Construction of orthonormal basis on self-similar sets by generalized permutative representations of the Cuntz algebras

Katsunori Kawamura ‡ and Osamu Suzuki §

‡*Research Institute for Mathematical Sciences*

Kyoto University, Kyoto 606-8502, Japan

§*Department of Computer Science and System Analysis*

Nihon University, Tokyo 156-8550, Japan

Abstract

A systematic construction method of orthonormal basis on self-similar sets is given by the use of representations of the Cuntz algebras. We introduce two kinds of orthonormal basis of self-similar sets including explicit description for the Cantor set, the Sierpiński gasket and the Sierpiński carpet.

1 Introduction

It is well known that Fourier analysis is one of the most important tools for the analysis and geometry on manifolds. In a sense of generalization of this, there are several studies of self-similar sets according to the aim of applications of analysis of Laplacians on them ([Ki1, Ki2, Sa, Tep]). On the other hand, we have interest in Fourier analysis on self-similar sets in a sense of harmonic analysis. In this paper we shall give a method of constructing orthonormal basis of the L_2-space with respect to the Hausdorff measure on the self-similar set systematically by using the representation theory of the Cuntz algebra. Our main result can be stated in the following:

Theorem 1.1. (Main theorem) *Let K be a self-similar set with contractions $\{\sigma_i\}_{i=1}^N$, $N \geq 2$, contraction ratios $\{\lambda_i\}_{i=1}^N$, the similarity dimension D, and the Hausdorff measure μ^D on K. Put $L_2(K, \mu^D)$ the Hilbert space of all complex valued square integrable functions on K.*

Choose a unitary matrix $g = (g_{ij})_{i,j=1}^N \in U(N)$ such that $g_{1j} = \lambda_j^{D/2}$ for $j = 1, \ldots, N$. Put a subset of multiindices consisting of $1, \ldots, N$ by

$$\Lambda_N \equiv \{1, \ldots, N\} \cup \bigcup_{k \geq 2} \left(\{1, \ldots, N\}^{k-1} \times \{2, \ldots, N\} \right) \tag{1.1}$$

and symbols $K_J \equiv \sigma_J(K)$, $\sigma_J = \sigma_{j_1} \circ \cdots \circ \sigma_{j_k}$, $\lambda_J = \lambda_{j_1} \cdots \lambda_{j_k}$ when $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, and denote χ_{K_J} the characteristic function of K_J for $J \in \{1, \ldots, N\}^k$, $k \geq 1$.

Then the followings hold:

‡e-mail : kawamura@kurims.kyoto-u.ac.jp.

§e-mail : osuzuki@am.chs.nihon-u.ac.jp.

(i) *There are two kinds of complete orthonormal basis(=CONB) on* $L_2(K,\mu^D)$*:*

(a) *A CONB* $\{E_I : I \in \Lambda_N\}$ *of* $L_2(K,\mu^D)$ *is given by*

$$E_I \equiv \sum_{J\in\{1,\dots,N\}^k} c_{I,J}\chi_{K_J}$$

for $I \in \Lambda_N \cap \{1,\dots,N\}^k$, $k \geq 1$ *where*

$$c_{I,J} \equiv \lambda_J^{-D/2} \prod_{l=1}^{k} g^*_{j_l,i_l}$$

when $I = (i_1,\dots,i_k) \in \Lambda_N$, $J = (j_1,\dots,j_k) \in \{1,\dots,N\}^k$, $k \geq 1$.

(b) *A CONB* $\{H_I : I \in \Lambda_N\}$ *of* $L_2(K,\mu^D)$ *is given by*

$$H_i \equiv \sum_{l=1}^{N} \lambda_l^{-D/2} g^*_{li}\chi_{K_l}, \quad H_{J,j} \equiv \lambda_J^{-D/2}\chi_{K_J} \cdot H_j \circ \sigma_J^{-1}$$

for $i = 1,\dots,N$ *and* $j = 2,\dots,N$, $J \in \{1,\dots,N\}^k$, $k \geq 1$.

(ii) *For* $\phi \in L_2(K,\mu^D)$*, we have the following expansions:*

(a)

$$\phi = \sum_{I\in\Lambda_N} a_I E_I, \tag{1.2}$$

where

$$a_I = \sum_{J\in\{1,\dots,N\}^k} \overline{c_{I,J}} \int_{K_J} \phi(x) d\mu^D(x) \tag{1.3}$$

when $I = (i_1,\dots,i_k) \in \Lambda_N$.

(b)

$$\phi = \sum_{I\in\Lambda_N} b_I H_I, \tag{1.4}$$

where

$$b_i = \sum_{l=1}^{N} \lambda_l^{-D/2} g_{il} \int_{K_l} \phi(x) d\mu^D(x), \quad b_{J,j} = \lambda_J^{-D/2} \sum_{l=1}^{N} \lambda_l^{-D/2} g_{jl} \int_{K_{J,l}} \phi(x) d\mu^D(x) \tag{1.5}$$

when $i = 1,\dots,N$, $j = 2,\dots,N$ *and* $J \in \{1,\dots,N\}^k$.

In § 2, we review generalized permutative representations and the Hausdorff representations of the Cuntz algebra on self-similar sets and show the construction of orthonormal basis. In § 3, we give examples of complete orthonormal basis by $g \in U(N)$ arising from the N-th root of unity for self-similar sets with the common contraction ratio. In the cases of the Cantor set, the Sierpiński gasket and the Sierpiński carpet, we give concrete formulae of basis in an explicit manner. In § 4, we construct basis for several examples by orthogonal matrix. In Appendix C, we show relations between states and representations associated with self-similar sets.

2 GP representations and Hausdorff representations of the Cuntz algebra and basis

For $N \geq 2$, a C^*-algebra with generators $s_1, \ldots, s_N$ which satisfy the following relations

$$s_i^* s_j = \delta_{ij} I \quad (i,j = 1, \ldots, N), \qquad \sum_{i=1}^{N} s_i s_i^* = I \tag{2.1}$$

is called *the Cuntz algebra*([C]) and is denoted by $\mathcal{O}_N$. $\mathcal{O}_N$ is non commutative, infinite dimensional, separable, simple and unique up to isomorphisms. Hence there is no finite dimensional representation except 0-representation. We notice that these relations give an algebraic description of the division of the total space into N-parts.

We denote

$$s_J \equiv s_{j_1} \cdots s_{j_k}, \quad s_J^* \equiv s_{j_k}^* \cdots s_{j_1}^*$$

for a multi index $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $k \geq 1$. In this paper, a representation always means a unital *-representation on a complex Hilbert space.

2.1 GP representations of $\mathcal{O}_N$ with 1-cycle

We show only the 1-cycle case about GP representations of Cuntz algebras with cycle in [Ka1]. Let $S(\mathbf{C}^N) \equiv \{z \in \mathbf{C}^N : \|z\| = 1\}$ be the complex sphere in a complex vector space $\mathbf{C}^N$.

Definition 2.1. *$(\mathcal{H}, \pi, \Omega)$ is the GP(= generalized permutative) representation of $\mathcal{O}_N$ by $z = (z_1, \ldots, z_N) \in S(\mathbf{C}^N)$ if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ and Ω is the unit cyclic vector which satisfies the following equation:*

$$\pi(z_1 s_1 + \cdots + z_N s_N)\Omega = \Omega. \tag{2.2}$$

We denote $GP(z) \equiv (\mathcal{H}, \pi, \Omega)$.

For two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathcal{O}_N$, $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$ means the unitary equivalence between $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$.

Theorem 2.2. *(Characterization of GP representation with 1-cycle)*

(i) *(Existence and Uniqueness) For any $z \in S(\mathbf{C}^N)$, $GP(z)$ exists uniquely up to unitary equivalences.*

(ii) *(Irreducibility) For any $z \in S(\mathbf{C}^N)$, $GP(z)$ is irreducible.*

(iii) *(Equivalence) For $z, z' \in S(\mathbf{C}^N)$, $GP(z) \sim GP(z')$ if and only if $z = z'$.*

(iv) *(State) For $z = (z_1, \ldots, z_N) \in S(\mathbf{C}^N)$, $GP(z)$ is equivalent to the GNS-representation by a state ρ of $\mathcal{O}_N$ (that is, ρ is a linear functional on $\mathcal{O}_N$ such that $\rho(I) = 1$, $\rho(x^*) = \overline{\rho(x)}$, $\rho(x^* x) \geq 0$, for $x \in \mathcal{O}_N$) which is defined by*

$$\rho(s_J s_{J'}^*) \equiv \overline{z_J} z_{J'} \tag{2.3}$$

where $J, J' \in \cup_{k \geq 0} \{1, \ldots, N\}^k$, $|J| + |J'| \geq 1$, $z_J \equiv z_{j_1} \cdots z_{j_k}$ when $J = (j_1, \ldots, j_k)$, and $s_J = I$, $z_J = 1$ when $J = \emptyset$.

Proof. For (i),(ii),(iii) see Appendix A.

(iv) Assume that $(\mathcal{H}, \pi, \Omega)$ is $GP(z)$. Let $\rho(x) \equiv < \Omega|\pi(x)\Omega >$ for $x \in \mathcal{O}_N$. Then ρ satisfies (2.3). On the other hand, a state ρ of $\mathcal{O}_N$ is uniquely determined by (2.3). By the uniqueness of GNS-representation and the cyclicity of $GP(z)$, the GNS-representation of ρ is equivalent to $GP(z)$. □

Next we proceed to two kinds of constructions of orthonormal basis for general GP representations.

Proposition 2.3. *For $z \in S(\mathbf{C}^N)$, if $(\mathcal{H}, \pi, \Omega) = GP(z)$, then we have the following two complete orthonormal basis(=CONB) of $\mathcal{H}$ which depends on $g \in U(N)$ under the condition $g_{1j} = \overline{z_j}$ for $j = 1, \ldots, N$.*

(i) *(Construction A) Put*

$$E_I \equiv \sum_{J\in\{1,\ldots,N\}^k} g^*_{J,I}\pi(s_J)\Omega \tag{2.4}$$

and

$$g^*_{J,I} \equiv \prod_{l=1}^{k} g^*_{j_l,i_l}$$

when $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$ and $I = (i_1, \ldots, i_k) \in \Lambda_N$. Then $\{E_I : I \in \Lambda_N\}$ is a CONB of $\mathcal{H}$.

(ii) *(Construction B) Put*

$$\begin{aligned} H_i \equiv & \sum_{l=1}^{N} g^*_{li}\pi(s_l)\Omega \qquad (i = 1, \ldots, N), \\ H_{J,j} \equiv & \ \pi(s_J)H_j \qquad (J \in \{1, \ldots, N\}^k,\ j = 2, \ldots, N,\ k \geq 1). \end{aligned} \tag{2.5}$$

Then $\{H_I : I \in \Lambda_N\}$ is a CONB of $\mathcal{H}$.

Proof. (i) By expanding results in Lemma A.5, we have (2.4). (ii) Note $H_i = E_i$ for $i = 1, \ldots, N$. Hence $H_1, \ldots, H_N$ are mutually orthonormal by (i). By (2.1), we can show $< \pi(s_J)H_j|\pi(s_{J'})H_{j'} >= \delta_{j,j'}\delta_{J,J'}$. □

Note that sums in (2.4) and (2.5) are always finite.

Remark 2.4. Both constructions of basis in Proposition 2.3 depend on the choice of $g \in U(N)$. On the other hand, the representation is unique up to unitary equivalences for any $g \in U(N)$ which satisfies (**??**). In general, g is taken from the mapping group $\{g : \mathbf{N} \to U(N)\}$ in [Ka4]. By using g in this set, we can make more complicated orthonormal basis for $GP(z)$. In this paper, we treat only two cases in them.

2.2 Hausdorff representations of $\mathcal{O}_N$ on self-similar sets

We proceed to representations of the Cuntz algebras on self-similar sets. For this purpose, we start from general construction of a representation of the Cuntz algebra on a measure space([BJ, QM1]).

Let (X, μ) be a measure space and $N \geq 2$.

Definition 2.5. *A family $f = \{f_i\}_{i=1}^N$ is a (measure theoretical)branching function system on (X, μ) if $f_i : X \to X$ is a measurable injective map such that there exists $\Phi_i \equiv \frac{d(\mu \circ f_i)}{d\mu} > 0$ a.e. on X for $i = 1, \ldots, N$, $\mu(f_i(X) \cap f_j(X)) = 0$ when $i \neq j$, and $\mu\left(X \setminus \bigcup_{i=1}^N f_i(X)\right) = 0$.*

Lemma 2.6. *For a branching function system $f = \{f_i\}_{i=1}^N$ on (X, μ), define operators $\pi_f(s_i) : L_2(X, \mu) \to L_2(X, \mu)$ by*

$$(\pi_f(s_i)\phi)(x) \equiv \begin{cases} \left\{\Phi_i(f_i^{-1}(x))\right\}^{-1/2} \phi\left(f_i^{-1}(x)\right) & (x \in f_i(X)), \\ 0 & (\textit{otherwise}) \end{cases} \tag{2.6}$$

for $\phi \in L_2(X, \mu)$ and $i = 1, \ldots, N$. Then $(L_2(X, \mu), \pi_f)$ is a representation of $\mathcal{O}_N$.

Proof. We can show that $\{\pi_f(s_i)\}_{i=1}^N$ satisfies relations (2.1) directly. □

Next we recall some basic facts on self-similar sets by [F, H].

We consider a self-similar set $(K, \{\sigma_i\}_{i=1}^N)$ with contraction ratios $\{\lambda_i : 0 < \lambda_i < 1,\ i = 1, \ldots, N\}$ which satisfies the following conditions: K is a compact subset of $\mathbf{R}^n$ and σ_i is a contraction on K with contraction ratio λ_i for $i = 1, \ldots, N$ with respect to the Euclid distance in $\mathbf{R}^n$ and this data satisfies

$$\bigcup_{i=1}^N \sigma_i(K) = K, \quad \mu^D(\sigma_i(K) \cap \sigma_j(K)) = 0 \quad (i \neq j), \tag{2.7}$$

where μ^D is the Hausdorff measure on K which satisfies $\mu^D(K) = 1$. A positive real number D which satisfies the following condition

$$\lambda_1^D + \cdots + \lambda_N^D = 1 \tag{2.8}$$

is called the *similarity dimension of* K. Here we put the following notations:

$$K_J \equiv \sigma_J(K) \tag{2.9}$$

where $\sigma_J \equiv \sigma_{j_1} \circ \cdots \circ \sigma_{j_k}$ for $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$ and $k \geq 1$. For $(K, \{\sigma_i\}_{i=1}^N)$, let $L_2(K, \mu^D)$ be the Hilbert space of complex valued square integrable functions on K by μ^D. Because a family $\sigma \equiv \{\sigma_i\}_{i=1}^N$ on a measure space (K, μ^D) satisfies the condition of branching function system in Definition 2.5, we have the following representation $(L_2(K, \mu^D), \pi_\sigma)$ of $\mathcal{O}_N$:

$$(\pi_\sigma(s_i)\phi)(x) \equiv \begin{cases} \lambda_i^{-D/2} \phi\left(\sigma_i^{-1}(x)\right) & (x \in \sigma_i(K)), \\ 0 & (\text{otherwise}), \end{cases} \quad (i = 1, \ldots, N) \tag{2.10}$$

for $\phi \in L_2(K,\mu^D)$. From this, we have $(\pi_\sigma(s_i)^*\phi)(x) = \lambda_i^{D/2}\phi(\sigma_i(x))$ for $i = 1,\ldots,N$, $x \in K$ and $\phi \in L_2(K,\mu^D)$.

The representation $(L_2(K,\mu^D),\pi_\sigma)$ of $\mathcal{O}_N$ which is defined in (2.10) is called *the Hausdorff representation of $\mathcal{O}_N$ on $(K,\{\sigma_i\}_{i=1}^N)$*([MSW] and see Appendix B). Let $\mathbf{1}$ be the constant function on K with value 1. By assumption $\mu^D(K) = 1$, $\|\mathbf{1}\| = 1$. Note

$$\pi_\sigma(s_i)\mathbf{1} = \lambda_i^{-D/2}\chi_{K_i} \quad (i = 1,\ldots,N). \tag{2.11}$$

From this we have

$$\pi_\sigma(s_J)\mathbf{1} = \lambda_J^{-D/2}\chi_{K_J} \tag{2.12}$$

for $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$ where $\lambda_J = \lambda_{j_1}\cdots\lambda_{j_k}$.

Lemma 2.7. *The Hausdorff representation $(L_2(K,\mu^D),\pi_\sigma)$ is a cyclic representation of $\mathcal{O}_N$ with the unit cyclic vector $\mathbf{1}$.*

Proof. By (2.12), $\{\chi_{K_J} : J \in \{1,\ldots,N\}^k,\ k \geq 1\} \subset \mathrm{Lin} < \{\pi_\sigma(s_I)\mathbf{1} : I \in \Lambda_N\} >$. Hence $\pi_\sigma(\mathcal{O}_N)\mathbf{1} = L_2(K,\mu^D)$. □

Theorem 2.8. *Let $(L_2(K,\mu^D),\pi_\sigma)$ be the Hausdorff representation of $\mathcal{O}_N$ on $(K,\{\sigma_i\}_{i=1}^N)$ with contraction ratios $\{\lambda_i\}_{i=1}^N$ and the similarity dimension D. Then the followings hold:*

(i) *$(L_2(K,\mu^D),\pi_\sigma,\mathbf{1})$ is $GP(z)$ for $z = (\lambda_1^{D/2},\ldots,\lambda_N^{D/2})$.*

(ii) *$(L_2(K,\mu^D),\pi_\sigma)$ is irreducible.*

Proof. By (2.8), $z = (\lambda_1^{D/2},\ldots,\lambda_N^{D/2}) \in S(\mathbf{C}^N)$. According to (2.11), the following holds:

$$\pi_\sigma(s(z))\mathbf{1} = \left(\sum_{i=1}^N \lambda_i^{D/2}\pi_\sigma(s_i)\right)\mathbf{1} = \sum_{i=1}^N \chi_{K_i} = \mathbf{1}.$$

By Lemma 2.7, $(L_2(K,\mu^D),\pi_\sigma,\mathbf{1})$ is $GP(z)$. Hence (i) is proved. By Theorem 2.2 (ii) and Theorem 2.8, $(L_2(K,\mu^D),\pi_\sigma)$ is irreducible. Therefore (ii) is proved. □

Remark 2.9. (Contraction ratios and invariants of representations) In this paper, we treat only the case of self-similar sets with constant contraction ratios. By Theorem 2.8 and [MSW], it seems that constant ratios are always important to characterize representation as the parameter z of GP representation. However, we can show that a representation of $\mathcal{O}_2$ associated with non constant case in [QM1] is equivalent to some GP representation. In this sense, the contraction ratio is independent in the invariant of representation of the Cuntz algebra in general.

Relations between Hausdorff representations and states of $\mathcal{O}_N$ are explained in Appendix C.

2.3 Construction of orthonormal basis on self-similar sets

We shall give a construction method of orthonormal basis on self-similar sets and prove Theorem 1.1.

Lemma 2.10. *Let $(K, \{\sigma_i\}_{i=1}^N)$ be a self-similar set with contraction ratios $\{\lambda_i\}_{i=1}^N$ and the similarity dimension D. Take $g = (g_{ij}) \in U(N)$ which satisfies*

$$g_{1j} = \lambda_j^{D/2} \quad (j = 1, \ldots, N). \tag{2.13}$$

We have two kinds of complete orthonormal basis(=CONB) of $L_2(K, \mu^D)$ as follows:

(i) *A CONB $\{E_I : I \in \Lambda_N\}$ of $L_2(K, \mu^D)$ is given by*

$$E_I \equiv \sum_{J \in \{1,\ldots,N\}^k} \lambda_J^{-D/2} g^*_{J,I}\, \chi_{K_J} \tag{2.14}$$

for $I \in \Lambda_N \cap \{1, \ldots, N\}^k$. Specially, $E_1 = \mathbf{1}$.

(ii) *A CONB $\{H_I : I \in \Lambda_N\}$ of $L_2(K, \mu^D)$ is given by*

$$H_i \equiv \sum_{l=1}^N \lambda_l^{-D/2} g^*_{li}\, \chi_{K_l}, \quad H_{J,j} \equiv \lambda_J^{-D/2} \chi_{K_J} \cdot H_j \circ \sigma_J^{-1} \tag{2.15}$$

for $J \in \{1, \ldots, N\}^k$, $i = 1, \ldots, N$ and $j = 2, \ldots, N$. Specially, $H_1 = \mathbf{1}$.

Proof. (i) By Proposition 2.3 (i) and Theorem 2.8, we have a CONB $\{E_I : I \in \Lambda_N\}$ for $L_2(K, \mu^D)$. A function E_I on K is computed by (2.4) and (2.12) as follows:

$$E_I = \sum_{J \in \{1,\ldots,N\}^k} g^*_{J,I} \pi_\sigma(s_J)\mathbf{1} = \sum_{J \in \{1,\ldots,N\}^k} g^*_{J,I} \lambda_J^{-D/2} \chi_{K_J}$$

for $I \in \Lambda_N \cap \in \{1, \ldots, N\}^k$.

(ii) By Proposition 2.3 (ii) and Theorem 2.8, we have a CONB $\{H_I : I \in \Lambda_N\}$ for $L_2(K, \mu^D)$. □

Proof of Theorem 1.1

(i) is shown by Lemma 2.10. The second assertion is a direct consequence of the orthonormality condition: When we expand a function $\phi \in L_2(K, \mu^D)$ by $\{E_I : I \in \Lambda_N\}$:

$$\phi = \sum_{I \in \Lambda_N} a_I E_I, \tag{2.16}$$

we have

$$a_I = < E_I | \phi > = \int_K \phi(x) \overline{E_I(x)} d\mu^D(x).$$

Note $g^*_{J,I} = \overline{g_{I,J}}$. Hence (ii) (a) holds. (ii) (b) follows in the same way. □

3 Examples of orthonormal basis by N-th root of unity

In this section and next, we give several examples of orthonormal basis of $L_2(K,\mu^D)$ in Theorem 1.1 (i) by choosing a unitary matrix $g \in U(N)$. In this section, we treat g arising from N-th root of unity for self-similar sets with common contraction ratio.

3.1 Construction of unitary matrix arising from N-th root of unity and basis

Assume that $(K, \{\sigma_i\}_{i=1}^N)$ is a self-similar set with common contraction ratio λ with the similarity dimension D. Then automatically, $\lambda^D = \frac{1}{N}$ by (2.8). By taking the N-th root $\xi = e^{2\pi\sqrt{-1}/N}$ of unity, we can make the following unitary matrix $g = (g_{ij}) \in U(N)$ in Theorem 2.8 by

$$g_{ij} \equiv \frac{1}{\sqrt{N}}\xi^{-(i-1)(j-1)} \quad (i, j = 1, \ldots, N). \tag{3.1}$$

Note that g is symmetric in this case. For example, when $N = 5$, $\xi = e^{2\pi\sqrt{-1}/5}$,

$$g^* = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & \xi^3 & \xi^4 \\ 1 & \xi^2 & \xi^4 & \xi & \xi^3 \\ 1 & \xi^3 & \xi & \xi^4 & \xi^2 \\ 1 & \xi^4 & \xi^3 & \xi^2 & \xi \end{pmatrix}.$$

Denote

$$(I|J) \equiv \sum_{l=1}^{k}(i_l - 1)(j_l - 1) \tag{3.2}$$

for $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$. Then E_I in Theorem 1.1 (i) (a) by g is given by

$$E_I = \sum_{J\in\{1,\ldots,N\}^k} \xi^{(I|J)}\chi_{K_J} \tag{3.3}$$

for $I \in \Lambda_N \cap \{1, \ldots, N\}^k$. We discuss meaning of this basis in § 3.5 in detail.

Let $\mathbf{Z}_N \equiv \{e^{2\pi\sqrt{-1}L/N} : L = 0, \ldots, N-1\}$.

Proposition 3.1. *Let $(K, \{\sigma_i\}_{i=1}^N)$ be a self-similar set with common contraction ratio and the Hausdorff measure μ^D. Then there is a complete orthonormal basis of $L_2(K,\mu^D)$ which consists of $\mathbf{Z}_N$-valued functions on K.*

This proposition is a special case of Theorem 1.1. However it is effective to explain the non-triviality of our construction of basis. In general, it is difficult to construct a basis which consists of $\mathbf{Z}_N$-valued functions in a L_2-space according to ordinary orthogonalization(Gram-Schmidt and so on) of vectors in a Hilbert space.

3.2 The Cantor set

The Cantor set is defined as a unique compact set K such that $K = f_1(K) \cup f_2(K)$ where contractions f_1, f_2 are defined by $f_1(x) \equiv \frac{1}{3}x, \quad f_2(x) \equiv \frac{1}{3}(x+2)$ on $\mathbf{R}$. Specially K is a subset of a closed interval $[0,1]$. Then $K_J = K \cap [a_J, b_J]$ where $a_J \equiv f_J(0)$ and $b_J \equiv f_J(1)$. Note $\lambda_1^D = \lambda_2^D = \frac{1}{2}$. Take $\xi = -1$. In this case, $\Lambda_2 = \{1,2\} \cup \{(1,2),(2,2)\} \cup \{(1,1,2),(1,2,2),(2,1,2),(2,2,2)\} \cup \cdots$.

Then we have the following functions in (3.3):

$$E_I = \sum_{J\in\{1,2\}^k} (-1)^{(I|J)} \chi_{K_J} \quad (I \in \Lambda_2 \cap \{1,2\}^k).$$

From this, we have

$$E_1 = \mathbf{1}, \quad E_2 = \chi_{K_1} - \chi_{K_2},$$

$$E_{12} = \chi_{K_{1,1}} - \chi_{K_{12}} + \chi_{K_{2,1}} - \chi_{K_{22}}, \quad E_{22} = \chi_{K_{11}} - \chi_{K_{12}} - \chi_{K_{21}} + \chi_{K_{22}}.$$

3.3 The Sierpiński gasket I

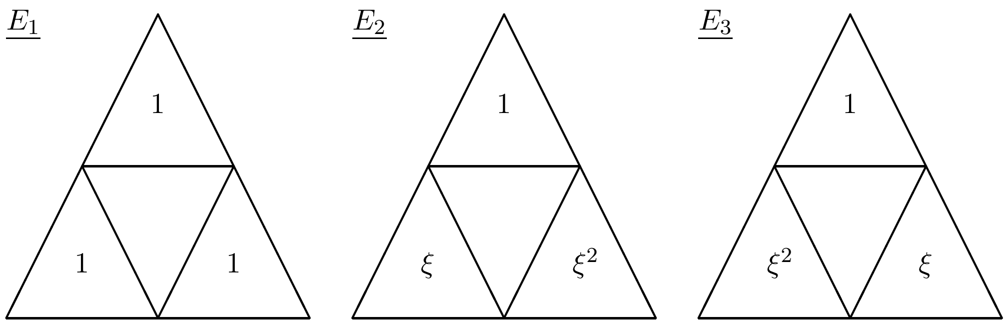
The Sierpiński gasket is defined as a unique compact set K such that $K = f_1(K) \cup f_2(K) \cup f_3(K)$ where the branching function system $f = \{f_i\}_{i=1}^3$ is defined by $f_i : \mathbf{R}^3 \to \mathbf{R}^3; \quad f_i(x) \equiv \frac{1}{2}(x + c_i)$ for $i = 1,2,3$ where $\{c_i\}_{i=1}^3$ is the canonical basis of $\mathbf{R}^3$. Specially, K is a subset of the 2-simplex $\Delta_2 \equiv \{(x_1,x_2,x_3) \in \mathbf{R}^3 : 0 \le x_i,\ x_1+x_2+x_3 = 1\}$. Then $\lambda_i^D = \frac{1}{3}$ for each $i = 1,2,3$. For $\xi \equiv e^{2\pi\sqrt{-1}/3}$, g in (3.1) is

$$g = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi^2 & \xi \\ 1 & \xi & \xi^2 \end{pmatrix}.$$

The basis $\{E_I : I \in \Lambda_3\}$ in (3.3) is given by

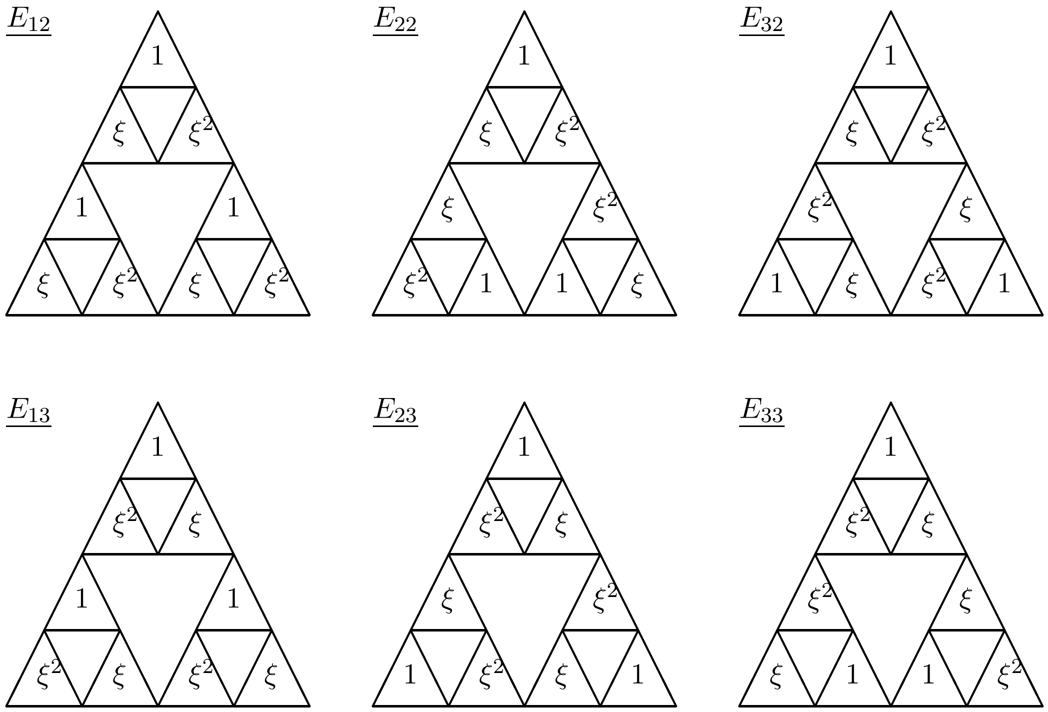
$$E_1 = \mathbf{1}, \quad E_2 = \chi_{K_1} + \xi\chi_{K_2} + \xi^2\chi_{K_3}, \quad E_3 = \chi_{K_1} + \xi^2\chi_{K_2} + \xi\chi_{K_3}.$$

We illustrate them by using the identification K_J and $f_J(\Delta_2)$ for $J \in \Lambda_3$ as follows:



In the same way, we have the followings:

$$
\begin{aligned}
E_{12} = \quad & \chi_{K_{11}} + \xi\chi_{K_{12}} + \xi^2\chi_{K_{13}} + \chi_{K_{21}} + \xi\chi_{K_{22}} + \xi^2\chi_{K_{23}} + \chi_{K_{31}} + \xi\chi_{K_{32}} + \xi^2\chi_{K_{33}}, \\
E_{22} = \quad & \chi_{K_{11}} + \xi\chi_{K_{12}} + \xi^2\chi_{K_{13}} + \xi\chi_{K_{21}} + \xi^2\chi_{K_{22}} + \chi_{K_{23}} + \xi^2\chi_{K_{31}} + \chi_{K_{32}} + \xi\chi_{K_{33}}, \\
E_{32} = \quad & \chi_{K_{11}} + \xi\chi_{K_{12}} + \xi^2\chi_{K_{13}} + \xi^2\chi_{K_{21}} + \chi_{K_{22}} + \xi\chi_{K_{23}} + \xi\chi_{K_{31}} + \xi^2\chi_{K_{32}} + \chi_{K_{33}}, \\
E_{13} = \quad & \chi_{K_{11}} + \xi^2\chi_{K_{12}} + \xi\chi_{K_{13}} + \chi_{K_{21}} + \xi^2\chi_{K_{22}} + \xi\chi_{K_{23}} + \chi_{K_{31}} + \xi^2\chi_{K_{32}} + \xi\chi_{K_{33}}, \\
E_{23} = \quad & \chi_{K_{11}} + \xi^2\chi_{K_{12}} + \xi\chi_{K_{13}} + \xi\chi_{K_{21}} + \chi_{K_{22}} + \xi^2\chi_{K_{23}} + \xi^2\chi_{K_{31}} + \xi\chi_{K_{32}} + \chi_{K_{33}}, \\
E_{33} = \quad & \chi_{K_{11}} + \xi^2\chi_{K_{12}} + \xi\chi_{K_{13}} + \xi^2\chi_{K_{21}} + \xi\chi_{K_{22}} + \chi_{K_{23}} + \xi\chi_{K_{31}} + \chi_{K_{32}} + \xi^2\chi_{K_{33}}.
\end{aligned}
$$



3.4 The Sierpiński carpet

The Sierpiński carpet is defined as a unique compact set K such that $K = f_1(K) \cup \cdots \cup f_8(K)$ where the branching function system $\{f_i\}_{i=1}^8$ is defined by $f_i : \mathbf{R}^2 \to \mathbf{R}^2; \quad f_i(x) \equiv \frac{1}{3}(x + c_i)$ for $i = 1, \ldots, 8$ where $(c_i)_{i=1}^8 = ((0,0),(1,0),(2,0),(2,1),(2,2),(1,2),(0,2),(0,1))$. Specially K is a subset of $[0,1]^2 \equiv [0,1] \times [0,1]$. Let $a \equiv e^{\pi\sqrt{-1}/4}$. We have the basis which is illustrated as follows:

$\underline{E_1}$

1	1	1
1		1
1	1	1

$\underline{E_2}$

a^6	a^5	a^4
a^7		a^3
1	a	a^2

$\underline{E_3}$

a^4	a^2	1
a^6		a^6
1	a^2	a^4

$\underline{E_4}$

a^2	a^7	a^4
a^5		a
1	a^3	a^6

$\underline{E_5}$

1	a^4	1
a^4		a^4
1	a^4	1

$\underline{E_6}$

a^6	a	a^4
a^3		a^7
1	a^5	a^2

$\underline{E_7}$

a^4	a^6	1
a^2		a^2
1	a^6	a^4

$\underline{E_8}$

a^2	a^3	a^4
a		a^5
1	a^7	a^6

In this case, the Hausdorff representation $(L_2(K,\mu^D),\pi_f)$ of $\mathcal{O}_8$ on K is $GP(z)$ for $z = \left(\frac{1}{2\sqrt{2}},\ldots,\frac{1}{2\sqrt{2}}\right) \in S(\mathbf{C}^8)$.

3.5 Fourier analysis on self-similar sets by the Cuntz algebra

Harmonic analysis of self-similar sets is studied in [Ki1, Ki2]. We consider this subject in the point of view of representation theory of the Cuntz algebra. More precisely, we try to consider the meaning of our basis which are obtained by representations of the Cuntz algebra as harmonic analysis style.

Recall § 3.1. We construct a basis $\{E_I : I \in \Lambda_N\}$ of $L_2(K,\mu^D)$ for a self-similar set $(K,\{\sigma_i\}_{i=1}^N)$ with common contraction ratio. E_I in (3.3) can be rewritten by

$$E_I(x) = e^{2\pi\sqrt{-1}\delta_I(x)} \tag{3.4}$$

for $x \in K$ where

$$\delta_I : K \to \left\{\frac{i}{N} : i = 0,1,2,\dots\right\}; \quad \delta_I(x) \equiv \frac{1}{N}\sum_{J\in\{1,\dots,N\}^k} (I|J)\cdot \chi_{K_J}(x)$$

for $I \in \Lambda_N \cap \{1,\dots,N\}^k$. By this parameterization, every value of (3.4) belongs to a finite set $\mathbf{Z}_N$. According to the expansion formula (2.16) in this case, we have a unitary transformation

$$\hat{}: L_2(K,\mu^D) \to l_2(\Lambda_N); \quad \hat{\phi}(I) \equiv \int_K e^{-2\pi\sqrt{-1}\delta_I(x)}\phi(x)\,d\mu^D(x)$$

which seems Fourier transformation on K in usual harmonic analysis by replacing the ordinary symmetry of group by that of the Cuntz algebra $\mathcal{O}_N$, for example, this situation is similar to the harmonic analysis between $\mathbf{T}^1 \equiv \{z \in \mathbf{C} : |z| = 1\}$ and $\mathbf{Z}$.

This is an answer of our question in § 1. In this way, a self-similar set K and the discrete space Λ_N of symbols are dual in a sense of classical harmonic analysis. In consequence, the basis E_I seems as a character of K with index $I \in \Lambda_N$ which is a map from K to $\mathbf{Z}_N$. On the other hand, the inverse transformation of $\hat{}$ is given by

$$\check{}: l_2(\Lambda_N) \to L_2(K,\mu^D); \quad \check{\psi}(x) \equiv \sum_{J\in\Lambda_N} e^{-2\pi\sqrt{-1}\delta_J(x)}\psi(J).$$

By summarizing these consideration and (3.3), we have the following:

Theorem 3.2. *Let $(K,\{\sigma_i\}_{i=1}^N)$ be a self-similar set with common contraction ratio and the Hausdorff measure μ^D. Then any $\phi \in L_2(K,\mu^D)$ has the following expansion*

$$\phi(x) = c_0 + \sum_{k\geq 0}\sum_{I\in\{1,\dots,N\}^k\times\{2,\dots,N\}} c_I e^{2\pi\sqrt{-1}\delta_I(x)}$$

where

$$c_0 \equiv \int_K \phi(x)\,d\mu^D(x), \quad c_I \equiv \int_K e^{-2\pi\sqrt{-1}\delta_I(x)}\phi(x)\,d\mu^D(x).$$

The transformation rule of the basis $\{E_I : I \in \Lambda_N\}$ by the branching function system $\{\sigma_i\}_{i=1}^N$ is given as follows:

Proposition 3.3. *Let $\{E_I : I \in \Lambda_N\}$ be the basis of $L_2(K,\mu^D)$ by $e^{2\pi\sqrt{-1}/N}$ for a self-similar set $(K,\{\sigma_i\}_{i=1}^N)$ with common contraction ratio in (3.3). Then*

$$E_i \circ \sigma_j = e^{2\pi\sqrt{-1}(i-1)(j-1)/N}\mathbf{1}, \quad E_I \circ \sigma_j = e^{2\pi\sqrt{-1}(i_1-1)(j-1)/N}E_{I'}$$

for $i,j = 1,\dots,N$ and $I = (i_1,\dots,i_k) \in \Lambda_N$, $k \geq 2$ where $I' = (i_2,\dots,i_k)$.

Proof. Let $I = (i_1, \ldots, i_k) \in \Lambda_N$. Assume that $k \geq 2$. By (3.4),

$$E_I \circ \sigma_i = e^{2\pi\sqrt{-1}\delta_I \circ \sigma_i}, \quad \delta_I \circ \sigma_i = \frac{1}{N} \sum_{J \in \{1,\ldots,N\}^k} (I|J) \cdot \chi_{K_J} \circ \sigma_i.$$

Note $\sigma_i(x) \in K_J \quad \Leftrightarrow \quad x \in \sigma_i^{-1}(K_J) \quad \Leftrightarrow \quad j_1 = 1$ when $J = (j_1, \ldots, j_k)$. Hence the sum in the rhs in the above remains only J which satisfies $j_1 = i$, and $\chi_{K_J} \circ \sigma_i = \chi_{K_{J'}}$ when $J' = (j_2, \ldots, j_k)$. By definition of $(I|J)$ in (3.2),

$$(I|J) = \sum_{l=1}^{k} (i_1 - 1)(j_1 - 1) = (i_1 - 1)(j_1 - 1) + (I'|J').$$

Therefore

$$\begin{aligned} \delta_I \circ \sigma_i = & \frac{1}{N} \left\{ \sum_{J' \in \{1,\ldots,N\}^{k-1}} \Big((i_1 - 1)(i - 1) + (I'|J') \Big) \chi_{K_{J'}} \right\} \\ = & \frac{1}{N} \left\{ (i_1 - 1)(i - 1) \times \sum_{J' \in \{1,\ldots,N\}^{k-1}} \chi_{K_{J'}} + \sum_{J' \in \{1,\ldots,N\}^{k-1}} (I'|J') \chi_{K_{J'}} \right\} \\ = & \frac{1}{N} \left\{ (i_1 - 1)(i - 1)\mathbf{1} + \delta_{I'} \right\}. \end{aligned}$$

Hence

$$E_I \circ \sigma_i = e^{2\pi\sqrt{-1}(i_1-1)(i-1)/N} e^{2\pi\sqrt{-1}\delta_{I'}} = e^{2\pi\sqrt{-1}(i_1-1)(i-1)/N} E_{I'}.$$

In the same way, $k = 1$ case follows. $\square$

Corollary 3.4.

$$E_I \circ \sigma_J = \begin{cases} e^{2\pi\sqrt{-1}(I_1|J)/N} E_{I_2} & (|I| > |J|,\ I = (I_1, I_2),\ |I_1| = |J|), \\ e^{2\pi\sqrt{-1}(I|J)/N} \mathbf{1} & (|I| = |J|), \\ e^{2\pi\sqrt{-1}(I|J_1)/N} \mathbf{1} & (|I| < |J|,\ J = (J_1, J_2),\ |I| = |J_1|), \end{cases}$$

where $|I|$ *means the length of* $I \in \Lambda_N$. *Specially* $E_1 \circ \sigma_J = E_1 = \mathbf{1}$ *for each* $J \in \{1, \ldots, N\}^k, k \geq 1$.

Proposition 3.3 shows that the transformation by the branching function system brings the multiplication of a phase factor belonging to $\mathbf{Z}_N$ and transformation rule gives the recursive construction of the basis $\{E_I : I \in \Lambda_N\}$ from single function $\mathbf{1}$ by $\{\sigma_i\}_{i=1}^N$. In this sense, we can always construct these basis without information of a representation of the Cuntz algebra.

4 Examples of orthonormal basis by orthogonal matrix

We give examples in Theorem 1.1 (i) (b) where $g \in U(N)$ is chosen in the orthogonal group $O(N)$.

4.1 Construction of orthogonal matrix associated with contraction ratios and basis

Assume that $(K, \{\sigma_i\}_{i=1}^N)$ is a self-similar set with contraction ratios $\{\lambda_i\}_{i=1}^N$ and the similarity dimension D. We construct orthogonal matrix $g \in O(N)$ under the condition (2.13). If $N = 2$, then put

$$g = \begin{pmatrix} \lambda_1^{D/2} & \lambda_2^{D/2} \\ \lambda_2^{D/2} & -\lambda_1^{D/2} \end{pmatrix}.$$

Assume that $N \geq 3$. In this case, put $g = (g_{ij}) \in O(N)$ by

$$g_{1j} \equiv \lambda_j^{D/2},$$

$$g_{2j} \equiv \begin{cases} \varepsilon_1 \lambda_1^{D/2} & (j = 1), \\ -\varepsilon_1' \lambda_j^{D/2} & (2 \leq j \leq N), \end{cases}$$

$$g_{ij} \equiv \begin{cases} 0 & (1 \leq j \leq i-2), \\ \varepsilon_{i-1} \lambda_{i-1}^{D/2} & (j = i-1), \qquad (3 \leq i \leq N), \\ -\varepsilon_{i-1}' \lambda_j^{D/2} & (i \leq j \leq N), \end{cases}$$

for $j = 1, \ldots, N$ where

$$\varepsilon_i \equiv \sqrt{\frac{\lambda_{i+1}^D + \cdots + \lambda_N^D}{\lambda_i^D \left(\lambda_i^D + \cdots + \lambda_N^D\right)}}, \quad \varepsilon_i' \equiv \sqrt{\frac{\lambda_i^D}{\left(\lambda_i^D + \cdots + \lambda_N^D\right)\left(\lambda_{i+1}^D + \cdots + \lambda_N^D\right)}}$$

for $1 \leq i \leq N-1$.

Then H_I in Theorem 1.1 (i) (b) by g in the above is given by

$$H_1 = \mathbf{1}, \quad H_j = \varepsilon_{j-1} \chi_{K_j} - \varepsilon_{j-1}' \sum_{l=j+1}^{N} \chi_{K_l}, \quad H_{J,j} = \lambda_J^{-D/2} \cdot H_j \circ \sigma_J^{-1} \tag{4.1}$$

for $j = 2, \ldots, N$ and $J \in \{1, \ldots, N\}^k$. In this way, we have a complete orthonormal basis $\{H_I : I \in \Lambda_N\}$ by g. We see that every coefficients in (4.1) is real. Therefore $\{H_I : I \in \Lambda_N\}$ is a basis consisting of real valued functions on K.

Proposition 4.1. *$\{H_I : I \in \Lambda_N\}$ defined in (4.1) is a complete orthonormal basis of $L_2(K, \mu^D; \mathbf{R})$ which is the Hilbert space of all real valued square integrable functions on K.*

4.2 Interval dynamical systems

We consider interval dynamical systems as examples of self-similar set with the integral similarity dimension 1.

In case of $N = 2$, let $K = [0,1]$ and $f \equiv \{f_1, f_2\}$, $f_1(x) \equiv \frac{1}{2}x$ and $f_2(x) \equiv \frac{1}{2}x + \frac{1}{2}$. Then $([0,1], \{f_1, f_2\})$ is a self similar set with common contraction ratio $\frac{1}{2}$ and the similarity dimension 1. In this case, we have so called *the Haar basis* on the interval by (4.1), the following $g \in U(2)$ and $\varepsilon_1, \varepsilon_1^{'}$:

$$g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \varepsilon_1 = \sqrt{2}, \quad \varepsilon_1^{'} = \sqrt{2}.$$

This construction is same in the case of § 3.1 for $N = 2$. The Hausdorff representation of $\mathcal{O}_2$ by f is equivalent to that in § 3.2.

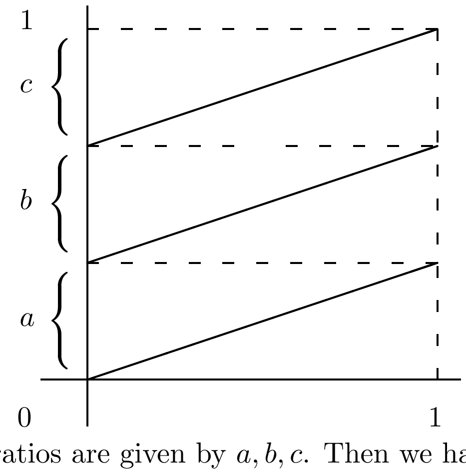
We generalize this as follows: Fix $0 < a < 1$. Let $f = \{f_1, f_2\}$ on $[0,1]$ by $f_1(x) \equiv ax, \quad f_2(x) \equiv (1-a)x + a$. Then their contraction ratios are a and $1-a$. Data for construction of basis in § 4.1 are given by

$$g = \begin{pmatrix} \sqrt{a} & \sqrt{1-a} \\ \sqrt{1-a} & -\sqrt{a} \end{pmatrix}, \quad \varepsilon_1 = \sqrt{\frac{1-a}{a}}, \quad \varepsilon_1^{'} = \sqrt{\frac{a}{1-a}}.$$

Note that the representation $(L_2[0,1], \pi_f)$ of $\mathcal{O}_2$ by f is $GP(z)$ for $z = (\sqrt{a}, \sqrt{1-a}) \in S(\mathbf{C}^2)$. Hence $a \in (0,1)$ is corresponded to an equivalence class of irreducible representations of $\mathcal{O}_2$ and representations associated with any two different points in the open interval $(0,1)$ are inequivalent by Theorem 2.2 (iii).

Next we treat $N = 3$ case. Fix $0 < a, b, c < 1$ such that $a + b + c = 1$. Consider a branching function system $f = \{f_i\}_{i=1}^{3}$ on $[0,1]$ defined by

$$f_1(x) \equiv ax, \quad f_2(x) \equiv bx + a, \quad f_3(x) \equiv cx + 1 - c.$$



Then their contraction ratios are given by a, b, c. Then we have

$$g = \begin{pmatrix} \sqrt{a} & \sqrt{b} & \sqrt{c} \\ \sqrt{1-a} & -\sqrt{\frac{ab}{1-a}} & -\sqrt{\frac{ac}{1-a}} \\ 0 & \sqrt{\frac{c}{b+c}} & -\sqrt{\frac{b}{b+c}} \end{pmatrix},$$

$$\varepsilon_1 = \sqrt{\frac{1-a}{a}}, \quad \varepsilon_1^{'} = \sqrt{\frac{a}{1-a}}, \quad \varepsilon_2 = \sqrt{\frac{c}{b(b+c)}}, \quad \varepsilon_2^{'} = \sqrt{\frac{b}{c(b+c)}}.$$

The representation $(L_2[0,1], \pi_f)$ of $\mathcal{O}_3$ by f is $GP(z)$ for $z = (\sqrt{a}, \sqrt{b}, \sqrt{c}) \in S(\mathbf{C}^3)$. $a = b = c = \frac{1}{3}$ if and only if $(L_2[0,1], \pi_f)$ is equivalent to the representation of $\mathcal{O}_3$ in § 3.3.

We treat a representation of $\mathcal{O}_2$ arising from an interval dynamical system by a real quadratic transformation in [QM1]. Interval dynamical systems by piecewise linear transformations are treated in [CKR] in detail.

4.3 The Sierpiński gasket II

In § 4.1, when $N \geq 3$ and $\lambda_j^D = \frac{1}{N}$ for each $j = 1, \ldots, N$, we have the following orthogonal matrix $g = (g_{ij})$ for construction of basis:

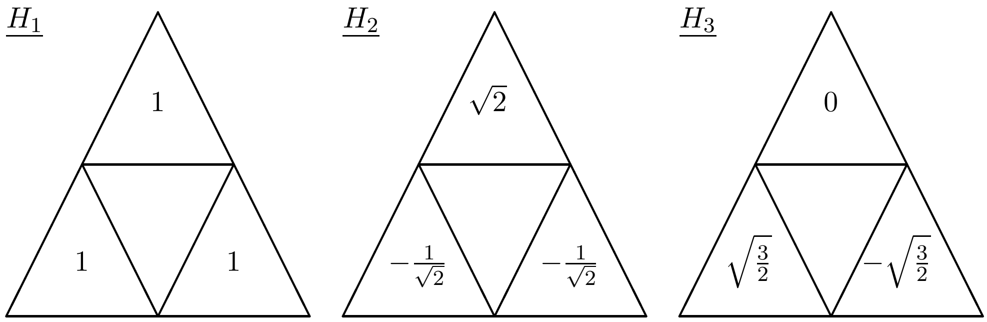
$$g_{1j} = \frac{1}{\sqrt{N}} \qquad (j = 1, \ldots, N), \qquad g_{2j} = \begin{cases} \sqrt{\dfrac{N-1}{N}} & (j = 1), \\[2ex] \dfrac{-1}{\sqrt{(N-1)N}} & (2 \leq j \leq N), \end{cases}$$

$$g_{ij} = \begin{cases} 0 & (1 \leq j \leq i-2), \\[2ex] \sqrt{\dfrac{N-i+1}{N-i+2}} & (j = i-1), \\[2ex] \dfrac{-1}{\sqrt{(N-i+1)(N-i+2)}} & (i \leq j \leq N), \end{cases}$$

for $3 \leq i \leq N$. For example, when $N = 3$,

$$g = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}, \quad \varepsilon_1 = \sqrt{2}, \quad \varepsilon_1^{'} = \frac{1}{\sqrt{2}}, \quad \varepsilon_2 = \varepsilon_2^{'} = \sqrt{\frac{3}{2}}.$$

The basis on the Sierpiński gasket are given as follows by the same style of illustration in § 3.3:



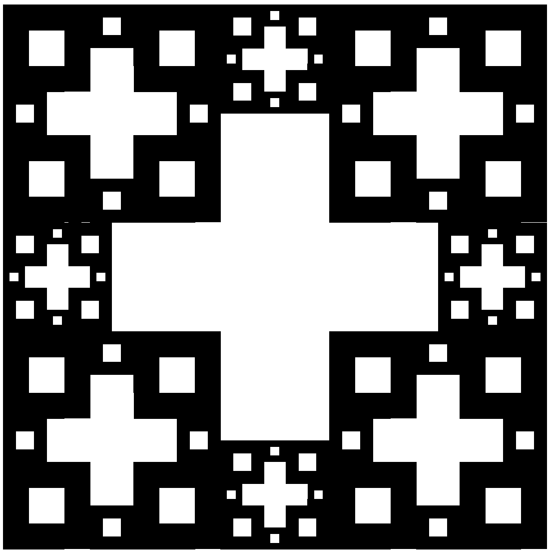
Note that basis in the above are different in that in § 3.3 clearly. This difference occurs that of two constructions and parameter g.

4.4 A kind of the Sierpiński carpet

We consider a self-similar set $(K, \{f_i\}_{i=1}^8)$ with two different contraction ratios $\lambda_i = \frac{2}{5}$, $i = 1, 2, 3, 4$ and $\lambda_i = \frac{1}{5}$, $i = 5, 6, 7, 8$. Define $f_i : [0, 1]^2 \to [0, 1]^2$;

$$f_i(x) \equiv \frac{1}{5}(2x + c_i) \quad (i = 1, 2, 3, 4), \qquad f_i(x) \equiv \frac{1}{5}(x + c_i) \quad (i = 5, 6, 7, 8),$$

where $(c_i)_{i=1}^8 = ((0,\, 0), (3,\, 0), (3,\, 3), (0,\, 3), (2,\, 0), (4,\, 2), (2,\, 4), (0,\, 2))$. The following figure(correctly, the limit of this kind of figure) is a unique set K such that $K = f_1(K) \cup \cdots \cup f_8(K)$:



The similarity dimension D of $(K, \{f_i\}_{i=1}^8)$ is a solution of the equation $4(2^D + 1) = 5^D$.

Ingredients $g \in O(8)$, $\{\varepsilon_i, \varepsilon_i^{'}\}_{i=1}^7$ to construct orthonormal basis of $L_2(K, \mu^D)$ by (4.1) are given by

$$g = \begin{pmatrix} a & a & a & a & b & b & b & b \\ \varepsilon_1 a & -\varepsilon_1^{'} a & -\varepsilon_1^{'} a & -\varepsilon_1^{'} a & -\varepsilon_1^{'} b & -\varepsilon_1^{'} b & -\varepsilon_1^{'} b & -\varepsilon_1^{'} b \\ 0 & \varepsilon_2 a & -\varepsilon_2^{'} a & -\varepsilon_2^{'} a & -\varepsilon_2^{'} b & -\varepsilon_2^{'} b & -\varepsilon_2^{'} b & -\varepsilon_2^{'} b \\ 0 & 0 & \varepsilon_3 a & -\varepsilon_3^{'} a & -\varepsilon_3^{'} b & -\varepsilon_3^{'} b & -\varepsilon_3^{'} b & -\varepsilon_3^{'} b \\ 0 & 0 & 0 & \varepsilon_4 a & -\varepsilon_4^{'} b & -\varepsilon_4^{'} b & -\varepsilon_4^{'} b & -\varepsilon_4^{'} b \\ 0 & 0 & 0 & 0 & \varepsilon_5 b & -\varepsilon_5^{'} b & -\varepsilon_5^{'} b & -\varepsilon_5^{'} b \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_6 b & -\varepsilon_6^{'} b & -\varepsilon_6^{'} b \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_7 b & -\varepsilon_7^{'} b \end{pmatrix},$$

$$\varepsilon_1 = \frac{\sqrt{1-a^2}}{a}, \quad \varepsilon_1^{'} = \frac{a}{\sqrt{1-a^2}}, \quad \varepsilon_2 = \frac{\sqrt{1-2a^2}}{a\sqrt{1-a^2}}, \quad \varepsilon_2^{'} = \frac{a}{\sqrt{(1-a^2)(1-2a^2)}},$$

$$\varepsilon_3 = \frac{\sqrt{1-3a^2}}{a\sqrt{1-2a^2}}, \quad \varepsilon_3^{'} = \frac{a}{\sqrt{(1-2a^2)(1-3a^2)}}, \quad \varepsilon_4 = \frac{2b}{a\sqrt{1-3a^2}}, \quad \varepsilon_4^{'} = \frac{a}{2b\sqrt{1-3a^2}},$$

$$\varepsilon_5 = \frac{\sqrt{3}}{2b}, \quad \varepsilon_5^{'} = \frac{1}{2\sqrt{3}b}, \quad \varepsilon_6 = \frac{\sqrt{2}}{\sqrt{3}b}, \quad \varepsilon_6^{'} = \frac{1}{\sqrt{3}b}, \quad \varepsilon_7 = \varepsilon_7^{'} = \frac{1}{\sqrt{2}b}$$

where $a \equiv (\frac{2}{5})^{D/2}$, $b \equiv (\frac{1}{5})^{D/2}$. In this case, the Hausdorff representation $(L_2(K, \mu^D), \pi_f)$ of $\mathcal{O}_8$ on K is $GP(z)$ for $z = (a, a, a, a, b, b, b, b) \in S(\mathbf{C}^8)$.

Appendix

A Proof of Theorem 2.2

Results in Theorem 2.2 are included in [Ka1]. For convenience, we show the proof of Theorem 2.2 here.

A.1 The standard representation of $\mathcal{O}_N$

We show examples of GP representations in order to prove Theorem 2.2. Let $l_2(\mathbf{N})$ be the Hilbert space with the canonical basis $\{e_n : n \in \mathbf{N}\}$, $\mathbf{N} = \{1, 2, 3, \ldots\}$, and make the following representation $(l_2(\mathbf{N}), \pi_S)$ of the Cuntz algebra $\mathcal{O}_N$ which is called *the standard representation of* $\mathcal{O}_N$([AK]):

$$\pi_S(s_i)e_n \equiv e_{N(n-1)+i} \quad (i = 1, \ldots, N,\, n \in \mathbf{N}). \tag{A.1}$$

From this, we have $\pi_S(s_i)^* e_{N(n-1)+j} = \delta_{ij} e_n$ for $i, j = 1, \ldots, N$ and $n \in \mathbf{N}$. Note that this is a permutative representation of $\mathcal{O}_N$ by [BJ]. By (A.1),

$$\pi_S(s_1)e_1 = e_1. \tag{A.2}$$

Proposition A.1. *Let $(l_2(\mathbf{N}), \pi_S)$ be the standard representation of $\mathcal{O}_N$ which is defined in (A.1).*

(i) $(l_2(\mathbf{N}), \pi_S, e_1)$ *is $GP(z)$ for $z = (1, 0, \ldots, 0) \in S(\mathbf{C}^N)$.*

(ii) $(l_2(\mathbf{N}), \pi_S)$ *is irreducible.*

Proof. (i) Since (A.2) and the cyclicity of $(l_2(\mathbf{N}), \pi_S)$, the statement holds by the comparison with (2.2) in Definition 2.1. (ii) By [BJ], this representation is irreducible. □

Lemma A.2. *Let $(\mathcal{H}, \pi, \Omega)$ be $GP(1, 0, \ldots, 0)$. Then $\{\pi(s_I)\Omega : I \in \Lambda_N\}$ is an orthonormal family in $\mathcal{H}$.*

Proof. We denote $|I|$ the length of $I \in \Lambda_N$. Put $I, J \in \Lambda_N$.
If $|I| = |J|$, then $<\pi(s_I)\Omega|\pi(s_J)\Omega> = \delta_{I,J}$.
Assume that $|I| > |J|$. Put $I = (i_1, \ldots, i_{k+l})$ and $J = (j_1, \ldots, j_k)$, $k, l \geq 1$. Then

$$<\pi(s_I)\Omega|\pi(s_J)\Omega> = \delta_{I_1,J} <\pi(s_{I_2})\Omega|\Omega> = \delta_{I_1,J} <\pi(s_{I_2})\Omega|\pi(s_1^l)\Omega>$$

where we use $\pi(s_1^l)\Omega = \Omega$ and $I_1 = (i_1, \ldots, i_k)$ and $I_2 = (i_{k+1}, \ldots, i_{k+l})$. By choice of I, $I_2 \neq (1, \ldots, 1)$. Hence $<\pi(s_{I_2})\Omega|\pi(s_1^l)\Omega> = \delta_{I_2,J_0} = 0$ where $J_0 \equiv (\underbrace{1, \ldots, 1}_{l})$. Therefore

$< \pi(s_I)\Omega|\pi(s_J)\Omega >= 0$. From this, we obtain $< \pi(s_I)\Omega|\pi(s_J)\Omega >= \delta_{I,J}$ for each $I, J \in \Lambda_N$. $\Box$

Proposition A.3. *$GP(1, 0, \ldots, 0)$ is unique up to unitary equivalences.*

Proof. Let $(\mathcal{H}, \pi, \Omega)$ be $GP(1, 0, \ldots, 0)$. Because Ω is a cyclic vector, the linear span of $\{\pi(s_I s_J^*)\Omega : I, J\}$ is dense in $\mathcal{H}$. Because $\pi(s_1)\Omega = \Omega$, $\{\pi(s_I s_J^*)\Omega : I, J\} = \{\pi(s_I)\Omega : I \in \Lambda_N\}$. Hence $\{\pi(s_I)\Omega : I \in \Lambda_N\}$ is a complete orthonormal basis of $\mathcal{H}$ by Lemma A.2. In this way, we know that any $GP(1, 0, \ldots, 0)$ always has such basis. Therefore we have a natural unitary between any two representations which are $GP(1, 0, \ldots, 0)$. Therefore $GP(1, 0, \ldots, 0)$ is unique up to unitary equivalences. $\Box$

A.2 Existence, uniqueness and irreducibility

For $g = (g_{ij}) \in U(N)$,

$$\alpha_g(s_i) \equiv \sum_{j=1}^{N} g_{ji} s_j \quad (i = 1, \ldots, N) \tag{A.3}$$

gives an action α of $U(N)$ on $\mathcal{O}_N$. Note that for a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ and $g \in U(N)$, $(\mathcal{H}, \pi \circ \alpha_g)$ is a representation of $\mathcal{O}_N$, too. For $z = (z_1, \ldots, z_N) \in S(\mathbf{C}^N)$, put $s(z) \equiv z_1 s_1 + \cdots + z_N s_N$.

Let $(l_2(\mathbf{N}), \pi_S)$ be the standard representation of $\mathcal{O}_N$ in (A.1).

Lemma A.4. *Fix $z = (z_1, \ldots, z_N) \in S(\mathbf{C}^N)$. Put $g = (g_{ij}) \in U(N)$ such that $g_{1j} = \overline{z_j}$ for $j = 1, \ldots, N$. Then $(l_2(\mathbf{N}), \pi_S \circ \alpha_g, e_1)$ is $GP(z)$.*

Proof. Since $(l_2(\mathbf{N}), \pi_S)$ is irreducible, $(l_2(\mathbf{N}), \pi_S \circ \alpha_g)$ is irreducible, too. Therefore $(l_2(\mathbf{N}), \pi_S \circ \alpha_g)$ is cyclic. Hence

$$(\pi_S \circ \alpha_g)(s(z))e_1 = (\pi_S \circ \alpha_g)(\alpha_{g^*}(s_1))e_1 = \pi_S(s_1)e_1 = e_1.$$

Therefore $(l_2(\mathbf{N}), \pi_S \circ \alpha_g, e_1)$ satisfies the condition of $GP(z)$. $\Box$

By this lemma, we finish to show the existence of $GP(z)$ for each $z \in S(\mathbf{C}^N)$.

Lemma A.5. *If $(\mathcal{H}, \pi, \Omega)$ is $GP(z)$, then there is $g \in U(N)$ such that $\{(\pi \circ \alpha_{g^*})(s_I)\Omega : I \in \Lambda_N\}$ is a complete orthonormal basis of $\mathcal{H}$.*

Proof. Choose $g \in U(N)$ such that $g_{1j} = \overline{z_j}$ for $j = 1, \ldots, N$. Put $\pi' \equiv \pi \circ \alpha_{g^*}$. By Lemma A.4, we know that $(\mathcal{H}, \pi', \Omega)$ satisfies $\pi'(s_1)\Omega = \Omega$. Since $(\mathcal{H}, \pi, \Omega)$ is cyclic, $(\mathcal{H}, \pi', \Omega)$ is $GP(1, 0, \ldots, 0)$. Then $(\mathcal{H}, \pi', \Omega)$ has a complete orthonormal basis $\{\pi'(s_I)\Omega : I \in \Lambda_N\} = \{(\pi \circ \alpha_{g^*})(s_I)\Omega : I \in \Lambda_N\}$ by the proof of Proposition A.3. $\Box$

By Lemma A.5 and the similar argument in the proof of Proposition A.3, $GP(z)$ is unique up to unitary equivalences. By Lemma A.1, Lemma A.4 and uniqueness, $GP(z)$ is irreducible. We finish to show Theorem 2.2 (i), (ii).

A.3 Equivalence

Lemma A.6. *Let* $(\mathcal{H}, \pi)$ *be a representation of* $\mathcal{O}_N$*. For* $z, z' \in S(\mathbf{C}^N)$*, assume that there are cyclic vectors* $\Omega, \Omega' \in \mathcal{H}$ *which satisfy* $\pi(s(z))\Omega = \Omega$ *and* $\pi(s(z'))\Omega' = \Omega'$*. Then* $z = z'$*.*

Proof. Note $<\Omega|\Omega'> = <\pi(s(z)))\Omega|\pi(s(z'))\Omega'> = <z|z'><\Omega|\Omega'>$. Hence $<\Omega|\Omega'> = (<z|z'>)^n <\Omega|\Omega'>$ for each $n \in \mathbf{N}$. $<\Omega|\Omega'> \neq 0$ if and only if $z = z'$ since $z, z' \in S(\mathbf{C}^N)$.

Assume that $z \neq z'$. Then $<\Omega|\Omega'> = 0$. Because of cyclicity and Lemma A.5, there is $J \in \{1, \ldots, N\}^k$ such that $<\pi(s_J)\Omega|\Omega'> \neq 0$. If $J = (j_1, \ldots, j_l)$, $l \geq 1$, then

$$<\pi(s_J)\Omega|\Omega'> = <\pi(s_J)\Omega|\{\pi(s(z'))\}^l\Omega'> = z'_{j_1} \cdots z'_{j_l} <\Omega|\Omega'> = 0.$$

This contradicts the choice of J. Hence $z = z'$. ☐

Lemma A.7. *If* $z \in S(\mathbf{C}^N)$*,* $z \neq (1, 0, \ldots, 0)$*, then* $GP(z)$ *is not equivalent to the standard representation.*

Proof. Let $(\mathcal{H}, \pi, \Omega)$ be $GP(z)$ for $z \neq (1, 0, \ldots, 0)$. Assume that $(\mathcal{H}, \pi)$ is equivalent to the standard representation $(l_2(\mathbf{N}), \pi_S)$. By unitary equivalence, we can identify $(\mathcal{H}, \pi) = (l_2(\mathbf{N}), \pi_S)$ and $\Omega \in l_2(\mathbf{N})$. By applying Lemma A.6 for $\Omega' = e_1 \in l_2(\mathbf{N})$ and $z' = (1, 0, \ldots, 0)$, we have $z = z'$. This is contradiction. Therefore $(\mathcal{H}, \pi)$ is not equivalent to the standard representation. ☐

Proof of Theorem 2.2 (iii).

It is sufficient to show that $GP(z) \not\sim GP(z')$ when $z \neq z'$. Assume that $z, z' \in S(\mathbf{C}^N)$ and $z \neq z'$.

We show by reduction to absurdity. Assume that $GP(z)$ and $GP(z')$ are unitarily equivalent. Then there is a unitary U such that $\mathrm{Ad}U \circ \pi_2 = \pi_1$ where $(\mathrm{Ad}U \circ \pi_2)(x) \equiv U\pi_2(x)U^*$ for $x \in \mathcal{O}_N$. Put $g, g' \in U(N)$ such that $g_{1j} = \overline{z_j}$, $g'_{1j} = \overline{z'_j}$, $j = 1, \ldots, N$, $\pi_1 \sim \pi_S \circ \alpha_g$ and $\pi_2 \sim \pi_S \circ \alpha_{g'}$ by Lemma A.4. Then there is a unitary U' such that $\mathrm{Ad}U' \circ \pi_S \circ \alpha_{g'} = \pi_S \circ \alpha_g$. From this, $\mathrm{Ad}U' \circ \pi_S \circ \alpha_{g'g^*} = \pi_S$. On the other hand,

$$(g'g^*)_{1j} = \sum_{k=1}^{N} g'_{1k} g^*_{kj} = \sum_{k=1}^{N} \overline{z'_k} g^*_{kj} \quad (j = 1, \ldots, N).$$

By Lemma A.7, $(\mathcal{H}, \pi_S \circ \alpha_{g'g^*})$ is equivalent to the standard representation $\Leftrightarrow (g'g^*)_{1j} = \delta_{1j}$ $\Leftrightarrow z = z'$. This is contradiction. Therefore $GP(z) \not\sim GP(z')$. ☐

About GP representations in more detail, see [Ka1].

B Another proof of the equivalence of Hausdorff representations

We give another proof of the following proposition in [MSW] by results of GP representations.

Proposition B.1. *(The Kakutani's dichotomy theorem)([MSW])*
Let K_i be a self-similar set with contraction ratios $\{\lambda_{i,j}\}_{j=1}^N$ and the similarity dimension D_i for $i = 1,2$, respectively. Assume that $(L_2(K_i, \mu^{D_i}), \pi_i)$ is the Hausdorff representation of $\mathcal{O}_N$ on K_i for $i = 1,2$, respectively. Then $(L_2(K_1, \mu^{D_1}), \pi_1)$ and $(L_2(K_2, \mu^{D_2}), \pi_2)$ are equivalent if and only if the following conditions hold:

$$(\lambda_{1,j})^{D_1} = (\lambda_{2,j})^{D_2} \quad (j = 1, \ldots, N). \tag{B.1}$$

Proof. By Theorem 2.8 (i), $(L_2(K_i, \mu^{D_i}), \pi_i, \mathbf{1})$ is $GP(z^{(i)})$ for $z^{(i)} = ((\lambda_{i,1})^{D_i/2}, \ldots, (\lambda_{i,N})^{D_i/2})$ for $i = 1,2$, respectively. By Theorem 2.2 (iii), $(L_2(K_1, \mu^{D_1}), \pi_1) \sim (L_2(K_2, \mu^{D_2}), \pi_2)$ $\Leftrightarrow GP(z^{(1)}) \sim GP(z^{(2)}) \Leftrightarrow ((\lambda_{1,1})^{D_1/2}, \ldots, (\lambda_{1,N})^{D_1/2}) = ((\lambda_{2,1})^{D_2/2}, \ldots, (\lambda_{2,N})^{D_2/2}) \Leftrightarrow$ $(\lambda_{1,j})^{D_1} = (\lambda_{2,j})^{D_2}$ for each $j = 1, \ldots, N$. □

Remark B.2. The equivalence does not imply that only the condition $D_1 = D_2$ holds in Proposition B.1. A set of all invariants of Hausdorff representations is a proper subset of invariants of GP representations with 1-cycle by Theorem 2.8. In fact, $GP(z)$ by $z = (1, 0, \ldots, 0)$ never appear as Hausdorff representation. In this sense, Hausdorff representation is a kind of GP representation as equivalence class of representations of the Cuntz algebra.

C States, Hausdorff representations and Hausdorff measures

Let $(K, \{\sigma_i\}_{i=1}^N)$ be a self-similar set with contraction ratios $\{\lambda_i\}_{i=1}^N$ and the similarity dimension D. Put X_N a compact Hausdorff space which consists of infinite sequences of symbols $1, \ldots, N$. Then there is the canonical surjective continuous map φ from X_N onto K associated with $\{\sigma_i\}_{i=1}^N$([Dev]). From this, we have an injective $*$-homomorphism φ_* from $C(K)$ to $C(X_N)$ where $C(K)$ and $C(X_N)$ are C*-algebras of complex valued continuous functions on K and X_N, respectively. Note that $\mathcal{O}_N$ has an abelian subalgebra $\mathcal{A} \equiv C^* < \{s_I s_I^* : |I| = k,\, k \geq 1\} >$ and there is the canonical isomorphism $C(X_N) \cong \mathcal{A}$ which is derived from $\chi_{K_J} \mapsto s_J s_J^*$. Therefore we have a $*$-embedding $\hat{\varphi}$ of $C(K)$ into $\mathcal{O}_N$ by identifying $C(X_N)$ and $\mathcal{A}$. The following is commutative:

$$\begin{array}{ccc} C(K) & \overset{\hat{\varphi}}{\hookrightarrow} & \mathcal{O}_N \\ {\scriptstyle \varphi_*} \downarrow & & \uparrow \\ C(X_N) & \cong & \mathcal{A}. \end{array}$$

Theorem C.1. (i) *The Hausdorff representation of* $\mathcal{O}_N$ *on* $(K, \{\sigma_i\}_{i=1}^N)$ *is equivalent to the GNS-representation by a state* ρ *defined by*

$$\rho(s_I s_J^*) \equiv (\lambda_I \lambda_J)^{D/2} \tag{C.1}$$

where $I, J \in \cup_{k\geq 0}\{1,\ldots,N\}^k$ *and the notation is same in Theorem 2.2 (iv).*

(ii) *Under the identification of* $C(K)$ *as a subalgebra of* $\mathcal{O}_N$ *by* $\hat{\varphi}$*, the restriction of* ρ *in (C.1) on* $C(K)$ *is a state* ρ_0 *of* $C(K)$ *which is given by*

$$\rho_0(\phi) = \int_K \phi(x)\, d\mu^D(x) \quad (\phi \in C(K)).$$

(iii) ρ *in (C.1) is pure.*

Proof. Put $(L_2(K,\mu^D), \pi_\sigma)$ the Hausdorff representation of $\mathcal{O}_N$ on $(K, \{\sigma_i\}_{i=1}^N)$. Let ρ be a state of $\mathcal{O}_N$ defined by $\rho(\cdot) \equiv < \mathbf{1}|\pi_\sigma(\cdot)\mathbf{1} >$. By Theorem 2.2 (iv) and Theorem 2.8, (i) follows. Note that any element in $C(K)$ is approximated by step functions over $\{K_J : J \in \{1,\ldots,N\}^k,\ k\geq 1\}$. If $I = J$, then $\rho(s_I s_I^*) = \lambda_I^D$. When $\phi = \sum_I a_I \chi_{K_I}$ is a step function, then

$$\rho(\phi) = \sum_I a_I \rho(\chi_{K_I}) = \sum_I a_I \rho(s_I s_I^*) = \sum_I a_I \lambda_I^D = \int_K \phi(x)\, d\mu^D(x) = \rho_0(\phi).$$

Hence $\rho(\phi) = \rho_0(\phi)$ for each $\phi \in C(K)$. We obtain (ii). Since any Hausdorff representation is irreducible by Theorem 2.8 (ii), ρ is pure. (iii) is proved. □

By Theorem C.1, the Hausdorff representation is uniquely determined by ρ up to unitary equivalences.

Specially, if every contraction ratio of $(K, \{\sigma_i\}_{i=1}^N)$ is same, then we have

$$\rho(s_I s_J^*) = \frac{1}{N^{(|I|+|J|)/2}}.$$

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