from an individual problem setting. For example, suppose that $f_W(x)$ denotes the total income of the firms obtained by assignment x between workers and firms, and that $\mathrm{dom}\, f_M$ is the set of assignments acceptable for workers and f_M is identically zero on $\mathrm{dom}\, f_M$. Then, for any $f_M f_W$-stable solution x and for a flexible element v with $x(v) > 0$ we have $p(v) \ge 0$ because $f_M[+p](x) \ge f_M[+p](x - \chi_v)$ and $f_M(x) = f_M(x - \chi_v) = 0$. □

Remark 3: In the case when $V = F$, $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is an $f_M f_W$-stable solution if there exists $p \in \mathbf{R}^V$ such that

$$f_M[+p](x) = \max\{f_M[+p](y) \mid y \in \mathbf{Z}^V\}, \tag{2.20}$$

$$f_W[-p](x) = \max\{f_W[-p](y) \mid y \in \mathbf{Z}^V\}. \tag{2.21}$$

It is a direct consequence of the following theorem that the set of all $f_M f_W$-stable solutions coincides with that of all maximizers of $f_M + f_W$.

Theorem 2.2 ([12]): *For $M^\natural$-concave functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ and a point $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$, we have*

$$f_1(x^*) + f_2(x^*) \geq f_1(x) + f_2(x) \quad (\forall x \in \mathbf{Z}^V) \tag{2.22}$$

if and only if there exists $p^* \in \mathbf{R}^V$ such that

$$f_1[+p^*](x^*) \quad \geq \quad f_1[+p^*](x) \quad (\forall x \in \mathbf{Z}^V), \tag{2.23}$$

$$f_2[-p^*](x^*) \quad \geq \quad f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V), \tag{2.24}$$

and furthermore, we have

$$\arg\max(f_1 + f_2) = \arg\max(f_1[+p^*]) \cap \arg\max(f_2[-p^*]) \tag{2.25}$$

for such p^*.

Since (A) guarantees that dom $(f_M + f_W)$ is nonempty and bounded, $f_M + f_W$ has a maximizer, which implies the existence of an $f_M f_W$-stable solution with respect to $(V, \emptyset)$. We also give an algorithm for finding an $f_M f_W$-stable solution with respect to $(V, \emptyset)$ in Section 4.2. □

3. Existing Special Models

In this section we explain some existing models that are special cases of our model. In these models there are two disjoint sets of agents $M = \{1, \cdots, m\}$ and $W = \{1, \cdots, w\}$. The pairs of agents in M and W may be recognized as men/women, workers/firms, and so on. We denote by V the set of all pairs of agents of M and W, i.e., $V = M \times W$. For each pair $(i, j) \in V$, a pair (a_{ij}, b_{ij}) is given, where a_{ij} and b_{ij} can be interpreted as utilities (or profits) of i and j, respectively, provided that they are paired. Here, we assume that $a_{ij} \geq 0$ if j is *acceptable* to i, otherwise $a_{ij} = -\infty$, and $b_{ij} \geq 0$ if i is *acceptable* to j, otherwise $b_{ij} = -\infty$.

3.1. The Stable Marriage Model and Its Extensions

Even if there are several variations of stable marriage model, we explain one of comprehensive variations. In this model each agent ranks the agents on the opposite side, where unacceptability and indifference are allowed. In our context, agent $i \in M$ *prefers* j_1 to j_2 if $a_{ij_1} > a_{ij_2}$, and j_1 and j_2 are *indifferent* for agent i if $a_{ij_1} = a_{ij_2}$ (similarly, preferences of each $j \in W$ are defined from b_{ij}'s). The model deals with the stability of matchings, where a matching is a subset of V such that every agent appears at most once in the subset. Given a matching X, $i \in M$ (resp. $j \in W$) is called *unmatched in* X if there exists no $j \in W$ (resp. $i \in M$) such that $(i, j) \in X$. A pair $(i, j) \notin X$ is said to be a *blocking pair* for X if i and j prefer each other to their partners or being alone in X. A matching X is called *stable* if each pair (i, j) in X is acceptable for i and j, and if there is no blocking pair for X. The stability of matchings is interpreted as follows. Agent $i \in M$ is assigned value q_i and $j \in W$ is assigned r_j. A matching X is stable if and only if

(m1) $q_i = a_{ij} > -\infty$ and $r_j = b_{ij} > -\infty$ for any $(i, j) \in X$,

(m2) $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X,

(m3) $q_i \geq a_{ij}$ or $r_j \geq b_{ij}$ for any $(i, j) \in V$.

It is well-known that any instance of the above model has a stable matching, originally proved by Gale and Shapley [8].

Recently, Fleiner [4] has generalized the above model to matroids. A triple $\mathcal{M} = (V, \mathcal{I}, >)$ is called an *ordered matroid*, if $(V, \mathcal{I})$ is a matroid on ground set V with family $\mathcal{I}$ of independent sets and $>$ is a linear order on V. A subset X of V *dominates* element $v \in V$ if $v \in X$ or there exists an independent set $Y \subseteq X$ such that $\{v\} \cup Y \notin \mathcal{I}$ and $u > v$ for all $u \in Y$. The set of elements dominated by X is denoted by $D_{\mathcal{M}}(X)$. Given two ordered matroids $\mathcal{M}_M = (V, \mathcal{I}_M, >_M)$ and $\mathcal{M}_W = (V, \mathcal{I}_W, >_W)$ on the same ground set V, a subset X of V is called an $\mathcal{M}_M\mathcal{M}_W$*-kernel* if X is a common independent set of $\mathcal{M}_M$ and $\mathcal{M}_W$, and if any element $v \in V$ is dominated by X in $\mathcal{M}_M$ or $\mathcal{M}_W$, that is, if the following condition holds:

(m4) $X \in \mathcal{I}_M \cap \mathcal{I}_W$ and $D_{\mathcal{M}_M}(X) \cup D_{\mathcal{M}_W}(X) = V$.

For example, given a stable marriage instance $(M, W, \{a_{ij}\}, \{b_{ij}\})$ without indifferent preferences, we can construct an equivalent instance in terms of matroids as follows. Let V be the set of pairs (i, j) with $a_{ij}, b_{ij} > -\infty$. Assume that $(V, \mathcal{I}_M)$ is the partition matroid on V defined by the sets V_i of pairs containing each $i \in M$ and that $(V, \mathcal{I}_W)$ is the partition matroid on V defined by the sets V_j of pairs containing each $j \in W$. Thus, X is a matching if and only if $X \in \mathcal{I}_M \cap \mathcal{I}_W$. We next define linear orders $>_M$ and $>_W$ on V so that $(i, j_1) >_M (i, j_2)$ whenever $a_{ij_1} > a_{ij_2}$, and that $(i_1, j) >_W (i_2, j)$ whenever $b_{i_1 j} > b_{i_2 j}$. By the definitions of the linear orders, matching X is an $\mathcal{M}_M\mathcal{M}_W$-kernel if and only if for any pair $(i, j) \notin X$ there exists either (i, j') or (i', j) in X such that either $(i, j') >_M (i, j)$ or $(i', j) >_W (i, j)$. Thus, the set of $\mathcal{M}_M\mathcal{M}_W$-kernels coincides with the set of stable matchings. The matroidal model also includes a many-to-many stable matching model, called stable b-matching model. We remark that the matroidal model can easily be modified so that indifference in preferences is admissible. Fleiner [4] showed that any instance of the matroidal model has an $\mathcal{M}_M\mathcal{M}_W$-kernel.

Quite recently, Eguchi and Fujishige [2] proposed a model in terms of M$^\natural$-concavity, which is a set version of our model with $V = R$ and $\mathrm{dom}\, f_M, \mathrm{dom}\, f_W \subseteq \{0,1\}^V$. For convenience, we idenfity a subset of V and its characteristic vector. The matroidal model above can be recognized as a special case of this model with linear utility functions. Let $\mathcal{M}_M = (V, \mathcal{I}_M, >_M)$ and $\mathcal{M}_W = (V, \mathcal{I}_W, >_W)$ be an instance of the matroidal model. We describe linear orders $>_M$ and $>_W$ by positive numbers $\{a_v\}$ and $\{b_v\}$ such that $a_u > a_v \iff u >_M v$ and $b_u > b_v \iff u >_W v$, and define functions f_M and f_W by

$$f_M(X) = \begin{cases} \sum_{v\in X} a_v & (X \in \mathcal{I}_M) \\ -\infty & (X \notin \mathcal{I}_M), \end{cases} \qquad f_W(X) = \begin{cases} \sum_{v\in X} b_v & (X \in \mathcal{I}_W) \\ -\infty & (X \notin \mathcal{I}_W), \end{cases} \tag{3.1}$$

which are M$^\natural$-concave because these are linear on independence families of matroids. For an independent set X of $\mathcal{M}_M$ and $Z \subseteq V$ with $X \subseteq Z$, we have that $X \in \arg\max\{f_M(Y) \mid Y \subseteq Z\}$ if and only if $Z \subseteq D_{\mathcal{M}_M}(X)$ by the optimality criterion of maximum weight independent sets of a matroid (the same statement for $\mathcal{M}_W$ also holds). Thus, a subset X of V is an $\mathcal{M}_M\mathcal{M}_W$-kernel if and only if it is f_Mf_W-stable. Eguchi and Fujishige [2] showed that any instance of the above model has an f_Mf_W-stable solution.

Therefore, our model with $V = R$ includes all of the above models. Moreover, our model admits multiplicity for each element of V. For example, our model naturally deals with the following problem. The same numbers of men and women attend a dance party at which each person dances a waltz k times and he/she can dance with the same person of the opposite gender time after time. The problem is to find an "agreeable" assignment of dance partners, in which each person is assigned at most k persons of the opposite gender with possible repetition. If preferences of assignments of dance partners for each person can be expressed by an M$^\natural$-concave function (see Remark 1 in Section 2), then our model gives a solution.

3.2. The Assignment Model and Its Extensions

The assignment model includes side payments, which is different from the stable marriage model. An *outcome* is a triple of payoff vectors $q = (q_i \mid i \in M) \in \mathbf{R}^M$, $r = (r_j \mid j \in W) \in \mathbf{R}^W$, and a subset $X \subseteq V$, denoted by $(q, r; X)$. An outcome $(q, r; X)$ is called *stable* if

(a1) X is a matching,

(a2) $q_i + r_j = a_{ij} + b_{ij}$ for any $(i, j) \in X$,

(a3) $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X,

(a4) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i + r_j \geq a_{ij} + b_{ij}$ for any $(i, j) \in V$,

where $p_{ij} = b_{ij} - r_j = q_i - a_{ij}$ means a side payment from j to i for each $(i, j) \in X$. The stability says that no pair $(i, j) \notin X$ will be better off by making a partnership. Shapley and Shubik [21] proved existence of stable outcomes by linear programming duality. The maximum weight bipartite matching problem with weights $(a_{ij} + b_{ij})$ and its dual problem are formulated by linear programs as follows:

$$
\begin{array}{ll}
\text{Maximize} & \sum_{(i,j) \in V} (a_{ij} + b_{ij}) x_{ij} \\
\text{subject to} & \sum_{j \in W} x_{ij} \leq 1 \quad (i \in M) \\
& \sum_{i \in M} x_{ij} \leq 1 \quad (j \in W) \\
& x_{ij} \geq 0 \quad ((i, j) \in V),
\end{array}
\tag{3.2}
$$

$$
\begin{array}{ll}
\text{Minimize} & \sum_{i \in M} q_i + \sum_{j \in W} r_j \\
\text{subject to} & q_i + r_j \geq a_{ij} + b_{ij} \quad ((i, j) \in V) \\
& q_i \geq 0 \quad (i \in M) \\
& r_j \geq 0 \quad (j \in W).
\end{array}
\tag{3.3}
$$

Thus, $(q, r; X)$ is a stable outcome if and only if $x = \chi_X$, q and r are optimal solutions of the above problems, because (a1) and (a4) require the primal and dual feasibility and because (a2) and (a3) mean the complementary slackness. Furthermore, in our model, by defining M$^\natural$-concave functions f_M and f_W as

$$
f_M(x) = \begin{cases} \sum_{(i,j) \in V} a_{ij} x_{ij} & (\text{if } x \in \{0,1\}^V \text{ and } \forall i \in M : \sum_{j \in W} x_{ij} \leq 1) \\ -\infty & (\text{otherwise}), \end{cases}
\tag{3.4}
$$

$$
f_W(x) = \begin{cases} \sum_{(i,j) \in V} b_{ij} x_{ij} & (\text{if } x \in \{0,1\}^V \text{ and } \forall j \in W : \sum_{i \in M} x_{ij} \leq 1) \\ -\infty & (\text{otherwise}), \end{cases}
\tag{3.5}
$$

a stable outcome $(q, r; X)$ gives an $f_M f_W$-stable solution $x = \chi_X$ together with p such that $p_{ij} = b_{ij} - r_j$ for all $(i, j) \in V$. Conversely, an $f_M f_W$-stable solution $x = \chi_X$ with p leads us to a stable outcome $(q, r; X)$ such that $q_i = a_{ij} + p_{ij}$ and $r_j = b_{ij} - p_{ij}$ for $(i, j) \in X$ and $q_i = r_j = 0$ for i and j unmatched in X.

Sotomayor [22] showed the existence of a stable outcome in a two-sided market model in which each agent can form several partnerships with agents of the opposite set without repetition of the same pair. Recently, Sotomayor [24] proposed an extension of the model in which M and W denote a set of firms and that of workers, respectively, and each firm $i \in M$ can employ $\alpha_i > 0$ units of labor-time, and each worker $j \in W$ can supply $\beta_j > 0$ units. The pair (i,j) can earn $c_{ij}(= a_{ij} + b_{ij})$ per unit time. Instead of considering matchings, let x_{ij} be the number of time units for which i hires j, and we call x a *labor allocation*. A labor allocation $x \in \mathbf{Z}^{M \times W}$ is called *feasible* if $x \geq \mathbf{0}$ and the following two hold:

$$\sum_{j \in W} x_{ij} \leq \alpha_i \qquad (i \in M), \tag{3.6}$$

$$\sum_{i \in M} x_{ij} \leq \beta_j \qquad (j \in W). \tag{3.7}$$

For any subsets $M' \subseteq M$ and $W' \subseteq W$, let $P(M', W')$ denote the maximum of $\sum_{i \in M'} \sum_{j \in W'} c_{ij} x_{ij}$ over all feasible labor allocations x, that is, the payoff of coalition $M' \cup W'$. On the other hand, we say that the pair of $q \in \mathbf{R}^M$ and $r \in \mathbf{R}^W$ is a *money allocation*, and that it is *feasible* if $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q(M) + r(W) \leq P(M, W)$, where $q(M) = \sum_{i \in M} q_i$ and $r(W) = \sum_{j \in W} r_j$. A money allocation (q, r) is said to be in the *core* if it is feasible and if $q(M') + r(W') \geq P(M', W')$ for all coalitions $M' \subseteq M$ and $W' \subseteq W$. Sotomayor [24] showed that an element of the core is derived from a dual optimal solution of the transportation problem:

$$\text{Maximize} \sum_{(i,j) \in V} c_{ij} x_{ij} \quad \text{subject to} \quad (3.6),\ (3.7),\ x \geq \mathbf{0}, \tag{3.8}$$

which implies the nonemptiness of the core. Therefore, in our context, by defining M$^\natural$-concave functions f_M and f_W as

$$f_M(x) = \begin{cases} \sum_{(i,j) \in V} c_{ij} x_{ij} & (\text{if } x \in \mathbf{Z}^V \text{ satisfies (3.6) and } x \geq \mathbf{0}) \\ -\infty & (\text{otherwise}), \end{cases} \tag{3.9}$$

$$f_W(x) = \begin{cases} 0 & (\text{if } x \in \mathbf{Z}^V \text{ satisfies (3.7) and } x \geq \mathbf{0}) \\ -\infty & (\text{otherwise}), \end{cases} \tag{3.10}$$

an $f_M f_W$-stable solution x together with p gives a money allocation (q, r) in the core defined by

$$q_i = \sum_{j : x_{ij} > 0} (c_{ij} + p_{ij}) x_{ij} \quad (i \in M), \tag{3.11}$$

$$r_j = \sum_{i : x_{ij} > 0} (-p_{ij}) x_{ij} \quad (j \in W). \tag{3.12}$$

However, the converse does not necessarily hold, as Sotomayor [24] pointed out that the core may be strictly greater than the set of dual optimal solutions (see [24, Example 2]).

Kelso and Crawford [11] introduced a many-to-one labor market model in which a utility function of each firm has the gross substitutability and a utility function of each worker is strictly increasing (not necessarily linear) in a salary. Our model is closely related to this model since both adopt M$^\natural$-concavity.

3.3. A Hybrid Model

Eriksson and Karlander [3] proposed a hybrid model of the stable marriage model and the assignment model. In this hybrid model, the agents are partitioned into two categories, called *flexible agents* and *rigid agents*, that is, M and W are partitioned into (M_F, M_R) and (W_F, W_R). According to the partitions of agents, the set V of all pairs is partitioned into F and R as

$$R = \{(i,j) \in V \mid i \in M_R \text{ or } j \in W_R\}, \tag{3.13}$$

$$F = \{(i,j) \in V \mid i \in M_F \text{ and } j \in W_F\}. \tag{3.14}$$

An outcome $(q, r; X)$ is called *stable* if

(h1) X is a matching,

(h2) $q_i + r_j = a_{ij} + b_{ij}$ for any $(i,j) \in X$,

(h3) $q_i = a_{ij} > -\infty$ and $r_j = b_{ij} > -\infty$ for any $(i,j) \in X \cap R$,

(h4) $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X,

(h5) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i + r_j \geq a_{ij} + b_{ij}$ for any $(i,j) \in F$,

(h6) $q_i \geq a_{ij}$ or $r_j \geq b_{ij}$ for any $(i,j) \in R$.

When $V = R$ (or $V = F$), Conditions (h1)$\sim$(h6) are obviously equivalent to (m1)$\sim$(m3) (or (a1)$\sim$(a4)). Eriksson and Karlander [3], and Sotomayor [23] showed the existence of a stable outcome. As is seen from the discussion in the previous subsections, our model includes this hybrid model as a special case.

4. Proof

In this section we prove our main theorem, Theorem 2.1. We divide arguments into the following three cases: (i) a case that includes the stable marriage model (i.e., $F = \emptyset$), (ii) a case that includes the assignment model (i.e., $R = \emptyset$), and (iii) the general case (i.e., $F, R \neq \emptyset$).

4.1. The Stable Marriage Case

In this subsection we give an algorithm for finding $x_M, x_W \in \mathbf{Z}^V$ and $z_M, z_W \in \mathbf{Z}^R$ such that

$$z|_R = z_M \vee z_W, \tag{4.1}$$

$$x_M \in \arg\max\{f_M(y) \mid y \in \mathbf{Z}^V,\ y|_R \leq z_M\}, \tag{4.2}$$

$$x_W \in \arg\max\{f_W(y) \mid y \in \mathbf{Z}^V,\ y|_R \leq z_W\}, \tag{4.3}$$

$$x_M|_R = x_W|_R. \tag{4.4}$$

Here it should be noted that $\{F, R\}$ can be any partition of V and that if $F = \emptyset$ and if there exist $z_M, z_W \in \mathbf{Z}^V$ satisfying (4.1), (4.2), and (4.3) with $x_M = x_W = x$, then $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is an $f_M f_W$-stable solution. Hence the algorithm proposed below can find an $f_M f_W$-stable solution with respect to $(\emptyset, V)$.

Before describing the algorithm, we show three fundamental properties of M$^\natural$-concave functions as Lemmas 4.1, 4.2, and 4.3, which hold without Assumption (A).

Lemma 4.1 ([14], see also [16]): *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and U be a nonempty subset of V. Define a function $f^U : \mathbf{Z}^U \to \mathbf{R}\cup\{\pm\infty\}$ by*

$$f^U(x) = \sup\{f(y) \mid y \in \mathbf{Z}^V,\ y|_U = x\} \quad (x \in \mathbf{Z}^U). \tag{4.5}$$

If $f^U(x) < +\infty$ for each $x \in \mathbf{Z}^U$, then f^U is an M$^\natural$-concave function. In particular, if $\mathrm{dom}\, f$ is bounded, then f^U is M$^\natural$-concave.

Lemma 4.2: *Let $f : \mathbf{Z}^V \to \mathbf{R}\cup\{-\infty\}$ be an M$^\natural$-concave function and $z_1, z_2 \in \mathbf{Z}^V$ be such that $z_1 \geq z_2$, $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$, and $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$.*

(a) *For any $x_1 \in \arg\max\{f(y) \mid y \leq z_1\}$, there exists x_2 such that*

$$x_2 \in \arg\max\{f(y) \mid y \leq z_2\} \quad \textit{and} \quad z_2 \wedge x_1 \leq x_2. \tag{4.6}$$

(b) *For any $x_2 \in \arg\max\{f(y) \mid y \leq z_2\}$, there exists x_1 such that*

$$x_1 \in \arg\max\{f(y) \mid y \leq z_1\} \quad \textit{and} \quad z_2 \wedge x_1 \leq x_2. \tag{4.7}$$

Proof. (a): Let x_2 be an element in $\arg\max\{f(y) \mid y \leq z_2\}$ that minimizes $\sum\{x_1(v) - x_2(v) \mid v \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\}$. We show $z_2 \wedge x_1 \leq x_2$. Suppose, to the contrary, that there exists $u \in V$ with $\min\{z_2(u), x_1(u)\} > x_2(u)$. Then $u \in \mathrm{supp}^+(x_1 - x_2)$. By ($-M^\natural$-EXC), there exists $v \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that

$$f(x_1) + f(x_2) \leq f(x_1 - \chi_u + \chi_v) + f(x_2 + \chi_u - \chi_v). \tag{4.8}$$

If $v \neq 0$, then $x_1(v) < x_2(v) \leq z_2(v) \leq z_1(v)$. Hence we have $x_1 - \chi_u + \chi_v \leq z_1$, which implies $f(x_1) \geq f(x_1 - \chi_u + \chi_v)$. This together with (4.8) yields $f(x_2) \leq f(x_2 + \chi_u - \chi_v)$. Moreover, since $z_2(u) > x_2(u)$, we have $x_2' = x_2 + \chi_u - \chi_v \leq z_2$. It follows that $x_2' \in \arg\max\{f(y) \mid y \leq z_2\}$ and $x_2'(v) \geq \min\{z_2(v), x_1(v)\}$ if $v \neq 0$, which contradicts the minimality condition of x_2.

(b): Let x_1 be an element in $\arg\max\{f(y) \mid y \leq z_1\}$ that minimizes $\sum\{x_1(u) - x_2(u) \mid u \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\}$. We show $z_2 \wedge x_1 \leq x_2$. Suppose, to the contrary, that there exists $u \in V$ with $\min\{z_2(u), x_1(u)\} > x_2(u)$. Then $u \in \mathrm{supp}^+(x_1 - x_2)$. By ($-M^\natural$-EXC), there exists $v \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that

$$f(x_1) + f(x_2) \leq f(x_1 - \chi_u + \chi_v) + f(x_2 + \chi_u - \chi_v). \tag{4.9}$$

Since $x_2(u) < z_2(u)$, we have $x_2 + \chi_u - \chi_v \leq z_2$, which implies $f(x_2) \geq f(x_2 + \chi_u - \chi_v)$. This together with (4.9) yields $f(x_1) \leq f(x_1 - \chi_u + \chi_v)$. Obviously $x_1' = x_1 - \chi_u + \chi_v \leq z_1$. However, this contradicts the minimality condition of x_1 because $x_2(v) \geq \min\{z_2(v), x_1'(v)\}$ if $v \neq 0$. ∎

Lemma 4.3: *For an M$^\natural$-concave function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ and a vector $z_1 \in \mathbf{Z}^V$ suppose that* $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$*. For any $x \in \arg\max\{f(y) \mid y \leq z_1\}$ and any $z_2 \in \mathbf{Z}^V$ such that* (1) *$z_2 \geq z_1$ and* (2) *if $x(v) = z_1(v)$, then $z_2(v) = z_1(v)$, we have $x \in \arg\max\{f(y) \mid y \leq z_2\}$.*

Proof. Assume to the contrary that the assertion is not satisfied. Let x' be a point such that $x' \leq z_2$, $f(x') > f(x)$, and x' minimizes $\sum\{x'(v) - z_1(v) \mid v \in \mathrm{supp}^+(x' - z_1)\}$ among such points. By the assumption, there exists $u \in V$ with $x'(u) > z_1(u) > x(u)$. By ($-$M$^\natural$-EXC) for x', x, and u, there exists $v \in \mathrm{supp}^-(x' - x) \cup \{0\}$ such that

$$f(x') + f(x) \leq f(x' - \chi_u + \chi_v) + f(x + \chi_u - \chi_v). \tag{4.10}$$

Since $x + \chi_u - \chi_v \leq z_1$, we have $f(x) \geq f(x + \chi_u - \chi_v)$, which implies $f(x') \leq f(x' - \chi_u + \chi_v)$. Obviously, $x' - \chi_u + \chi_v \leq z_2$, However, this contradicts the minimality condition of x' because if $v \neq 0$, then $z_1(v) \geq x(v) > x'(v)$. ∎

It should be noted that Lemma 4.3 holds for any function f on $\mathbf{Z}^V$ that has a concave extension on $\mathbf{R}^V$.

To describe an algorithm for finding $x_M, x_W \in \mathbf{Z}^V$ and $z_M, z_W \in \mathbf{Z}^R$ satisfying (4.1), (4.2), (4.3), and (4.4), we assume that we are initially given $x_M, x_W \in \mathbf{Z}^V$ and $z_M, z_W \in \mathbf{Z}^R$ satisfying (4.1) and the following:

$$\begin{aligned} x_M &\in \arg\max\{f_M(y) \mid y|_R \leq z_M\}, & (4.11)\\ x_W &\in \arg\max\{f_W(y) \mid y|_R \leq z_W \vee x_M|_R\}, & (4.12)\\ x_W|_R &\leq x_M|_R. & (4.13)\end{aligned}$$

In the case when $R = V$, we can easily compute such vectors by setting $z_M = z$, $z_W = \mathbf{0}$, and by finding x_M and x_W such that

$$x_M \in \arg\max\{f_M(y) \mid y \leq z_M\}, \quad x_W \in \arg\max\{f_W(y) \mid y \leq x_M\}. \tag{4.14}$$

The algorithm is given as follows.

Algorithm_1$(f_M, f_W, x_M, x_W, z_M, z_W)$
Input: M$^\natural$-concave functions f_M, f_W and
x_M, x_W, z_M, z_W satisfying (4.1), (4.11), (4.12), (4.13) ;
Step 1: **repeat** {
let x_M be any element in $\arg\max\{f_M(y) \mid x_W|_R \le y|_R \le z_M\}$;
let x_W be any element in $\arg\max\{f_W(y) \mid y|_R \le x_M|_R\}$;
for each $v \in R$ with $x_M(v) > x_W(v)$ {
$z_M(v) \leftarrow x_W(v)$;
$z_W(v) \leftarrow z(v)$;
} ;
} **until** $x_M|_R = x_W|_R$;
return $(x_M, x_W, z_M, z_W \vee x_M|_R)$.

It should be noted here that because of Assumption (A) x_M and x_W are well-defined within the effective domains and that Algorithm_1 terminates after at most $\sum_{v\in R} z(v)$ iterations, because $\sum_{v\in R} z_M(v)$ is strictly decreased at each iteration. In order to show that the outputs of Algorithm_1 satisfy (4.1), (4.2), (4.3), and (4.4), we will show two lemmas, Lemmas 4.4 and 4.5.

Let $x_M^{(i)}$, $x_W^{(i)}$, $z_M^{(i)}$, and $z_W^{(i)}$ be x_M, x_W, z_M, and z_W obtained after the ith iteration in Step 1 of Algorithm_1 for $i = 1, 2, \cdots, t$, where t is the last to get the outputs. For convenience, let us assume that $x_M^{(0)}$, $x_W^{(0)}$, $z_M^{(0)}$, and $z_W^{(0)}$ are the input vectors.

Lemma 4.4: *For each* $i = 0, 1, \cdots, t$, *we have*

$$x_M^{(i+1)} \in \arg\max\left\{f_M(y) \mid y|_R \le z_M^{(i)}\right\}. \tag{4.15}$$

Proof. We prove (4.15) by induction on i. For $i = 0$, (4.15) holds from (4.11) and (4.13). We assume that for some l with $0 \le l < t$ (4.15) holds for any $i \le l$, and we show (4.15) for $i = l+1$. Since $x_M^{(l+1)} \in \max\{f_M(y) \mid y|_R \le z_M^{(l)}\}$ and $z_M^{(l)} \ge z_M^{(l+1)}$, Lemma 4.2 (a) guarantees the existence of an $x \in \arg\max\{f_M(y) \mid y|_R \le z_M^{(l+1)}\}$ with $z_M^{(l+1)} \wedge x_M^{(l+1)}|_R \le x|_R$, which implies (4.15) for $i = l+1$ because $z_M^{(l+1)} \wedge x_M^{(l+1)}|_R = x_W^{(l+1)}|_R$ by the modification of z_M. ∎

Lemma 4.5: *For each* $i = 0, 1, \cdots, t$, *we have*

$$x_W^{(i)} \in \arg\max\left\{f_W(y) \mid y|_R \le z_W^{(i)} \vee x_M^{(i)}|_R\right\}. \tag{4.16}$$

Proof. We show (4.16) by induction on i. For $i = 0$, (4.16) holds by (4.12). We assume that for some l with $0 \le l < t$ (4.16) holds for any $i \le l$, and we show (4.16) for $i = l+1$. By the definition of x_M, we have

$$x_M^{(l+1)}|_R \ge x_W^{(l)}|_R. \tag{4.17}$$

By Lemma 4.2 (b) and the assumption, there exists x such that

$$x \in \arg\max\left\{f_W(y) \mid\ y|_R \le z_W^{(l)} \vee (x_M^{(l)}|_R) \vee (x_M^{(l+1)}|_R)\right\} \tag{4.18}$$

and

$$\left(z_W^{(l)} \vee x_M^{(l)}|_R\right) \wedge x|_R \le x_W^{(l)}|_R. \tag{4.19}$$

From (4.17), (4.18), and (4.19), we have $x|_R \le x_M^{(l+1)}|_R$ and hence we have $f_W(x) = f_W(x_W^{(l+1)})$. If $z_W^{(l+1)} = z_W^{(l)}$, then we immediately obtain (4.16) for $i = l+1$. So, we assume that $z_W^{(l+1)} \ne z_W^{(l)}$. By the modification of z_W, we have $z_W^{(l)}(v) < z_W^{(l+1)}(v)$ if and only if $x_W^{(l+1)}(v) < x_M^{(l+1)}(v)$. Hence it follows from Lemma 4.3 that (4.16) holds for $i = l+1$. ∎

The correctness of Algorithm_1 follows from Lemmas 4.4 and 4.5.

Theorem 4.6: *The outputs of* Algorithm_1 *satisfy* (4.1), (4.2), (4.3), *and* (4.4).

Proof. From Lemmas 4.4 and 4.5 we have for $i = t$

$$x_M \in \arg\max\left\{f_M(y) \mid\ y|_R \le z_M^{(t)}\right\}, \tag{4.20}$$

$$x_W \in \arg\max\left\{f_W(y) \mid\ y|_R \le z_W^{(t)} \vee x_M^{(t)}|_R\right\}, \tag{4.21}$$

$$x_M|_R = x_W|_R. \tag{4.22}$$

By the way of modifying z_M, z_W, and x_M, we have

$$z_M^{(t)} \vee \left(z_W^{(t)} \vee x_M^{(t)}|_R\right) = z|_R. \tag{4.23}$$

This completes the proof of this theorem. ∎

The following is a direct consequence of Theorem 4.6 when $F = \emptyset$.

Theorem 4.7: *For any* $M^\natural$*-concave functions* $f_M, f_W : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *satisfying* (A)*, there always exists an* $f_M f_W$*-stable solution with respect to* $(\emptyset, V)$.

4.2. The Assignment Case

In this subsection we explain a successive shortest path algorithm for finding a maximizer of $f_M + f_W$, i.e., an $f_M f_W$-stable solution with respect to $(V, \emptyset)$ (cf. the discussion in Section 3.2). The algorithm presented here will give a basic procedure for finding a stable solution for our general model.

Before describing the algorithm, we give several known results on $M^\natural$-concave functions.

For an M$^\natural$-concave function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, we define $\hat{f} : \mathbf{Z}^{\{0\}\cup V} \to \mathbf{R} \cup \{-\infty\}$ by

$$\hat{f}(y_0, y) = \begin{cases} f(y) & (\text{if } y_0 = -y(V)) \\ -\infty & (\text{otherwise}) \end{cases} \qquad ((y_0, y) \in \mathbf{Z}^{\{0\}\cup V}), \tag{4.24}$$

where $y(V)$ denotes the sum of all components of y. Function $\hat{f}$ is called an *M-concave function* and can be characterized by the following exchange property [12, 13]:

($-$M-EXC) $\forall x, y \in \mathrm{dom}\,\hat{f}$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:

$$\hat{f}(x) + \hat{f}(y) \le \hat{f}(x - \chi_u + \chi_v) + \hat{f}(y + \chi_u - \chi_v). \tag{4.25}$$

In particular, an M-concave function is also M$^\natural$-concave. Here we denote $\{0\} \cup V$ by $\hat{V}$. For any vector $x \in \mathbf{R}^V$, we denote by $\hat{x}$ the vector $(-x(V), x) \in \mathbf{R}^{\hat{V}}$. For a vector $(p_0, p) \in \mathbf{R}^{\hat{V}}$, we have

$$x \in \arg\max f[p - p_0\mathbf{1}] \iff \hat{x} \in \arg\max \hat{f}[(p_0, p)]. \tag{4.26}$$

Thus, the problem of finding an $f_M f_W$-stable solution with respect to $(V, \emptyset)$ is equivalent to that of finding a maximizer of $\hat{f}_M + \hat{f}_W$ (see Remark 3 and Theorem 2.2 in Section 2).

The maximizers of an M-concave function have a good characterization.

Theorem 4.8 ([12, 13]): *For an M-concave function $\hat{f} : \mathbf{Z}^{\hat{V}} \to \mathbf{R} \cup \{-\infty\}$ and $x \in \mathrm{dom}\,\hat{f}$, $x \in \arg\max \hat{f}$ if and only if $\hat{f}(x) \ge \hat{f}(x - \chi_u + \chi_v)$ for all $u, v \in \hat{V}$.*

The following property is a direct consequence of ($-$M-EXC).

Lemma 4.9: *For any M-concave function $\hat{f}$, $\arg\max \hat{f}$ satisfies:*

(B-EXC) $\forall x, y \in \arg\max \hat{f}$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:

$$x - \chi_u + \chi_v,\ y + \chi_u - \chi_v \in \arg\max \hat{f}. \tag{4.27}$$

A set B of integral vectors satisfying (B-EXC) is called an *M-convex set*, (the set of integer points of) an integral base polyhedron. It is known that an M-convex set has a property called "no-short cut lemma" below.

Lemma 4.10 ([6]): *Suppose that B is an M-convex set, $x \in B$, and that $u_1, v_1, u_2, v_2, \cdots, u_r, v_r$ are distinct. If $x - \chi_{u_i} + \chi_{v_i} \in B$ for $i = 1, \ldots, r$ and $x - \chi_{u_i} + \chi_{v_j} \notin B$ for $i < j$, then $y = x - \sum_{i=1}^{r}(\chi_{u_i} - \chi_{v_i}) \in B$.*

Let $\hat{x}_M$ and $\hat{x}_W$ be arbitrary maximizers of $\hat{f}_M$ and $\hat{f}_W$, respectively. We construct a directed graph $G = (\hat{V}, A)$ and an arc length $\ell \in \mathbf{R}^A$ as follows. Arc set A consists of two disjoint parts:

$$A_M = \{(u,v) \mid u,v \in \hat{V},\ u \neq v,\ \hat{x}_M - \chi_u + \chi_v \in \mathrm{dom}\, \hat{f}_M\}, \qquad (4.28)$$
$$A_W = \{(v,u) \mid u,v \in \hat{V},\ u \neq v,\ \hat{x}_W - \chi_u + \chi_v \in \mathrm{dom}\, \hat{f}_W\}, \qquad (4.29)$$

and $\ell \in \mathbf{R}^A$ is defined by

$$\ell(a) = \begin{cases} \hat{f}_M(\hat{x}_M) - \hat{f}_M(\hat{x}_M - \chi_u + \chi_v) & (a = (u,v) \in A_M) \\ \hat{f}_W(\hat{x}_W) - \hat{f}_W(\hat{x}_W - \chi_u + \chi_v) & (a = (v,u) \in A_W). \end{cases} \qquad (4.30)$$

The length function ℓ is nonnegative due to Theorem 4.8.

For a specified vertex s of $\hat{V}$, let $d : \hat{V} \to \mathbf{R} \cup \{+\infty\}$ denote the shortest distance from s to each vertex with respect to ℓ. Let t be an arbitrary vertex of $\hat{V}$ reachable from s and define $p \in \mathbf{R}^{\hat{V}}$ by $p(v) = \min\{d(v), d(t)\}$ for each $v \in \hat{V}$. It follows from the nonnegativity of ℓ that

$$\ell(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \qquad (4.31)$$

for every arc $a \in A$, where $\partial^+ a$ and $\partial^- a$ denote the initial vertex and the terminal vertex of a, respectively. The system of inequalities (4.31) is equivalent to

$$\begin{array}{l} \hat{f}_M(\hat{x}_M) - \hat{f}_M(\hat{x}_M - \chi_u + \chi_v) + p(u) - p(v) \geq 0 \\ \hat{f}_W(\hat{x}_W) - \hat{f}_W(\hat{x}_W - \chi_u + \chi_v) - p(u) + p(v) \geq 0 \end{array} \quad (\forall u, v \in \hat{V}), \qquad (4.32)$$

which is further equivalent to

$$\hat{x}_M \in \arg\max \hat{f}[+p], \quad \hat{x}_W \in \arg\max \hat{f}[-p], \qquad (4.33)$$

by Theorem 4.8. Note that $\ell_p(a) = \ell(a) + p(\partial^+ a) - p(\partial^- a)$ is the length of arc a in a directed graph defined in the same way as above for $\hat{f}_M[+p]$, $\hat{f}_W[-p]$, $\hat{x}_M$, and $\hat{x}_W$. Also note that we have $\ell_p(a) = 0$ for any arc a in a shortest path from s to t.

Let P be a shortest path from s to t in G with a minimum number of arcs. Since $\ell_p(a) = 0$ for any $a \in P$, we have

$$\begin{array}{ll} \hat{x}_M - \chi_u + \chi_v \in \arg\max \hat{f}_M[+p] & (\forall (u,v) \in P \cap A_M), \\ \hat{x}_W - \chi_u + \chi_v \in \arg\max \hat{f}_W[-p] & (\forall (v,u) \in P \cap A_W). \end{array} \qquad (4.34)$$

Since P has a minimum number of arcs, we have

$$\hat{x}_M - \chi_u + \chi_w \notin \arg\max \hat{f}_M[+p], \quad \hat{x}_W - \chi_w + \chi_u \notin \arg\max \hat{f}_W[-p] \qquad (4.35)$$

for any vertices u and w of P such that $(u, w) \notin P$ and u appears earlier than w in P. Furthermore, arcs of A_M and A_W alternatively appear in P. For otherwise assume that consecutive two arcs $(u,v),(v,w) \in P$ belong to A_M and then, by (−M-EXC) we have

$$\hat{f}_M(\hat{x}_M + \chi_u - \chi_v) + \hat{f}_M(\hat{x}_M + \chi_v - \chi_w) \leq \hat{f}_M(\hat{x}_M) + \hat{f}_M(\hat{x}_M + \chi_u - \chi_w), \quad (4.36)$$

which yields

$$\ell(u,v) + \ell(v,w) \geq \ell(u,w), \quad (4.37)$$

a contradiction to the minimality of P. Consequently,

$$\begin{array}{l} a_1, a_2 \in P \cap A_M,\ a_1 \neq a_2 \implies \partial^+ a_1 \neq \partial^- a_2, \\ a_1, a_2 \in P \cap A_W,\ a_1 \neq a_2 \implies \partial^+ a_1 \neq \partial^- a_2. \end{array} \quad (4.38)$$

From Lemmas 4.9 and 4.10 together with (4.34), (4.35), and (4.38), we have

$$\hat{x}_M - \sum_{(u,v)\in P\cap A_M} (\chi_u - \chi_v) \in \arg\max \hat{f}_M[+p], \quad (4.39)$$

$$\hat{x}_W - \sum_{(v,u)\in P\cap A_W} (\chi_u - \chi_v) \in \arg\max \hat{f}_W[-p]. \quad (4.40)$$

Now, a successive shortest path algorithm for finding a maximizer of $\hat{f}_M + \hat{f}_W$ is described as below.

Algorithm_2

Step 0: find $\hat{x}_M \in \arg\max \hat{f}_M$ and $\hat{x}_W \in \arg\max \hat{f}_W$, $\quad p \leftarrow \mathbf{0}$;
Step 1: **if** $\hat{x}_M = \hat{x}_W$ **then** stop ;
Step 2: construct G and compute ℓ for $\hat{f}_M[+p]$, $\hat{f}_W[-p]$, $\hat{x}_M$, and $\hat{x}_W$;
  $S \leftarrow \mathrm{supp}^+(\hat{x}_M - \hat{x}_W)$, $\quad T \leftarrow \mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$;
Step 3: compute the shortest distance $d(v)$ from S to each $v \in \hat{V}$ in G
    with respect to ℓ ;
  let P be a shortest path from S to T with a minimum number of arcs ;
Step 4: **for** each $v \in \hat{V}$
    $p(v) \leftarrow p(v) + \min\{d(v), \sum_{a\in P} \ell(a)\}$;
  for each arc $a \in P$ {
    $a \in A_M \Rightarrow \hat{x}_M(\partial^+ a) \leftarrow \hat{x}_M(\partial^+ a) - 1, \quad \hat{x}_M(\partial^- a) \leftarrow \hat{x}_M(\partial^- a) + 1,$
    $a \in A_W \Rightarrow \hat{x}_W(\partial^- a) \leftarrow \hat{x}_W(\partial^- a) - 1, \quad \hat{x}_W(\partial^+ a) \leftarrow \hat{x}_W(\partial^+ a) + 1$;
  }
  goto Step 1 ;

Under Assumption (A), a shortest path P in Step 3 always exists because if there is no shortest path from $\mathrm{supp}^+(\hat{x}_M - \hat{x}_W)$ to $\mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$, then $\mathrm{dom}\, \hat{f}_M \cap \mathrm{dom}\, \hat{f}_W$ must be empty (see [16]). By (4.39) and (4.40), the algorithm preserves

$$\hat{x}_M \in \arg\max \hat{f}_M[+p], \quad \hat{x}_W \in \arg\max \hat{f}_W[-p]. \quad (4.41)$$

Thus, if Algorithm_2 terminates, then it finds a maximizer of $\hat{f}_M + \hat{f}_W$, that is, an $f_M f_W$-stable solution with respect to $(V, \emptyset)$. Furthermore, since P is a path from $\mathrm{supp}^+(\hat{x}_M - \hat{x}_W)$ to $\mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$, $||\hat{x}_M - \hat{x}_W||_1$ is decreased by two after each execution of Step 4, which guarantees the termination of Algorithm_2.

For the case when $R = \emptyset$, we can relax (A) into that $\mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is nonempty and bounded as we see from the above discussion.

Theorem 4.11: *For any* M$^\natural$*-concave functions* $f_M, f_W : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *such that* $\mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ *is nonempty and bounded, there always exists an* $f_M f_W$*-stable solution with respect to* $(V, \emptyset)$.

4.3. The General Case

In this subsection, we give an algorithm for finding an $f_M f_W$-stable solution in the case when both F and R are nonempty. The algorithm consists of two phases:

(1) Phase 1 finds $x_M, x_W \in \mathbf{Z}^V$, $p \in \mathbf{R}^V$, and $z_M, z_W \in \mathbf{Z}^R$ satisfying (2.14), (2.17), (4.4), and the following:

$$x_M \in \arg\max\{f_M[+p](y) \mid y|_R \leq z_M\}, \tag{4.42}$$

$$x_W \in \arg\max\{f_W[-p](y) \mid y|_R \leq z_W\}, \tag{4.43}$$

$$x_M \leq x_W. \tag{4.44}$$

(Note that if we further get $x_M = x_W$ in (4.44), then $x = (x_M = x_W)$ is an $f_M f_W$-stable solution.)

(2) Phase 2 finds an $f_M f_W$-stable solution by applying Phase 1 in a modified way. By interchanging the roles of M and W Phase 2 executes part of Phase 1 with the inputs being the outputs of Phase 1. The detail will be given below.

We first give a lemma.

Lemma 4.12: *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *be an* M$^\natural$*-concave function. For an element* $u \in V$ *let* $z_1, z_2 \in \mathbf{Z}^V$ *be vectors such that* $z_1 = z_2 + \chi_u$, $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$, *and* $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$. *Then, the following two statements hold:*

(a) *For any* $x \in \arg\max\{f(y) \mid y \leq z_1\}$, *there exists* $v \in \{0\} \cup V$ *such that*

$$x - \chi_u + \chi_v \in \arg\max\{f(y) \mid y \leq z_2\}. \tag{4.45}$$

(b) *For any* $x \in \arg\max\{f(y) \mid y \leq z_2\}$, *there exists* $v \in \{0\} \cup V$ *such that*

$$x + \chi_u - \chi_v \in \arg\max\{f(y) \mid y \leq z_1\}. \tag{4.46}$$

Proof. (a) If $x \in \arg\max\{f(y) \mid y \le z_2\}$, then it suffices to set $v = u$. Hence we assume that $x \notin \arg\max\{f(y) \mid y \le z_2\}$. Then we have $x(u) = z_1(u) = z_2(u) + 1$. Let x' be any element of $\arg\max\{f(y) \mid y \le z_2\}$. By ($-M^\natural$-EXC) for x, x', and u, there exists $v \in \{0\} \cup \mathrm{supp}^-(x - x')$ such that

$$f(x) + f(x') \le f(x - \chi_u + \chi_v) + f(x' + \chi_u - \chi_v). \tag{4.47}$$

Since $x' + \chi_u - \chi_v \le z_1$ and $x \in \arg\max\{f(y) \mid y \le z_1\}$, the above inequality implies $f(x') \le f(x - \chi_u + \chi_v)$, that is, $x - \chi_u + \chi_v \in \arg\max\{f(y) \mid y \le z_2\}$.

(b) If $x \in \arg\max\{f(y) \mid y \le z_1\}$, then it suffices to set $v = u$. Hence we assume that $x \notin \arg\max\{f(y) \mid y \le z_1\}$, i.e., there exists $x' \in \arg\max\{f(y) \mid y \le z_1\}$ with $x'(u) = z_1(u)$. Then, by ($-$M$^\natural$-EXC) for x', x, and u, there exists $v \in \{0\} \cup \mathrm{supp}^-(x' - x)$ such that

$$f(x') + f(x) \le f(x' - \chi_u + \chi_v) + f(x + \chi_u - \chi_v). \tag{4.48}$$

Since $x' - \chi_u + \chi_v \le z_2$ and $x \in \arg\max\{f(y) \mid y \le z_2\}$, the above inequality implies $f(x') \le f(x + \chi_u - \chi_v)$, that is, $x + \chi_u - \chi_v \in \arg\max\{f(y) \mid y \le z_1\}$. ∎

Next, we give the procedure for Phase 1, where $f_M^{z_M}$ and $f_W^{z_W}$ are M$^\natural$-concave functions defined by

$$f_M^{z_M}(x) = \begin{cases} f_M(x) & (\text{if } x|_R \le z_M) \\ -\infty & (\text{otherwise}), \end{cases} \tag{4.49}$$

$$f_W^{z_W}(x) = \begin{cases} f_W(x) & (\text{if } x|_R \le z_W) \\ -\infty & (\text{otherwise}). \end{cases} \tag{4.50}$$

Phase 1

Step 0: $z_M \leftarrow z|_R, \quad z_W \leftarrow \mathbf{0}, \quad p \leftarrow \mathbf{0}$;
let x_M be any element in $\arg\max\{f_M(y) \mid y|_R \le z_M\}$;
let x_W be any element in $\arg\max\{f_W(y) \mid y|_R \le x_M|_R\}$;
Step 1: $(x_M, x_W, z_M, z_W) \leftarrow$ Algorithm_1$(f_M[+p], f_W[-p], x_M, x_W, z_M, z_W)$;
Step 2: **if** $x_M \le x_W$ **then** return (x_M, x_W, z_M, z_W, p) ;
Step 3: construct G and compute ℓ for $\hat{f}_M^{z_M}[+(0,p)]$, $\hat{f}_W^{z_W}[-(0,p)]$, $\hat{x}_M$, and $\hat{x}_W$;
$S \leftarrow \mathrm{supp}^+(\hat{x}_M - \hat{x}_W) \setminus \{0\}, \quad T \leftarrow \{0\} \cup R \cup \mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$;
Step 4: compute the shortest distance $d(v)$ from S to each $v \in \hat{V}$ in G
with respect to ℓ ;
let P be a shortest path from S to T with a minimum number of arcs ;
Step 5: **for** each $v \in \hat{V}$
$p(v) \leftarrow p(v) + \min\{d(v), \ell(P)\} - \ell(P) \quad$ (where $\ell(P) = \sum_{a \in P} \ell(a)$) ;
for each arc $a \in P$ {
$a \in A_M \;\Rightarrow\; \hat{x}_M(\partial^+ a) \leftarrow \hat{x}_M(\partial^+ a) - 1, \quad \hat{x}_M(\partial^- a) \leftarrow \hat{x}_M(\partial^- a) + 1,$

$a{\in}A_W \;\Rightarrow\; \hat{x}_W(\partial^- a) \leftarrow \hat{x}_W(\partial^- a){-}1, \quad \hat{x}_W(\partial^+ a) \leftarrow \hat{x}_W(\partial^+ a){+}1$;
}
Step 6: **if** the terminal u of P is in R **then** {
if the last arc of P is in A_M **then** {
choose $v \in \{0\} \cup V$ such that
$x_W + \chi_u - \chi_v \in \arg\max\{f_W(y) \mid\ y|_R \le z_W \vee x_M|_R\}$;
$x_W \leftarrow x_W + \chi_u - \chi_v$;
if $v \in \{0\} \cup F$ **then** goto Step 2 ;
else $(v \in R)$ goto Step 1 ;
}
goto Step 1 ;
}
goto Step 2 ;

In Step 6 of Phase 1 we can choose $v \in \{0\} \cup V$ such that $x_W + \chi_u - \chi_v \in \arg\max\{f_W(y) \mid\ y|_R \le z_W \vee x_M|_R\}$, due to Lemma 4.12 (b).

We will discuss the correctness of Phase 1. Step 0 constructs (x_M, x_W, z_M, z_W) satisfying (4.1), (4.11), (4.12), and (4.13) for $f_M[+\mathbf{0}]$ and $f_W[-\mathbf{0}]$. Therefore, Step 1 at the first time finds (x_M, x_W, z_M, z_W) satisfying (4.1)$\sim$(4.4). Steps 3 through 5 are the same as Algorithm_2 except for the definitions of S, T, and p. The difference is to achieve $p|_R = \mathbf{0}$ and $p_0 = 0$ that are required by (2.14) and (4.26). Since $\{0\} \cup R \subseteq T$, the shortest distances from S to vertices in $\{0\} \cup R$ is greater than or equal to $\ell(P)$, and hence, (p_0, p) updated in Step 5 satisfies $p|_R = \mathbf{0}$ and $p_0 = 0$. Hence (2.14) is preserved during Phase 1, where note that Algorithm_1 does not change p. Furthermore, x_M and x_W modified in Step 5 satisfy (4.42) and (4.43) because $\hat{x}_M(\hat{V}) = \hat{x}_W(\hat{V}) = 0$ guarantees that replacing (p_0, p) by $(p_0, p) + \mathbf{1}$ does not destroy the optimality of $\hat{x}_M$ and $\hat{x}_W$. From the discussion of the previous subsection, (4.42) and (4.43) are preserved through Steps 3$\sim$5.

Next, we consider the case when the terminal u of P belongs to R. In this case, at the beginning of Step 6 we have $x_M(u) = x_W(u) + 1$ and $x_M(v) = x_W(v)$ for any $v \in R \backslash \{u\}$. Hence, conditions (4.1), (4.11), and (4.13) are satisfied for $f_M[+p]$ and $f_W[-p]$. If the last arc a of P is in A_W, then $z_W \vee x_M|_R$ is unchanged, and hence, (4.12) also holds at the beginning of Step 6, which guarantees that Algorithm_1 correctly works when we go from Step 6 to Step 1. On the other hand, if $a \in A_M$, then (4.12) may not hold because $z_W \vee x_M|_R$ is modified. In this case, we modify x_W as $x_W + \chi_u - \chi_v$ in Step 6. If $v \in \{0\} \cup F$, then succeeding Steps 2 through 5 can be executed because $x_M|_R = x_W|_R$; otherwise $(v \in R)$, succeeding Step 1 can be executed because (4.12) holds. Hence, conditions (2.17), (4.4), (4.42), and (4.43) are satisfied in Step 2.

The rest is to show that Phase 1 terminates in a finite number of iterations. When we go from Step 6 to Step 1, we have either $x_M(u) > x_W(u)$ or (when v was chosen in Step 6) $x_M(v) > x_W(v)$, so that we can decrease either $z_M(u)$ or $z_M(v)$ by at least one by performing Algorithm_1 in Step 1. Therefore, Step 1 strictly decreases z_M and hence Step 1 is executed finitely many times, due to Assumption (A). Moreover, every execution of Steps 3∼5 strictly reduces the value of $\sum\{x_M(v) - x_W(v) \mid v \in S\}$. So, the cycle of Steps 2∼6 formed by going from Step 6 to Step 2 is executed consecutively finitely many times, and then Phase 1 terminates or we go to Step 1. Hence, Phase 1 must terminate in a finite number of iterations with outputs (x_M, x_W, z_M, z_W, p) satisfying (2.14), (2.17), (4.4), (4.42), (4.43), and (4.44).

Phase 2 is the same as Phase 1 except that the roles of M and W are interchanged and that Phase 2 starts from Step 2, where the inputs to Phase 2 are the outputs of Phase 1. Since we have $x_W \geq x_M$ for the outputs of Phase 1, to show that Phase 2 finds an $f_M f_W$-stable solution, it suffices to verify that Phase 1 preserves

$$x_M \geq x_W \tag{4.51}$$

once we have this relation. Note that if relation (4.51) is preserved by Phase 1, then Phase 2 with $x_W \geq x_M$ terminates in a finite number of iterations with outputs $x = x_M = x_W$, z_M, z_W, and p satisfying (2.14), (2.17), (2.18), and (2.19).

It follows from the discussion of Section 4.2 that (4.51) holds after the execution of Steps 3 through 5. We will show that we keep relation (4.51) while executing Algorithm_1. We now consider the case when Step 1 is executed after Step 6. In this case, at the beginning of Algorithm_1 we have $x_M(u') = x_W(u') + 1$ for some $u' \in R$ and $x_M(v) = x_W(v)$ for any $v \in R \setminus \{u'\}$. In Algorithm_1, we can apply Lemma 4.12 (a) to update x_M and Lemma 4.12 (b) to update x_W, respectively. Hence, at the end of the l-th iteration (but not the final one) of Algorithm_1, we have for some $u \in R$

$$x_M^{(l)}(u) = x_W^{(l)}(u) + 1, \quad x_M^{(l)}(w) = x_W^{(l)}(w) \qquad (w \in R \setminus \{u\}). \tag{4.52}$$

Moreover, Algorithm_1 terminates when for v in Lemma 4.12 we have $v \in \{0\} \cup F$. If $v \in F$, then either $x_M(v)$ is increased by one or $x_W(v)$ is decreased by one. Hence, relation (4.51) is preserved by Algorithm_1. We have thus shown our main result.

Theorem 2.1: *For any $M^\natural$-concave functions $f_M, f_W : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ satisfying* (A) *and for any partition (F, R) of V, there always exists an $f_M f_W$-stable solution with respect to (F, R).*

References

[1] V. Danilov, G. Koshevoy, and K. Murota, Discrete convexity and equilibria in economies with indivisible goods and money, *Math. Social Sci.* **41** (2001) 251–273.

[2] A. Eguchi and S. Fujishige, An extension of the Gale-Shapley matching algorithm to a pair of $\mathrm{M}^\natural$-concave functions, Discrete Mathematics and Systems Science Research Report 02-05, Osaka University, (2002).

[3] K. Eriksson and J. Karlander, Stable matching in a common generalization of the marriage and assignment models, *Discrete Math.* **217** (2000) 135–156.

[4] T. Fleiner, A matroid generalization of the stable matching polytope, in: B. Gerards and K. Aardal, eds., *Proceedings of the 8th International IPCO Conference*, LNCS **2081** (Springer-Verlag, Berlin, 2001) 105–114.

[5] T. Fleiner, Some results on stable matchings and fixed points, EGRES Technical Report 2002-08, (2002), `http://www.cs.elte.hu/egres`.

[6] S. Fujishige, *Submodular Functions and Optimization*, Annals of Discrete Mathematics 47 (North-Holland, Amsterdam, 1991).

[7] S. Fujishige and Z. Yang, A note on Kelso and Crawford's gross substitutes condition, *Math. Oper. Res.* to appear.

[8] D. Gale and L. S. Shapley, College admissions and the stability of marriage, *Amer. Math. Monthly* **69** (1962) 9–15.

[9] F. Gul and E. Stacchetti, Walrasian equilibrium with gross substitutes, *J. Econom. Theory* **87** (1999) 95–124.

[10] M. Kaneko, The central assignment game and the assignment markets, *J. Math. Econom.* **10** (1982) 205–232.

[11] A. S. Kelso, Jr. and V. P. Crawford, Job matching, coalition formation, and gross substitutes, *Econometrica* **50** (1982) 1483–1504.

[12] K. Murota, Convexity and Steinitz's exchange property, *Adv. Math.* **124** (1996) 272–311.

[13] K. Murota, Discrete convex analysis, *Math. Programming* **83** (1998) 313–371.

[14] K. Murota, Discrete convex analysis (in Japanese), in: S. Fujishige, ed., *Discrete Structures and Algorithms, vol. V* (Kindai-Kagakusha, Tokyo, 1982) 51–100.

[15] K. Murota, *Discrete Convex Analysis — An Introduction* (in Japanese) (Kyoritsu Publishing Co., Tokyo, 2001).

[16] K. Murota, *Discrete Convex Analysis*, (Society for Industrial and Applied Mathematics, Philadelphia, 2003).

[17] K. Murota and A. Shioura, M-convex function on generalized polymatroid, *Math. Oper. Res.* **24** (1999) 95–105.

[18] K. Murota and A. Shioura, Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati–Tardella, *Discrete Appl. Math.* **115** (2001) 151–176.

[19] K. Murota and A. Tamura, New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities, *Discrete Appl. Math.* to appear.

[20] A. E. Roth and M. A. O. Sotomayor, *Two-Sided Matching — A Study in Game-Theoretic Modeling and Analysis*, (Cambridge University Press, Cambridge, 1990).

[21] L. S. Shapley and M. Shubik, The assignment game I: The core, *Internat. J. Game Theory* **1** (1972) 111–130.

[22] M. Sotomayor, The lattice structure of the set of stable outcomes of the multiple partners assignment game, *Internat. J. Game Theory* **28** (1999) 567–583.

[23] M. Sotomayor, Existence of stable outcomes and the lattice property for a unified matching market, *Math. Social Sci.* **39** (2000) 119–132.

[24] M. Sotomayor, A labor market with heterogeneous firms and workers, *Internat. J. Game Theory* **31** (2002) 269–283.