A Unified Scheme for Generalized Sectors based on Selection Criteria*
–Order parameters of symmetries and of thermality and physical meanings of adjunctions–

Dedicated to the memory of Moshé Flato

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Abstract

A unified scheme for treating generalized superselection sectors is proposed on the basis of the notion of selection criteria to characterize states of relevance to each specific domain in quantum physics, ranging from the relativistic quantum fields in the vacuum situations with unbroken and spontaneously broken internal symmetries, through equilibrium and non-equilibrium states to some basic aspects in measurement processes. This is achieved by the help of $c \to q$ and $q \to c$ channels: the former determines the states to be selected and to be parametrized by the order parameters, and the latter provides the physical interpretations of selected states in terms of order parameters. This formulation extends the traditional range of applicability of the Doplicher-Roberts construction method for recovering the field algebra and the gauge group (of the first kind) from the data of group invariant observables to the situations with spontaneous symmetry breakdown: in use of the machinery proposed, the physical and mathematical meaning of basic structural ingredients associated with the spontaneously broken symmetry are re-examined, such as the degenerate vacua parametrized by the variable belonging to the relevant homogeneous space, the Goldstone modes and condensates, etc. The geometrical meaning of the space of order parameters is naturally understood in relation with the adjunction as the classifying space of a sector structure. As further examples of applications, some basic notions arising in the mathematical framework of quantum theory are reformulated and examined in connection with control theory.

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1 Introduction

The standard way of treating the microscopic world on the basis of quantum field theory (QFT for short) is to introduce first the *quantum fields* whose characterization is given by means of their behaviours under the various kinds of *symmetries*; e.g., the internal symmetry groups such as colour $SU(3)$, chiral $SU(2)$, electromagnetic $U(1)$, or any other bigger (super)groups of grand unifications (and their corresponding versions of local gauge symmetries), in combination with spacetime symmetry groups such as Poincaré group in Minkowski spacetime, conformal groups in massless theories, or isometry groups of curved spacetimes, and so on. In a word, basic objects of such a system can essentially be found in an algebra $\mathfrak{F}$ of quantum fields (called a *field algebra* for short) acted upon by two kinds of symmetries, *internal* and *spacetime* (whose unification has been pursued as one of the ultimate goals of microscopic physics). With respect to the group of an internal symmetry denoted generically by G, the generating elements of $\mathfrak{F}$ (usually called *basic* or *fundamental fields*) are assumed (*by hand*) to belong to certain multiplet(s) transforming covariantly under the action of G, which defines mathematically an action τ of G on $\mathfrak{F}$: $G \underset{\tau}{\curvearrowright} \mathfrak{F}$.

Contrary to this kind of *theoretical* setting, what can be observed (experimentally) in the real world is believed (or, can be proved under a certain setup; for instance, see [37]) to be only elements in $\mathfrak{F}$ *invariant under* G and are usually called *observables* which constitute the algebra $\mathfrak{A}$ of observables:

$$\mathfrak{A} := \mathfrak{F}^G = \{A \in \mathfrak{F}; \tau_g(A) = A \text{ for } \forall g \in G\}, \tag{1}$$

the fixed-point subalgebra of $\mathfrak{F}$ under the action τ of G. Thus, what we can directly check experimentally is supposed to be only those data described in terms of $\mathfrak{A}$ (and its derived objects) and the rest of the notions appearing in our framework are just mathematical devices whose pertinence can be justified only through the information related to $\mathfrak{A}$. Except for the systematic approaches [16, 19, 25] undertaken by the pioneers in algebraic QFT [25], however, there have so far been no serious attempts to understand the basic mechanism pertaining to this point as to how a particular choice of $\mathfrak{F}$ and G can be verified, with the problems of this sort left aside just to the heuristic arguments based on trials and errors. While a particularly chosen combination $G \underset{\tau}{\curvearrowright} \mathfrak{F}$ is no doubt meaningless without good agreements of its consequences with the observed data described in terms of $\mathfrak{A}$, the attained agreements support the postulated theoretical assumption only as *one of many possible candidates* of explanations, without justifying it as a *unique* inevitable solution. (Does it not look quite strange that such a kind of problems as this have hardly been examined in the very sophisticated discussions about the unicity of the unification models at the Planck scale?)

Just when restricted to the cases with G an *unbroken* global gauge symmetry (or, gauge symmetry of the first kind), a satisfactory framework in this context has been established in the *superselection theory* of Doplicher-Haag-Roberts (DHR) [16] and of Doplicher-Roberts (DR) [18, 19] (DHR-DR sector theory for short) in algebraic QFT, whose physical essence has, unfortunately, not been recognized widely (which may be partly due to its mathematical sophistication, but mainly due to the lack of common understanding of the importance of the above-mentioned problem). What is marvelous about this theory is that it enables one to *recover both* $\mathfrak{F}$ *and* G starting only from the data encoded in $\mathfrak{A}$ when supplemented by the so-called DHR *selection criterion* [5, 16] to choose physically relevant states with localizable charges (which need, in the case of topological charges, be modified as in [12]). Then the vacuum representation of the so constructed field algebra $\mathfrak{F}$ is decomposed into mutually disjoint irreducible representations of $\mathfrak{A} = \mathfrak{F}^G$, called *superselection sectors* (or *sectors* for short), in one-to-one correspondence with mutually disjoint irreducible unitary representations of the internal symmetry group G which is found to be *compact Lie*. While the essence of this theory will be briefly summarized in Sec.3 in a reformulated form convenient for the present context, it may be meaningful to mention some general aspects of it for the sake of explaining the reason why we think the analysis of spontaneous symmetry breakdown (SSB for short) as important.

Among the most important consequences of the DHR-DR sector theory, we mention here that the familiar Bose/Fermi statistics of the basic fields is automatically derived from the *local net structure* $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ consisting of *local subalgebras* $\mathfrak{A}(\mathcal{O})$ *of observables* in spacetime regions $\mathcal{O}$ satisfying the *local commutativity* (i.e., Einstein causality), without necessity of introducing from the outset *unobservable* field operators such as *fermionic fields* subject to local *anti*commutativity *violating Einstein causality*; this shows that fermionic fields are, in a sense, simple mathematical devices for bookkeeping of half-integer spin states. Then all the non-trivial spacetime behaviours are described here by the observable net $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$, while the internal symmetry aspects are encoded in the *sector structure*, which also originates from the observable net.

The symmetry arising from this beautiful theory is, however, destined to be *unbroken*, excluding the situation of SSB, which poses a question about the "**stability**" of this method, as remarked by the late Moshé Flato [20]. Indeed we know that many (actually, almost all) of the "sacred symmetries" in nature can be broken (explicitly or spontaneously) in various situations: e.g., SSB's of chiral symmetry in the electro-weak theory based upon $SU(2) \times U(1)$, electromagnetic $U(1)$ in the superconductivity, and the rotation symmetry $SO(3)$ in ferromagnetism, etc. So, the question as to whether or not this theory can incorporate systematically the cases of SSB is a real challenge to it, deserving serious examination, and if the answer is yes, what kind of sector structure is realized in that case is another non-trivial interesting question. This sort of investigation is expected also to give us some important hints for getting rid of another restriction of *global* gauge symmetries so as to incorporate *local* gauge symmetries.

In the following, we give an affirmative answer to the above question, clarifying the sector structure emerging from SSB. For this purpose, we note that the traditional notion of sector structures has hinged strongly to the essential features of unbroken symmetry, allowing only the *discrete sectors* which are parametrized by the discrete $\hat{G}$, the dual of a compact group defined as the set of all equivalence classes of finite-dimensional continuous unitary irreducible representations of G. When we start to extend this formalism to the situations with SSB, we encounter the presence of *continuous sectors* (or, "degenerate vacua" in the traditional terminology) parametrized by continuous *macroscopic order parameters*, as is seen in Sec.4.

This forces us to extend the notion of sectors to incorporate the continuous ones. Once we take this viewpoint, then we notice that unexpectedly wide perspectives open out before us: aside from such very fundamental issues as the "ultimate" unifications, we have so far faced with so many different levels and domains of physical nature in the directions from microscopic worlds to macroscopic ones, ranging from the vacuum situations (the standard QFT relevant to particle physics), thermal equilibria (QFT at finite temperatures or quantum statistical mechanics), non-equilibrium ones and so on, where we find a huge supply of examples of continuous sectors. In [13], a general framework is proposed for defining non-equilibrium local states in relativistic QFT and for describing their thermodynamic properties in terms of the associated macroscopic observables found in the *centre* of a (kind of) "universal" representation containing all the representations of observables relevant to the context. From this general standpoint, one easily notices that the thermal equilibria at different temperatures can be seen to constitute families of continuous sectors parametrized by such thermodynamic variables as temperatures, chemical potentials and pressure and so on. In view of such roles of central observables associated with continuous sectors appearing in SSB cases as well as the above various kinds of thermal states, it is appropriate now to try the possibility of unified ways of treating these different cases, just regarding the traditional discrete ones as special cases; this is simply parallel to the extension of the traditional *eigenvalue* problems for linear operators with *discrete spectra* to the general *spectral decompositions* admitting the appearance of *continuous spectra*.

Thus the aim of the present paper is threefold: to incorporate SSB into the sector theory, we extend the notion of discrete sectors to continuous ones, through which we are led to a unified scheme to treat such a generalized notion of sectors. The key roles are played by the *selection criterion* set up at the starting point of theory in defining and choosing *physically relevant family of states* as well as in providing a systematic way for *describing* and *interpreting* relevant physical properties.

We introduce the necessary ingredients for formulating the scheme through the discussions on the basic structures found in thermal situations of equilibrium and of the extension to non-equilibrium (Sec.2) and in an operational reformulation of DHR-DR sector theory (Sec.3). Here the general mathematical meaning of selection criterion is found in the *adjunction* as a tool for controlling the mutual relations between generic objects to be characterized and special objects (such as temperatures or order parameters, in general) serving as standard reference systems in the context of classification and interpretation.

In Sec.4, we apply our unified method of treating generalized sectors to the situation with SSB by combining the discussion in Sec.2 for continuous sectors and one in Sec.3 for discrete ones. What is interesting physically and mathematically is the duality relation (states $\longleftrightarrow$ algebra) between the above degenerate vacua (as *states*) with global classical parameters and the local quantum Goldstone modes found in the dual-net *algebra* $\mathfrak{A}^d$ of extended observables in a vacuum representation, in view of its close relation to such a physical picture of Goldstone modes to search degenerate vacua in a virtual way (i.e., while the order parameter G/H of SSB from G down to H is exhibited as a macroscopic quantity by the degenerate vacua, the quantum Golstone modes φ related to G/H represent in a fixed irreducible representation the virtual transitions from a vacuum to another, as seen in the analysis of the Goldstone commutators, $\omega(\delta_X(\varphi)) \neq 0$, as the infinitesimal form of symmetry breaking $\omega(\tau_g(\varphi)) \neq \omega(\varphi)$ for $g \in G \diagdown H$). At the same time, this is also related with the mathematical notion of duality between a homogeneous space G/H and its representations, as a natural extension of Tannaka-Krein duality of compact groups [28]. Then the basic structural features of the theory with spontaneously broken symmetry are clarified, establishing mutual relationship among degenerate vacua, order parameters, Goldstone modes and condensates responsible for SSB (see Sec.4.3 and 4.4). Since these constitute the starting points for the systematic approach, there are many things to be settled and developed further as is indicated.

In Sec.5, we explain the general mathematical meaning of the proposed scheme, in relation with the categorical adjunctions, especially with the geometric notions of classifying spaces and classifying maps. As further examples of applications of the method, we examine also some basic notions supporting the physical and operational meanings of the mathematical framework of quantum theory; as the standard probabilistic interpretation is naturally understood as something arising from physical processes between measured objects and measuring systems, we can formulate and examine the measurability problem of particular physical quantities as the *realizability* of certain physical dynamical processes suited for the purpose, which is among the typical problems appearing in the context of control theory. In the same context, the problem of state preparation can be treated as a *reachability* problem to examine whether there is a process to bring the system into any desired condition. In both cases, we find that what to be selected is not always states but can also be channels.

2 Selection criteria and $c\to q$ & $q\to c$ channels in thermal situations

2.1 Equilibrium states and thermal interpretations

To draw a clear picture of the idea, we briefly sketch the essense of the scheme proposed in [13] for defining and describing non-equilibrium local states in a relativistic QFT. From the present standpoint, it can be reformulated as follows according to [40, 41]. To characterize an unknown state ω as a non-equilibrium local state, we prepare the following basic ingredients.

i) Candidates of such states are sought within the set E of states ω (understood as an *expectation functional*, mathematically formulated as a normalized positive linear functional on the algebra $\mathfrak{A}$ of observables of the system under consideration) with locally finite energy characterized by the energy-bound condition [21]

$$\omega((\mathbf{1}+H_{\mathcal{O}})^{2m})<\infty \tag{2}$$

valid in some spacetime local region $\mathcal{O}$ and some $m>0$ with $H_{\mathcal{O}}$ a *local Hamiltonian* playing the role of Hamiltonian in $\mathcal{O}$ (whose definition is justified under the assumption of the nuclearity condition [9]). This choice is so designed that the comparison is fully meaningful between an unknown state $\omega\in E$ and known reference states $\in K$ specified in the next ii) as statistical mixtures of thermal equilibria, *in infinitesimally small neighbourhoods of a spacetime point* x by means of observables $\in\mathcal{T}_x$ defined in iii).) We denote $E_{\mathcal{O}}$ the totally of states ω satisfying Eq.(2) with a suitable $m>0$,

$$E_{\mathcal{O}}:=\{\omega;\omega\text{: state of }\mathcal{A}\text{ and }\exists m>0\text{ s.t. }\omega((\mathbf{1}+H_{\mathcal{O}})^{2m})<\infty\}, \tag{3}$$

whose pointlike limit (projective limit)

$$E_x(=\varprojlim_{\mathcal{O}\to x}E_{\mathcal{O}}) \tag{4}$$

is given by the set of equivalence classes in $\cup_{\mathcal{O}}E_{\mathcal{O}}$ with respect to the equivalence relation $\sim$ defined by

$$\omega_1\sim\omega_2\overset{\text{def}}{\Longleftrightarrow}\exists\mathcal{O}\text{: neighbourhood of }x\text{ s.t. }\omega_1\restriction_{\mathcal{O}}=\omega_2\restriction_{\mathcal{O}}. \tag{5}$$

Proposition 1 *If the local Hamiltonians $H_{\mathcal{O}}$ are positive, the family $\mathcal{O} \longmapsto E_{\mathcal{O}}$ constitutes a presheaf of state germs [26] whose stalk at x is given by E_x.*

(Proof is simple and omitted.)

ii) The set K of *thermal reference states* consisting of all global thermal equilibrium states defined as the relativistic KMS states ω_β [8] (with inverse temperature 4-vectors $\beta = (\beta^\mu) \in V_+ := \{x \in \mathbb{R}^4; x^0 > 0, x^2 = (x^0)^2 - \vec{x}^2 > 0\}$) and of their suitable convex combinations: K plays the role of a *model space* whose analogue in the definition of a manifold M can be found in a Euclidean space $\mathbb{R}^n$ as the value space of local charts. Any states belonging to this set K is seen to belong to the above $E_{\mathcal{O}}$ with any arbitrary finite spacetime region $\mathcal{O}$: $K \subset E_{\mathcal{O}} \subset E$.

iii) The linear space $\mathcal{T}_x$ of *local thermal observables*[1] is defined as linear forms on states in E_x satisfying the regularity (2) which makes meaningful the notion of *quantum fields at a point* x [13, 6]:

$$\mathcal{T}_x := \sum_{p,q} \mathcal{N}(\hat{\phi}_0^p)_{q,x}\,, \tag{6}$$

where $\hat{\phi}_0$ generically denotes the basic quantum fields defining our QFT. The notion of *normal products* $\mathcal{N}(\hat{\phi}_0^p)_{q,x}$ enters here to recover effectively the product structure of quantum fields lost through the process of pointlike limit, arising from the operator product expansion (OPE) of $\hat{\phi}_0(x+\zeta_1)\cdots\hat{\phi}_0(x+\zeta_p)$ in the limit of $\zeta_i \to 0$, $\sum_j \zeta_j = 0$ reformulated recently by [6] in a mathematically rigorous form. The simplest case, $\mathcal{N}(\hat{\phi}^2)_{q,x}$, with $p = 2$ can be understood as the linear space spanned by the coefficients $\hat{\Phi}_j(x)$ of c-number singular functions $c_j(\zeta)$ in ζ in

$$||(\mathbf{1}+H_{\mathcal{O}})^{-n}\left[\hat{\phi}(x+\zeta)\hat{\phi}(x-\zeta) - \sum_{j=1}^{J(q)} c_j(\zeta)\,\hat{\Phi}_j(x)\right](\mathbf{1}+H_{\mathcal{O}})^{-n}|| \leq c'\,|\zeta|^q, \tag{7}$$

valid for sufficiently large $n \in \mathbb{N}$, which serve as substitutes for the ill-defined $\hat{\phi}(x)^2$, and similarly $\mathcal{N}(\hat{\phi}^p)_{q,x}$ for higher power $\hat{\phi}(x)^p$. What is important about $\mathcal{T}_x$ is its natural *hierarchical structure* ordered by

[1] While the set $\mathcal{T}_x$ is designed for detecting local thermal properties in use of quantum observables, it is defined as a suitable subset of the point-like fields dual to the state germs [26, 6], relying essentially on the criterion [5] to select states with moderate energy contents. The elements of *thermality* enters here only in excluding certain point-like observables irrelevant to thermal contexts, and hence, the naming with "*thermal*" may somehow be misleading, as remarked by Prof. R. Haag.

the indices p, q related to energy bound and OPE, starting from scalar multiples of identity to higher powers $\mathcal{N}(\hat{\phi}^p)_{q,x}$ with the larger p providing the finer resolution.

Along the above analogy to a manifold M in differential geometry, their role is to relate our unknown state $\omega \in E$ to the known reference states in K, just in parallel to the local coordinates which relate locally a generic curved space M to the known space $\mathbb{R}^n$. As explained just below, the physical interpretations of local thermal observables $\hat{A}$ are given by macroscopic *thermal functions* A corresponding to $\hat{A}$,[2] through which our unknown ω can be *compared* with thermal reference states in K.

Before going into the discussion of non-equilibrium, we need first to establish the physical roles of the above ingredients for describing the relevant thermal properties of states and quantum observables in the realm K of generalized thermal equilibria. To this end, we introduce

Definition 2 ***Thermal functions*** *are defined for each quantum observables $\hat{A}(\in \mathcal{T}_x)$ by the map*

$$\mathcal{C} : \hat{A} \longmapsto \mathcal{C}(\hat{A}) \in C(B_K) \\ \text{with } \mathcal{C}(\hat{A})(\beta, \mu) := \omega_{\beta,\mu}(\hat{A}) \quad \text{for} \quad (\beta, \mu) \in B_K, \tag{8}$$

where B_K is the classifying space to parameterize thermodynamic pure phases, consisting of inverse temperature 4-vectors $\beta \in V_+$ in addition to any other thermodynamic parameters (if any) generically denoted by μ (e.g., chemical potentials) necessary to exhaust and discriminate all the thermodynamic pure phases.

Since the map $\mathcal{C}$ is easily seen to be unital and positive linear, $\mathcal{C}(\mathbf{1}) = 1, \mathcal{C}(\hat{A}^*\hat{A}) \geq 0$, it is a *completely positive* map characterized by the condition $\sum_{ij=1}^n \bar{f}_i \mathcal{C}(\hat{A}_i^* \hat{A}_j) f_j \geq 0$ for $\forall n \in \mathbb{N}, \forall f_1, \cdots, \forall f_n \in C(B_K)$ (and $\forall \hat{A}_i$'s belonging to a suitable C*-algebra $\mathfrak{A}$ to which the operator space $\mathcal{T}_x$ is affiliated). As the dual of a completely positive map, $\mathcal{C}^*$ on states becomes a ***c**lassical-**q**uantum (c→q) channel* [36] $\mathcal{C}^* : Th \ni \rho \longmapsto \mathcal{C}^*(\rho) \in K$ given by

$$\mathcal{C}^*(\rho)(\hat{A}) = \rho(\mathcal{C}(\hat{A})) = \int_{B_K} d\rho(\beta, \mu)\mathcal{C}(\hat{A})(\beta, \mu) = \int_{B_K} d\rho(\beta, \mu)\omega_{\beta,\mu}(\hat{A}), \\ \Longrightarrow \mathcal{C}^*(\rho) := \int_{B_K} d\rho(\beta, \mu)\omega_{\beta,\mu} = \omega_\rho \in K. \tag{9}$$

Here $Th := M_1(B_K)$ is the space of classical thermal states identified with probability measures ρ on B_K describing the mean values of thermodynamic

[2]Whenever convenient without fear of confusions, we adopt here a physicist's convention to indicate the correspondence and distinction between a quantum observable $\hat{A}$ and a classical one A in a suitable correspondence to the former.

parameters (β, μ) together with their fluctuations. One can see that thermal interpretation of local quantum thermal observables $\hat{A} \in \mathcal{T}_x$ is given in all thermal reference states of the form $\mathcal{C}^*(\rho) = \omega_\rho \in K$ by the corresponding thermal function $\mathcal{C}(\hat{A})$ evaluated with the classical probability ρ describing the thermodynamic configurations of ω_ρ through the relation

$$\omega_\rho(\hat{A}) = \int_{B_K} d\rho(\beta, \mu)\, \omega_{\beta,\mu}(\hat{A}) = \rho(\mathcal{C}(\hat{A})). \tag{10}$$

This applies to the case where ρ is already known. What we need to ask in the actual situations is how to determine the unknown ρ from the given data set $\Phi \longmapsto \rho(\Phi)$ of expectation values of thermal functions Φ (which is the problem of state estimation): this problem can be solved if $\mathcal{T}_x$ has sufficiently many local thermal observables so that the totality $\mathcal{C}(\mathcal{T}_x)$ of the corresponding thermal functions can approximate arbitrary continuous functions of $(\beta, \mu) \in B_K$. In this case ρ is given as the unique solution to a (generalized) "moment problem". Thus we see:

★ If the set $\mathcal{T}_x$ of local thermal observables is large enough to discriminate all the thermal reference states in K, then any reference state $\in K$ can be written as $\mathcal{C}^*(\rho)$ in terms of a uniquely determined probability measure ρ on B_K describing the statistical fluctuations of thermal parameters in the state in question. Then local thermal observables $\hat{\Phi} \in \mathcal{T}_x$ provide the same information on the thermal properties of states in K as that provided by the corresponding classical macroscopic thermal functions $\Phi = \mathcal{C}(\hat{\Phi})$ [e.g., internal energy, entropy density, etc.]: $\omega_\rho(\hat{\Phi}) = \rho(\Phi)$.

In this situation, *any continuous function* F in B_K can be approximated by thermal functions $\Phi_x = \mathcal{C}(\hat{\Phi}(x))$ with arbitrary precision, *even if* F itself is *not* an image of $\mathcal{C}$:

$$\overline{\mathcal{C}(\mathcal{T}_x)}^{\|\cdot\|} = C(B_K). \tag{11}$$

For instance, the *entropy density* $s(\beta)$ can be treated as such an *approximate* thermal function in spite of the absence of quantum observables $\hat{s}(x) \in \mathcal{T}_x$ s.t. $\omega_\beta(\hat{s}(x)) = s(\beta)$. What the above (★) says is the equality and the equivalence,

$$K = \mathcal{C}^*(Th); \tag{12}$$

$$\omega_{\rho_1} \underset{\mathcal{T}_x}{\equiv} \omega_{\rho_2} \Longleftrightarrow \rho_1 \underset{\mathcal{C}(\mathcal{T}_x)}{\equiv} \rho_2, \tag{13}$$

for $\rho_i \in Th$, $\omega_{\rho_i} = \mathcal{C}^*(\rho_i) = \int_{B_K} d\rho_i(\beta,\mu)\omega_{\beta,\mu} \in K$, where $\underset{\mathcal{T}_x}{\equiv}$ and $\underset{\mathcal{C}(\mathcal{T}_x)}{\equiv}$ denote the equivalence relations in K and Th given respectively by

$$\omega_1 \underset{\mathcal{T}_x}{\equiv} \omega_2 \Longleftrightarrow (\omega_1 - \omega_2)(\mathcal{T}_x) = \{0\}, \tag{14}$$

$$\rho_1 \underset{\mathcal{C}(\mathcal{T}_x)}{\equiv} \rho_2 \Longleftrightarrow (\rho_1 - \rho_2)(\mathcal{C}(\mathcal{T}_x)) = \{0\}. \tag{15}$$

So, it ensures the existence of *inverse of* $c{\to}q$ *channel* $\mathcal{C}^*$ on K:

$$K \ni \omega_\rho = \mathcal{C}^*(\rho) \longleftrightarrow (\mathcal{C}^*)^{-1}(\omega_\rho) = \rho \in Th, \tag{16}$$

and the thermal interpretation of thermal reference states $\in K$ is just given by this $\boldsymbol{q} \to \boldsymbol{c}$ ***channel*** $(\mathcal{C}^*)^{-1} : K \ni \omega \longmapsto \rho \in Th$ s.t. $\omega = \mathcal{C}^*(\rho)$ [40]. In the parallelism between the integral representation in Eq.(9) and the Fourier decomposition of a function, we note that $(\mathcal{C}^*)^{-1}$ acting on $\omega_\rho \in K$ corresponds to the Fourier transform.

To adapt to our discussion of local thermal situations, we summarize the above points in such a form of *adjunction* [30] as

$$K/\mathcal{T}_x(\omega, \mathcal{C}^*(\rho)) \overset{q\rightleftarrows c}{\simeq} Th/\mathcal{C}(\mathcal{T}_x)((\mathcal{C}^*)^{-1}(\omega), \rho), \tag{17}$$

with a quantum state $\omega \in E$ and a probability measure $\rho \in Th$. Since the adjunction turns out to be a convenient tool in formulating a scheme for attaining simultaneously the selection of relevant objects (on the left) and interpreting the selected objects (on the right), we make a slight detour for explaining it here. While its most general formulation should be given in the context of categories and functors (see Sec.5), we concentrate here on our present context of treating equivalence relations given by Eqs.(14) and (15), according to which the sets K and Th become groupoids.

Roughly speaking, a groupoid Γ is such a generalization of a group that there are many unit elements which constitute a set Γ_0 and that the product $\gamma_1\gamma_2$ of two elements γ_1, γ_2 is defined only conditionally in the following sense: Each element (also called an "arrow") $\gamma \in \Gamma$ has its source $s(\gamma)$ and target $r(\gamma)$ in Γ_0 and these points are thought to be connected by γ, $s(\gamma) \overset{\gamma}{\to} r(\gamma)$ (dented also as $\gamma : s(\gamma) \to r(\gamma)$), in an invertible way: $r(\gamma) \overset{\gamma^{-1}}{\to} s(\gamma)$. Two arrows $\gamma_1, \gamma_2 \in \Gamma$ are composable to yield a product $\gamma_1\gamma_2 \in \Gamma$ if and only if $r(\gamma_2) = s(\gamma_1)$ and the product is to be associative: $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$. There is a one-to-one and onto correspondence between an *equivalence relation* $\sim$ on a set Γ_0 and a groupoid Γ through $[a \sim b$ for $a, b \in \Gamma_0] \Longleftrightarrow [\exists\gamma \in \Gamma$ s.t. $a = s(\gamma)$ and $b = r(\gamma)]$, according to which the characterization of equivalence relation $[a \sim a]$, $[a \sim b \Longrightarrow b \sim a]$, $[a \sim b, b \sim c \Longrightarrow a \sim c]$ is translated into the basic properties of Γ as the presence of unit $(a \overset{\iota_a}{\to} a) \in \Gamma$ (for $\forall a \in \Gamma_0$), the invertibility of any γ: $\Gamma \ni (a \overset{\gamma}{\to} b) \Longrightarrow (b \overset{\gamma^{-1}}{\to} a) \in \Gamma$, and the composition: $(a \overset{\gamma_1}{\to} b), (b \overset{\gamma_2}{\to} c) \in$

$\Gamma \Longrightarrow (a \overset{\gamma_2\gamma_1}{\to} c) \in \Gamma$. Collecting all the arrows from $a \in \Gamma_0$ to $b \in \Gamma_0$, we denote $\Gamma(a,b) := \{\gamma \in \Gamma; s(\gamma) = a \text{ and } r(\gamma) = b\}$. Viewed as a category, Γ is one with Γ_0 as the set of objects and with all its arrows being invertible. Corresponding to the equivalence relations $\underset{\mathcal{T}_x}{\equiv}$ and $\underset{\mathcal{C}(\mathcal{T}_x)}{\equiv}$ defined by Eqs. (14) and (15) on K and Th, we can consider the groupoids denoted respectively by $K/\mathcal{T}_x$ and $Th/\mathcal{C}(\mathcal{T}_x)$. Then

Proposition 3 *Under the condition of (★), the groupoids $K/\mathcal{T}_x$ and $Th/\mathcal{C}(\mathcal{T}_x)$ are isomorphic with the $c{\to}q$ channel $\mathcal{C}^* : Th \to K$ as a groupoid isomorphism preserving the structures as in (13).*

For an arbitrary state $\omega \in E_x$ at x, the existence of a non-empty set $K/\mathcal{T}_x(\omega, \mathcal{C}^*(\rho))$ of arrows in $K/\mathcal{T}_x$ identifies it with a uniquely determined member $\mathcal{C}^*(\rho)$ of K through the relation $\omega \underset{\mathcal{T}_x}{\equiv} \mathcal{C}^*(\rho)$, which can be transmitted by the $q{\to}c$ channel $(\mathcal{C}^*)^{-1}$ meaningful on K to the right-side $Th/\mathcal{C}(\mathcal{T}_x)((\mathcal{C}^*)^{-1}(\omega), \rho)$ of Eq.(17) to provide the thermal interpretation of the selected ω by $(\mathcal{C}^*)^{-1}(\omega) \underset{\mathcal{C}(\mathcal{T}_x)}{\equiv} \rho \in Th$ in terms of a probability distribution ρ (of temperature, etc.).

While the use of adjunction may look like something unnecessarily pedantic in the present simple situation treating equivalence relations, the essence of (★) in this form (17) can be generalized to wider contexts as a selection criterion to choose states of relevance. Then we will encounter more involved cases where the arrows of relevant categories are not necessarily invertible and the $c{\to}q$ and $q{\to}c$ channels need be replaced by *functors* constituting *adjoint pairs* and so on. One of the merits of the use of adjunctions is that it clearly shows the characteristic features, essence, and basic ingredients common to all the problems to *select* objects with specific properties from generic ones and to *describe, interpret* and *classify* the features of all what to be selected by comparing them with special standard reference objects. In this setup, for instance, it is evident and conceptually important that we have here *two different levels or domains*, quatum statistical mechanics with family K of mixtures of KMS states and macroscopic thermodynamics described by Th of probability measures of fluctuating thermal parameters on the parameter space B_K, which are so interrelated by the two channels, $c \to q$ ($\mathcal{C}^*$) and $q \to c$ ($(\mathcal{C}^*)^{-1}$), that the following two points are simultaneously attained:

a) characterization of thermal reference states K as image of $\mathcal{C}^*$, $\omega_\rho = \mathcal{C}^*(\rho)$: *selection criterion* for K,

b) *thermal interpretation* of selected states in K in terms of classical data, $\Phi_x = \mathcal{C}(\hat{\Phi}(x))$ and $\rho = (\mathcal{C}^*)^{-1}(\omega_\rho)$.

To implement this sort of machineries in the actual situations, the most non-trivial steps are the pertinent choices of pair of maps (adjoint pair of functors) corresponding to (and, generalizing) the $c\to q$ and $q\to c$ channels together with the ***standard reference systems*** for comparison.

Going back to the original context, the problem is now boiled down into how to select suitable classes of *non-equilibrium states* $\omega \notin K$ in such a way that some thermal interpretations are still guaranteed. This is what to be answered in the next subsection.

2.2 Selection criterion for non-equilibrium states

Selection criterion and thermal interpretation of non-equilibrium local states based on hierarchized zeroth law of local thermodynamics [40]: To meet simultaneously the two requirements of characterizing an unknown state ω as a non-equilibrium local state and of establishing its thermal interpretations in a similar way to the above a) and b), we now compare ω with thermal reference states $\in K = \mathcal{C}^*(Th)$ by means of some local thermal observables at x whose physical meanings are exhibited by the associated thermal functions as seen above. In view of the above conclusion [$q\to c$ channel $(\mathcal{C}^*)^{-1}$ on K] = [thermal interpretation of quantum states] and also of the hierarchical structure in $\mathcal{T}_x$, we relax the requirement for ω to agree with $\exists\omega_{\rho_x} := \mathcal{C}^*(\rho_x) \in K$ up to some suitable *sub*space $\mathcal{S}_x$ of local thermal observables $\mathcal{T}_x$. Then we characterize ω as a non-equilibrium local state by

iii) a *selection criterion* for ω to be $\mathcal{S}_x$*-thermal* at x, requiring the existence of $\rho_x \in Th$ s.t.

$$\omega(\hat{\Phi}(x)) = \mathcal{C}^*(\rho_x)(\hat{\Phi}(x)) \quad \text{for } \forall\hat{\Phi}(x) \in \mathcal{S}_x, \tag{18}$$

or, $\omega \underset{\mathcal{S}_x}{\equiv} \mathcal{C}^*(\rho_x)$, for short. In terms of thermal functions $\Phi := \mathcal{C}(\hat{\Phi}(x)) \in \mathcal{C}(\mathcal{S}_x)$, this can be rewritten as

$$\omega(\Phi)(x) := \omega(\hat{\Phi}(x)) = \rho_x(\Phi), \qquad \Phi \in \mathcal{C}(\mathcal{S}_x). \tag{19}$$

So, ω: $\mathcal{S}_x$*-thermal* implies that the selection criterion $\omega \underset{\mathcal{S}_x}{\equiv} \mathcal{C}^*(\rho_x)$ can be "solved" conditionally in favour of ρ_x as "$(\mathcal{C}^*)^{-1}$"$(\omega) \underset{\mathcal{C}(\mathcal{S}_x)}{\equiv} \rho_x$, which provides the local thermal interpretation of ω [40]. Physically this means the state ω looks like a statistical mixture $\mathcal{C}^*(\rho_x)$ of thermal equilibria *locally* at x to within a level controlled by a subset $\mathcal{S}_x$ of thermal observables.

To be precise mathematically, we should be careful here about the meaning of such a heuristic expression as "$(\mathcal{C}^*)^{-1}$"(ω) for $\omega \notin K$ in relation to our observation above: $\omega \notin K = \mathcal{C}^*(Th)$. Physically this is related to the deviations of ω from $\mathcal{C}^*(\rho_x)$ revealed by the finer resolutions which exhibit the extent of ω being *away from equilibrium* even locally. As we shall see below, "$(\mathcal{C}^*)^{-1}$" outside of K is certainly *not* a $q\to c$ channel preserving the positivity, whereas it can be seen to be still definable on the states ω selected out by the above criterion Eq.(18), by means of its equivalent reformulation given by:

Proposition 4 *[13] For a subspace $\mathcal{S}_x$ of $\mathcal{T}_x$ containing* $\mathbf{1}$*, a state $\omega \in E_x$ is $\mathcal{S}_x$-thermal iff there is a compact set $B \subset V_+$ of inverse temperatures s.t.*

$$|\omega(\hat{\Phi}(x))| \le \tau_B(\hat{\Phi}(x)) := \sup_{(\beta,\mu)\in B_K, \beta\in B} |\omega_{\beta,\mu}(\hat{\Phi}(x))| = \left|\left|\mathcal{C}(\hat{\Phi}(x))\right|\right|_B, \quad \textit{for } \hat{\Phi}(x) \in \mathcal{S}_x. \tag{20}$$

(For the above semi-norm to be well-defined, $B_K \ni (\beta,\mu) \longmapsto \omega_{\beta,\mu} \in K$ should be (weakly) continuous, which requires singularities of critical points to be excluded from our considerations.)

Since the requirement for "$(\mathcal{C}^*)^{-1}$"(ω) to be a probability measure forces ω to belong to K, it is incompatible with our premise $\omega \notin K$. However, the above inequality (20) combined with the Hahn-Banach extension theorem (under the assumption for τ_B to be a norm) allows us to extend $\mathcal{C}(\mathcal{S}_x) \ni \mathcal{C}(\hat{\Phi}(x)) \longmapsto \omega(\hat{\Phi}(x))$ as a *linear functional* defined on $\mathcal{C}(\mathcal{S}_x)$ to one ν defined on $\overline{\mathcal{C}(\mathcal{T}_x)} = C(B_K)$, which should *not* be a positive-definite measure but is allowed to be a *signed* measure: $\nu = \nu_+ - \nu_-$, $0 \le \nu_\pm \in C(B_K)^*_+$, $\nu_- \ne 0$, $\nu_- \upharpoonright_{\mathcal{C}(\mathcal{S}_x)} = 0$, $\mathcal{C}^*(\nu_+) \upharpoonright_{\mathcal{S}_x} = \omega \upharpoonright_{\mathcal{S}_x}$. (See the similar argument in [13] for the existence of an $\mathcal{S}_x$-thermal state ω showing deviations from K for observables outside of a finite-dimensional $\mathcal{S}_x$ as well as the treatment of the case with τ_B being a *semi*-norm.) Thus, understanding the meaning of $(\mathcal{C}^*)^{-1}(\omega)$ as the set of inverse images of ω under $\mathcal{C}^*$ in the space $C(B_K)^*$ of linear functionals,

$$(\mathcal{C}^*)^{-1}(\omega) := \{\nu \in C(B_K)^*; \nu = \nu_+ - \nu_-, \nu_\pm \ge 0, \nu_- \upharpoonright_{\mathcal{C}(\mathcal{S}_x)} = 0, \mathcal{C}^*(\nu_+) \upharpoonright_{\mathcal{S}_x} = \omega \upharpoonright_{\mathcal{S}_x}\}, \tag{21}$$

we can put Eq.(18) into the similar form to Eq.(17) as

iv) The characterization and local thermal interpretation of a non-equilibrium local state:

Proposition 5 *[41] The following isomorphism holds for $\omega \in E_x$, $\rho_x \in Th$ and a subspace $\mathcal{S}_x \subset \mathcal{T}_x$,*

$$E_x/\mathcal{S}_x(\omega, \mathcal{C}^*(\rho_x)) \overset{q\rightleftarrows c}{\simeq} Th/\mathcal{C}(\mathcal{S}_x)((\mathcal{C}^*)^{-1}(\omega), [\rho_x]), \tag{22}$$

where $[\rho_x] := \{\sigma \in Th; \sigma \upharpoonright_{\mathcal{C}(\mathcal{S}_x)} = \rho_x \upharpoonright_{\mathcal{C}(\mathcal{S}_x)}\}$. The existence of ρ_x to make the sets of arrows non-empty is equivalent to the $\mathcal{S}_x$-thermality of ω.

This relation can be viewed as a form of "*hierarchized zeroth law of local thermodynamics*"; the reason for mentioning the "zeroth law" here is due to the implicit relevance of measuring processes of local thermal observables validating the above equalities, which require the *contacts of two bodies*, measured object(s) and measuring device(s), in a local thermal equilibrium, conditional on the chosen $\mathcal{S}_x$. The transitivity of this contact relation just corresponds to the localized and hierarchized version of the standard zeroth law of thermodynamics.

We can use the relation

$$\exists \nu = \nu_+ - \nu_- \in (\mathcal{C}^*)^{-1}(\omega) \text{ with } \nu_- = 0 \Longleftrightarrow (\mathcal{C}^*)^{-1}(\omega) = \{\nu\} \subset Th \\ \Longleftrightarrow \omega \in K \Longleftrightarrow [\text{maximal choice of } \mathcal{S}'_x \text{ s.t. } \mathcal{C}^*(\nu_+) \upharpoonright_{\mathcal{S}'_x} = \omega \upharpoonright_{\mathcal{S}'_x}] = \mathcal{T}_x, \tag{23}$$

for specifying the extent to which a non-equilibrium $\mathcal{S}_x$-thermal ω deviates from equilibria belonging to K by the *failure of positivity* ($\nu_- \neq 0$) and can also measure it by the *maximal size* of $\mathcal{S}'_x$ within the hierarchy of subspaces $\mathcal{S}'_x$ in $\mathcal{T}_x$ such that $\mathcal{S}'_x \supset \mathcal{S}_x$, $\nu_- \upharpoonright_{\mathcal{C}(\mathcal{S}'_x)} = 0$ with all the possible choices of $\nu \in (\mathcal{C}^*)^{-1}(\omega)$: owing to the presence of ν_-, ω ceases to be $\mathcal{S}'_x$-thermal when $\mathcal{S}'_x$ is so enlarged that $\nu_- \upharpoonright_{\mathcal{C}(\mathcal{S}'_x)} = 0$ is invalidated, which shows that ω shares with reference states in K only gross thermal properties described by smaller $\mathcal{S}'_x$. In this sense, the hierarchy of $\mathcal{S}'_x$ in $\mathcal{T}_x$ should have a close relationship with the thermodynamic hierarchy at various scales appearing in the transitions between non-equilibrium and equilibrium controlled by certain family of *coarse graining* procedures. Thus, we see that our selection criterion can give a characterization of states identifiable as non-equilibrium ones and, at the same time, provide associated relevant physical interpretations of the selected states in a systematic way.

The two goals of identifying non-equilibrium local states admitting local thermal interpretation and of describing their specific thermodynamic properties are solved simultaneously by the above selection criterion based upon a *localized and hierarchized form of the zeroth law* of thermodynamics. In this framework, we can identify at least three different kinds of sources of derivations of an $\mathcal{S}_x$-thermal non-equilibrium local state $\omega \in E_x$ from the genuine equilibrium states ω_β as

a) *spacetime dependence* of thermal parameters such as temperature distributions $x \longmapsto \beta(x)$,

b) *statistical fluctuations* of thermal parameters at x described by probability distributions $d\rho_x(\beta) \in Th$,

and

c) essential deviations of local states $\omega \in E_x$ from states in K expressed by the *positivity-violating* term $\nu_- \neq 0$ in $\nu = \nu_+ - \nu_- \in (\mathcal{C}^*)^{-1}(\omega) \subset C(B_K)^*$ with $\nu_- \upharpoonright_{\mathcal{C}(\mathcal{S}_x)} = 0, \mathcal{C}^*(\nu_+) \upharpoonright_{\mathcal{S}_x} = \omega \upharpoonright_{\mathcal{S}_x}$.

3 Reformulation of DHR-DR sector theory

3.1 Basic results of DHR-DR theory

According to the discussion in the previous section, we now try to reformulate the essence of the DHR-DR sector theory into a physically more understandable form. As the mathematical essence of the theory itself is very sophisticated and complicated, it is not our aim here to reproduce it faithfully, for which purpose interested readers are advised to look into their original papers starting from [19, 18]. Before taking our approach to it in Sec.3.2, however, we need to introduce the basic ingredients and to summarize the most essential results of the DHR-DR sector theory. The starting point of the theory with *localizable charges* [16, 25] is as follows:

- A net $\mathcal{K} \ni \mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ of von Neumann algebras $\mathfrak{A}(\mathcal{O})$ of *local observables* is defined on the set, $\mathcal{K} := \{(a+V_+)\cap(b-V_+); a, b \in \mathbb{R}^4\}$, of all double cones in the Minkowski spacetime $\mathbb{R}^4$; it is assumed to satisfy
  - isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$, allowing the global (or, quasi-local) algebra of observables $\mathfrak{A} := C^*\text{-}\varinjlim_{\mathcal{K}\ni\mathcal{O}\nearrow\mathbb{R}^4} \mathfrak{A}(\mathcal{O})$ to be defined as the C*-inductive limit,
  - relativistic covariance under the action of the Poincaré group $\mathcal{P}_+^{\uparrow} := \mathbb{R}^4 \rtimes L_+^{\uparrow} \ni (a, \Lambda) \longmapsto \alpha_{(a,\Lambda)} \in Aut(\mathfrak{A})$, $\alpha_{(a,\Lambda)}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\Lambda(\mathcal{O})+a)$, and
  - local commutativity (or locality for short): $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$ for $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ spacelike separated (i.e., $\forall x \in \mathcal{O}_1, \forall y \in \mathcal{O}_2$, $(x-y)^2 < 0$).

- **DHR criterion**: A physically relevant state $\omega \in E_{\mathfrak{A}}$(: the set of all states of $\mathfrak{A}$ defined as normalized positive linear functionals on $\mathfrak{A}$) around a pure vacuum $\omega_0 \in E_{\mathfrak{A}}$ is selected by the Doplicher-Haag-Roberts (DHR) criterion[3] which requires the GNS reprepresentation π_ω corresponding to ω to be unitarily equivalent to the vacuum representation $\pi_{\omega_0} =: \pi_0$ in spacelike distance; i.e., $\exists\mathcal{O} \in \mathcal{K}$ s.t. for $\forall a \in \mathbb{R}^4$ with $\mathcal{O}_a := \mathcal{O} + a \in \mathcal{K}$

$$\pi_\omega \upharpoonright_{\mathfrak{A}(\mathcal{O}_a')} \cong \pi_0 \upharpoonright_{\mathfrak{A}(\mathcal{O}_a')}, \tag{24}$$

  where $\mathcal{O}' := \{x \in \mathbb{R}^4; (x-y)^2 < 0 \text{ for } \forall y \in \mathcal{O}\}$ is the causal complement of $\mathcal{O}$ and $\mathfrak{A}(\mathcal{O}') := C^*\text{-}\varinjlim_{\mathcal{K}\ni\mathcal{O}_1\subset\mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)$.

[3] In view of the local normality, this criterion can be imposed on *any* $\mathcal{O} \in \mathcal{K}$ as seen in [19], but, we adopt this original form presented in [16] in relation to the notion of support of ρ in the next item.

- **Local endomorphisms**: In the GNS representation $(\pi_0, \mathfrak{H}_0)$ corresponding to ω_0, the validity of Haag duality,

$$\pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))'', \tag{25}$$

is assumed. On the basis of the standard postulates [16], the selection criterion (24) can be shown to be equivalent to the existence of a *local endomorphism* $\rho \in End(\mathfrak{A})$ such that $\pi_\omega = \pi_0 \circ \rho$, localized in some $\mathcal{O} \in \mathcal{K}$ in the sense of

$$\rho(A) = A \quad \text{for } \forall A \in \mathfrak{A}(\mathcal{O}'). \tag{26}$$

In this situation, we say (in a rather sloppy way) that the *support of* ρ is (contained in) $\mathcal{O}$: $\mathrm{supp}(\rho) \subset \mathcal{O}$. Note that an endomorphism ρ preserves all the algebraic structure on $\mathfrak{A}$ but that its image set $\rho(\mathfrak{A})$ can be strictly smaller than $\mathfrak{A}$, $\rho(\mathfrak{A}) \subsetneqq \mathfrak{A}$ which is possible only for an infinite-dimensional algebra $\mathfrak{A}$.

- **Transportability** (of charges associated with an *internal* symmetry): The above spacetime dependence of ρ coming from its localization region $\mathcal{O}$ can be absorbed into its *transportability*, namely, for any translation $a \in \mathbb{R}^4$, there exists $\rho_a \in End(\mathfrak{A})$ with support in $\mathcal{O} + a$ and $\rho \cong \rho_a = Ad(u_a) \circ \rho$ with a unitary $u_a \in \mathfrak{A}$. We denote

$$\Delta(\mathcal{O}) := \{\rho \in End(\mathfrak{A}); \rho\text{: transportable and localized in } \mathcal{O}\}. \tag{27}$$

- **DR-category** [18]: Then a **C*-tensor category** $\mathcal{T}$ which we call here a DR-category is defined as a full subcategory of $End(\mathfrak{A})$ consisting of *objects* $\rho \in \Delta := \cup_{\mathcal{O} \in \mathcal{K}} \Delta(\mathcal{O})$ and with *morphisms* (or, *arrows*) given by intertwiners $T \in \mathfrak{A}$ between $\rho, \sigma \in \Delta$ s.t. $T\rho(A) = \sigma(A)T$. $\mathcal{T}$ has the *permutation symmetry* due to the locality, and is closed under *direct sums* and *subobjects* (due to the Property B following from the spectrum condition, locality and weak additivity) [19].

As promised in Sec.2, we encounter here a category $\mathcal{T}$, a mathematical notion more general than a groupoind (corresponding to an equivalence relation) in that its arrows are not necessarily invertible. By $\mathcal{T}$ being a *C*-tensor category* we mean

i) [*C*-category*]: all the sets $\mathcal{T}(\rho, \sigma)$ of arrows in $\mathcal{T}$ are Banach spaces over complex numbers $\mathbb{C}$ such that a hermitian conjugation $\mathcal{T}(\rho, \sigma) \ni T \longmapsto T^* \in \mathcal{T}(\sigma, \rho)$ is so defined that the C*-norm property $||T^*T|| = ||T||^2$ holds for the norms of arrows (which is straightforward from $T \in \mathcal{T}(\rho, \sigma) \subset \mathfrak{A}$: C*-algebra), and

ii) [*tensor category*]: a tensor-product structure is defined on the set of objects $\rho, \sigma, \cdots$, etc., by $\rho \otimes \sigma := \rho\sigma$ and also on that of arrows. $S \otimes T$ is defined for $S \in \mathcal{T}(\rho_1, \rho_2), T \in \mathcal{T}(\sigma_1, \sigma_2)$ by $S \otimes T := S\rho_1(T) = \rho_2(T)S$, and satisfies $S\rho_1(T)\rho_1\sigma_1(A) = \rho_2(\sigma_2(A)T)S = \rho_2\sigma_2(A)S\rho_1(T)$, which means $S \otimes T \in \mathcal{T}(\rho_1 \otimes \sigma_1, \rho_2 \otimes \sigma_2) = \mathcal{T}(\rho_1\sigma_1, \rho_2\sigma_2)$ and also $(S_1 \otimes T_1)(S_2 \otimes T_2) = S_1S_2 \otimes T_1T_2$.

According to i), a C*-category with only one object is just a C*-algebra as a Banach space equipped with product structure and the C*-norm. What is remarkable is the tensor structure ii) which is shared by the category Rep_G of unitary representations (γ, V_γ) of a group G (i.e., $\gamma(g)$: unitary operators in the inner product space V_γ s.t. $\gamma(g_1g_2) = \gamma(g_1)\gamma(g_2), \gamma(g^{-1}) = \gamma(g)^*, \gamma(e) = Id_{V_\gamma}$) whose arrows are intertwiners between pairs of such representations, i.e., $T \in Rep_G(\gamma_1, \gamma_2) \Longleftrightarrow T\gamma_1(g) = \gamma_2(g)T$ for $\forall g \in G$.

In a word, the mathematical essence of Doplicher-Roberts theory is to verify that, in spite of its abstract form as a certain category of local endomorphisms ρ on the observable algebra $\mathfrak{A}$, the DR-category $\mathcal{T}$ determined by the DHR criterion for relevant states is isomorphic to the category Rep_G of group representations with a certain uniquely determined group G to be identified with the gauge group (of the 1st kind). Up to the technical details, the essential contents can be summarized in the following basic results due to the structure of $\mathcal{T}$ as a C*-tensor category having the *permutation symmetry*, *direct sums*, *subobjects* and *conjugates*:

- Unique existence of an *internal symmetry group* G such that

$$\mathcal{T} \simeq Rep_G \overset{\text{Tannaka-Krein duality}}{\longleftrightarrow} G = End_\otimes(V), \tag{28}$$

where $End_\otimes(V)$ is defined as the group of natural unitary transformations $g = (g_\rho)_{\rho\in\mathcal{T}} : V \overset{\cdot}{\to} V$ from the C*-tensor functor $V : \mathcal{T} \hookrightarrow Hilb$ to itself [18, 30] as characterized by $g_{\rho\sigma} = g_\rho \otimes g_\sigma$ and the commutativity $Tg_{\rho_1} = g_{\rho_2}T$ of the diagram:

$$\begin{array}{ccccc} \rho_1 & & V_{\rho_1} & \overset{g_{\rho_1}}{\to} & V_{\rho_1} \\ T\downarrow & & T\downarrow & \circlearrowleft & \downarrow T \\ \rho_2 & & V_{\rho_2} & \overset{g_{\rho_2}}{\to} & V_{\rho_2} \end{array} . \tag{29}$$

Here, V embeds $\mathcal{T}$ into the category $Hilb$ of Hilbert spaces and its image turns out to be just the category Rep_G of unitary representations (γ, V_γ) of a compact Lie group $G \subset SU(d)$ (owing to the presence of *conjugates* in $\mathcal{T}$), where the dimensionality d is intrinsically defined in $\mathcal{T}$ by the generating element $\rho \in \mathcal{T}$ [18][4]. In this formulation, the essence of Tannaka-Krein duality [28] is found in the one-to-one correspondence,

$${}_{\mathcal{I}}\backslash^{\Delta} \ni [\rho] = [\rho_\gamma] \longleftrightarrow \gamma = \gamma_\rho \in \hat{G}, \tag{30}$$

[4]For the unique existence of G, the functor V should be so chosen that it maps the 'bosonized' form [19] of permutation symmetry of $\mathcal{T}$ onto the unique permutation symmetry of $Hilb$, as emphasized by Prof. Roberts.

with $\rho \in \Delta$ satisfying $\rho(\mathfrak{A})' \cap \mathfrak{A} = \mathbb{C}\mathbf{1}$ (corresponding to the *irreducibility* of γ_ρ), and the identification $g_\rho = \gamma_\rho(g)$ for $g \in G$, where ${}_{\mathcal{I}}\backslash^{\Delta}$ is the set of equivalence classes $\{Ad(v) \circ \rho; \mathcal{O} \in \mathcal{K}, v \in \mathfrak{A}(\mathcal{O})\} \subset \Delta$ of ρ w.r.t. the action of inner automorphism group $\mathcal{I} = \{Ad(v); \mathcal{O} \in \mathcal{K}, v \in \mathcal{U}(\mathfrak{A}(\mathcal{O}))$: unitary operators$\}$ and the group dual $\hat{G}$ is defined by the totality of equivalence classes of continuous unitary irreducible representations of G.
Once these are known, the relation $g_{\rho\sigma} = g_\rho \otimes g_\sigma$ can be understood as representating the tensor structure of representations γ_ρ of G (i.e., a representation of representations, which is sometime expressed by the word "birepresentation" corresponding to the bidual $\hat{\hat{\Gamma}}$ of an abelian group Γ isomorphic to Γ itself by the Pontryagin duality) and the relation $Tg_{\rho_1} = g_{\rho_2}T$ rewritten by $T\gamma_{\rho_1}(g) = \gamma_{\rho_2}(g)T$ simply shows that the intertwiner T from an endomorphism ρ_1 to another such ρ_2 is just the one from γ_{ρ_1} to γ_{ρ_2} in the context of group representations.

Remark 6 *The appearance of group structure here is due to the permutation symmetry encoded in $\mathcal{T}$ coming from the local commutativity in the four dimensional spacetime. In the two dimensional case, the permutation symmetry is to be replaced by the braid group symmetry, as a consequence of which quantum group symmetry arises instead of the familiar group (see [24] for wide perspectives of the relevant problems involving "quantum categories").*

- Unique existence of a *field algebra* such that

$$\mathfrak{F} := \mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d \qquad \curvearrowleft G = Aut_{\mathfrak{A}}(\mathfrak{F}) = Gal(\mathfrak{F}/\mathfrak{A}) \tag{31}$$
$$:= \{\tau \in Aut(\mathfrak{F}); \tau(A) = A, \ \forall A \in \mathfrak{A}\} \text{ (: Galois group)},$$

with $\mathfrak{A} = \mathfrak{F}^G$ (fixed-point algebra), where $\mathcal{O}_d$ is the Cuntz algebra [14] defined as the unique simple C*-algebra generated by a d-dimensional Hilbert space h_d of d isometries ψ_i, $i = 1, 2, \cdots, d$,

$$\psi_i^* \psi_j = \delta_{ij}\mathbf{1}, \qquad \sum_{i=1}^{d} \psi_i \psi_i^* = \mathbf{1}, \tag{32}$$

whose fixed-point subalgebra $\mathcal{O}_d^G$ is embedded into $\mathfrak{A}$, $\mu : \mathcal{O}_d^G \hookrightarrow \mathfrak{A}$, satisfying the relation $\mu \circ \sigma = \rho \circ \mu$ with respect to the canonical endomorphism σ of $\mathcal{O}_d$: $\sigma(C) := \sum_{i=1}^{d} \psi_i C \psi_i^*$ for $C \in \mathcal{O}_d$. As a linear space, $\mathfrak{F}$ is uniquely defined as a tensor product of $\mathfrak{A}$ as a right $\mathcal{O}_d^G$-module via μ and of $\mathcal{O}_d$ as a left $\mathcal{O}_d^G$-module, and its product structure is defined [18] by

$$
\begin{aligned}
&(A_1 \underset{\mathcal{O}_d^G}{\otimes} \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^*)(A_2 \underset{\mathcal{O}_d^G}{\otimes} C) \\
&= [(-1)^{d-1}\sqrt{d}]^s A_1 \rho^r (\underbrace{R^* \rho^{d-1}(\cdots (R^* \rho^{d-1}}_{s}(A_2)) \cdots) \\
&\quad \underset{\mathcal{O}_d^G}{\otimes} \psi_{i_1} \cdots \psi_{i_r} \hat{\psi}_{j_1} \cdots \hat{\psi}_{j_s} C
\end{aligned} \tag{33}
$$

for $A_i \in \mathfrak{A}$, $\psi_i \in h_d$, $C \in \mathcal{O}_d$, where

$$
\hat{\psi}_i = 1/\sqrt{(d-1)!} \sum_{p \in \mathbb{P}_d(i)} sgn(p) \psi_{p(2)} \cdots \psi_{p(d)} \tag{34}
$$

with $\mathbb{P}_d(i)$ the subset of permutations p of $1, 2, \cdots, d$ s.t. $p(1) = i$ and $R = 1/\sqrt{d}\mu(\sum_{i=1}^d \psi_i \hat{\psi}_i) \in \mathcal{T}(\iota, \rho^d)$. (For the unique existence of C*-norm see [18].)

- The local net structure of $\mathfrak{F}$ is provided consistently by the local W*-algebras $\mathfrak{F}(\mathcal{O})$ generated from the family of Hilbert spaces H_ρ, $\rho \in \Delta(\mathcal{O})$, in $\mathfrak{F}$,

$$
H_\rho := \{\psi \in \mathfrak{F}; \psi A = \rho(A)\psi \text{ for } \forall A \in \mathfrak{A}\} \subset \mathfrak{F}, \tag{35}
$$

  whose inner product structure is due to the basic structural relation $\mathfrak{A}' \cap \mathfrak{F} = \mathbb{C}\mathbf{1}$ [17, 19] (equivalent to the condition for all G-representations to be contained in $\mathfrak{F}$). Mathematically, the uniqueness of $\mathfrak{F}$ and of $\mathfrak{F}(\mathcal{O})$ comes from the fact that they are the solutions of the *universality* problem to make the following diagram commutative, which automatically ensures the uniqueness and consistency of the constructions of $\mathfrak{F}$ and $\mathfrak{F}(\mathcal{O})$ from $\mathfrak{A}$ and $\mathfrak{A}(\mathcal{O})$, respectively:

$$
\begin{array}{ccc}
\mathfrak{A} & \hookrightarrow & \mathfrak{F} \\
\uparrow & & \uparrow \\
\mathfrak{A}(\mathcal{O}) & \hookrightarrow & \mathfrak{F}(\mathcal{O}) \\
\mu_{\mathcal{O}}\uparrow & & \uparrow\zeta_{\mathcal{O}} \\
\mathcal{O}_d^G & \hookrightarrow & \mathcal{O}_d
\end{array} . \tag{36}
$$

- The *sector structure* in the irreducible vacuum representation $(\pi, \mathfrak{H})$ of the constructed field algebra $\mathfrak{F}$ is understood as follows: first, the group G of symmetry arising in this way is **unbroken** with a unitary implementer $U : G \to \mathcal{U}(\mathfrak{H})$, $\pi(\tau_g(F)) = U(g)\pi(F)U(g)^*$ and is *global* (i.e., gauge symmetry of the 1st kind) [due to the transportability in spacetime imposed on each $\rho \in \mathcal{T}$]. This representation is realized as the induced representation of $\mathfrak{F}$ from the pure vacuum representation

$(\pi_0, \mathfrak{H}_0)$ of $\mathfrak{A}$ through the conditional expectation of G-average m : $\mathfrak{F} \to \mathfrak{A}$ defined by

$$\mathfrak{F} \ni F \longmapsto m(F) := \int_G dg \tau_g(F) \in \mathfrak{A}, \tag{37}$$

arising from the vacuum state $\bar{\omega}$ of $\mathfrak{F}$ given by $\bar{\omega}(F) := \omega_0(m(F))$, $\pi = \pi_{\bar{\omega}}$, $\mathfrak{H} = \mathfrak{H}_{\bar{\omega}}$. Then, $\mathfrak{H}$ contains the starting Hilbert space $\mathfrak{H}_0$ of the vacuum representation π_0 of $\mathfrak{A}$ as a cyclic G-fixed-point subspace, $\mathfrak{H}_0 = \mathfrak{H}^G = \{\xi \in \mathfrak{H};\, U(g)\xi = \xi \text{ for } \forall g \in G\}$, $\overline{\pi(\mathfrak{F})\mathfrak{H}_0} = \mathfrak{H}$. Then $\mathfrak{H}$ is decomposed into a direct sum in the following form [16],

$$\mathfrak{H} = \bigoplus_{\gamma \in \hat{G}} (\mathfrak{H}_\gamma \otimes V_\gamma), \tag{38}$$

$$\pi(\mathfrak{A}) = \bigoplus_{\gamma \in \hat{G}} (\pi_\gamma(\mathfrak{A}) \otimes \mathbf{1}_{V_\gamma}), \qquad U(G) = \bigoplus_{\gamma \in \hat{G}} (\mathbf{1}_{\mathfrak{H}_\gamma} \otimes \gamma(G)), \tag{39}$$

where **superselection sectors** defined as equivalence classes of irreducible representations $(\pi_\gamma, \mathfrak{H}_\gamma)$ of $\mathfrak{A}$ are *in one-to-one correspondence*, $\pi_\gamma = \pi_0 \circ \rho_\gamma \longleftrightarrow [\rho_\gamma] \in_{\mathcal{I}} \backslash^{\Delta} \longleftrightarrow (\gamma, V_\gamma)$, with equivalence classes of irreducible unitary representations $(\gamma, V_\gamma) \in \hat{G}$ of G.

3.2 Centre and central decompositions

What is important about (39) is the existence of a *non-trivial centre* of $\pi(\mathfrak{A})''$,

$$\begin{aligned} \mathfrak{Z}_\pi(\mathfrak{A}) &:= \mathfrak{Z}(\pi(\mathfrak{A})'') = \pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})' = \mathfrak{Z}(U(G)'') \\ &= \bigoplus_{\gamma \in \hat{G}} \mathbb{C}(\mathbf{1}_{\mathfrak{H}_\gamma} \otimes \mathbf{1}_{V_\gamma}) = l^\infty(\hat{G}), \end{aligned} \tag{40}$$

which implies that points $\gamma \in \hat{G}$ or (generalized) observables $(f_\gamma)_{\gamma \in \hat{G}} \in l^\infty(\hat{G})$ belonging to the centre of $\pi(\mathfrak{A})''$ as G-invariants are *order parameters* to distinguish among different sectors carrying different G-representations (in parallel with the similar role of Casimir operators in the enveloping algebra of Lie algebra $\mathfrak{g}$). From our viewpoint, the physical essence of the long and complicated mathematical story involved in the DHR-DR sector theory can be summarized as follows: a pure state $\omega \in E_{\mathfrak{A}}$ of the observable algebra $\mathfrak{A}$ is characterized as one carrying a localized charge by the DHR selection criterion, Eq.(24), for $\pi_\omega \in Rep_{\mathfrak{A}}$, which is equivalent to the existence of $\rho \in \mathcal{T}$[: DR category $(\subset End(\mathfrak{A}))$] s.t. $\pi_\omega = \pi_0 \circ \rho$. Via the Doplicher-Roberts categorical equivalence $\mathcal{T} \simeq Rep_G$, this data is further transformed into a G-charge $\gamma = \gamma_\rho \in \hat{G} \subset Rep_G$ describing the G-behaviour of the state $\omega \circ m$ of the field algebra $\mathfrak{F}$ induced from $\mathfrak{A}$ through the conditional expectation m, as a result of which the sector structure of states of $\mathfrak{A}$ selected by the DHR-criterion (DHR-selected states for short) is parametrized and classified by $Spec(\mathfrak{Z}_\pi(\mathfrak{A})) \simeq \hat{G}$. Namely, we can draw such a flow chart:

a DHR-selected state $\omega \in E_{\mathfrak{A}} \overset{\text{GNS-rep.}}{\Longrightarrow} [\pi_\omega \in \{\pi_0 \circ \rho; \rho \in \mathcal{T}\}(\subset Rep\mathfrak{A})]$ $\overset{\text{DHR}}{\Longleftrightarrow} [\rho \in \mathcal{T}(\subset End(\mathfrak{A})) \overset{\text{DR}}{\simeq} Rep_G] \Longleftrightarrow [\gamma_\rho \in \hat{G}(\subset Rep_G)]$ $\Longrightarrow$ [sectors of $\mathfrak{A}$ parametrized by $Spec(\mathfrak{Z}_\pi(\mathfrak{A})) \simeq \hat{G}$ in the irreducible vacuum representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$].

While the similarity to the scheme in Sec.2 starts now to emerge, we note that the relation of the mathematical notion of *representations* to the actual physical situations is rather indirect in comparison to that of *states*, in view of which it is desirable to reformulate the above scheme into such a form that the parallelism with Sec.2 becomes more evident. So, we need to examine here as to how one can physically attain the information on the G-charge contents of a given state ω of $\mathfrak{A}$ encoded in $\mathfrak{Z}_\pi(\mathfrak{A})$ as in Eq.(40), which has not been discussed in the traditional context of the sector theory. For this purpose, starting from a generic mixture ω of DHR-selected states, we aim at an expression for it of *Fourier-decomposition type* similar to Eq.(9), $\omega_\rho = \int_{B_K} d\rho(\beta, \mu)\omega_{\beta,\mu} = \mathcal{C}^*(\rho)$, for a thermal reference state $\omega_\rho \in K$ in Sec.2.1.

We consider now the mutual relation between states and representations of $\mathfrak{A}$. In the direction from states to representations, the GNS construction, $E_{\mathfrak{A}} \ni \omega \overset{\text{GNS}}{\longmapsto} (\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ s.t. $\omega(A) = \langle \Omega_\omega \mid \pi_\omega(A)\Omega_\omega \rangle$, $\mathfrak{H}_\omega = \overline{\pi_\omega(\mathfrak{A})\Omega_\omega}$ induces a canonical map $E_{\mathfrak{A}} \ni \omega \longmapsto (\pi_\omega, \mathfrak{H}_\omega) \in Rep_{\mathfrak{A}}$ (well-defined up to unitary equivalence). The opposite direction, however, involves the inevitable many-valuedness which necessitates the treatment of suitable sets of states, for instance, the set of vector states, $(\eta, \mathfrak{H}_\eta) \longmapsto \mathfrak{V}_\eta \equiv \{\omega_\Psi \in E_{\mathfrak{A}}; \omega_\Psi(A) := \langle \Psi \mid \eta(A)\Psi \rangle$ for $\forall A \in \mathfrak{A}$, $\Psi \in \mathfrak{H}_\eta\}$, or that of density-matrix states in $(\eta, \mathfrak{H}_\eta)$. Since the latter choice has a natural connection with the von Neumann algebra $\eta(\mathfrak{A})''$ of the representation η, it has a name, a *folium*[5] associated to $(\eta, \mathfrak{H}_\eta)$, which we denote by

$$\mathfrak{f}(\eta) := \{\omega \in E_{\mathfrak{A}}; \exists \sigma : \text{density operator in } \mathfrak{H}_\eta \text{ s.t. } \omega(A) = Tr_{\mathfrak{H}}[\sigma\eta(A)]\}, \tag{41}$$

and is also related to a state ω by $\mathfrak{f}(\omega) := \mathfrak{f}(\pi_\omega)$ by means of the corresponding GNS representation π_ω. A state in $\mathfrak{f}(\eta)$ is also called a η-normal state of $\mathfrak{A}$.

Then a state $\omega \in E_{\mathfrak{A}}$ of $\mathfrak{A}$ is a mixture of DHR-selected states if and only if $\omega \in \mathfrak{f}(\pi)$ (with π the restriction to $\mathfrak{A}$ of the vacuum representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ induced from the vacuum representation $(\pi_0, \mathfrak{H}_0)$ of $\mathfrak{A}$), which is also equivalent to the existence of an extension $\tilde{\omega}$ of ω to the von Neumann algebra $\pi(\mathfrak{A})''$ given by $\tilde{\omega}(\tilde{A}) = Tr_{\mathfrak{H}}(\sigma_\omega \tilde{A})$ for $\tilde{A} \in \pi(\mathfrak{A})''$ with such a density operator σ_ω in $\mathfrak{H}$ that $\omega(A) = Tr_{\mathfrak{H}}(\sigma_\omega \pi(A))$. Through the *central decomposition* for the "simultaneous diagonalization" of centre $\mathfrak{Z}_\pi(\mathfrak{A}) = l^\infty(\hat{G})$, such

[5] A folium $\mathfrak{f}(\eta)$ is related with the von Neumann algebra $\eta(\mathfrak{A})''$ in such a way that its linear hull $Lin(\mathfrak{f}(\eta))$ consisting of linear combinations of states in $\mathfrak{f}(\eta)$ is the *predual* $\eta(\mathfrak{A})''_*$ of $\eta(\mathfrak{A})''$, $Lin(\mathfrak{f}(\eta)) = \eta(\mathfrak{A})''_*$ uniquely characterized by the relation $(\eta(\mathfrak{A})''_*)^* = \eta(\mathfrak{A})''$.

a state $\omega \in \mathfrak{f}(\pi)$ can be uniquely decomposed into the sum of factor states ω_γ corresponding to $\gamma \in \hat{G}$:

$$\omega(A) = \sum_{\gamma \in \hat{G}} \mu_\omega(\gamma)\omega_\gamma(A). \tag{42}$$

Thus, we have a *q→c channel* $\omega \longmapsto \mu_\omega$ transforming quantum states into probability distributions over the spectrum $\hat{G}$ of $\mathfrak{Z}_\pi(\mathfrak{A})$, which describes G-charge contents of each such quantum state $\omega \in \mathfrak{f}(\pi)$ in terms of a probability distribution $\mu_\omega = \{\mu_\omega(\gamma)\}_{\gamma \in \hat{G}}$ over $\hat{G}$. This is in parallel with the integral decomposition Eq.(9). However, one important difference should be noted here: within a sector $(\pi_\gamma, \mathfrak{H}_\gamma)$ of the same G-charge γ, there exist *many* different states ω_γ showing different behaviours under $\mathfrak{A}$, e.g., with different localization or different energy-momentum spectrum, as energy-momentum (tensor) is invariant under G.[6]. Thus, in contrast to the thermal situation with fixed choice of $\omega_{\beta,\mu}$, each factor state ω_γ appearing on the right-hand side of Eq.(42) may vary depending upon $\omega \in \mathfrak{f}(\pi)$. In the former case, different factor KMS states $\omega_{\beta,\mu}$ are always disjoint [7] corresponding to different order parameters (because of the uniqueness of a KMS state within its folium), whereas what is shared in common by all the pure states ω_γ within a sector is just the unitary equivalence class $[\pi_{\omega_\gamma}]$ of the corresponding GNS *representation* π_{ω_γ} of $\mathfrak{A}$ in terms of which all the above equivalent expressions starting from the DHR criterion (24) are given. Since this point is related to the equivalence of endomorphisms $\rho \cong Ad(u) \circ \rho$ for $\rho \in \Delta$ w.r.t. $Ad(u) \in \mathcal{I} = Inn(\mathfrak{A})$ [16], we should resolve this ambiguity to extract internal symmetry aspects of a given *state*. In view of the fact that local subalgebras $\mathfrak{A}(\mathcal{O})$ are *factor* von Neumann algebras *without centres* from which the non-trivial centre $\mathfrak{Z}_\pi(\mathfrak{A})$ arises only in the weak closure $\pi(\mathfrak{A})''$ of the *global* algebra $\mathfrak{A}$, it is also interesting to ask a related question as to how we can attain *locally* and *minimally*[7] the above solution in physical situations according to the spirit of local quantum physics [25].

This is consistently achieved in use of $\rho \in \mathcal{T}$ as follows, in parallel with the previous section. Choose a *representative* ρ_γ from each equivalence class $[\rho_\gamma] \in_{\mathcal{I}} \backslash^{\Delta}$, which amounts to a choice of a cross section $\hat{G} \ni \gamma \longmapsto \rho_\gamma \in [\rho_\gamma] \subset \Delta$ of a bundle $\Delta \twoheadrightarrow_{\mathcal{I}} \backslash^{\Delta} \simeq \hat{G} = Spec(\mathfrak{Z}_\pi(\mathfrak{A}))$. After identifying a compact Lie group G, such a choice can be achieved, e.g., by choosing one ρ_{γ_0} corresponding to the *fundamental representation* γ_0 of G; ρ_γ for arbitrary $\gamma \in \hat{G}$ can be extracted from $\rho_{\gamma_0}^n$ with suitable $n \in \mathbb{N}$ as a direct-sum component, by means of Clebsch-Gordan coefficients. In view of the

[6]This situation was carelessly overlooked in the original version of this paper, as pointed out by Prof. J. E. Roberts, to whom I am very grateful.

[7]If we are allowed to collect and to introduce *all* the information concerning $\mathfrak{A}$, then the "ambiguities" trivially disappear, because of their origins coming from the choices of states within a given sector and from that of a representative ρ among equivalent ones $Ad(u) \circ \rho$, $Ad(u) \in Inn(\mathfrak{A})$.

physical meaning of ρ's, this choice can be interpreted as a specification of procedures to create G-charges from the vacuum.

3.3 Physical interpretation by conditional expectation as $c\to q$ channel and its "inverse"

Then choosing an everywhere non-vanishing probability distribution μ_G over $\hat{G}$, $\mu_G = (\mu_\gamma)_{\gamma\in\hat{G}} \in (0,1)^{\hat{G}}$, $\sum_{\gamma\in\hat{G}} \mu_\gamma = 1$, we can define a *central measure* μ on $E_{\mathfrak{A}}$ with support $\{\omega_\gamma := \omega_0 \circ \rho_\gamma ; \gamma \in \hat{G}\}$ in the state space $E_{\mathfrak{A}}$ whose barycentre ω_μ is given by

$$\omega_\mu(A) := \sum_{\gamma\in\hat{G}} \mu_\gamma \omega_0 \circ \rho_\gamma(A). \tag{43}$$

This allows us also to define, in a similar way to the thermal situation, a *conditional expectation* $\Lambda_\mu : \mathfrak{A} \to \mathfrak{Z}_\pi(\mathfrak{A})$ as a $c\to q$ channel s.t. $\Lambda_\mu(A) := [\hat{G} \ni \gamma \longmapsto \omega_0 \circ \rho_\gamma(A)] \in \mathfrak{Z}_\pi(\mathfrak{A})$, $\Lambda_\mu^*(\nu)(A) = \sum_{\gamma\in\hat{G}} \nu_\gamma [\Lambda_\mu(A)](\gamma) = \sum_{\gamma\in\hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma(A)$. Here the definition of Λ_μ depends on the choice of a cross section $\hat{G} \ni \gamma \longmapsto \rho_\gamma \in [\rho_\gamma] \subset \Delta$ but is independent of the particular assignment of a probability weight μ_γ to each $\gamma \in \hat{G}$. In use of this freedom we see now that, similarly to the discussion in Sec.2.1, the central measure μ as a *$q\to c$ channel* allows physical interpretation w.r.t. G of all states of such forms as $\Lambda_\mu^*(\nu) = \sum_{\gamma\in\hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma \in E_{\mathfrak{A}}$ with $\nu = (\nu_\gamma)_{\gamma\in\hat{G}} \in M_1(\hat{G}) := \{(\nu'_\gamma)_{\gamma\in\hat{G}};\ \nu'_\gamma \geq 0, \sum_{\gamma\in\hat{G}} \nu'_\gamma = 1\}$. Defining a map W by

$$W : End(\mathfrak{A}) \ni \rho \longmapsto \omega_0 \circ \rho \in E_{\mathfrak{A}}, \tag{44}$$

we see the relations

$$\begin{aligned} &[\Lambda_\mu(A)](\gamma) = \omega_0 \circ \rho_\gamma(A) = [W(\rho_\gamma)](A); \\ &\Lambda_\mu^*(\nu)(A) = \nu(\Lambda_\mu(A)) = \sum_{\gamma\in\hat{G}} \nu_\gamma [\Lambda_\mu(A)](\gamma) = (\sum_{\gamma\in\hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma)(A) \\ &\Longrightarrow \Lambda_\mu^*(\nu) = \sum_{\gamma\in\hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma = \sum_{\gamma\in\hat{G}} \nu_\gamma W(\rho_\gamma). \end{aligned} \tag{45}$$

Therefore, the map Λ_μ^* extends W to "convex combinations" of ρ_γ's, and acts as a "*charging* map" to create from the vacuum ω_0 a state $\Lambda_\mu^*(\nu) = \sum_{\gamma\in\hat{G}} \nu_\gamma (\omega_0 \circ \rho_\gamma)$ whose charge contents are described by the charge distribution $\nu = (\nu_\gamma)_{\gamma\in\hat{G}} \in M_1(\hat{G})$ over the group dual $\hat{G}$. The role of the chosen cross section $\gamma \longmapsto \rho_\gamma$ and the state family $E_\mu := \Lambda_\mu^*(M_1(\hat{G})) = \{\sum_{\gamma\in\hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma ; \nu_\gamma \geq 0, \sum_{\gamma\in\hat{G}} \nu_\gamma = 1\} \subset E_{\mathfrak{A}}$ is just to make the $c\to q$ channel Λ_μ^* *invertible* on E_μ, $E_\mu \ni \omega = \Lambda_\mu^*(\nu) \longmapsto \nu \in M_1(\hat{G})$, to give a physical interpretation of ω w.r.t. G in terms of ν.

As far as the internal symmetry aspect is concerned, we see that this setup is already sufficient for providing any given state $\omega \in \mathfrak{f}(\pi)$ with its physical interpretation owing to the above observation and the simple relation between central observables and folia: any states, $\varpi_\gamma \in \mathfrak{f}(\omega_\gamma)$, in a folium of the *factorial* state $\omega_\gamma = \omega_0 \circ \rho_\gamma$ yield the same expectation value $\varpi_\gamma(f) = f_\gamma$ to each central observable $f = (f_\gamma)_{\gamma\in\hat{G}} \in l^\infty(\hat{G}) = \mathfrak{Z}_\pi(\mathfrak{A})$ which is "diagonalized" in the central decomposition. Therefore, we arrive at a similar formula to Eq.(17) in Sec.2.1 as

Proposition 7 *Selection and interpretation of G-charges:*

$$
\begin{aligned}
&(\mathfrak{f}(\pi)/\mathfrak{Z}_\pi(\mathfrak{A}))(\omega, \Lambda_\mu^*(\nu)) \simeq M_1(\hat{G})(\mu_\omega, \nu) \\
&\Longleftrightarrow \mathfrak{f}(\omega) = \mathfrak{f}(\Lambda_\mu^*(\nu)) \Longleftrightarrow \mu_\omega(\gamma) = \nu_\gamma \ \textit{(for } \forall\gamma \in \hat{G}\textit{)}. \qquad (46)
\end{aligned}
$$

To obtain a formula of Fourier-decomposition type similar to Eq.(9), however, we need to exhibit the additional elements appearing in the many to one correspondence between states and representations $[E_\mathfrak{A} \ni \omega \overset{\text{GNS}}{\Longleftrightarrow} (\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega) \underset{\leftarrow--}{\overset{\text{many to one}}{\longrightarrow}} (\pi_\omega, \mathfrak{H}_\omega) \in Rep_\mathfrak{A}]$, in order to relate an arbitrary state $\phi = \sum_{\gamma\in\hat{G}} \nu_\gamma \varpi_\gamma \in \mathfrak{f}(\pi)$, $\varpi_\gamma \in \mathfrak{f}(\omega \circ \rho_\gamma)$ to the family E_μ. Since each pure state belonging to $\mathfrak{f}(\omega \circ \rho_\gamma)$ is written as $\omega \circ \sigma_\gamma$ with σ_γ related to ρ_γ through $\sigma_\gamma(A) = u_\gamma^* \rho_\gamma(A) u_\gamma$, $u_\gamma \in \mathcal{U}(\mathfrak{A})$, we have, for $\forall\phi \in \mathfrak{f}(\pi)$ and $\forall A \in \mathfrak{A}$,

$$
\begin{aligned}
\phi(A) &= \sum_{\gamma\in\hat{G}} \nu_\gamma \sum_{i\in I_\gamma} p_i^\gamma \omega_0 \circ Ad(u_{\gamma,i}^*) \circ \rho_\gamma(A) \qquad (47)\\
&= \sum_{\gamma\in\hat{G}} \nu_\gamma \sum_{i\in I_\gamma} p_i^\gamma \langle u_{\gamma,i}\Omega_0 \mid \pi_0 \circ \rho_\gamma(A) u_{\gamma,i}\Omega_0\rangle, \qquad (48)
\end{aligned}
$$

with $p_i^\gamma \in [0,1]$, $\sum_{i\in I_\gamma} p_i^\gamma = 1$, $u_{\gamma,i} \in \mathcal{U}(\mathfrak{A})$ for $\forall\gamma \in \hat{G}$, $\forall i \in I_\gamma$. Here ν_γ is the probability to find the sector with G-charge $\gamma \in \hat{G}$ in the state ϕ and p_i^γ is the conditional probability to find the state $\langle u_{\gamma,i}\Omega_0 \mid \pi_0 \circ \rho_\gamma(-) u_{\gamma,i}\Omega_0\rangle$ associated to the vector $u_{\gamma,i}\Omega_0$, knowing that the system is already in the sector with γ.

Since the "gap" between $u_{\gamma,i}\Omega_0$ and Ω_0 is due to $u_{\gamma,i} \in \mathcal{U}(\mathfrak{A})$, its "observability" should enable one to find some physical processes to identify it, for instance, involving *energy-momentum* (*as observables*) by some limits of taking the *lowest energy state* among $\{u_\gamma\Omega_0; u_\gamma \in \mathcal{U}(\mathfrak{A})\}$, etc. (Actually this is the same problem as discussed above concerning the choice of a section of $\Delta \twoheadrightarrow_\mathcal{I} \backslash^\Delta \simeq \hat{G}$ in a different disguise. If we combine the data of relevant observables, such as energy-momentum, from the beginning, this can be totally absorbed into the choice of a section.) Once this is done, any other states $\phi = \sum_{\gamma\in\hat{G}} \nu_\gamma \sum_{i\in I_\gamma} p_i^\gamma \omega_0 \circ Ad(u_{\gamma,i}^*) \circ \rho_\gamma \in \mathfrak{f}(\pi)$ can be related to the corresponding $\Lambda_\mu^*(\nu) \in E_\mu$ through the measurement of relevant observables (e.g., energy momentum) and/or the limiting procedures to pick up Ω_0 as the lowest energy state among $u_\gamma\Omega_0$ with $u_\gamma \in \mathcal{U}(\mathfrak{A})$.

If $\phi = \sum_{\gamma\in\hat{G}} \nu_\gamma\varpi_\gamma \in \mathfrak{f}(\pi)$ has such a decomposition into sectors that its component factorial states $\varpi_\gamma \in \mathfrak{f}(\omega\circ\rho_\gamma)$ are all *pure*, there is a different but equivalent formulation in use of a reducible representation $\gamma_\nu := \oplus_{\gamma\in\hat{G},\nu_\gamma\neq 0}\gamma \in RepG$, which may look more familiar for treating the same situation. For this purpose, we use the invariance of the vacuum state under $U(G)$ which implies the following relations in terms of the conditional expectation $m : \mathfrak{F} \to \mathfrak{A} = \mathfrak{F}^G$, $m(F) = \int_G dg\tau_g(F)$:

$$(\omega_0 \circ \rho_\gamma)(m(F))) = \langle \Omega_0 \mid \sum_i \psi_i^\gamma m(F)\psi_i^{\gamma *}\Omega_0\rangle, \tag{49}$$

where the last expression is understood in the representation space $\mathfrak{H}$ of $\mathfrak{F}$ and $\psi_i^\gamma \in \mathfrak{F}$ are such that $\psi_i^\gamma\pi(A) = \pi\circ\rho_\gamma(A)\psi_i^\gamma$ for $\forall A \in \mathfrak{A}$ (coming from the Cuntz algebra $\mathcal{O}_d$). Then owing to the disjointness among different sectors, the state $\Lambda_\mu^*(\nu)$ can be rewritten as an induced state $\Lambda_\mu^*(\nu)\circ m$ of $\mathfrak{F}$ by

$$\begin{aligned}\Lambda_\mu^*(\nu)(m(F)) &= \sum_{\gamma\in\hat{G}} \nu_\gamma\omega_0\circ\rho_\gamma(m(F)) = \sum_{\gamma\in\hat{G}}\sum_i \langle\sqrt{\nu_\gamma}\psi_i^{\gamma *}\Omega_0 \mid m(F)\sqrt{\nu_\gamma}\psi_i^{\gamma *}\Omega_0\rangle \\ &= \langle\Psi \mid m(F)\Psi\rangle = \langle\Psi \mid F\Psi\rangle, \end{aligned}\tag{50}$$

with a vector

$$\Psi := \sum_{\gamma\in\hat{G}}\sum_i \sqrt{\nu_\gamma}\psi_i^{\gamma *}\Omega_0 \in \mathfrak{H} \tag{51}$$

belonging to the above mentioned reducible representation $\gamma_\nu := \oplus_{\gamma\in\hat{G},\nu_\gamma\neq 0}\gamma$ of G.

In either formulation, we attain operational interpretations of the basic results of DHR-DR theory, which provide the physical interpretation of any state $\omega \in \mathfrak{f}(\pi)$ as a mixture of the DHR-selected states, with respect to their internal-symmetry aspects, specifying its ***G-charge contents*** understood as the G-representation contents. Since the spacetime behaviours of quantum fields are expressed by the observable net $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ and since the internal symmetry aspects are described in the above machinery also encoded in $\mathfrak{A}$, the role of the field algebra $\mathfrak{F}$ and the internal symmetry group G becomes now quite subsidiary, simply providing comprehensible vocabulary based on the covariant objects under the symmetry transformations. Thus, we have arrived at an physical and operational picture for the sector theory showing the parallelism with the previous discussion of the thermal interpretation based upon the $c\to q$ channel $\mathcal{C} : \mathcal{A} \to C(B_K)$. While, in the latter case, the reference system to provide the vocabulary for the interpretation is already known at the beginning, it is remarkable that the corresponding one, $\mathfrak{Z}_\pi(\mathfrak{A}) \simeq l^\infty(\hat{G})$, in the DHR-DR theory naturally emerges from the basic ingredients of the theory written in terms of the algebra $\mathfrak{A}$ of observables, through the chain of equivalence starting from the DHR criterion: [DHR-selected representations of $\mathfrak{A}$] $\Longleftrightarrow$ [Doplicher-Roberts category $\mathcal{T}$] $\Longleftrightarrow$ [Rep_G and G] $\Longrightarrow$ [$\hat{G} = Spec(\mathfrak{Z}_\pi(\mathfrak{A}))$].

From the above observation that the ambiguity in the choice of a cross section $\hat{G} \ni \gamma \longmapsto \rho_\gamma \in [\rho_\gamma] \subset \Delta$ which picks up one ρ_γ to each $\gamma \in \hat{G}$ among the equivalence class $\{Ad(u) \circ \rho_\gamma; u \in \mathfrak{A}$: unitary$\}$ is essentially due to observables in $\mathfrak{A}$ related to the spacetime symmetry, i.e., the energy contents of sectors, we realize that it is important to understand the mutual relations between the energy-momentum spectrum and the sectors as internal-symmetry spectrum, in such a form as the ***energy contents of sectors***: for instance, the contents of the sectors parametrized by $\gamma \in \hat{G}$, $\gamma \neq \iota$(: the trivial representation corresponding to the vacuum sector) are *excited states* above the vacuum. Since only the sector with trivial representation $\iota \in \hat{G}$ contains the vacuum state with the *minimum* energy 0 and since all other sectors consist of the excited states, the above picture suggests the following results to be expected to hold (under the assumption of the existence of a mass gap):

$$\min\{Spec(\hat{P}_0 \upharpoonright_{\mathfrak{H}_0})\} = 0, \qquad \inf\{Spec(\hat{P}_0 \upharpoonright_{\mathfrak{H}_0^\perp})\} > 0. \tag{52}$$

In the treatment of thermal functions in Sec.2, it is easily seen that, while the entropy density $s(\beta)$ is not contained in the image set $\mathcal{C}(\mathcal{T}_x)$ due to the absence of such a quantum observable $\hat{s}(x) \in \mathcal{T}_x$ that $\omega_\beta(\hat{s}(x)) = s(\beta)$, it can be *approximated* by the thermal functions in $\mathcal{C}(\mathcal{T}_x)$. In order to facilitate the above discussions of mutual relations between spacetime and internal symmetries, it is important to have those observables freely at hand which detect the G-charge contents in $\mathfrak{Z}_\pi(\mathfrak{A}) = l^\infty(\hat{G})$, and, for this purpose, we need also here to consider the problem as to how such observables can be supplied from the *local* observables belonging to $\mathfrak{A}(\mathcal{O})$, i.e., the approximation of global order parameters by local order fields or central sequences. For this purpose, the analyses of point-like fields and the rigorous method of their operator-product expansions developed by Bostelmann in [6] would be quite useful in these contexts. All the above sort of considerations (with the modifications of the DHR selection criterion necessitated by the possible presence of the long range forces, such as of Buchholz-Fredenhagen type) will be crucially relevant to the approach to the colour confinement problem, and, especially the latter one (to find a suitably modified criterion) seems to be quite a non-trivial issue there.

4 SSB-vacua as continuous sectors with order parameter whose quantum precursor is Goldstone mode

4.1 Dual net $\mathfrak{A}^d$ and unbroken remaining symmetry H

To treat physically more interesting cases of *spontaneous symmetry breakdown (SSB)*, we need to extend the original DR sector theory where the internal symmetry is unbroken with unitary implementers as long as the Haag duality $\mathfrak{A}^d(\mathcal{O}) := \pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))$ (for $\mathcal{O} \in \mathcal{K}$) holds to play the crucial roles. It can be shown that this property is also a necessary condition for the field system with normal statistics and with unbroken symmetry (see, [16, 19]). As pointed out by Roberts [45], SSB does not take place without the breakdown of the Haag duality.

In the previous case with unbroken symmetry, the superselection sectors are parametrized by the *discrete* variables belonging to the dual $\hat{G}$ of a compact group G. In the situation with SSB, one anticipates physically the appearance of *continuous* macroscopic *order parameters*, as typically exemplified by the continuous directions of magnetization in the ferromagnetism, which strongly suggests the appearance of *continuous superselection sectors*, parametrized by macroscopic order parameters. This will be shown actually to be the case in the following.

For the sake of convenience, we change the notation adopted in Sec.3 in the unbroken symmetry case, so that the observable algebra $\mathfrak{A}$ and the symmetry group G in Sec.3 are replaced, respectively, by the dual net $\mathfrak{A}^d$ (of the genuine observable algebra $\mathfrak{A}$) and the group H of *unbroken remaining symmetry* in the present context. To begin with, the correspondence between physically relevant states ω around the vacuum ω_0 and such an endomorphism ρ as $\omega = \omega_0 \circ \rho$ can be maintained when all the ingredients here are understood in relation to the dual net $\mathfrak{A}^d$ under the natural assumption of *essential duality*

$$\mathfrak{A}^{dd} = \mathfrak{A}^d \tag{53}$$

which is equivalent to the local commutativity of the dual net and is valid whenever some Wightman fields are underlying the theory [10]. First, in view of the relation $\pi_0(\mathfrak{A}^d(\mathcal{O}'))'' = \pi_0(\mathfrak{A}(\mathcal{O}'))''$ [45], the starting vacuum state and representation, ω_0 and $(\pi_0, \mathfrak{H}_0)$, can safely be extended from $\mathfrak{A}$ to $\mathfrak{A}^d$ (meaning both the local nets and the global algebras). Then the DHR selection criterion is understood for the states ω of $\mathfrak{A}^d$, as $\pi_\omega \upharpoonright_{\mathfrak{A}^d(\mathcal{O}')} = \pi_0 \upharpoonright_{\mathfrak{A}^d(\mathcal{O}')}$, and is equivalent to the existence of $\rho \in \mathcal{T} \subset End(\mathfrak{A}^d)$ such that $\pi_\omega = \pi_0 \circ \rho$. On the basis of these items, we can repeat the same procedure of constructing the field algebra $\mathfrak{F}$ and the group H of unbroken symmetry according to the general method [18, 19]:

$$\mathfrak{F} = \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0}, \qquad H = Gal(\mathfrak{F}/\mathfrak{A}^d). \tag{54}$$

4.2 Spontaneously broken symmetry

Now we start to clarify the sector structure associated with a spontaneously broken symmetry described by the Galois group $G := Gal(\mathfrak{F}/\mathfrak{A}) \supset H$. First we consider the irreducible H-covariant vacuum representation $(\pi, U, \mathfrak{H})$ of the system $\mathfrak{F} \underset{\tau}{\curvearrowleft} H$, $\pi(\tau_h(F)) = U(h)\pi(F)U(h)^*$ for $\forall F \in \mathfrak{F}$, $\forall h \in H$, containing the original representation $(\pi_0, \mathfrak{H}_0)$ of $\mathfrak{A}$ and of $\mathfrak{A}^d$ as the cyclic fixed-point subspace under $U(H)$: $\mathfrak{H}_0 = \{\xi \in \mathfrak{H};\, U(h)\xi = \xi \text{ for } \forall h \in H\}$, $\overline{\pi(\mathfrak{F})\mathfrak{H}_0} = \mathfrak{H}$. Then according to the DHR sector structure in the unbroken case [16], we have

$$\mathfrak{Z}_\pi(\mathfrak{A}^d) = \mathfrak{Z}(U(H)'') = \underset{\eta \in \hat{H}}{\oplus} \mathbb{C}(\mathbf{1}_{\mathfrak{H}_\eta} \otimes \mathbf{1}_{W_\eta}) = l^\infty(\hat{H}). \tag{55}$$

Since this group H is the maximal group of unbroken symmetry in the irreducible vacuum situation, the group G bigger than H cannot be unitarily implemented in the above representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$, which is just the precise meaning of the SSB of G in the present situation. To cover more general situations we propose a general definition of SSB in the following form:

Definition 8 *A symmetry described by a (strongly continous) automorphic action τ of G on the field algebra $\mathfrak{F}$ is said to be* ***unbroken*** *in a given representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ if the spectrum of the centre $\mathfrak{Z}_\pi(\mathfrak{F}) = \mathfrak{Z}(\pi(\mathfrak{F})'')$ is pointwise invariant under the action of G induced on $Spec(\mathfrak{Z}_\pi(\mathfrak{F}))$ (almost everywhere w.r.t. the central measure μ which appears in the central decomposition of π into factor representations). If the symmetry is not unbroken in $(\pi, \mathfrak{H})$, it is said to be* ***broken spontaneously*** *there.*

In particular, G acting on $\mathfrak{F}$ is unbroken if each **factor** subrepresentation $(\sigma, \mathfrak{H}_\sigma)$, $\sigma(\mathfrak{F})' \cap \sigma(\mathfrak{F})'' = \mathbb{C}\mathbf{1}_{\mathfrak{H}_\sigma}$, appearing in the central decomposition of $(\pi, \mathfrak{H})$ admits a **covariant representation** of the system $G \overset{\tau}{\curvearrowright} \mathfrak{F}$ in terms of a (strongly continuous) unitary representation $(U_\sigma, \mathfrak{H}_\sigma)$ of G verifying the relation $\sigma(\tau_g(F)) = U_\sigma(g)\sigma(F)U_\sigma(g)^*$ for $\forall g \in G, \forall F \in \mathfrak{F}$.

Remark 9 *In essence, SSB means the conflict between unitary implementability and factoriality (=triviality of centres) [38]. The situation with SSB is seen to exhibit the features of the so-called "infrared instability" under the action of G, because G does not stabilize the spectrum of centre which can be viewed physically as macroscopic order parameters emerging in the infrared (=low energy) regions.*

Remark 10 *Since the above definition of SSB still allows the mixture of unbroken and broken subrepresentations of a given* π*, we need to decompose* $Spec(\mathfrak{Z}_\pi(\mathfrak{F}))$ *into* G*-invariant domains which cannot be further decomposed. It is easily seen that each such minimal domain is characterized by the ergodicity under* G *which is nothing but the notion of central ergodicity. Then* π *is decomposed into the direct sum (or, direct integral) of unbroken factor representations and broken non-factor representations, each component of which is stable under* G*. In this way we obtain a phase diagram on the spectrum of the centre.*

As indicated above, the natural physical picture of *order parameters arising from the SSB* from G down to H is realized in connection with the sector structure of the whole theory involving the presence of *continuous sectors* parametrized by $\dot{g} := Hg \in H\backslash G$. Here we need to combine the above two formulations of discrete sectors of unbroken internal symmetry (Sec.3) and of continuous sectors (Sec.2) in the following way. One important point to be mentioned is that our motivation for treating here the *centre*s at various levels of representations is always coming from the natural and inevitable occurrence of *disjoint* representations which leads to the appearance of *macroscopic order parameters to classify different modes of* macroscopic manifestations of microscopic systems; this should be properly contrasted to a mathematical pursuit of generalizing the pre-existing machinery involving factor algebras to non-factorial ones.

According to this formulation, we should find such a covariant representation of the system $(\mathfrak{F} \underset{\tau}{\curvearrowleft} G)$ as implementing *minimally* the broken G in the sense of *central ergodicity* under G. Since the subgroup H is unbroken in the irreducible covariant representation $(\pi, U, \mathfrak{H})$ of $\mathfrak{F} \underset{\tau}{\curvearrowleft} H$, what we seek for can actually be provided by the representation $(\hat{\pi}, \hat{\mathfrak{H}})$, induced from $(\pi, U, \mathfrak{H})$, of the crossed product $\hat{\mathfrak{F}} := \mathfrak{F} \rtimes (H\backslash G) = \Gamma(G \times_H \mathfrak{F})$ of $\mathfrak{F}$ with the homogeneous space $H\backslash G$ (having the right G-action being transitive, and hence, trivially G-ergodic), which can be identified with the algebra of H-equivariant norm-continuous functions $\hat{F} : G \to \mathfrak{F}$ satisfying

$$\hat{F}(hg) = \tau_h(\hat{F}(g)). \tag{56}$$

The action $\hat{\tau}$ of G on $\hat{F} \in \hat{\mathfrak{F}}$ is defined by

$$[\hat{\tau}_g(\hat{F})](g_1) = \hat{F}(g_1 g), \tag{57}$$

with $g, g_1 \in G$, consistently with (56). For the technical reason, we need here the assumption that G should be a *locally compact* group equipped with a left-invariant Haar measure, although the general definition as a Galois group $G := Gal(\mathfrak{F}/\mathfrak{A})$ does not ensure it. Then denoting $d\xi$ the left-invariant Haar measure on G/H (equipped with the *left* G-action), we define a Hilbert space $\hat{\mathfrak{H}}$ of L^2-sections of $G \times_H \mathfrak{H}$ by

$$\hat{\mathfrak{H}} = \int_{\xi \in G/H}^{\oplus} (d\xi)^{1/2} \mathfrak{H} = \Gamma_{L^2}(G \times_H \mathfrak{H}, d\xi), \tag{58}$$

which can also be identified with the L^2-space of $\mathfrak{H}$-valued (U, H)-equivariant functions ψ on G,

$$\psi(gh) = U(h^{-1})\psi(g) \quad \text{ for } \psi \in \hat{\mathfrak{H}},\ g \in G,\ h \in H. \tag{59}$$

On this $\hat{\mathfrak{H}}$, representations $\hat{\pi}$ and $\hat{U}$ of $\hat{\mathfrak{F}}$ and G are defined, respectively, by

$$(\hat{\pi}(\hat{F})\psi)(g) := \pi(\hat{F}(g^{-1}))(\psi(g)) \quad \text{ for } \hat{F} \in \hat{\mathfrak{F}},\ \psi \in \hat{\mathfrak{H}},\ g \in G, \tag{60}$$

$$(\hat{U}(g_1)\psi)(g) := \psi(g_1^{-1}g) \quad \text{ for } g, g_1 \in G, \tag{61}$$

which are compatible with the above equivariance condition (59):

$$\begin{aligned}
&(\hat{\pi}(\hat{F})\psi)(gh) = \pi(\hat{F}(h^{-1}g^{-1}))(\psi(gh)) \\
&= U(h^{-1})\pi(\hat{F}(g^{-1}))U(h)U(h^{-1})(\psi(g)) = U(h^{-1})(\hat{\pi}(\hat{F})\psi)(gh); \quad (62)\\
&(\hat{U}(g_1)\psi)(gh) = \psi(g_1^{-1}gh) = U(h^{-1})\psi(g_1^{-1}g) = U(h^{-1})(\hat{U}(g_1)\psi)(g), \quad (63)
\end{aligned}$$

and satisfies the covariance relation:

$$\hat{\pi}(\hat{\tau}_g(\hat{F})) = \hat{U}(g)\hat{\pi}(\hat{F})\hat{U}(g)^{-1}. \tag{64}$$

We consider an embedding $\hat{\imath}_{H\backslash G} : \mathfrak{F} \hookrightarrow \hat{\mathfrak{F}}$ of $\mathfrak{F}$ into $\hat{\mathfrak{F}}$ defined by

$$[\hat{\imath}_{H\backslash G}(F)](g) := \tau_g(F), \tag{65}$$

which intertwines the G-actions τ on $\mathfrak{F}$ and $\hat{\tau}$ on $\hat{\mathfrak{F}}$,

$$\hat{\imath}_{H\backslash G} \circ \tau_g = \hat{\tau}_g \circ \hat{\imath}_{H\backslash G} \quad (\forall g \in G). \tag{66}$$

Combining $\hat{\imath}_{H\backslash G}$ with $\hat{\pi}$, we obtain a covariant representation $(\bar{\pi}, \hat{U}, \hat{\mathfrak{H}})$, $\bar{\pi} := \hat{\pi} \circ \hat{\imath}_{H\backslash G}$, of $\mathfrak{F} \underset{\tau}{\curvearrowleft} G$ defined on $\hat{\mathfrak{H}}$ by

$$(\bar{\pi}(F)\psi)(g) := \pi(\tau_{g^{-1}}(F))\psi(g) \qquad (F \in \mathfrak{F}, \psi \in \hat{\mathfrak{H}})$$

and satisfying

$$\bar{\pi}(\tau_g(F)) = \hat{U}(g)\bar{\pi}(F)\hat{U}(g)^{-1}.$$

All the above operations are compatible with the constraints of H-equivariance.

4.3 Sector structures and $c \to q$ channel

The crucial information for determining the sector structure is the *centres* of $\mathfrak{A}^d, \mathfrak{A}$ and $\mathfrak{F}$ in the representation $(\bar{\pi}, \hat{\mathfrak{H}})$. The mutual relations among the relevant C*-algebras can be summarized in the following commuting diagram:

$$
\begin{array}{ccccc}
 & & \hat{\mathfrak{F}} = \Gamma(G \underset{H}{\times} \mathfrak{F}) & & \\
 & \hat{i}_{H\backslash G} \nearrow & & \nwarrow \hat{i}_{H\backslash G} & \\
\mathfrak{A}^d = \mathfrak{F}^H & & \overset{m_H}{\underset{i_H}{\leftrightarrows}} & & \mathfrak{F} = \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0} \; , \\
 & m_{G/H} \searrow \nwarrow i_{G/H} & & i_G \nearrow \swarrow m_G & \\
 & & \mathfrak{A} \subset \mathfrak{F}^G & &
\end{array}
$$

where the maps i_G and m_G, etc. are, respectively, the embedding maps (of a C*-algebra into another) and the conditional expectations, such as

$$
m_{G/H} : \mathfrak{A}^d = \mathfrak{F}^H \ni B \longmapsto m_{G/H}(B) := \int_{G/H} d\dot{g} \; \tau_g(B) \in \mathfrak{F}^G. \qquad (67)
$$

Using the relations, $\bar{\pi}(\mathfrak{A}) = \int_{\dot{g} \in H\backslash G}^{\oplus} d\dot{g} \;\; \pi(\mathfrak{A}) = \mathbf{1}_{L^2(G/H, d\dot{g})} \otimes \pi(\mathfrak{A})$ and $\pi(\mathfrak{A})'' = \pi(\mathfrak{A}^d)''$ (following from $\pi(\mathfrak{A}^d) = \pi(\mathfrak{A})'' \cap \pi(\mathfrak{F})$ [10]), we obtain

Proposition 11 *[42]*

$$
\begin{aligned}
\mathfrak{Z}_{\bar{\pi}}(\mathfrak{F}) &= L^\infty(H\backslash G; d\dot{g}) = \mathfrak{Z}_{\hat{\pi}}(\hat{\mathfrak{F}}); \\
\mathfrak{Z}_{\bar{\pi}}(\mathfrak{A}) &= \mathbf{1}_{L^2(G/H, d\dot{g})} \otimes \mathfrak{Z}_{\pi}(\mathfrak{A}) = \mathbf{1}_{L^2(G/H, d\dot{g})} \otimes l^\infty(\hat{H}); \\
\mathfrak{Z}_{\bar{\pi}}(\mathfrak{A}^d) &= L^\infty(H\backslash G; d\dot{g}) \otimes \mathfrak{Z}_{\pi}(\mathfrak{A}^d) = L^\infty(H\backslash G; d\dot{g}) \otimes l^\infty(\hat{H}).
\end{aligned}
$$

(The first line follows from the disjointness $\pi \overset{|}{\circ} (\pi \circ \tau_g)$ (for $\forall g \in G \diagdown H$) and the definition of $\bar{\pi} = \hat{\pi} \circ \hat{\imath}_{H\backslash G}$, and hence, $\mathfrak{Z}_{\bar{\pi}}(\mathfrak{F}) = L^\infty(H\backslash G; d\dot{g}) \subset \bar{\pi}(\mathfrak{F})'' \cap \hat{U}(H)' = \bar{\pi}(\mathfrak{A}^d)''$ from which the third line follows.)

Remark 12 *It is remarkable that only the centre of von Neumann algebra $\bar{\pi}(\mathfrak{A}^d)''$ representing $\mathfrak{A}^d$ in $(\bar{\pi}, \hat{\mathfrak{H}})$ carries full information on* ***both*** *aspects of order parameters of broken $H\backslash G$ and of unbroken $\hat{H}$, whereas centres of any other von Neumann algebras representing $\hat{\mathfrak{F}}$, $\mathfrak{F}$ or $\mathfrak{A}$ in $(\pi, \mathfrak{H})$ or $(\bar{\pi}, \hat{\mathfrak{H}})$ carry only partial information.*

On the basis of these structures of relevant centres of representations, we define a *$c \to q$ channel* Ψ as follows:

$$
\begin{aligned}
&\Psi : \mathfrak{A}^d \ni B \longmapsto \Psi(B) \in C(H\backslash G) \otimes \mathfrak{Z}_{\pi}(\mathfrak{A}^d), \\
&[\Psi(B)](\dot{g}, \eta) := (\omega_0 \circ \rho_\eta \circ m_H)(\tau_{g^{-1}}(B)) \\
&\qquad\qquad \text{for } (\dot{g}, \eta) \in (H\backslash G) \times Spec(\mathfrak{Z}_{\pi}(\mathfrak{A}^d)). \qquad (68)
\end{aligned}
$$

Here, $\rho_\eta \in \mathcal{T}$ is a local endomorphism of $\mathfrak{A}^d$ belonging to the DR-category $\mathcal{T}_{\mathfrak{A}^d}$ on $\mathfrak{A}^d$ and $g \in G$ is an arbitrary representative of $\dot{g} = Hg \in H\backslash G$.

Remark 13 *It is due to the* ***non-invariance*** *of* π *under* G *that the validity of such relations as* $\pi(\mathfrak{A})'' = \pi(\mathfrak{A}^d)''$, $\mathfrak{Z}_\pi(\mathfrak{A}) = \mathfrak{Z}_\pi(\mathfrak{A}^d) = l^\infty(\hat{H})$ *is consistent with the* G*-invariance of* $\mathfrak{A}(\subset \mathfrak{F}^G)$ *and the non-invariance of* $\mathfrak{A}^d = \mathfrak{F}^H$*.*

Before a field algebra $\mathfrak{F}$ is constructed, the Doplicher-Roberts method based on the local endomorphisms and the related Cuntz algebras [14] seems to be the only possible path starting from $\mathfrak{A}$ to the pair of $\mathfrak{F}$ and G *without knowing* either of them, which has necessarily led us to an *unbroken* and *compact* symmetry group. However, in the situation of SSB with possible presence of massless spectrum, there is *no* reason *nor* guarantee for the broken group $G = Gal(\mathfrak{F}/\mathfrak{A})$ to be compact, as shown in [10] through the counter-examples. Fortunately, once $\mathfrak{F}$ is so constructed from the dual net $\mathfrak{A}^d$ and the DR category $\mathcal{T}_{\mathfrak{A}^d}$ as to show certain kinds of stability properties (as will be discussed later), we need not stick any more to the original line of thought inherent to the Doplicher-Roberts theory: having at hand the information on the group $G = Gal(\mathfrak{F}/\mathfrak{A})$, we can control the mutual relations among $\mathfrak{F}$, G, $\mathfrak{A}^d$ and $\mathfrak{A}$ by means of various versions of crossed products applicable to G, *irrespective* of whether it is compact or not [32] (as long as it is assumed to be locally compact).

When the group $G = Gal(\mathfrak{F}/\mathfrak{A})$ of spontaneously broken symmetry is compact, as is common in the physical examples of SSB (such as the case of chiral $SU(2) \times SU(2)$ down to the vectorial $SU(2)$), we can get more information. We see here the important roles played by the *finite-dimensional induction/reduction* between H and G on the algebra $\mathfrak{F}$ according to the following results under the assumption of the compactness of G:

1. *Finite-dimensional induction* for a compact pair $H \hookrightarrow G$:
   Any finite-dimensional unitary representation (η, W) of H can be extended to a representation (γ, V) of G by taking a direct sum $\gamma|_H \cong \eta \oplus \eta'$ with a suitable representation (η', W') of H (for proof, see [46]). At the level of a field algebra, this kind of induction is sufficient, in contrast to the situations of states for which the genuine Mackey induction involving infinite dimensional spaces is indispensable.

2. Stability and consistency of field algebra construction in SSB cases:
   In use of the above result, one can verify the *stability* of the crossed product construction of the field algebra under the change of Cuntz algebras.

   Proposition 14 *[34] If the dual-net algebra* $\mathfrak{A}^d$ *is properly infinite* C^**-algebra, the field algebra* $\mathfrak{F}$ *due to the original DR construction* *from $\mathfrak{A}^d$ and a Cuntz algebra $\mathcal{O}_{d_0}$ is isomorphic to the crossed product of $\mathfrak{A}^d$ with a Cuntz algebra $\mathcal{O}_d$ for any $d > d_0$:*

$$\mathfrak{F} := \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0} \cong \mathfrak{A}^d \underset{\mathcal{O}_d^H}{\otimes} \mathcal{O}_d. \tag{69}$$

With this freedom, we can naturally let $\mathfrak{F}$ be acted upon by a compact Lie group G whose fundamental representation is d-dimensional, bigger than the corresponding dimensionality d_0 of the unbroken H. In this situation, while the relation $g(\mathfrak{A}^d) = \mathfrak{A}^d = \mathfrak{F}^{\tau(H)}$ for $g \in G$ forces g to belong to the normalizer $N_H = \{s \in G; sHs^{-1} \subset H\}$ of unbroken H in G, the equality

$$g(\mathfrak{A}^d) \underset{\mathcal{O}_d^{gHg^{-1}}}{\otimes} g(\mathcal{O}_d) = g(\mathfrak{A}^d \underset{\mathcal{O}_d^H}{\otimes} \mathcal{O}_d) = \mathfrak{F} \tag{70}$$

can be verified [34] even for such $g \in G$ that $g \notin N_H$, which shows the *consistency* of the construction method with the action of G bigger than H.
(In [11] where the relation $Gal(\mathfrak{A}^d/\mathfrak{A}) = N_H/H$ was verified, the analysis of degenerate vacua was restricted to N_H in order to avoid $g(\mathfrak{A}^d) \neq \mathfrak{A}^d$. In the physically interesting situations involving *Lie groups*, however, the reductivity of a compact Lie group H implies that N_H/H is abelian and/or discrete with vanishing Lie brackets, which does not seem to be relevant to the physically meaningful contexts.)

3. *Duality for homogeneous spaces*: corresponding to the relevance of the Tannaka-Krein duality between a compact group and the category of its representations in the DR construction, we encounter here its extended version to a homogeneous space $H\backslash G$.

   For a compact group pair $H \hookrightarrow G$, the definition of $Rep_{H\backslash G}$ and the mutual relations among Rep_G, Rep_H and $Rep_{H\backslash G}$ can be described in terms of a *homotopy-fibre category* Rep_G over Rep_H with $Rep_{H\backslash G}$ as homotopy fibre ([31]): over $\eta \in Rep_H$ a homotopy fibre (h-fibre for short) is given by a category η/Rep_G (which is called a comma category under η [30] whose objects are pairs (γ, T) of $\gamma \in Rep_G$ and $T \in Rep_H(\eta, \gamma|_H)$ and whose morphisms $\phi : (\gamma, T) \to (\gamma', T')$ are given by $\phi \in Rep_G(\gamma, \gamma')$ s.t. $T' = \phi \circ T$:

$$\begin{array}{ccccc} & & \eta & & \\ & \swarrow T & & T' \searrow & \\ \gamma|_H & \to & \underset{\phi|_H}{\to} & \to & \gamma'|_H \\ r_H \uparrow & & & & \uparrow r_H \\ \gamma & \to & \underset{\phi}{\to} & \to & \gamma' \end{array} \tag{71}$$

(To be more precise, the comma category η/Rep_G is to be understood as η/r_H where the functor $r_H : Rep_G \to Rep_H$ is the restriction of G-representations to the subgroup H of G.)

The h-fibre over the trivial representation $\eta = \iota \in Rep_H$ of H is nothing but the category of *linear representations of* $H\backslash G$ due to Iwahori-Sugiura [27], to which any other h-fibres can be shown to be homotopically equivalent [31].

4. Generalizing a theorem in [18], we obtain a result to construct the extended field algebra $\hat{\mathfrak{F}}$ for implementing the broken G also as a crossed product with a Cuntz algebra:

Proposition 15 *[42] If* $\mathfrak{Z}(\mathfrak{A}^d) = \mathbb{C}\mathbf{1}$ *(as a C*-algebra), we have the following relations*

$$\mathfrak{A}^d \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d = \Gamma(G \underset{H}{\times} (\mathfrak{A}^d \underset{\mathcal{O}_d^H}{\otimes} \mathcal{O}_d)) = \hat{\mathfrak{F}}, \tag{72}$$

$$\hat{\mathfrak{F}}^{\hat{\tau}(G)} = \mathfrak{A}^d \underset{\mathcal{O}_d^G}{\otimes} \mathbf{1}, \tag{73}$$

$$Spec(\mathfrak{Z}(\mathfrak{A}^d \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d)) = H\backslash G. \tag{74}$$

The original version [18] of the above relation was formulated for $G = SU(d)$ and was used to detect the unbroken H as the stabilizer of the factorial subrepresentations. As for the physical significance of the extended algebra $\hat{\mathfrak{F}}$ see the next subsection.

4.4 Interpretation of sector structure: degenerate vacua with order parameters, Goldstone modes and condensates

Combining the previous two cases with purely continuous sectors (Sec.2) and with purely discrete sectors (Sec.3), we can now adapt the present scheme for treating generalized sectors to the situation with SSB in order to clarify the sector structure involved there and the physical meaning of each ingredient appearing so far in our fomulation of SSB.

The map

$$\Psi^* : M_1(H\backslash G) \otimes M_1(\hat{H}) \to E_{\mathfrak{A}^d},$$

defined as the dual of $\Psi : \mathfrak{A}^d \ni B \longmapsto \Psi(B)$, $[\Psi(B)](\dot{g}, \eta) = (\omega_0 \circ m_{H\backslash G} \circ \rho_\eta \circ m_H)(\tau_{g^{-1}}(B))$, gives a *c→q channel*, whose inverse $(\Psi^*)^{-1}$ exists on the mixtures $\in \mathfrak{f}(\bar{\pi})$ of states on $\mathfrak{A}^d$ selected by the DHR criterion, as a *q→c channel* to provide the physical interpretations of such states in terms of the order parameters in $\dot{g} \in H\backslash G$, H-charge $\eta \in \hat{H}$. In view of our starting premise of the observable algebra $\mathfrak{A}$, however, it looks natural to take $A \in$

$\mathfrak{A}$ as the argument of Ψ, instead of $B \in \mathfrak{A}^d$. Because of G-invariance of $A \in \mathfrak{A} \subset \mathfrak{F}^G$, however, $\Psi \upharpoonright_{\mathfrak{A}}$ is independent of $\dot{g} \in H\backslash G$, $[\Psi(A)](\dot{g}, \eta) = (\omega_0 \circ m_{H\backslash G} \circ \rho_\eta \circ m_H)(\tau_{g^{-1}}(A)) = (\omega_0 \circ m_{H\backslash G} \circ \rho_\eta)(A)$, failing to pick up the information on $H\backslash G$, so we here take $\mathfrak{A}^d$ as our *extended observables*. This standpoint is justified by the natural physical meaning of $\mathfrak{A}^d$ as the *maximal local net generated by the original net* $\mathfrak{A}$, which is just a version, adapted to the observable net, of the notion of the Borchers classes [4] consisting of all the *relatively local* fields to absorb the *arbitrariness in the "interpolating fields"* [33]. What we see here is that, in spite of their G-noninvariance property, the *Goldstone modes* related to the homogeneous space $H\backslash G$ are allowed to appear here with the qualification as such extended observables belonging to $\mathfrak{A}^d$ and they detect the information concerning the position of a pure vacuum $\dot{g} \in H\backslash G$ among the degenerate vacua, as is exhibited through the $\dot{g}$-dependence of $\Psi(B)$ for $B \in \mathfrak{A}^d$.

To understand the situation, we first consider the physical meaning of the obtained structures related to order parameters:

i) $H\backslash G$ as *order parameters* to parametrize the degenerate vacua: in the decomposition of the representation space $\hat{\mathfrak{H}}$ of $\hat{\mathfrak{F}}$ to pure vacuum representations of $\mathfrak{F}$ in $\mathfrak{H}$, we get the centre $L^\infty(H\backslash G, d\dot{g}) = \mathfrak{Z}_{\bar{\pi}}(\mathfrak{F}) = \mathfrak{Z}_{\hat{\pi}}(\hat{\mathfrak{F}})$ with the spectrum $H\backslash G$ which parametrizes the **degenerate vacua** with minimum energy 0 generated by the SSB of G up to the unbroken remaining H. Mathematically, the Mackey induction from H to G is relevant here. The physical meaning of $H\backslash G$ is seen through such examples as the directions of magnetization in the Heisenberg ferromagnets, or, as the *Josephson effect* where the difference of the phases of Cooper pair condensates between adjacent vacua across a junction exhibits such eminent physical effects as the resistance-free *Josephson current* (see, e.g., [39]). In the context of *static* structures of sectors, the field algebra $\hat{\mathfrak{F}}$ bigger than $\mathfrak{F}$ looks redundant, whereas it becomes relevant in the situations with the *order parameters behaving dynamically* as in the above cases through the couplings with external fields (see, e.g., [23]). Also the non-trivial centre $C(H\backslash G)$ in $\hat{\mathfrak{F}}$ *in the C*-version* with a *continuous* G-action resolves a puzzling conflict between the disjointness $\omega \overset{\shortmid}{\circ} (\omega \circ \tau_g)$ (for *any* $g \in G$ s.t. $g \notin H$) along the G-orbit of any pure vacuum ω of $\mathfrak{F}$ and the continuous behaviours of the order parameter $\dot{g} \in H\backslash G$ under G.

ii) Internal spectrum $\hat{H}$ of *excited states* on a chosen pure vacuum specified by a fixed $\dot{g} = Hg \in H\backslash G$: in the representation space $\mathfrak{H}$ of $\mathfrak{F}$, we see the standard picture of sectors $(\pi_\eta, \mathfrak{H}_\eta)$ with respect to $\mathfrak{A}^d$ parametrized by $\eta \in \hat{H}$, which describes the internal symmetry aspects of excited states in terms of the unbroken H (to be precise, $g^{-1}Hg \simeq H$ at $\dot{g}$) just in the same way as the situations discussed in

Sec. 3. For the description of this aspect, we find no essential difference between $\mathfrak{A}$ and $\mathfrak{A}^d$, because of the relations $\pi(\mathfrak{A})'' = \pi(\mathfrak{A}^d)''$, $\mathfrak{Z}_\pi(\mathfrak{A}) = \mathfrak{Z}_\pi(\mathfrak{A}^d) = l^\infty(\hat{H})$, valid in the factor representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ carrying no explicit information on the order parameters i) of SSB.

iii) *Goldstone modes* responsible for the gap $\mathfrak{A}(\mathcal{O}) \subsetneqq \mathfrak{A}^d(\mathcal{O})$ at the *local* level, whose *global* manifestation is found in $\bar{\pi}(\mathfrak{A})'' \subsetneqq \bar{\pi}(\mathfrak{A}^d)''$ (both involving $H\backslash G$): in sharp contrast with the above ii), we are concerned here with the algebraic dual objects of the physically relevant order parameters of SSB appearing in i). The origin of these gaps can be understood naturally in the following context: for a co-action δ of $\hat{G}$ on $\mathfrak{F}^G$ the relations

$$\mathfrak{F} = \mathfrak{F}^G \rtimes_\delta \hat{G},$$
$$\mathfrak{A}^d = \mathfrak{F}^H = \mathfrak{F}^G \rtimes_\delta \widehat{(H\backslash G)}, \tag{75}$$

are known to hold in the W*-version of crossed products [32]. If its C*-version is verified, the latter relation (75) shows that the gap between $\mathfrak{A}(\subset \mathfrak{F}^G)$ and $\mathfrak{A}^d$ comes from the G-non-invariant elements in $\mathfrak{A}^d$ related to $H\backslash G$, which can be interpreted properly as an abstract algebraic form of the (would-be) *Goldstone modes* (whose full-fledged form as the massless Goldstone spectrum can be absent in the representation Hilbert space $\mathfrak{H}$ depending upon the decay rates of long-range correlations, as shown in [10]). To understand this, recall that the standard picture of Goldstone modes is given by the physical degrees of freedom φ responsible for the non-invariance of a chosen pure vacuum ω_0 under the action of broken G, $\omega_0(\tau_g(\varphi)) \neq \omega_0(\varphi)$ $(g \in G\diagdown H)$, which yields, as in i), the orbit $\{\omega_0 \circ \tau_g \ ; \ g \in G\}$ constituting of the "degenerate vacua" parametrized by G/H as the spectrum of the centre of $\mathfrak{A}^d$: $\mathfrak{Z}_{\bar{\pi}}(\mathfrak{A}^d) = L^\infty(G/H) \otimes \mathfrak{Z}_\pi(\mathfrak{A}) = L^\infty(G/H) \vee \mathfrak{Z}_{\bar{\pi}}(\mathfrak{A})$. While neither $\mathfrak{A}^d$ nor its local subalgebras contain non-trivial central elements, there should exist some sequences of local elements (central sequences or order fields) in $\mathfrak{A}^d$ tending to global central elements belonging to $L^\infty(G/H) \subset \mathfrak{Z}_{\bar{\pi}}(\mathfrak{A}^d)$, which is to be identified with the Goldstone modes describing virtual transitions from one specific vacuum to another among degenerate vacua. In view of the transformation property under G, it is clear that this kind of sequences cannot be supplied by $\mathfrak{F}^G(\supset \mathfrak{A})$, but should be found in the second component of $\mathfrak{A}^d = \mathfrak{F}^G \rtimes \widehat{(H\backslash G)}$. Thus, the main cause of the gap between $\mathfrak{A}^d$ and $\mathfrak{A}$ can be found in the presence of the above sequences identified with Goldstone modes. Then the relation (75) can be interpreted as an algebraic version of the Goldstone and/or low-energy theorems in the sense that they give a *dual* description of the SSB-sector structure with degenerate vacua in i) in a local and/or algebraic virtual form (appearing already in a pure vacuum); this will fully justify such a heuristic and physical expression that "Goldstone degrees of freedom related to $H\backslash G$ search the degenerate vacua in a virtual way".

In the standard approach focusing on discrete sectors in ii), the continuous sectors appearing in i) fail to be recognized as genuine sectors, as a consequence of which the situations with continuous sectors only are regarded as the *absence of sectors* (cf. [10]). From the above discussion, however, we find both physical and mathematical reasons for treating them as sectors, in view of the physically important roles played by the associated order parameters as seen in i) and also of the mathematically interesting interrelationship between i) and iii). It may be also instructive to compare the above i) and ii) with the results in [11]; analysis was restricted there to the factor representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ without touching on the larger one $(\bar{\pi}, \hat{\mathfrak{H}})$ implementing the broken G, and hence, what is found as the degenerate vacua is only along N_H/H (with vanishing Lie algebra), failing to find the whole $H\backslash G$-orbit. To recover the full information on the degenerate vacua, one should not avoid the complications due to the instability, $g(\mathfrak{A}^d) \neq \mathfrak{A}^d$, of the dual-net algebra $\mathfrak{A}^d$ which moves around inside $\mathfrak{F}$ under the action of $G(\supsetneqq N_H)$.

Remark 16 *Since our focus in the above iii) is just as to how Goldstone modes appear in $\mathfrak{A}^d$ as extended observables in spite of their G-non-invariance, the question whether $\mathfrak{A}$ is Galois-closed or not, $\mathfrak{A} \overset{?}{=} \mathfrak{F}^G$, is irrelevant, with the relation $\mathfrak{A} \subset \mathfrak{F}^G$ following from $G := Gal(\mathfrak{F}/\mathfrak{A})$ being sufficient. In the problem of the intrinsic characterization of the observable net $\mathfrak{A}$ itself, however, the problems as to whether this property holds or not, and, as to how it is ensured, are interesting questions to be examined.*

To avoid possible confusions on the various notions appearing in SSB, one needs to be careful about the distinctions and mutual relations among the following four levels involving Goldstone modes and order parameters:

1) degenerate vacua as continuous sectors parametrized by the *order parameters* $\dot{g} \in H\backslash G$ which is a global notion,

2) *Goldstone modes* belonging to $\mathfrak{A}^d$, whose massless spectrum (if any) is responsible for the validity of Goldstone theorem,

3) *Goldstone multiplet* belonging to $\mathfrak{F}$ and consisting of Goldstone modes together with *condensates* responsible for the above 1); this field multiplet transforms under G according to a *linear* representation, which is nothing but a "*linear representation of a homogeneous space*" according to the definition of [27]. What is most confusing is the mutual relation between the Goldstone modes and the condensates; in

the simplest example of SSB from $G = SO(3)$ to $H = SO(2)$ with $H\backslash G = S^2$ (e.g., Heisenberg ferromagnet), a pure vacuum among degenerate vacua is parametrized and geometrically depicted by a point $p \in S^2$, a condensate by a radius from the centre of the unit ball to p, and the Goldstone modes geometrically expressed by *tangent vectors* at p *tangential to* S^2 and *orthogonal to the condensate.* The Goldstone multiplet is an entity in $\mathfrak{F}$ which is behaving as a three-dimensional covariant vector under $SO(3)$.

4) There is a useful physical notion called "nonlinear realization" of Goldstone bosons [47], expressing the above situation in a geometric way and serving as very effective tools in the derivation of the so-called low energy theorems, such as the soft-pion theorem, to describe the low energy scattering processes involving Goldstone bosons associated with SSB. While its functional role is very akin to our Goldstone modes in 2), it may not be so straightforward to accommodate it literally into the present context, because of the nonlinear transformation law exhibited in its transformation property under G. In the attempt to incorporate the general essence of low energy theorems into the present context, however, this notion is expected to play some useful roles.

5 Selection criteria as categorical adjunctions and their operational meanings

Here we emphasize the important roles played by the categorical adjunctions underlying our discussions so far, in achieving the systematic organizations of various domains in physics: the essence of the three formulae (17), (22) and (46) encountered in Sec.2 and Sec.3 can be summarized as follows:

$X\ (=q)$: to be classified	$q \leftarrow c$	$A\ (=c)$: to classify
$x \underset{X}{\equiv} G(a)$	$\underset{F}{\overset{G}{\leftrightarrows}}$	$F(x) \underset{A}{\equiv} a$
selection criterion	$q \rightarrow c$	interpretation

with X a quantum domain of generic states to be characterized and classified, A a classical classifying space identified with the spectrum of centre, G the $c{\rightarrow}q$ *channel* and with F the $q{\rightarrow}c$ *channel* to provide the interpretation of X in terms of the vocabulary in A. This scheme exhibits the essential meaning and the pertinence of this notion to our discussion of selecting, classifying and interpreting physically interesting classes of states.

These cases, however, share such special features that the relevant categories are groupoids of equivalence relations with all arrows invertible and that the mapping between quantum and classical domains are groupoid isomorphisms, and hence, the essence of adjunctions in our context is found in such quantitative form as above. Perhaps this is because the category consisting of states of C*-algebras is a rather rigid one, allowing only few meaningful morphisms among different objects, requiring strict equalities or equivalence relations. As we have seen above, however, once the contents of imposed selection criteria are paraphrased into different languages, such as thermal functions in Sec.2, the category of DHR-selected representations π_ω, the DR-category of local endomorphisms ρ and that of group representations γ_ρ in Sec.3, then the machinery stored in the category theory starts to work. In such contexts, objects are not always states, and arrows between objects (taking such forms as intertwiners among local endomorphisms or among group representations) or functors between different categories need not necessarily be invertible. In the next subsections we also find that what to be selected need not always be states but can be channels as well.

So we should not to mistake these special features of our examples as the universal essence of the adjunction, especially because what is important about categorical notions is their flexibility allowing to look at the same object in many different ways and to unify objects with different appearances in one and the same notion. Although we do not use it systematically here, we give, for convenience, the general definition of adjunction [30],

$$X(x, G(a)) \simeq A(F(x), a), \tag{76}$$

which involves four levels of notions, objects, arrows, a pair of functors $G : A \to X$, $F : X \to A$ and a pair of natural transformations (= arrows between functors) η and ε between two functors, $\eta : 1_X \dot{\to} GF$, $\varepsilon : FG \dot{\to} 1_A$, in such a way that the relation $\simeq$ is specified by $\varepsilon_{F(x)} \circ F(\eta_x) = 1_{F(x)}$, $G(\varepsilon_a) \circ \eta_{G(a)} = 1_{G(a)}$. This is equivalent to a natural family of bijections $\nu_{x,a} : X(x, G(a)) \simeq A(F(x), a)$ where "naturality" is characterized by the relations $\nu_{y,b}(fgF(\psi)) = G(f)\nu_{x,a}(g)\psi$ for $\forall x, y$: objects in X, $\forall a, b$: objects in A and $\forall \psi \in X(y, x)$, $\forall f \in A(a, b)$. Their mutual relations are given by $\eta_x = \nu_{x,F(x)}(1_{F(x)})$, $\varepsilon_a = \nu^{-1}_{G(a),a}(1_{G(a)}) \iff \nu(\psi) = \varepsilon F(\psi)$, $\nu^{-1}(f) = G(f)\eta$. (When η is invertible, the adjunction is called an isomorphism and the obstruction for η to be isomorphism yields a cohomology theory.) Identifying $A = Th/\mathcal{C}(\mathcal{S}_x)$, $X = E_x/\mathcal{S}_x$, $x = \omega \in E_x$, $a = \rho_x \in Th$, $G = \mathcal{C}^*$, we apply this to the case (22) of non-equilibrium local states. Then $F(\omega)$ can be understood as (the restriction to $\mathcal{S}_x$ of) the Hahn-Banach extension $\nu = \nu_+ - \nu_- \in C(B_K)^*$ of $\mathcal{C}(\mathcal{S}_x) \ni \mathcal{C}(\hat{A}) \longmapsto \omega(\hat{A})$ to $\mathcal{T}_x$ and $FG = F\mathcal{C}^* = 1_{Th}$. Then $\nu_- \neq 0$ for $\omega \notin K$ signals the deviation of $GF = \mathcal{C}^*F$ from 1_{E_x}. Therefore, we encounter the *hierarchical family of adjunctions* according to the choice of $\mathcal{S}_x(\subset \mathcal{T}_x)$, in which not only the validity of adjunctions with a suitable $\mathcal{S}_x$ but also its breakdown for a bigger $\mathcal{S}'_x(\supset \mathcal{S}_x)$ are physically meaningful.

Next, we recall another important aspect of the adjunction. In decoding the deep messages encoded in a selection criterion, what plays the decisive roles at the first stage is the identification of the *centre* of a representation containing universally all the selected quantum states; its spectrum provides us with the information on the associated sector structure, which serves as the vocabulary to be used when the interpretations of a given quantum state are presented. The necessary bridge between the selected generic quantum states and the classical familar objects living on the above centre is provided, in one direction, by the *c→q channel* which embeds all the known classical states (=probability measures) into the form of quantum states constituting the totality of the selected states by the starting selection criterion. The achieved identification between what is selected and what is embedded from the known world is nothing but the most important consequence of the categorical adjunction formulated in the form of selection criterion. This automatically enables us to take the inverse of the *c→q channel* which brings in another most important ingredient, the *q→c channel* to decode the physical contents of selected states from the viewpoint of those aspects selected out by the starting criterion. Mathematically speaking, the spectrum of the above centre is nothing but the *classifying space* universally appearing in the geometrical contexts; for instance, in Sec.3 of DR sector theory of unbroken symmetry described by a compact Lie group G, its dual $\hat{G}$ (of all the equivalence classes of irreducible unitary representations) is such a case, $\hat{G} = B_{\mathcal{T}}$ for $\mathcal{T}$ the DR category of local endomorphisms of the observable net, where our *q→c channel* $(\Lambda_\mu^*)^{-1}$ plays the role of the *classifying map* by embedding the G-representation contents of a given quantum state into the subset of $\hat{G}$ consisting of its irreducible components. For an arbitrary $(\mathfrak{A},G)$-module $E = \bigoplus_{\gamma \in M} \mathfrak{H}_\gamma$ (corresponding to a choice of state of $\mathfrak{A}$ as in Sec.3) whose G-representation structure is specified by a subset M of $\hat{G}$, we obtain the following relation in parallel with the definition of classifying maps of G-bundles:

$$\begin{array}{ccccl} \bigoplus_{\gamma \in M} \mathfrak{H}_\gamma = E & & \bigoplus_{\gamma \in \hat{G}} \mathfrak{H}_\gamma : & \text{universal bundle of all the sectors} \\ \downarrow Rep_G & & \downarrow Rep_G & \\ (\hat{G} \supset) M & \overset{\mathrm{supp} \circ (\Lambda_\mu^*)^{-1}}{\hookrightarrow} & \hat{G} = B_{\mathcal{T}} & : \text{classifying space} \end{array} .$$

Corresponding to the relevance of *homotopy* to the situations where classifying maps appear to reproduce the bundle structure up to homotopy, everything here is up to multiplicities, since the *G-charge contents* of a selected generic state ω are examined on the basis of the data coming from the centre which neglects all the information concerning the multiplicities. In this way, the present scheme can easily be related with many current topics concerning the geometric and classification aspects of commutative as well as non-commutative geometry based upon the (homotopical) notions of classifying spaces, K-theory and so on.

5.1 Spectral decomposition and probabilistic interpretation in quantum measurements

In view of the importance of the interpretations above, we pick up some relevant points here from the quantum measurement processes, in regard to the following basic points:

i) The operator-theoretical notion of spectral decomposition of a self-adjoint observable A to be measured is equivalent to the algebraic homomorphism (so-called the map of "functional calculus"):

$$\hat{A} : L^{\infty}(Spec(A)) \ni f \mapsto \hat{A}(f) = f(A) := \int_{a \in Spec(A)} f(a)\ E_A(da) \in \mathfrak{A}'' \subset B(\mathfrak{H}), \tag{77}$$

where $\mathfrak{H}$ is the Hilbert space of the defining representation of the observable algebra $\mathfrak{A}$ to which our observable A belongs. Here we omit the symbol for discriminating the original C*-algebra $\mathfrak{A}$ and its representation in $\mathfrak{H}$, and hence, we will freely move between C*- and W*-versions without explicit mention. This fits quite well to the common situations of discussing measurements owing to the absence of disjoint representations in the purely quantum side $\mathfrak{A}$ with *finite* degrees of freedom (due to Stone-von Neumann theorem). In such cases, the non-trivial existence of a centre comes only from the classical system coupled to quantum one (, the former of which need to be derived from the quantum system with infinite degrees of freedom at the "ultimate" levels, though).

ii) To give this homomorphism $\hat{A}$ is (almost) equivalent to giving a spectral measure E_A by

$$E_A : \mathcal{B}(Spec(A)) \ni \Delta \mapsto E_A(\Delta) := \hat{A}(\chi_\Delta) = \chi_\Delta(A) \in \mathrm{Proj}(\mathfrak{H}), \tag{78}$$

on the σ-algebra $\mathcal{B}(Spec(A))$ on $Spec(A)$ of Borel sets Δ, identified with the indicator function χ_Δ, taking values in the set $\mathrm{Proj}(\mathfrak{H})$ of orthogonal projections in $\mathfrak{H}$. Then the dual map $\hat{A}^*$ defines a mapping from a quantum state ω to a *probability distribution*, $p^A(\cdot|\ \omega) : \mathcal{B}(Spec(A)) \ni \Delta \mapsto p^A(\Delta|\omega) = \mathrm{Prob}(A \in \Delta \mid \omega) := \omega(E_A(\Delta))$, of measured values in the measurements of A performed in the state ω. The above reservation "(almost) equivalent" is due to the fact that the reverse direction from a probability distribution to a spectral decomposition admits a sligthly more general notion, positive-operator valued measure (POM), which corresponds to a unital completely positive map instead of a homomorphism and which becomes relevant for treating the set of mutually non-commutative observables. In any case, the operational meaning of the mathematical notion of spectral decomposition is exhibited by this $\hat{A}^*$ (or, the dual of POM) as a simplest sort of $q \to c$ *channel* providing the familiar probabilistic interpretation.

iii) To implement physically the spectral decomposition, however, we need some *physical interaction processes* between the system and the apparatus through the coupling term of the observable $A \in \mathfrak{A}$ to be measured and an external field J belonging to the apparatus. While one of the most polemic issues in the measurement theory is as to how this "contraction of wave packets" is realized consistently with the "standard" formulaion of quantum theory, we here avoid this issue, simply taking such a "phenomenological" standpoint that our purpose will be attained if the composite system consisting of the object system and the classical system involving J is effectively (Fourier- or Legendre-) transformed through this coupled dynamical process into $\mathfrak{A} \otimes C^*\{A\} =: \mathfrak{A}_A = C(Spec(A), \mathfrak{A})$, the centre of which is just the commutative C*-algebra $C^*\{A\} \simeq C(Spec(A))$ generated by a self-adjoint operator A: $C^*\{A\} \overset{\iota}{\hookrightarrow} \mathfrak{Z}(\mathfrak{A}_A) \hookrightarrow \mathfrak{A}_A$. So the *sector structure* comes in here with sectors parametrized by the spectrum of the observable A to be measured. (It was the important contribution of Machida and Namiki [29] that shed a new light on the notion of continuous superselection rules, where the focus was, unfortunately, upon sectors related to *irrelevant unobservable* variables, in sharp contrast to those discussed here.)

5.2 Measurement scheme and its realizability

Then the basic measurement scheme [43] reduces to the requirement that all the information on the probability distribution in ii) should be recorded in and can be read out from this classical part $\{A\}'' = L^\infty(Spec(A))$ as a mathematical representative of the measuring apparatus:

$$\omega(E_A(\Delta)) = p^A(\Delta|\omega) = (\omega \otimes \mu_0)[\hat{\tau}(\mathbf{1} \otimes \chi_\Delta)], \tag{79}$$

where μ_0 is some initial state of $\{A\}''$ and $\hat{\tau} \in Aut(\mathfrak{A}_A)$ describes the effects of dynamics of the composite system of $\mathfrak{A}$ and $C^*\{A\}$ (or, more generally, a dissipative dynamics of a completely positive map also to be allowed).

We are interested here in examining how the problem of a selection criterion according to our general formulation becomes relevant to the present context. Applying to any state $\hat{\omega} \in E_{\mathfrak{A}_A}$ the uniquely determined *central decomposition*, we have

$$\hat{\omega} = \int_{Spec(A)} d\mu(a)(\omega_a \otimes \delta_a), \tag{80}$$

with some family of states $\{\omega_a\} \subset E_{\mathfrak{A}}$ (which can be universally chosen by $\omega_a(B) := \langle \psi_a \mid B\psi_a \rangle$ with $A\psi_a = a\psi_a$ *if* A has only discrete spectrum without multiplicity). What plays important roles here is the instrument $\mathcal{J}_{A,\tau}$ [15] depending on $A \in \mathfrak{A}$ and on the composite-system dynamics $\hat{\tau}$

defined by

$$\mathcal{I}_{A,\hat{\tau}}:\mathfrak{A}_A\ni\hat{B}\longmapsto\mathcal{I}_{A,\hat{\tau}}(\hat{B}):=\int d\mu_0(a)(\hat{\tau}(\hat{B}))(a)$$
$$=\int d\mu_0(a)\delta_a(\hat{\tau}(\hat{B}))\in\mathfrak{A}, \tag{81}$$
$$\mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(B):=[\mathcal{I}^*_{A,\hat{\tau}}(\omega)](B\otimes\chi_\Delta))=\omega(\mathcal{I}_{A,\hat{\tau}}(B\otimes\chi_\Delta))$$
$$=(\omega\otimes\mu_0)[\hat{\tau}(B\otimes\chi_\Delta)]. \tag{82}$$

In terms of these notions, Eq.(79) can be rewritten as

$$\hat{A}^*(\omega)=(\mathcal{I}_{A,\hat{\tau}}\circ\iota')^*(\omega)$$
$$\Longrightarrow\hat{A}^*=\iota'^*\circ\mathcal{I}^*_{A,\hat{\tau}}, \tag{83}$$

where $\iota'^*:E_{\mathfrak{A}_A}\to M_1(Spec(A))$ defined by the dual of

$$\iota':\{A\}''\ni f\longmapsto\mathbf{1}\otimes f\in\mathfrak{A}''_A \tag{84}$$

is the standard (tautological) $q{\to}c$ *channel* to allow the data read-out from the system-apparatus composite system. Eq.(83) selects out an observable A(, or its corresponding $q{\to}c$ channel $\hat{A}^*$ describing the probabilistic interpretation of A) according to a criterion as to whether it can be factorized into the standard tautological $q{\to}c$ *channel* ι'^* and some instrument $\mathcal{I}_{A,\hat{\tau}}$. In view of the formal similarity between the relation $\pi_\omega=\pi_0\circ\rho$ coming from the DHR criterion and Eq.(83), it is interesting to note that what are examined here is $q{\to}c$ channels, $\hat{A}^*$ and ι'^*, the latter of which is a fixed standard one. This criterion is just for examining whether the measurement of A can actually be materialized by means of the coupling $\hat{\tau}$ between the system containing A and some measuring apparatus constituting the composite system $\mathfrak{A}_A=\mathfrak{A}\otimes\{A\}''$. In this sense, the criterion examines the *realization problem* in the context of control theory [3], asking whether a suitable choice of an apparatus and a choice of dynamical coupling can correctly describe the input-output behaviour of the system. Once this criterion is valid, its experimental observation is most conveniently described by the instrument $\mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(B)$ whose interpretation is given [43] by

1) the probability distribution of the measured value of A in a state ω is given by $\mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(\mathbf{1})=p_A(\Delta|\omega)$,

2) the final state realized (in the repeatable measurement) after the read-out $a\in\Delta$ is given by the Radon-Nikodym derivative $\mathcal{J}_{A,\hat{\tau}}(da|\omega)/p_A(da|\omega)$,

3) in combination of 1) and 2), the quantity $\mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(B)$ itself can be regarded as the expectation value of another observable $B\in\mathfrak{A}$ when the initial state ω goes into some final state whose A-values belong to $\Delta(\subset Spec(A))$.

5.3 Problem of state preparation as reachability problem

In the related context, we need to examine the problem of *reachability* to ask whether there is a controlled way to drive the (composite) system to any desired state starting from some initial state; this is nothing but the problem of **state preparation**, which has not been seriously discussed, in spite of its vital importance in the physical interpretation of quantum theory.

For this purpose, we need to define the *c→q channel* relevant to it. Fixing a family $(\omega_a)_{a\in Spec(A)} =: \phi$ of states on $\mathfrak{A}$ appearing in the central decomposition (80), we can define a *c→q channel* by

$$C_{A,\phi} : \mathfrak{A}_A \ni \hat{B} \longmapsto (Spec(A) \ni a \longmapsto \omega_a(\hat{B}(a))) \in C(Spec(A)), \tag{85}$$

and hence, $C^*_{A,\phi} : M_1(Spec(A)) \ni \rho \longmapsto C^*_{A,\phi}(\rho) \in E_{\mathfrak{A}_A}$, where

$$C^*_{A,\phi}(\rho)(\hat{B}) = \rho(C_{A,\phi}(\hat{B})) = \int d\rho(a)\omega_a(\hat{B}(a)) = \int d\rho(a)(\omega_a \otimes \delta_a)(\hat{B}),$$
$$\text{or, } C^*_{A,\phi}(\rho) = \int d\rho(a)(\omega_a \otimes \delta_a). \tag{86}$$

In terms of these, the reachability (or, preparability) criterion can be formulated as the problem to examine the validity of

$$\omega = \lim_{t\to\infty} (\iota^* \circ C^*_{A,\phi})(\mu_{\hat{\tau}_t}), \tag{87}$$

where $\iota^* : E_{\mathfrak{A}_A} \to E_{\mathfrak{A}}$ is the dual of $\iota : \mathfrak{A} \ni B \longmapsto B \otimes \mathbf{1} \in \mathfrak{A}_A$, and the measure $\mu^{\omega}_{\hat{\tau}_t} \in M_1(Spec(A))$ is defined through the central decomposition of $(\omega \otimes \mu_0) \circ \hat{\tau}_t = \int d\mu^{\omega}_{\hat{\tau}_t}(a)\omega_a \otimes \delta_a$ valid for such an observable A as with discrete spectrum. If we can find such a suitable coupled dynamics $\hat{\tau}_t$ and an initial and final probability measures $\mu_0, \mu_1 \in M_1(Spec(A))$ that $\lim_{t\to\infty}(\omega \otimes \mu_0) \circ \hat{\tau}_t(B \otimes \mathbf{1}) = (\omega \otimes \mu_1)(B \otimes \mathbf{1})$ for each $B \in \mathfrak{A}$, then a state ω can actually be prepared:

$$(\iota^* \circ C^*_{A,\phi})(\mu_{\hat{\tau}})(B) = \mu_{\hat{\tau}_t}(C_{A,\phi}(B \otimes \mathbf{1})) = \int d\mu_{\hat{\tau}_t}(a)\omega_a(B \otimes \mathbf{1})$$
$$= (\omega \otimes \mu_0) \circ \hat{\tau}_t(B \otimes \mathbf{1}) \underset{t\to\infty}{\longrightarrow} (\omega \otimes \mu_1)(B \otimes \mathbf{1}) = \omega(B), \tag{88}$$

in the sense that there is some operational means specified in terms of $A \in \mathfrak{A}$, a coupled dynamics $\hat{\tau}_t$ and an initial and final probability measures $\mu_0, \mu_1 \in M_1(Spec(A))$.

Here, the assumption of discreteness of the spectrum of A is no problem, since A plays here only a subsidiary role. However, this problem becomes crucial when we start to examine the *repeatability* of the measurement of the observable A itself. We compare the above *q→c channel* $(C^*_{A,\phi})^{-1}$ with another natural *q→c channel* $(\iota \circ \hat{A})^*$, which can be defined on all the states $\in E_{\mathfrak{A}_A}$, independently of a specific choice of a family $\phi = (\omega_a)_{a\in Spec(A)}$ of states on $\mathfrak{A}$, simply as the dual of the composed embedding maps, $C(Spec(A)) \overset{\hat{A}}{\hookrightarrow} \mathfrak{A} \overset{\iota}{\hookrightarrow} \mathfrak{A}_A$. As is seen from the relation,

$$
\begin{aligned}
&(\iota \circ \hat{A})^*(\int d\mu(a)(\omega_a \ \otimes \delta_a))(f) \\
&= \int d\mu(a)(\omega_a \ \otimes \delta_a))((\iota \circ \hat{A})(f)) = \int d\mu(a)(\omega_a \ \otimes \delta_a))(f(A) \otimes \mathbf{1}) \\
&= \int d\mu(a)\omega_a(f(A)) = \int d\mu(a) \int \omega_a(dE_A(b))f(b), \qquad (89)
\end{aligned}
$$

$(\iota \circ \hat{A})^*$ is, in general, not equal to $(C^*_{A,\phi})^{-1}$, nor has a simple interpretation. If we can choose such a family $(\omega_a)_{a\in Spec(A)}$ that $\int f(b)\omega_a(dE_A(b)) = f(a)$ for $\forall f \in C(Spec(A))$, or equivalently, $\omega_a(E_A(\Delta)) = \chi_\Delta(a)$ for $\forall\Delta$: measurable subset of $Spec(A)$, we can attain the equality between $(C^*_{A,\phi})^{-1}$ and $(\iota\circ\hat{A})^*$ on the image of $C^*_{A,\phi}$ in $E_{\mathfrak{A}_A}$, which can be extended to the whole $E_{\mathfrak{A}_A}$ by the use of the Hahn-Banach extension. As a result, we can attain universally the state preparations and physical interpretations (in relation to A), independently of a specific choice of the above family $(\omega_a)_{a\in Spec(A)}$. While such a choice is always possible for observables A with discrete spectrum, its impossibility for those A with *continuous spectra* forces us to consider the *approximate measurement scheme* (see [43]), which involves the essential dependence on the choice of the family $(\omega_a)_{a\in Spec(A)}$ and the selection of and restriction to preparable and interpretable states.

In this way, we have seen that this approach provides a simple unified scheme based upon instruments and channels for discussing various aspects in the measurement processes without being trapped in the depth of philosophical issues. So, it will be worthwhile to attempt the possible extension of the measurement scheme to more general situations involving QFT. It will be also interesting to examine the problems of state correlations in entanglements, of state estimation, and so on, in use of the notions of mutual entropy, channel capacities [35, 36], Cramér-Rao bounds, etc.

Through the above relation with the spectral decomposition of an observable A and the superselection sectors parametrized by $a \in Spec(A)$, we can reconfirm the naturalility of our extending the meaning of sectors from their traditional version of discrete one, to the present version including both: in SSB, order parameters of continuous family of disjoint states (of $\mathfrak{A}$) parametrized by $H\backslash G$ and in thermal situations, (inverse) temperatures $\beta[=(\beta^\mu)]$ discriminating pure thermodynamic phases corresponding also to disjoint KMS states (of $\mathfrak{A}$), and variety of non-equilibrium local states ([13]). Our way of unifying these various cases is seen to be quite similar to the unified treatment of discrete and continuous spectra of self-adjoint operators in the general theory of spectral decompositions.

We conclude this paper by mentioning some problems under investigation, which will be reported somewhere.

1. Treatment of a non-compact group of broken internal symmetry as remarked in Sec.4.3 and 4.4.

2. Reformulation of characterization of KMS states: in Sec.2, we have just relied on the known simplicial structure of the set of all KMS states. To be consistent with the spirit of the present scheme, we need also to find a version of selection criterion to characterize these KMS states, whose essence should be found in the **zeroth law of thermodynamics** from which the familiar parameter of temperature arises (in combination with the first and second laws in such a form as the passivity [44] or the Gibbs variational principle [2]). In any case, such a physically interesting problem as drawing a phase diagram to accommodate phase transitions just belongs to the analysis of sector structure in the present context.

3. To substantiate the above consideration, it is necessary to develop a systematic way of treating a chemical potential [1] as one of the order parameters to be added to temperature. This requires the local and systematic treatment of conserved currents such as $T_{\mu\nu}$(: energy-momentum tensor) and j_μ(: current density), extended to thermal situations just in a parallel way to the local thermal observables in [13]. To understand sectors in relation with spacetime structure, the notion of soliton sectors [22] seems also quite interesting.

4. It would be worthwhile to examine whether the notion of a field algebra $\mathfrak{F}$ is a simple mathematical device, convenient for making the interpretation easier from the viewpoint laid out by Klein's Erlangen programme and no more than that.

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References

[1] Araki, H., Haag, R., Kastler, D. and Takesaki, M., Extension of KMS states and chemical potential, Comm. Math. Phys. **53**, 97-134 (1977).

[2] Araki, H. and Ion, P.D.F., On the equivalence of KMS and Gibbs conditions for states of quantum lattice systems, Comm. Math. Phys. **35**, 1-12 (1974); Araki, H., On the equivalence of the KMS condition and the variational principle for quantum lattice systems, Comm. Math. Phys. **38**, 1-10 (1974); Araki, H. and Sewell, G.L., KMS conditions and local thermodynamical stability of quantum spin lattice systems, Comm. Math. Phys. **52**, 103-109 (1977); Araki, H. and Moriya, H., Equilibrium statistical mechanics of fermion lattice systems, to appear in Rev. Math. Phys.

[3] Arbib, M.A. and Manes, E.G., *Arrows, Structures, and Functors*, Academic Press (1975).

[4] Borchers, H.-J., Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix, Nuovo Cim. **15**, 784-794 (1960).

[5] Borchers, H.-J., Local rings and the connection of spin with statistics, Comm. Math. Phys. **1**, 281-307 (1965).

[6] Bostelmann, H., Lokale Algebren und Operatorprodukte am Punkt, PhD Thesis, Universität Göttingen, 2000.

[7] Bratteli, O. and Robinson, D.W., *Operator Algebras and Statistical Mechanics*, vol. 2, Springer-Verlag (1981).

[8] Bros, J. and Buchholz, D., Towards a relativistic KMS condition, Nucl. Phys. **B429**, 291-318 (1994).

[9] Buchholz, D., Doplicher, S. and Longo, R., On Noether's theorem in quantum field theory, Ann. Phys. (N.Y.), **170**, 1-17 (1986).

[10] Buchholz, D., Doplicher, S., Longo, R., and Roberts, J.E., A new look at Goldstone's theorem, Reviews in Math. Phys. Special Issue, 49-83 (1992).

[11] Buchholz, D., Doplicher, S., Longo, R., and Roberts, J.E., Extension of automorphisms and gauge symmetries, Comm. Math. Phys. **155**, 123-134 (1993)

[12] Buchholz, D. and Fredenhagen, K., Locality and the structure of particle states, Comm. Math. Phys. **84**, 1-54 (1982).

[13] Buchholz, D., Ojima, I. and Roos, H., Thermodynamic properties of non-equilibrium states in quantum field theory, Ann. Phys. (N.Y.) **297**, 219 - 242 (2002).

[14] Cuntz, J., Simple C*-algebras generated by isometries, Comm. Math. Phys. **57**, 173-185(1977).

[15] Davies, E.B., and Lewis, J., An operational approach to quantum probability, Comm. Math. Phys. **17**, 239-260 (1970); Davies, E.B., *Quantum Theory of Open Systems*, Academic Press (1976).

[16] Doplicher, S., Haag, R. and Roberts, J.E., Fields, observables and gauge transformations I & II, Comm. Math. Phys. **13,** 1-23 (1969); **15,** 173-200 (1969); Local observables and particle statistics I & II, **23,** 199-230 (1971); **35,** 49-85 (1974).

[17] Doplicher, S. and Roberts, J.E., Compact group actions on C*-algebras, J. Operator Theory **19**, 283-305 (1988).

[18] Doplicher, S. and Roberts, J.E., Endomorphism of C*-algebras, cross products and duality for compact groups, Ann. Math. **130,** 75-119 (1989); A new duality theory for compact groups, Inventiones Math. **98,** 157-218 (1989).

[19] Doplicher, S. and Roberts, J.E., Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics, Comm. Math. Phys. **131**, 51-107 (1990).

[20] Flato, M., private communication (1996).

[21] Fredenhagen, K. and Hertel, J., Comm. Math. Phys. **80**, 555 (1981); Driessler, W. and Fröhlich, J., Ann. Inst. H. Poincaré **27**, 221 (1997).

[22] Fröhlich, J., New superselection sectors (soliton-states) in two dimensional Bose quantum field models, Comm. Math. Phys. **47**, 269-310 (1976).

[23] Fröhlich, J., *Non-perturbative Quantum Field Theory.* (Advanced Math. Vol.15), World Sci. (1992).

[24] Fröhlich, J. and Kerler, T., *Quantum Groups, Quantum Categories and Quantum Field Theory.* Lecture Notes in Mathematics No.1542, Springer-Verlag (1993).

[25] Haag, R., *Local Quantum Physics* (2nd. ed.), Springer-Verlag (1996).

[26] Haag, R. and Ojima, I., On the problem of defining a specific theory within the frame of local quantum physics, Ann. Inst. H. Poincaré **64-4**, 385-393 (1996).

[27] Iwahori, N. and Sugiura, M., A duality theorem for homogeneous manifolds of compact Lie groups, Osaka J. Math. **3**, 139-153 (1966).

[28] See, for instance, Kirillov, A. A., *Elements of the Theory of Representations.* Springer-Verlag (1976).

[29] Machida, S. and Namiki, M., Theory of measurement in quantum mechanics: mechanism of reduction of wave packet. I & II, Prog. Theor. Phys. **63**, 1457-1473; 1833-1847 (1980).

[30] See, for instance, MacLane, S., *Categories for the Working Mathematician*, Springer-Verlag (1971).

[31] Maumary, S., Category dual of a compact group relative to a subgroup, in preparation.

[32] Nakagami, Y. and Takesaki, M., *Duality for Crossed Products of von Neumann Algebras*, Lecture Notes in Mathematics **731**, Springer-Verlag (1979).

[33] Nishijima, K., Arbitrariness in the choice of field operators, in *"High-Energy Physics and Elementary Particles"*, pp.137-146(1965).

[34] Nozawa, T. and Ojima, I., in preparation.

[35] Ohya, M., On compound state and mutual information in quantum information theory, IEEE Trans. Information Theory **29**, 770-777 (1983).

[36] Ohya, M. and Petz, D., *Quantum Entropy and Its Use*, Springer-Verlag (1993).

[37] Ojima, I., Observables and quark confinement in covariant canonical formalism of Yang-Mills theory, Nucl. Phys. **B143**, 340-352 (1978).

[38] Ojima, I., Symmetry breaking patterns –Spontaneous collapse of SUSY and others– pp.337–353 *in* "*Trends in Contemporary Infinite Dimensional Analysis and Quantum Probability*", eds. L. Accardi, H.-H. Kuo, N.Obata, K.Saito, Si Si and L. Streit, Italian School of East Asian Studies, 2000.

[39] Ojima, I., A Simple derivation of Josephson formulae in superconductivity, Lett. Math. Phys. **51**, 257-264 (2000).

[40] Ojima, I., Non-equilibrium local states in relativistic quantum field theory, to appear in Proc. of Japan-Italy Joint Workshop on Fundamental Problems in Quantum Mechanics, September 2001, Waseda University (available also at http://www.f.waseda.jp/stasaki/WS/Ojima.pdf).

[41] Ojima, I., How to formulate non-equilibrium local states in QFT?– General characterization and extension to curved spacetime–, to appear in "Garden of Quanta" (World Scientific Publ. Co.); e-print: cond-mat/0302283.

[42] Ojima, I., in preparation.

[43] Ozawa, M., Quantum measuring processes of continuous observables, J. Math. Phys. **25**, 79-87 (1984); Conditional probability and a posteriori states in quantum mechanics, Publ. RIMS, Kyoto Univ. **21**, 279-295 (1985); Ann. Phys. (N.Y.) **259**, 121-137 (1997).

[44] Pusz, W. and Woronowicz, S.L., Passive states and KMS states for general quantum systems, Comm. Math. Phys. **58**, 273-290 (1978).

[45] Roberts, J. E., Spontaneously broken gauge symmetries and superselection rules, in Proc. International School of Mathematical Physics, Camerino 1974, ed. G. Gallavotti, Università di Camerino, 1976.

[46] Toda, H. and Mimura, M., *Topology of Lie Groups*, Vol.2, (In Japanese), Kinokuniya-Shoten (1979); Mimura, M., *Topology of Lie groups*, AMS transl. (in English).

[47] Weinberg, S., Phys. Rev. Lett. **18**, 188 (1967); J. Schwinger, Phys. Lett. **24B**, 473 (1967); Coleman, S. Wess, J., and Zumino, B., Phys. Rev. **177**, 2239-2247 (1969); Callan, C.G., Coleman, S. Wess, J. and Zumino, B., Phys. Rev. **177**, 2247-2250 (1969).