Canonical curves of genus eight

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Let C be a smooth complete algebraic curve of genus g and $C_{2g-2} \subset \mathbb{P}^{g-1}$ the canonical model. It is generally difficult to describe its equations for higher genus. We restrict ourselves to the case of genus 8. If C has no g^2_7, then $C_{14} \subset \mathbb{P}^7$ is a transversal linear section $[G(2,6) \subset \mathbb{P}^{14}] \cap H_1 \cap \cdots \cap H_7$ of the 8-dimensional Grassmannian ([Muk2]). This is the case $\langle 8 \rangle$ of the flowchart below. In this article we study the case where C has a g^2_7 α. The system of defining equations of the canonical model is easily found from the following:

Theorem (i) *Assume that C has no g^1_4. If $\alpha^2 \cong K_C$, then C is the intersection of the* 6*-dimensional weighted Grassmannian w-$G(2,5) \subset \mathbb{P}(1^3 : 2^6 : 3)$ with a subspace $\mathbb{P}(1^3 : 2^2)$, where $w = (1,1,1,3,3)/2$ (Case $\langle 7 \rangle$ of Flowchart). Otherwise C is the complete intersection of three divisors of bidegree $(1,1), (1,2)$ and $(2,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (Case $\langle 6 \rangle$ of Flowchart).*

(ii) *Assume that C has a g^1_4 but no g^2_6. Then C is the complete intersection of four divisors of bidegree $(1,1), (1,1), (0,2)$ and $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ (Case $\langle 5 \rangle$ of Flowchart).*

Corollary *A curve of genus 8 is a complete intersection of divisors in a variety X which is either a non-singular toric variety or a weighted Grassmannian:*

Case	$\langle 1 \rangle$	$\langle 2 \rangle$	$\langle 3 \rangle$	$\langle 4 \rangle$	$\langle 5 \rangle$	$\langle 6 \rangle$	$\langle 7 \rangle$	$\langle 8 \rangle$
X	$\mathbb{F}_9$	$\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2$	$\mathbb{P}^1$-bundle over S_7	$Bl_p\ \mathbb{P}^3$	$\mathbb{P}^1 \times \mathbb{P}^4$	$\mathbb{P}^2 \times \mathbb{P}^2$	w-$G(2,5)$	$G(2,6)$
Cliff C	0	1	2			3		

Here S_7 is the blow-up of $\mathbb{P}^2$ at two points and the bottom row indicates the Clifford index of C.

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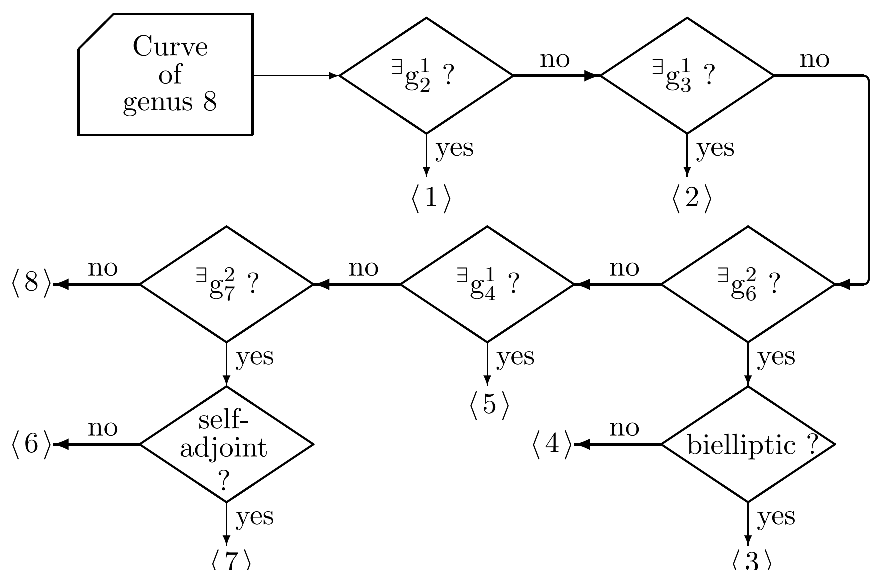
The essential part of the proof of Case $\langle 6 \rangle$ is the surjectivity of the multiplication map $H^0(\alpha) \otimes H^0(K_C\alpha^{-1}) \to H^0(K_C)$. In order to show this we investigate the linear section $[\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8] \cap H_1 \cap H_2$ of the Segre variety. (This is one of well-known descriptions of del Pezzo surfaces of degree 6.) We classify all degenerate cases in Appendix.

In the cases $\langle 6 \rangle$ and $\langle 7 \rangle$, the image of $\Phi_{|\alpha|} : C \to \mathbb{P}^2$ is a plane curve of degree 8 with 7 double points (counted with multiplicity). The blow-up $S \to \mathbb{P}^2$ at these points is very important for the theorem. In fact, in the former case, S is the minimal resolution of the intersection of two divisors $(1,1) \cap (2,1) \subset \mathbb{P}^2 \times \mathbb{P}^2$.

In the latter case $\langle 7 \rangle$ the 7 double points lie on the same conic. We use an extremal contraction of the log del Pezzo surface $(S, -K_S - \frac{1}{2}\Delta)$ to show the theorem, where Δ is the strict transform of the conic. The linear systems $|n(h + \frac{2}{3}\Delta)|$, $n = 1, 2, 3$, define a morphism of S to the weighted projective 5-space $\mathbb{P}(111223)$, where h is the total transform of a line. This contracts the (-3)-curve Δ to a point. Since C belongs to $|3h + 2\Delta|$, its image is contained in the 4-space $\mathbb{P}(11122)$. Moreover, $C \to \mathbb{P}(11122)$ is an embedding. In Section 3 we find five hypersurfaces containing the image of C explicitly and show that their equations are the five 4×4 Pfaffians of a skew symmetric matrix of size 5.



Flowchart

Remark. (1) A curve C of genus 7 is a complete intersection of divisors in a smooth variety X which is either toric or the 10-dimensional orthogonal

Grassmannian ([Muk3]).

Cliff$C$	0	1	2	3
X	$\mathbb{F}_8$	$\mathbb{F}_1, \mathbb{F}_3$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)), \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^3$	10-dim. orth. Grassmannian

(2)The corollary is applied to the K3-extension problem in [I].

Notation and convention: The canonical divisor of a variety X is denoted by K_X. A g^r_d is a line bundle of degree d and $h^0 \geq r+1$. For the tenser product $\alpha \otimes \beta$ of line bundles, we often omit the tensor symbol $\otimes$ and denote by $\alpha\beta$. Sometimes we denote by $\alpha + \beta$ as if they are (Cartier) divisors. The linear equivalence of divisors is denoted by $\sim$.

1 Curves with small Clifford index

In this section, we study the cases $\langle 1 \rangle, \cdots, \langle 5 \rangle$. For any $\mathbb{P}^r$-bundle $\mathbb{F} = \mathbb{P}(\mathcal{E})$ on a scheme X (where $\mathcal{E}$ is a vector bundle of rank $r+1$ on X), L denotes the tautological line bundle on $\mathbb{F}$ and π the structure morphism $\mathbb{F} \longrightarrow X$.

Case $\langle 1 \rangle$: hyperelliptic. There is a $2:1$-morphism

$$\Phi_{|K_C|} : C \longrightarrow \mathbb{P}^1$$

onto a rational curve. It is branched over 18 points, and these are distinct since C is smooth. Let $(s_0 : s_1)$ be a homogeneous coordinate of $\mathbb{P}^1$ and $f(s_0, s_1) = 0$ an equation of degree 18 defining branch loci. The double covering

$$\{y^2 - f(s_0, s_1) = 0\}$$

in the weighted projective space $\mathbb{P}(1:1:9)$ is isomorphic to C. Thus C is contained in the scroll $\mathbb{F}_9 = \mathbb{P}^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(9)) \longrightarrow \mathbb{P}(1:1:9)$ as a divisor linearly equivalent to $2L$.

Case $\langle 2 \rangle$: trigonal. The quadric hull

$$X := \bigcap_{\substack{Q:\text{quadric} \\ Q \supset C}} Q \subset \mathbb{P}^7$$

is a 2-dimensional rational normal scroll ([ACGH] III §3). Moreover, X is either $\mathbb{F}_{3,3} \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_{4,2} \cong \mathbb{P}^*(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.

Case $\langle 3\rangle$: bielliptic. There exists a two-to-one morphism

$$\pi : C \longrightarrow E_7 \subset \mathbb{P}^6$$

onto an elliptic curve E_7 of degree 7. E_7 is a hyperplane section of the del Pezzo surface $S := S_7 \subset \mathbb{P}^7$ of degree 7. The branch locus B of π is a sum of distinct 14 points, and therefore there exists a quadric hypersurface $Q \subset \mathbb{P}^7$ with $B = Q \cap E_7$. Since $S \subset \mathbb{P}^7$ is an embedding defined by the anti-canonical linear system of S, we have $E_7 \sim -K_S$ and $D := Q \cap E_7 \in |-2K_S|$. There is a double covering $\overline{S}$

$$\mathbb{P}^*(\mathcal{O}_S \oplus \mathcal{O}_S(-K_S)) \supset \overline{S} \xrightarrow{\pi} S$$

branched over D. C is the complete intersection of two divisors $\overline{S}$ and $\pi^{-1}(E_7)$.

Case $\langle 4\rangle$: tetragonal with a non-bielliptic g^2_6. Let α be a non-bielliptic g^2_6, and $\beta = K_C\alpha^{-1}$ its Serre adjoint, which is a g^3_8 by Riemann-Roch. Since the Clifford index of C is equal to 2, both α and β are base point free. The linear system $|\alpha|$ defines a birational morphism $\Phi_{|\alpha|} : C \longrightarrow C_6$ onto a plane sextic with two nodes, one of which may be infinitely near. Let

$$\pi : S \xrightarrow{\pi_2} S' \xrightarrow{\pi_1} \mathbb{P}^2$$

be the composite of the blow-ups at these nodes, h the pull-back of a line class of $\mathbb{P}^2$, e_1 the total transform of the exceptional divisor of π_1, and e_2 the exceptional divisor of π_2. Since $-K_S \sim 3h - e_1 - e_2$ and $C \sim 6h - 2e_1 - 2e_2$,

$$\begin{aligned} K_C &= (K_S + C)|_C = -K_S|_C \\ &= (3h - e_1 - e_2)|_C \\ &= h|_C + (2h - e_1 - e_2)|_C \end{aligned}$$

by the adjunction formula. Thus we have

$$\alpha = h|_C, \ \beta = (2h - e_1 - e_2)|_C, \text{ and } \beta\alpha^{-1} = (h - e_1 - e_2)|_C.$$

On the other hand, $|\beta|$ defines a birational morphism $\Phi_{|\beta|} : C \longrightarrow C_8 \subset \mathbb{P}^3$ onto a curve of degree 8. This morphism is extended to a morphism

$$\Phi_{|2h-e_1-e_2|} : S \longrightarrow S_2 \subset \mathbb{P}^3$$

from S onto a quadric surface S_2. If the two nodes of C_6 are distinct, then $|e_1 - e_2|$ is empty and S_2 is a smooth quadric. Otherwise, there is a unique

(-2)-curve $D \in |e_1 - e_2|$ with $D.C = 0$, and S_2 is the singular quadric. In each case, $\Phi_{|2h-e_1-e_2|}$ contracts a (-1)-curve $L \in |h - e_1 - e_2|$, which is the strict transform of the line passing through the two nodes of C_6, to a nonsingular point of S_2. Since $L.C = 2$, C_8 has one double point $q \in C_8$. Since C_8 is a curve of degree 8 and arithmetic genus $8 + 1 = 9$, it is a complete intersection of S_2 and a quartic surface $S_4 \subset \mathbb{P}^3$.

Let $\varphi : V \longrightarrow \mathbb{P}^3$ be the blow-up at q, H the pull-back of a hyperplane class of $\mathbb{P}^3$, and E the exceptional divisor. Then $-K_V \sim 4h - 2E$, and C is the complete intersection of two divisors of classes $2H - E \sim -\frac{1}{2}K_V$ and $4H - 2E \sim -K_V$ since $C + 2L \sim 4(2h - e_1 - e_2)$.

Case $\langle 5 \rangle$: tetragonal without g^2_6. Let α be a g^1_4 and $\beta = \omega_C \alpha^{-1}$ its Serre adjoint. Then the system $|\alpha|$ is base point free since C is not trigonal. Now β is a g^4_{10} by Riemann-Roch, and is very ample since C has no g^2_6.

Lemma 1.1. *The multiplication map*

$$\mu : H^0(\alpha) \otimes H^0(\beta) \longrightarrow H^0(K_C)$$

is surjective.

Proof. By the base point free pencil trick ([ACGH] III §3), the kernel of the multiplication map μ is $H^0(\alpha^{-1}\beta)$. If μ is not surjective, then the dimension of the kernel is at least 3. Therefore $\alpha^{-1}\beta$ is a g^2_6. This is a contradiction. $\square$

There is a commutative diagram of embeddings

$$\begin{array}{ccc}
\mathbb{P}^1 \times \mathbb{P}^4 & \xrightarrow{\text{Segre}} & \mathbb{P}^9 = \mathbb{P}^*(H^0(\alpha) \otimes H^0(\beta)) \\
{\scriptstyle \Phi_{|\alpha|} \times \Phi_{|\beta|}} \uparrow & & \uparrow {\scriptstyle \mu^*} \\
C & \xrightarrow{\text{canonical}} & \mathbb{P}^7 = \mathbb{P}^* H^0(\omega_C).
\end{array}$$

where μ^* is the linear embedding associated with the surjective multiplication map μ. By the lemma, the number of linearly independent $(1,1)$-forms vanishing on C is equal to 2. Therefore C is contained in the intersection Y of two divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^3$. Since every divisor of bidegree $(1,1)$ containing C is smooth, Y is smooth of dimension 3. Moreover $\operatorname{Pic} Y \cong \mathbb{Z}^2$ by Lefschetz theorem. By easy dimension count, there exists a divisor of bidegree $(1,2)$ and $(0,2)$ on Y which contain C. Since the degree of the complete intersection is

$$(a+b)^2.(a+2b).(2b).(a+b) = 14ab^3 = 14 = \deg C,$$

where $a = pr_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $b = pr_2^*\mathcal{O}_{\mathbb{P}^4}(1)$, the complete intersection $Y \cap (1,2) \cap (0,1)$ coincides with C.

2 Curve with a linear net of degree 7 (Case $\langle 6 \rangle$)

Assume that C has a g^2_7 α but no g^1_4. Let $\overline{C}$ be the image of the morphism $\Phi_{|\alpha|} : C \longrightarrow \mathbb{P}^2$ defined by α. Then $\overline{C}$ is a plane septic with no triple points, since C has no g^2_6 or g^1_4. By the genus formula, $\overline{C}$ has 7 double points, some of which may be infinitely near. Therefore, there are a composition π of seven one-point-blow-ups

$$S := S_7 \longrightarrow S_6 \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 = \mathbb{P}^2$$

from the projective plane and a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & \overline{C} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\pi} & \mathbb{P}^2. \end{array}$$

Let E_i, $1 \leq i \leq 7$, be the total transform on S of the exceptional divisor of the blow-up $S_i \longrightarrow S_{i-1}$, and let h be the pull back of a line. Then the curve C is a member of the linear system $|7h - 2\sum_{i=1}^{7} E_i|$. The canonical divisor of S is $K_S = -3h + \sum_{i=1}^{7} E_i$.

From the exact sequence

$$0 \to \mathcal{O}_S((n-7)h + \sum_{i=1}^{7} E_i) \to \mathcal{O}_S(nh - \sum_{i=1}^{7} E_i) \to \mathcal{O}_C(nh - \sum_{i=1}^{7} E_i) \to 0$$

and the vanishing

$$H^i(\mathcal{O}_S((n-7)h + \sum_{i=1}^{7} E_i)) = H^{2-i}(\mathcal{O}_S((4-n)h))^\vee = 0 \quad \text{for all } n \leq 6,$$

we have

Lemma 2.1. *The restriction map*

$$H^0(S, \mathcal{O}_S(nh - \sum_{i=1}^{7} E_i)) \longrightarrow H^0(C, \mathcal{O}_C(nh - \sum_{i=1}^{7} E_i))$$

is surjective for every n. Moreover, it is isomorphic for $n \leq 6$. $\square$

By the adjunction formula, we have

$$K_C = (K_S + C)|_C = (4h - \sum_{i=1}^{7} E_i)|_C = h|_C + (3h - \sum_{i=1}^{7} E_i)|_C,$$

and therefore the Serre adjoint $\beta = K_C\alpha^{-1}$ of the g^2_7 $\alpha \cong \mathcal{O}_C(h)$ is isomorphic to $\mathcal{O}_C(3h - \sum_{i=1}^{7} E_i)$. By Lemma 2.1, α is self adjoint, *i.e.*, $\alpha \cong \beta$, if and only if $|2h - \sum_{i=1}^{7} E_i| \neq \emptyset$. We discuss the case $\alpha \cong \beta$ in the next section, and now assume that $\alpha \ncong \beta$. The 7 double points of $\overline{C}$ does not lie on the same conic.

Proposition 2.2. *The multiplication map*

$$H^0(\alpha) \otimes H^0(\beta) \longrightarrow H^0(\alpha\beta) = H^0(K_C)$$

is surjective.

Proof. Assume the contrary. Then there are two independent $(1,1)$-forms on $\mathbb{P}^2 \times \mathbb{P}^2$. Let P be the pencil generated by them, and $X = X_P$ the base locus of P. If P contains a $(1,1)$-form of rank 1, then the image of $\Phi_{|\alpha|}$ is a line, which is a contradiction. Therefore P contains no $(1,1)$-forms of rank 1.

If P is a regular pencil, then by Appendix, X_P is irreducible. Let $\pi : X_P \longrightarrow \mathbb{P}^2$ be the restriction of the projection $\mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ onto the first factor. Then there is an effective divisor E such that

$$K_X = \pi^* K_{\mathbb{P}^2} + E.$$

On one hand, since $K_X = \mathcal{O}_X(-1,-1)$, $\pi^* K_{\mathbb{P}^2} = \mathcal{O}_X(-3,0)$, and $\mathcal{O}_X(C) = \mathcal{O}_X(7,7)$, we have

$$E.C = \mathcal{O}_X(2,-1).\mathcal{O}_X(7,7) = 7.$$

On the other hand, since I_P is of colength 3 (Proposition 4.3 of Appendix), we have a composition series

$$I_P = I_3 \subset I_2 \subset I_1 \subset \mathcal{O}_{\mathbb{P}^2}$$

of ideal sheaves. Therefore, there is a composition

$$\psi : X_3 := Bl_{I_3} X_2 \xrightarrow{\psi_3} X_2 := Bl_{I_2} X_1 \xrightarrow{\psi_2} X_1 := Bl_{I_1} \mathbb{P}^2 \xrightarrow{\psi_1} \mathbb{P}^2$$

of three one-point-blow-ups. The variety X_3 is clearly smooth, and has its canonical bundle

$$K_{X_3} = \psi^* K_{\mathbb{P}^2} + E_1 + E_2 + E_3,$$

where E_i is the total transform of the exceptional divisors of ψ_i. By the universal property of the blow-up ([H] II 7.14), there is a morphism

$$\psi' : X_3 \longrightarrow X_P = Bl_{I_P} \mathbb{P}^2.$$

(If P is general, then $V(I_P)$ consists of distinct three points, and ψ' is an isomorphism.) Since X_P is a complete linear section of $\mathbb{P}^2 \times \mathbb{P}^2$, it has at worst rational double points as its singularities. Thus ψ' is a crepant resolution, and we have

$$E_1 + E_2 + E_3 = \psi'^* E.$$

Therefore, for some $i = 1, 2, 3$, we have $E_i.C \geq 3$, and $\psi^* \mathcal{O}_{\mathbb{P}^2}(1) - E_i$ restricts to a g^1_d with $d \leq 4$ on C. This is a contradiction.

If P is singular, then by Appendix, X_P is either

I) $\Delta \cup (\mathbb{P}^1 \times \mathbb{P}^1)$, or

II) $\mathbb{F}_{3,2} \cup (p \times \mathbb{P}^2)$.

If X_P is of type I, then C is contained in either the diagonal Δ or $\mathbb{P}^1 \times \mathbb{P}^1$. The former means that α is isomorphic to β, and the latter means that the image of $\Phi_{|\alpha|}$ and $\Phi_{|\beta|}$ are both lines. This is a contradiction.

If X_P is of type II, then C is contained in either $\mathbb{F}_{3,2}$ or $p \times \mathbb{P}^2$. In the former case, the image of $\Phi_{|\beta|}$ is contained in a conic, and in the latter case, the image of $\Phi_{|\alpha|}$ is a point. Thus we have a contradiction. $\square$

Now, we first consider the commutative diagram

$$\begin{array}{ccccc} H^0(S,h) & \otimes & H^0(S, 3h - \sum_{i=1}^{7} E_i) & \longrightarrow & H^0(S, 4h - \sum_{i=1}^{7} E_i) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(C,\alpha) & \otimes & H^0(C,\beta) & \longrightarrow & H^0(C, K_C) \end{array}$$

whose horizontal and vertical maps are multiplication and restriction, respectively. The vertical maps are all isomorphisms by Lemma 2.1. Hence the image of the rational map

$$(\Phi_{|h|}, \Phi_{|3h - \sum E_i|}) : S \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$$

is contained in a divisor W of bidegree $(1,1)$ (This rational map is a morphism in the neighborhood of C). It is obvious that W is irreducible.

Next, we consider the commutative diagram

$$\begin{array}{ccccc} H^0(S,2h) & \otimes & H^0(S, 3h - \sum_{i=1}^{7} E_i) & \longrightarrow & H^0(S, 5h - \sum_{i=1}^{7} E_i) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(C,\alpha^2) & \otimes & H^0(C,\beta) & \longrightarrow & H^0(C, \alpha K_C). \end{array}$$

The vertical restriction maps are all isomorphisms. The dimension of the kernel of the horizontal multiplication map is at least

$$\binom{2+2}{2} \times 3 - h^0(\alpha K_C) = 4.$$

Hence there is a divisor W' of bidegree $(2,1)$, which does not contain W but contains the image of S. The intersection $W \cap W'$ contains $C \subset \mathbb{P}^2 \times \mathbb{P}^2$.

Next, we consider the divisor class of bidegree $(1,2)$. Its pull-back to S is

$$h + 2(3h - \sum_{i=1}^{7} E_i) = 7h - 2\sum_{i=1}^{7} E_i,$$

and therefore, is linearly equivalent to C. We now study the 15 quadrics which vanish on the canonical model $C_{14} \subset \mathbb{P}^7$ of C. First, C_{14} is contained in a hyperplane section of the Segre variety

$$[W \subset \mathbb{P}^7] = [\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8] \cap H,$$

and there are exactly 9 quadrics vanishing on W. Next, there are 3 quadrics which cut out $W \cap W'$ from W. Finally, since the pull-back of $\mathcal{O}_{\mathbb{P}^7}(2)$ to S is $\mathcal{O}_S(2(4h - \sum_{i=1}^{7} E_i)) = \mathcal{O}_S(C+h)$, and since we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_S(h) & \longrightarrow & \mathcal{O}_S(C+h) & \longrightarrow & \mathcal{O}_C(C+h) & \longrightarrow & 0, \\ & & & & \wr\Vert & & \wr\Vert & & \\ & & & & \mathcal{O}_S(2) & & \mathcal{O}_C(2) & & \end{array}$$

there are 3 more independent quadrics vanishing on C. Thus we have found $9+3+3 = 15$ independent quadrics vanishing on C. By Noether's theorem, they form a basis of $H^0(\mathbb{P}^7, \mathcal{I}_C(2))$, and by the Enriques-Petri theorem ([GH], Chap. 4), they define the canonical model $C_{14} \subset \mathbb{P}^7$ scheme-theoretically. Thus C is the complete intersection of divisors $(1,2)$ and $(2,1)$ in W.

3 Curves with a self adjoint net (Case $\langle 7 \rangle$)

Let Δ be the unique member of $|2h - \sum_{i=1}^{7} E_i|$ and $\bar{\Delta} \subset \mathbb{P}^2$ its image. Then $\bar{\Delta}$ is an irreducible conic. We choose homogeneous coordinates of $\Delta \cong \bar{\Delta} \cong \mathbb{P}^1$ and $\mathbb{P}^2$ such that the morphism $\Delta \to \mathbb{P}^2$ is given by $(s : t) \mapsto (x_0 : x_1 : x_2) = (s^2 : st : t^2)$. The surface S is the blow-up at seven points on $\bar{\Delta}$. Let $f(s,t) = 0$ be the equation of degree 7 over $\bar{\Delta}$ whose solutions are the center of this blow-up. We shall construct a key polynomial $F(x) \in H^0(S, \mathcal{O}_S(7h - 2\sum_{i=1}^7 E_i))$ which is determinantal in a certain sense. This will imply that the system of equations of $C \subset \mathbb{P}(1:1:1:2:2)$ is Pfaffian of skew-symmetric matrices.

We start with a pair of ternary quartic polynomials $A(x)$ and $B(x)$ such that

$$A(s^2, st, t^2) = sf(s,t) \quad \text{and} \quad B(s^2, st, t^2) = tf(s,t).$$

Such polynomials exist by the exact sequence

$$\begin{array}{ccccc} 0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(4)) \to & H^0(\bar{\Delta}, \mathcal{O}_{\bar{\Delta}}(4)) & \to 0. \\ & \wr\Vert & \\ & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)) & \end{array} \tag{1}$$

Since $tA(s^2, st, t^2) - sB(s^2, st, t^2)$ is zero, the quintic polynomials $x_1 A(x) - x_0 B(x)$ and $x_2 A(x) - x_1 B(x)$ are divisible by $\delta(x)$, the equation of $\bar{\Delta} \subset \mathbb{P}^2$. We put

$$\begin{cases} -x_0 B(x) \quad +x_1 A(x) & = \delta(x) D(x) \\ \qquad\qquad -x_1 B(x) \quad +x_2 A(x) & = \delta(x) E(x), \end{cases} \tag{2}$$

where $D(x)$ and $E(x)$ are cubic forms. Put

$$\begin{cases} D(x) = q_0(x) x_0 + q_1(x) x_1 + q_2(x) x_2 \\ E(x) = r_0(x) x_0 + r_1(x) x_1 + r_2(x) x_2 \end{cases} \tag{3}$$

for quadratic forms $q_i(x)$'s and $r_i(x)$'s and we obtain

$$\begin{cases} [B(x) + q_0(x)\delta(x)] x_0 + [-A(x) + q_1(x)\delta(x)] x_1 + q_2(x)\delta(x) x_2 = 0 \\ r_0(x)\delta(x) x_0 + [B(x) + r_1(x)\delta(x)] x_1 + [-A(x) + r_2(x)\delta(x)] x_2 = 0. \end{cases} \tag{4}$$

By Cramer's formula we have

$$\frac{\begin{vmatrix} -A + q_1\delta & q_2\delta \\ B + r_1\delta & -A + r_2\delta \end{vmatrix}}{x_0} = \frac{\begin{vmatrix} q_2\delta & B + q_0\delta \\ -A + r_2\delta & r_0\delta \end{vmatrix}}{x_1} = \frac{\begin{vmatrix} B + q_0\delta & -A + q_1\delta \\ r_0\delta & B + r_1\delta \end{vmatrix}}{x_2} := F(x). \tag{5}$$

Here $F(x)$ is a form of degree 7 since $x_i F(x)$ is a form of degree 8 for $i = 0, 1, 2$. Let y_0, y_1 and z be new indeterminates which are algebraically independent over $k(x_0, x_1, x_2)$. We consider the ring homomorphism

$$\varphi_S : k[x_0, x_1, x_2, y_0, y_1, z] \longrightarrow k\left[x_0, x_1, x_2, \frac{1}{\delta(x)}\right],$$

$$y_0 \mapsto \frac{A(x)}{\delta(x)},\ y_1 \mapsto \frac{B(x)}{\delta(x)},\ z \mapsto \frac{F(x)}{\delta(x)^2}$$

and its kernel I_S. Then I_S is a (quasi-)homogeneous ideal under the grading $\deg x_i = 1$, $\deg y_j = 2$ and $\deg z = 3$. By the equation (4), two cubic forms

$$\begin{aligned} &a_0(x,y)x_0 + a_1(x,y)x_1 + a_2(x,y)x_2, \text{ and} \\ &b_0(x,y)x_0 + b_1(x,y)x_1 + b_2(x,y)x_2 \end{aligned} \tag{6}$$

belong to I_S, where we put

$$\begin{array}{lll} a_0(x,y) = y_1 + q_0(x), & a_1(x,y) = -y_0 + q_1(x), & a_2(x,y) = q_2(x) \\ b_0(x,y) = r_0(x), & b_1(x,y) = y_1 + r_1(x), & b_2(x,y) = -y_0 + r_2(x). \end{array}$$

By (5), three quartic forms

$$\begin{aligned} &x_0 z - \begin{vmatrix} a_1(x,y) & a_2(x,y) \\ b_1(x,y) & b_2(x,y) \end{vmatrix}, \quad \begin{vmatrix} a_2(x,y) & a_0(x,y) \\ b_2(x,y) & b_0(x,y) \end{vmatrix} - x_1 z, \\ &x_2 z - \begin{vmatrix} a_0(x,y) & a_1(x,y) \\ b_0(x,y) & b_1(x,y) \end{vmatrix} \end{aligned} \tag{7}$$

belong to I_S. These five relations (6) and (7) are the five 4×4 Pfaffians of the skew-symmetric matrix

$$\begin{pmatrix} 0 & z & a_0(x,y) & a_1(x,y) & a_2(x,y) \\ & 0 & b_0(x,y) & b_1(x,y) & b_2(x,y) \\ & & 0 & x_2 & -x_1 \\ & & & 0 & x_0 \\ & \ominus & & & 0 \end{pmatrix}. \tag{8}$$

Now we relate the ideal I_S with the anti-canonical ring of a weak log del Pezzo surface. Let

$$R := \bigoplus_{n \geq 0} H^0(S, \lfloor n(-K_S - \frac{1}{3}\Delta) \rfloor) \cong \bigoplus_{n \geq 0} H^0(S, \lfloor n(h + \frac{2}{3}\Delta) \rfloor)$$

be the homogeneous coordinate ring of the $\mathbb{Q}$-divisor $-K_S - \frac{1}{3}\Delta$, which is linearly equivalent to $h + \frac{2}{3}\Delta$. For a global section $s \in H^0(S, n(h + \frac{2}{3}n)) = H^0(S, nh + a\Delta) = H^0((n+2a)h - a\sum_{i=1}^7 E_i)$, $a = \lfloor \frac{2}{3}n \rfloor$, its push-forward $\pi_* s \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n+2a))$ is a homogeneous polynomial of degree $n + 2a$. We identify R with the image of the injective ring homomorphism $\psi : R \longrightarrow k[x_0, x_1, x_2, 1/\delta(x)]$ defined by

$$H^0(S, nh + a\Delta) \ni s \mapsto \frac{\pi_* s}{\delta(x)^a} \in k[x_0, x_1, x_2, \frac{1}{\delta(x)}]_n.$$

The degree 1 part $H^0(S,h)$ is spanned by the homogeneous coordinates x_0, x_1, x_2. The degree 2 part $H^0(S, 2h+\Delta)$ contains $S^2\langle x_0, x_1, x_2\rangle$ as a subspace. The pull-back of the quartic forms $A(x)$ and $B(x)$ to S belong to $H^0(S, 4h - \sum_{i=1}^7 E_i)$ and $\{A(x)/\delta(x), B(x)/\delta(x)\}$ is a complementary basis of $S^2\langle x_0, x_1, x_2\rangle \subset H^0(S, \mathcal{O}_S(2h+\Delta))$ by the exact sequence (1). Consider the multiplication map

$$\begin{array}{ccc} H^0(S,h)\otimes H^0(S,2h+\Delta) \longrightarrow & H^0(S,3h+\Delta) & \cong H^0(S,5h-\sum_{i=1}^7 E_i) \\ & \cap & \\ & H^0(S,3h+2\Delta) & \cong H^0(S,7h-2\sum_{i=1}^7 E_i) \end{array} \tag{9}$$

from degree 1 and 2 to degree 3. Since the restriction maps $H^0(S,h) \longrightarrow H^0(\mathcal{O}_\Delta(h))$ and $H^0(S,2h+\Delta) \longrightarrow H^0(\mathcal{O}_\Delta(2h+\Delta))$ are surjective, so is this multiplication map. By the exact sequence

$$\begin{array}{c} 0 \to \mathcal{O}_S(5h - \sum_{i=1}^7 E_i) \to \mathcal{O}_S(7h - 2\sum_{i=1}^7 E_i) \to \mathcal{O}_\Delta(7h - 2\sum_{i=1}^7 E_i) \to 0, \\ \hspace{8em} \cong \\ \hspace{8em} \mathcal{O}_\Delta \end{array} \tag{10}$$

the degree 3 part $H^0(S, 3h+2\Delta)$ is generated by the image of (9) and $F(x)/\delta(x)^2$.

Now we relate the ideal I_S with the (semi-)canonical ring of $C \in |7h - 2\sum_{i=1}^7 E_i|$. Since C is disjoint from Δ and since $\mathcal{O}_C(h+\frac{2}{3}\Delta) \cong \alpha$, we have the restriction map

$$H^0(S, \lfloor n(h+\frac{2}{3}\Delta)\rfloor) \longrightarrow H^0(C, \alpha^n) \tag{11}$$

and the restriction homomorphism

$$R \longrightarrow R(C,\alpha) := \bigoplus_{n\geq 0} H^0(C,\alpha^n).$$

Since (11) is an isomorphism for $n=1$ and 2, the ring homomorphisms φ_S and ψ induce that

$$\varphi_C : k[x_0, x_1, x_2, y_0, y_1] \longrightarrow R(C,\alpha)$$

to the semi-canonical ring.

The equation of $\bar{C} \subset \mathbb{P}^2$, or $C \subset S$, is of the form $F(x) + \delta(x)G(x)$ for a quintic form $G(x) \in H^0(S, 5h - \sum_{i=1}^{7} E_i)$. There exists a cubic form $c(x, y_0, y_1)$ such that $c(x, A(x)/\delta(x), B(x)/\delta(x)) = G(x)/\delta(x)^2$ and there exists a commutative diagram

$$\begin{array}{ccc} k[x_0, x_1, x_2, y_0, y_1, z] & \xrightarrow{\varphi_S} & R \\ \downarrow & & \downarrow \\ k[x_0, x_1, x_2, y_0, y_1] & \xrightarrow[\varphi_C]{} & R(C, \alpha), \end{array}$$

where the left vertical map is the specialization of z to the degree 3 element $c(x, y)$. Hence the five 4×4 Pfaffians of

$$\begin{pmatrix} 0 & c(x,y) & a_0(x,y) & a_1(x,y) & a_2(x,y) \\ & 0 & b_0(x,y) & b_1(x,y) & b_2(x,y) \\ & & 0 & x_2 & -x_1 \\ & \ominus & & 0 & x_0 \\ & & & & 0 \end{pmatrix} \tag{12}$$

belongs to the kernel of φ_C.

Now we prove Theorem in case $\langle 7 \rangle$. Let $C \subset \mathbb{P}^8 = \mathbb{P}^* H^0(K_C)$ be the canonical model of C. Since $\mathrm{Sym}^2 H^0(\alpha) \subset H^0(K_C)$, C is contained in the join of the Veronese surface and a line. This join is nothing but the weighted projective space $\mathbb{P}(1:1:1:2:2)$ whose coordinates are x_0, x_1, x_2, y_0, y_1. Two polynomials (6) vanish on C. Multiplying these by x_0, x_1 and x_2, we obtain 6 relations of degree 4, which are linearly independent. Together with 3 relations (7) of degree 4, the five Pfaffians of (12) generate 9 quartic forms on $\mathbb{P}(1:1:1:2:2)$ vanishing on C. There are 6 quadratic forms vanishing on $\mathbb{P}(1:1:1:2:2) \subset \mathbb{P}^7$. Hence we have 15 quadratic forms vanishing on $C \subset \mathbb{P}^7$. These are all quadratic forms vanishing on C. Hence the five Pfaffians cut out C scheme-theoretically from $\mathbb{P}(1:1:1:2:2)$ by the Enriques-Petri theorem.

4 Appendix: Linear section of the Segre variety and a pencil of matrices

Let $M_3(k)$ be the space of 3×3-matrices over a field k. The 8-dimensional projective space $\mathbb{P}^8 = \mathbb{P}^* M_3(k)$ naturally contains the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$. For a linear subspace P of $M_3(k)$, we consider the linear section

$$X_P := (\mathbb{P}^2 \times \mathbb{P}^2) \cap H_P$$

of the Segre variety, where $H_P = \mathbb{P}^*(M_3(k)/P)$ is the linear subspace defined by P. If A is a non-zero matrix, we denote by X_A instead of $X_{\langle A \rangle}$.

For a non-zero matrix $A = (a_{ij})_{0 \le i,j \le 2}$, we define a $(1,1)$-form

$$f_A(x,y) = \sum_{0 \le i,j \le 2} a_{ij} x_i y_j,$$

where $(x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2)$ is the coordinate of $\mathbb{P}^2 \times \mathbb{P}^2$. Then X_P is the common zero locus of $f_A(x,y)$, $A \in P$, in $\mathbb{P}^2 \times \mathbb{P}^2$.

Lemma 4.1. *For a non-zero matrix A, we have*
1) $\operatorname{rank} A = 1$ *if and only if X_A is reducible,*
2) $\operatorname{rank} A = 2$ *if and only if X_A is singular at one point, and*
3) $\operatorname{rank} A = 3$ *if and only if X_A is smooth.*

We consider the case that P is a pencil, in other words, P is a 2-dimensional subspace of $M_3(k)$. If P contains an invertible matrix, we call it *regular*, otherwise we call *singular*. Let

$$\pi : X_P \longrightarrow \mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1, x_2]$$

be the restriction of the projection $\mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ onto the first factor. Let $\{A, B\}$ be a basis of P and

$$\begin{aligned} f_A(x,y) &= a_0(x)y_0 + a_1(x)y_1 + a_2(x)y_2, \\ f_B(x,y) &= b_0(x)y_0 + b_1(x)y_1 + b_2(x)y_2, \end{aligned}$$

the $(1,1)$-forms corresponding A and B respectively. Let I_P be the ideal of $k[x_0, x_1, x_2]$ (or, the ideal sheaf of $\mathcal{O}_{\mathbb{P}^2}$) generated by the three minors

$$D_0 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix},$$

of the 2×3-matrix

$$\begin{pmatrix} a_0(x) & a_1(x) & a_2(x) \\ b_0(x) & b_1(x) & b_2(x) \end{pmatrix}.$$

Then I_P is independent of the choice of the basis of P, and the common zeros $V(I_P) \subset \mathbb{P}^2$ is the locus where $\pi : X_P \longrightarrow \mathbb{P}^2$ is not isomorphic.

If P is regular, then the divisor $Y := X_A$, corresponding to an invertible matrix $A \in P$, is nonsingular and the projections $Y \longrightarrow \mathbb{P}^2$ onto each factor are $\mathbb{P}^1$-bundles. By the Lefshetz Theorem, the Picard number of Y is 2, and the Picard group is generated by $\mathcal{O}_Y(1,0)$ and $\mathcal{O}_Y(0,1)$. Thus if X_P

is reducible, it must be a sum of divisors of bidegree $(1,0)$ and $(0,1)$ on Y, *i.e.*, a section of Y by $(1,1)$-forms of rank 1. X_P is a union of two cubic scrolls

$$X_P = \mathbb{F}_{2,1} \cup \mathbb{F}_{1,2}.$$

So we have

Proposition 4.2. *If P is regular and contains no member of rank 1, then X_P is irreducible.*

It is well known that π is the blow-up at three points if X_P is smooth.

Proposition 4.3. *Let P and X_P be as in the above proposition. Then $V(I_P)$ is of dimension 0, and $I_P \subset \mathcal{O}_{\mathbb{P}^2}$ is of colength 3. Moreover $\pi : X_P \longrightarrow \mathbb{P}^2$ is the blow-up with center I_P.*

Proof. If the dimension of $V(I_P)$ is more than or equal to 1, then the inverse image $\pi^{-1}V(I_P)$ must have dimension more than or equal to 2. This is impossible since X_P is irreducible of dimension 2.

There is a natural embedding

$$\begin{array}{rcl} \varphi : Bl_{I_P}\mathbb{P}^2 & \hookrightarrow & \mathbb{P}^2 \times \mathbb{P}^2 \\ & & \cup\!\!\!\!\cup \\ & & (x_0 : x_1 : x_2) \times (D_0 : D_1 : D_2) \end{array}$$

of the blow-up $Bl_{I_P}\mathbb{P}^2$ into the Segre variety. Since

$$\begin{aligned} & a_0(x)D_0 + a_1(x)D_1 + a_2(x)D_2 \\ &= \begin{vmatrix} a_0 & a_1 & a_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0, \end{aligned}$$

and also $b_0D_0 + b_1D_1 + b_2D_2 = 0$, the image of φ is contained in X_P. Since X_P is irreducible and reduced, the image of φ coincides with X_P. $\square$

If P is singular, then by Kronecker's classification ([Ga] Chap. XII), we have three types (after suitable change of the coordinate and factors of

$\mathbb{P}^2 \times \mathbb{P}^2$),

$$\begin{aligned}
&\text{type I} \quad P = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle, \\
&\text{type II} \quad P = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle, \text{ and} \\
&\text{type III} \quad P = \left\langle \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle.
\end{aligned}$$

If P is of type I, then the defining equation of X_P is

$$x_0y_1 - x_1y_0 = x_0y_2 - x_2y_0 = 0,$$

and hence

$$X_P = \Delta \cup (\mathbb{P}^1 \times \mathbb{P}^1),$$

where $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the diagonal. In this case, all the non-zero members of P are of rank 2.

If P is of type II, then the defining equation of X_P in $\mathbb{P}^2 \times \mathbb{P}^2$ is

$$x_0y_0 - x_1y_1 = x_0y_0 - x_1y_2 = 0,$$

and hence X_P is the union

$$\begin{aligned}
X_P &= \overline{\{(1:\lambda:\mu) \times (\lambda^2:\lambda:1) | \lambda, \mu \in k\}} \cup (0:0:1) \times \mathbb{P}^2 \\
&= \mathbb{F}_{3,2} \cup (p \times \mathbb{P}^2),
\end{aligned}$$

of a quintic scroll and a plane. All non-zero members are of rank 2 in this case also.

If P is of type III, then by the classification of Jordan normal form of 2×2-matrices, the defining equation of X_P is either

$$\begin{aligned}
&x_0y_0 + x_1y_1 = x_iy_j = 0 \quad \text{for some } 0 \le i \le j \le 1, \text{ or} \\
&x_0y_0 = x_0y_1 = 0.
\end{aligned}$$

In the former case, X_P is the union of two $\mathbb{P}^2$'s and two $\mathbb{P}^1 \times \mathbb{P}^1$'s. In the latter case, X_P is the union of $\mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbb{P}^2 \times (0:0:1)$.

P	rank 1	X_P	degree
regular	×	$Bl_{I_P}\mathbb{P}^2$	6
	○	$\mathbb{F}_{2,1} \cup \mathbb{F}_{1,2}$	$3+3$
singular	×	$\Delta \cup (\mathbb{P}^1 \times \mathbb{P}^1)$	$4+2$
		$\mathbb{F}_{3,2} \cup (p \times \mathbb{P}^2)$	$5+1$
	○	$2\mathbb{P}^2 \cup 2(\mathbb{P}^1 \times \mathbb{P}^1)$	$1+1+2+2$
		$(\mathbb{P}^1 \times \mathbb{P}^2) \cup (\mathbb{P}^2 \times p)$	$(3+1)$

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