Representations of the Cuntz algebra $\mathcal{O}_3$ arising from real cubic transformations

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Abstract

We construct representations of the Cuntz algebra $\mathcal{O}_3$ from real cubic transformations on closed intervals. By intertwining relations of transformations, we have those of operators of representations of $\mathcal{O}_2$ and $\mathcal{O}_3$. By these relations, we show that such representations are unitarily equivalent to barycentric representations.

1 Introduction

We study representations of the Cuntz algebra $\mathcal{O}_N$ naturally arising from dynamical systems with non injective transformations. In [6], we show that a representation of $\mathcal{O}_2$ arising from a real quadratic transformation $x^2 - 2$ on a closed interval $[-2, 2]$ is equivalent to one of GP representation in [5] by using intertwining relations of operators corresponding to semiconjugacies of dynamical systems. This means that representations of $\mathcal{O}_N$ arising from non linear transformations are not always hard to treat.

In this paper, we consider a real cubic transformation T_3 on a closed interval $[-1, 1]$ defined by

$$T_3(x) \equiv 4x^3 - 3x \quad (x \in [-1, 1]). \tag{1.1}$$

Put a representation $(L_2[-1, 1], \pi_t)$ of $\mathcal{O}_3$ by

$$(\pi_t(s_i)\phi)(x) \equiv m_i(x)\phi(T_3(x)) \quad (i = 1, 2, 3) \tag{1.2}$$

for $\phi \in L_2[-1, 1]$ where $m_i(x) \equiv \chi_{D_i}(x)\sqrt{|12x^2 - 3|}$, χ_{D_i} the characteristic function on D_i, $i = 1, 2, 3$, $D_1 = [-1, -1/2]$, $D_2 = [-1/2, 1/2]$, $D_3 = [1/2, 1]$ and s_1, s_2, s_3 are generators of $\mathcal{O}_3$. Put $\mathbf{N} = \{1, 2, 3, \ldots\}$.

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Theorem 1.1 *The representation* $(L_2[-1,1],\pi_t)$ *in (1.2) is unitarily equivalent to the following representation* $(l_2(\mathbf{N}),\pi')$ *of* $\mathcal{O}_3$*:*

$$\frac{1}{\sqrt{3}}\sum_{i=1}^{3} e^{2\pi\sqrt{-1}\tau_{ij}}\pi'(s_j)e_n \equiv e_{3(n-1)+i} \quad (i=1,2,3,\ n\in\mathbf{N})$$

where

$$\tau_{ij} \equiv \frac{(i-1)(j-1)}{3} \quad (i,j=1,2,3). \tag{1.3}$$

Specially, $(L_2[-1,1],\pi_t)$ *is irreducible.*

$$\Omega(x) = \frac{1}{\sqrt{\pi}}\frac{1}{(1-x^2)^{1/4}}$$

is unique positive normalized eigen function of $\pi_t(s_1)+\pi_t(s_2)+\pi_t(s_3)$ *with eigen value* $\sqrt{3}$*.*

In § 2, we review the representation theory of the Cuntz algebra. In § 3, we show intertwining relations between cubic transformations and piecewise linear transformations by the cosine function. In § 4, we show Theorem 1.1 by using of the intertwining relation derived in § 3. In § 5, we compare these results with those in [6] and discuss about remaining problems.

2 Representations of $\mathcal{O}_N$

For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([3]), that is, it is a C*-algebra which is universally generated by generators $s_1,\ldots,s_N$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad (i,j=1,\ldots,N), \qquad \sum_{i=1}^{N} s_i s_i^* = I. \tag{2.1}$$

In this paper, any representation means a unital $*$-representation. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1,\ldots,S_N$ on an infinite dimensional Hilbert space which satisfy (2.1) in order to construct a representation of $\mathcal{O}_N$.

2.1 The barycentric representation of $\mathcal{O}_N$

In order to characterize the representation in (1.2), we introduce a representation of $\mathcal{O}_N$ which is defined by an eigen equation.

Definition 2.1 *$(\mathcal{H}, \pi, \Omega)$ is a barycentric representation of $\mathcal{O}_N$ if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with a cyclic vector Ω which satisfies the following condition:*

$$\frac{1}{\sqrt{N}}\pi(s_1 + \cdots + s_N)\Omega = \Omega.$$

Proposition 2.2 *Let $(\mathcal{H}, \pi, \Omega)$ be a barycentric representation of $\mathcal{O}_N$.*

(i) *The representation $(\mathcal{H}, \pi)$ exists uniquely up to unitary equivalences.*

(ii) *$(\mathcal{H}, \pi)$ is irreducible.*

Proof. See Lemma A.3 (i),(ii),(iii). ∎

The barycentric representation of $\mathcal{O}_N$ is the GP representation by the parameter $\left(\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}\right)$([5]). The naming of "barycentric" is that a representation of $\mathcal{O}_N$ on a simplex Δ_{N-1} which is equivalent to the barycentric representation, is naturally defined by using the barycenter of Δ_{N-1}([8]).

Proposition 2.3 *The representation $(l_2(\mathbf{N}), \pi', e_1)$ in Theorem 1.1 is the barycentric representation of $\mathcal{O}_3$.*

Proof. See Corollary A.4. ∎

2.2 Isometries arising from transformations on measure spaces

We give a method of construction of representation of $\mathcal{O}_N$ from a branching function system over a measure space([2, 6, 7]). We introduce an easy method to construct partial isometries from maps on a measure space.

Let (X, μ) be a measure space and $Y \subset X$ a measurable subset of X.

Definition 2.4 (i) *$RN(Y, X)$ is the set of measurable maps on Y defined by*

$$RN(Y,X) \equiv \left\{ f : Y \to X \;\middle|\; \begin{array}{c} f \text{ is injective and} \\ \text{there exists } \Phi_f \text{ and } \Phi_f > 0 \text{ a.e. } Y \end{array} \right\}$$

where Φ_f is the Radon-Nikodým derivative of $\mu \circ f$ with respect to μ on Y.

(ii) *$RN_{loc}(X) \equiv \bigcup_{Y \subset X} RN(Y, X)$ where Y is taken from all measurable subsets of X.*

For $f, g \in RN_{loc}(X)$, we denote the domain and the range of f by $D(f)$ and $R(f)$, respectively and $(f \circ g)(x) \equiv f(g(x))$ when $D(f) \supset R(g)$.

Lemma 2.5 (i) *If $f \in RN_{loc}(X)$, then $f^{-1} \in RN_{loc}(X)$.*

(ii) *For $f, g \in RN_{loc}(X)$, $f \circ g \in RN_{loc}(X)$ when $D(f) \supset R(g)$.*

(iii) *For $f, g \in RN_{loc}(X)$, $\Phi_{f\circ g} = \Phi_g \cdot ((\Phi_f) \circ g)$.*

Proof. These are easily checked by direct computation and property of Radon-Nikodým derivative. ∎

Note that $RN_{loc}(X)$ is a groupoid by Lemma 2.5 (ii).

Definition 2.6 *For $f \in RN_{loc}(X)$, define an operator $S(f) : L_2(X, \mu) \to L_2(X, \mu)$ by*

$$(S(f)\phi)(x) \equiv \begin{cases} \{\Phi_f\,(f^{-1}(x))\}^{-1/2}\,\phi(f^{-1}(x)) & (\textit{when } x \in R(f)\,), \\ 0 & (\textit{otherwise}) \end{cases} \tag{2.2}$$

for $\phi \in L_2(X, \mu)$ and $x \in X$.

We simply denote

$$S(f) = \mathcal{L}_f M_{\Phi_f^{-1/2}} \tag{2.3}$$

where M_g is the multiplication operator of $g \in L_\infty(X, \mu)$ and $\mathcal{L}_f$ is defined by

$$(\mathcal{L}_f\phi)(x) \equiv \chi_{R(f)}(x)\phi(f^{-1}(x)) \quad (x \in X)$$

and χ_Y is the characteristic function on $Y \subset X$.

Lemma 2.7 (i) *For $f \in RN_{loc}(X)$, $S(f)$ is a partial isometry on $L_2(X, \mu)$ with the initial projection $M_{\chi_{D(f)}}$ and the range projection $M_{\chi_{R(f)}}$.*

(ii) *For $f \in RN_{loc}(X)$, $S(f)^* = S(f^{-1})$.*

(iii) *$S(id_Y) = M_{\chi_Y}$.*

(iv) *For $f, g \in RN_{loc}(X)$, $\mathcal{L}_f\mathcal{L}_g = \mathcal{L}_{f\circ g}$ when $D(f) \supset R(g)$.*

(v) *For $f \in RN_{loc}(X)$ and $g \in L_\infty(X)$, $\mathcal{L}_f M_g = M_{\chi_{R(f)}} M_{g\circ f^{-1}}\mathcal{L}_f$.*

Proof. (i), (iv) and (v) follow by simple computation. (ii) Since $R(f^{-1}) = D(f)$, $D(f^{-1}) = R(f)$ and the property of Radon-Nikodým derivative, it follows. (iii) Since $\Phi_{id_Y} = \chi_Y$, it follows. ∎

Let $\mathrm{PIso}(L_2(X,\mu))$ be the groupoid of partial isometries on $L_2(X,\mu)$ by usual product of operators.

Lemma 2.8 *A map $S : RN_{loc}(X) \to \mathrm{PIso}(L_2(X,\mu))$ defined in (2.2) is a groupoid homomorphism, that is,*

$$S(f)S(g) = S(f \circ g) \tag{2.4}$$

when $f, g \in RN_{loc}(X)$ and $D(f) \supset R(g)$.

Proof. By Lemma 2.5 (iii) and Lemma 2.7 (iv),(v),

$$S(f)S(g) = \mathcal{L}_f M_{\Phi_f^{-1/2}} \mathcal{L}_g M_{\Phi_g^{-1/2}} = \mathcal{L}_{f\circ g} M_{((\Phi_f)\circ g)^{-1/2}} M_{\Phi_g^{-1/2}} = S(f \circ g).$$

∎

Remark that $f \circ g$ in rhs of (2.4) is only the composition of two transformations f and g but not special product of them. By Lemma 2.8, we see that a map S realizes the iteration of transformations on a measure space as the product of operators on a Hilbert space naturally.

The notion of branching function system was introduced in [2] in order to construct a representation of $\mathcal{O}_N$ from a family of maps.

Let $N \geq 2$.

Definition 2.9 (i) *$f = \{f_i\}_{i=1}^N$ is a branching function system over (X,μ) if $f_i \in RN_{loc}(X)$ and $D(f_i) = X$, put $R_i \equiv R(f_i)$, then $\mu(R_i \cap R_j) = 0$, $i \neq j$ and $\mu\left(X \setminus \bigcup_{i=1}^N R_i\right) = 0$.*

(ii) *F is the coding map of a branching function system $f = \{f_i\}_{i=1}^N$ on (X,μ) if $(F \circ f_i)(x) = x$ for a.e. $x \in X$ and $i = 1, \ldots, N$.*

Proposition 2.10 *For a branching function system $f = \{f_i\}_{i=1}^N$ on (X,μ),*

$$\pi_f(s_i) \equiv S(f_i) \quad (i = 1, \ldots, N),$$

defines a representation $(L_2(X,\mu), \pi_f)$ of $\mathcal{O}_N$.

Proof. It is straightforward to show that $S(f_1), \ldots, S(f_N)$ satisfy (2.1) by Lemma 2.7, Lemma 2.8 and Definition 2.9. ∎

We show several examples of branching function system associated with 1-dimensioanl dynamical systems in [8].

Lemma 2.11 *Let* (Y, ν) *be another measure space. Assume that* φ *from* X *to* Y *is a measurable bijection a.e.* X *and* Y *and its Radon-Nikodým derivative* Φ_φ *is positive a.e.* X*.*

(i) *An operator* $U_\varphi : L_2(X, \mu) \to L_2(Y, \nu)$ *defined by* $U_\varphi \equiv \mathcal{L}_\varphi M_{\Phi_\varphi^{1/2}}$ *is a unitary.*

(ii) *For* $f \in RN_{loc}(X)$*,* $\varphi \circ f \circ \varphi^{-1} \in RN_{loc}(Y)$ *and* $U_\varphi S(f) U_\varphi^* = S(\varphi \circ f \circ \varphi^{-1})$*.*

(iii) *If* $f = \{f_i\}_{i=1}^N$ *is a branching function system over* (X, μ)*, then* $\{\varphi \circ f_i \circ \varphi^{-1}\}_{i=1}^N$ *is a branching function system over* (Y, ν)*, too.*

Proof. These are easily proved by direct computation. ∎

Proposition 2.12 *Let* $f = \{f_i\}_{i=1}^N$ *and* $g = \{g_i\}_{i=1}^N$ *be branching function systems over measure spaces* (X, μ) *and* (Y, ν)*, respectively. Assume that there is a map* φ *from* X *to* Y *which satisfies the assumption in Lemma 2.11 and map identities* $g_i = \varphi \circ f_i \circ \varphi^{-1}$ *for* $i = 1, \ldots, N$ *hold. Then* $(L_2(X, \mu), \pi_f)$ *and* $(L_2(Y, \nu), \pi_g)$ *are unitarily equivalent.*

Proof. By Lemma 2.11 (ii), we can show $S(g_i) = U_\varphi S(f_i) U_\varphi^*$ for $i = 1, \ldots, N$. These relations induce a unitary equivalence between π_f and π_g in Proposition 2.10 immediately. ∎

3 The cosine function as intertwining map for cubic transformations

In order to show an intertwining relation of T_3 in (1.1), we prepare other transformations.

On a closed interval $[-1, 1]$, we consider transformations $V_3, C : [-1, 1] \to [-1, 1]$ by

$$V_3(x) \equiv \begin{cases} 3x + 2 & \left(-1 \le x \le -\frac{1}{3}\right), \\ -3x & \left(-\frac{1}{3} \le x \le \frac{1}{3}\right), \\ 3x - 2 & \left(\frac{1}{3} \le x \le 1\right), \end{cases} \qquad C(x) \equiv \cos \pi x.$$

We denote $T \equiv T_3$ and $V \equiv V_3$ here. Then the following intertwining relation holds:

$$C \circ V = T \circ C. \tag{3.1}$$

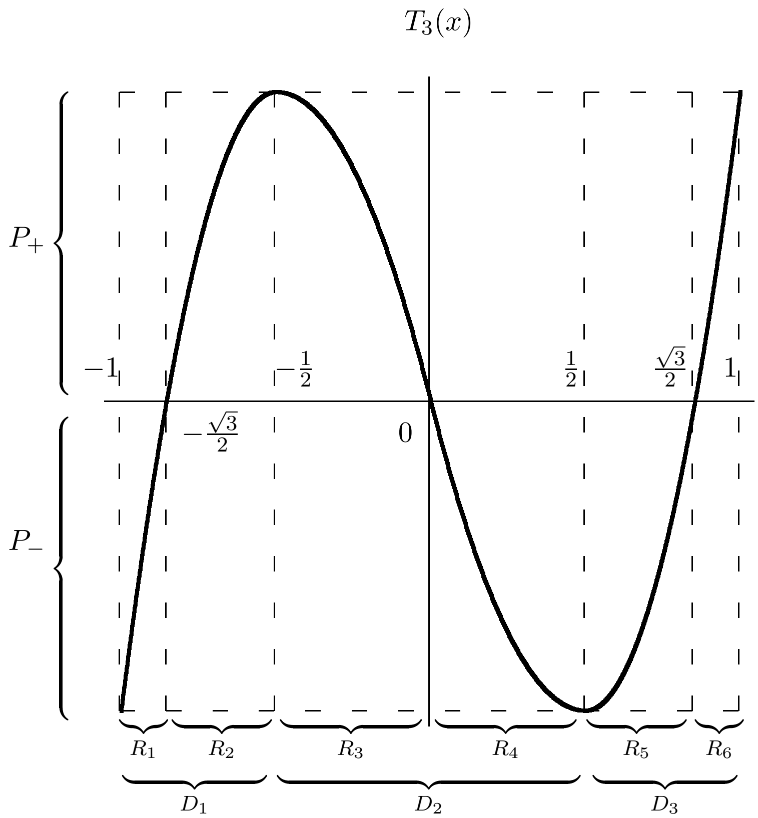
Remark that C is not injective. We can not say that T and V are conjugate. C is a semiconjugacy between V and T ([4]).

We divide the interval $[-1,1]$ by the following families of subintervals of $[-1,1]$:

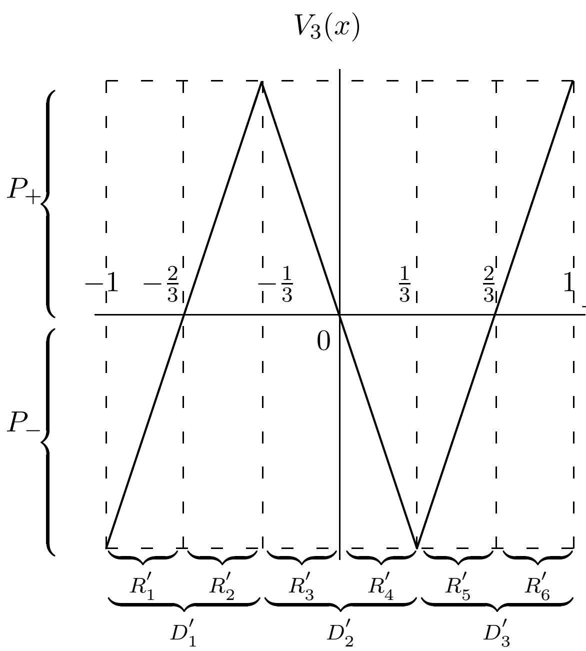
$$\begin{aligned}
\left(D_i'\right)_{i=1}^{3} &\equiv \left(\left[-1,-\tfrac{1}{3}\right], \left[-\tfrac{1}{3},\tfrac{1}{3}\right], \left[\tfrac{1}{3},1\right]\right), \\
(D_i)_{i=1}^{3} &\equiv \left(\left[-1,-\tfrac{1}{2}\right], \left[-\tfrac{1}{2},\tfrac{1}{2}\right] \left[\tfrac{1}{2},1\right]\right), \\
\left(R_i'\right)_{i=1}^{6} &\equiv \left(\left[-1,-\tfrac{2}{3}\right], \left[-\tfrac{2}{3},-\tfrac{1}{3}\right], \left[-\tfrac{1}{3},0\right], \left[0,\tfrac{1}{3}\right], \left[\tfrac{1}{3},\tfrac{2}{3}\right], \left[\tfrac{2}{3},1\right]\right), \\
(R_i)_{i=1}^{6} &\equiv \left(\left[-1,-\tfrac{\sqrt{3}}{2}\right], \left[-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2}\right], \left[-\tfrac{1}{2},0\right], \left[0,\tfrac{1}{2}\right], \left[\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right], \left[\tfrac{\sqrt{3}}{2},1\right]\right), \\
(P_-,P_+) &\equiv ([-1,0],\,[0,1]).
\end{aligned}$$

The relations between these divisions are followings:

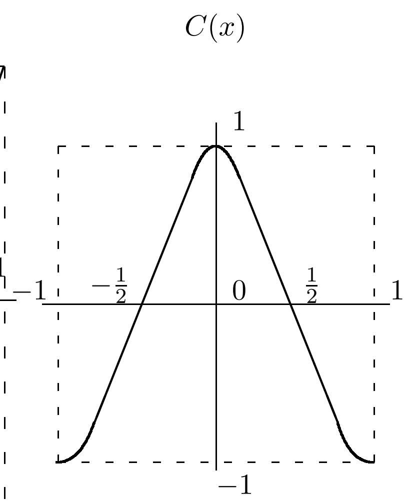
$$\begin{aligned}
\left(D_i'\right)_{i=1}^{3} &= \left(R_1'\cup R_2',\, R_3'\cup R_4',\, R_5'\cup R_6'\right), \\
(P_-,P_+) &= \left(R_1'\cup R_2'\cup R_3',\, R_4'\cup R_5'\cup R_6'\right).
\end{aligned}$$



$T_3(x)$
P_+
P_-
-1
$-\frac{1}{2}$
$\frac{1}{2}$
$\frac{\sqrt{3}}{2}$
1
$-\frac{\sqrt{3}}{2}$
0
R_1 R_2 R_3 R_4 R_5 R_6
D_1 D_2 D_3



$V_3(x)$
P_+
P_-
-1
$-\frac{2}{3}$
$-\frac{1}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
1
0
R'_1 R'_2 R'_3 R'_4 R'_5 R'_6
D'_1 D'_2 D'_3



$C(x)$
1
-1
$-\frac{1}{2}$
0
$\frac{1}{2}$
1
-1

The images of transformations T, V, C on these divisions are followings:

$$\Big(T(R_i)\Big)_{i=1}^{6} = (P_-, P_+, P_+, P_-, P_-, P_+),$$

$$\Big(C(R_i')\Big)_{i=1}^{6} = (D_1, D_2, D_3, D_3, D_2, D_1),$$

$$\Big(V(R_i')\Big)_{i=1}^{6} = (P_+, P_-, P_-, P_+, P_+, P_-).$$

For these divisions, we have the following six relations by (3.1):

$$C|_{P_+} \circ V|_{D_1'} = T|_{D_1} \circ C|_{P_-}, \qquad C|_{P_+} \circ V|_{D_2'} = T|_{D_3} \circ C|_{P_+},$$

$$C|_{P_-} \circ V|_{D_1'} = T|_{D_2} \circ C|_{P_-}, \qquad C|_{P_+} \circ V|_{D_3'} = T|_{D_2} \circ C|_{P_+},$$

$$C|_{P_-} \circ V|_{D_2'} = T|_{D_3} \circ C|_{P_-}, \qquad C|_{P_-} \circ V|_{D_3'} = T|_{D_1} \circ C|_{P_+}.$$

Note that restrictions $C|_{P_\pm}$, $V|_{D_i'}$ and $T|_{D_i}$, $i = 1, 2, 3$ are injective. Let

$$c_\pm \equiv (C|_{P_\pm})^{-1}, \quad v_i \equiv (V|_{D_i'})^{-1}, \quad t_i \equiv (T|_{D_i})^{-1} \tag{3.2}$$

for $i = 1, 2, 3$. Then

$$c_- \circ t_1 = v_1 \circ c_+, \qquad c_+ \circ t_3 = v_2 \circ c_+,$$

$$c_- \circ t_2 = v_1 \circ c_-, \qquad c_+ \circ t_2 = v_3 \circ c_+,$$

$$c_- \circ t_3 = v_2 \circ c_-, \qquad c_+ \circ t_1 = v_3 \circ c_-.$$

From these, we have the following relations:

$$v_1 = \begin{cases} c_- \circ t_1 \circ c_+^{-1} & (\text{on } P_+), \\ c_- \circ t_2 \circ c_-^{-1} & (\text{on } P_-), \end{cases} \qquad v_2 = \begin{cases} c_+ \circ t_3 \circ c_+^{-1} & (\text{on } P_+), \\ c_- \circ t_3 \circ c_-^{-1} & (\text{on } P_-), \end{cases}$$

$$v_3 = \begin{cases} c_+ \circ t_2 \circ c_+^{-1} & (\text{on } P_+), \\ c_+ \circ t_1 \circ c_-^{-1} & (\text{on } P_-). \end{cases} \tag{3.3}$$

4 Representations of $\mathcal{O}_3$ arising from dynamical systems

We show intertwining relations of operators in this section.

4.1 Intertwining relations of operators

By (3.2), $c \equiv \{c_+, c_-\}$ is a branching function system over $[-1,1]$ for $N=2$ and both $t \equiv \{t_1, t_2, t_3\}$ and $v \equiv \{v_1, v_2, v_3\}$ are those over $[-1,1]$ for $N=3$. Hence operators $\{S(c_\pm)\}$ gives a representation of $\mathcal{O}_2$ on $L_2[-1,1]$ and $\{S(t_i)\}_{i=1}^3$, $\{S(v_i)\}_{i=1}^3$ give those of $\mathcal{O}_3$ on $L_2[-1,1]$, too.

Then $\pi_t(s_i) = S(t_i)$ for $i = 1,2,3$ where π_t is the representation of $\mathcal{O}_3$ on $L_2[-1,1]$ in (1.2) and

$$\begin{cases} (S(v_1)\phi)(x) = & 3^{1/2}\chi_{[-1,-1/3]}(x)\phi(3x+2), \\ (S(v_2)\phi)(x) = & 3^{1/2}\chi_{[-1/3,1/3]}(x)\phi(-3x), \\ (S(v_3)\phi)(x) = & 3^{1/2}\chi_{[1/3,1]}(x)\phi(3x-2), \end{cases} \tag{4.1}$$

$$\begin{cases} (S(c_+)\phi)(x) = & \sqrt{\pi \sin\pi|x|}\,\chi_{[0,1]}(x)\phi(\cos\pi x), \\ (S(c_-)\phi)(x) = & \sqrt{\pi \sin\pi|x|}\,\chi_{[-1,0]}(x)\phi(\cos\pi x), \end{cases}$$

for $\phi \in L_2[-1,1]$ and $x \in [-1,1]$.

By Lemma 2.8 and (3.3), we have the following operator relations:

$$\begin{cases} S(v_1) = & S(c_-)S(t_1)S(c_+)^* + S(c_-)S(t_2)S(c_-)^*, \\ S(v_2) = & S(c_+)S(t_3)S(c_+)^* + S(c_-)S(t_3)S(c_-)^*, \\ S(v_3) = & S(c_+)S(t_2)S(c_+)^* + S(c_+)S(t_1)S(c_-)^*. \end{cases} \tag{4.2}$$

Note that (4.2) is a relation of representations π_c, π_t, π_v and it is derived from (3.1) essentially. We explain these relations in § 5. Let

$$A_1 \equiv \sum_{i=1}^3 S(v_i), \quad A_2 \equiv \sum_{i=1}^3 S(t_i), \quad B \equiv S(c_+) + S(c_-).$$

Then we have the following intertwining relations of operators:

$$B^* A_1 = A_2 B^*. \tag{4.3}$$

Let $\mathbf{1}$ be the constant function of $[-1,1]$ with value 1. Note $A_1\mathbf{1} = \sqrt{3}\mathbf{1}$. Let $\Omega_0 \equiv B^*\mathbf{1}$. By (4.3),

$$A_2\Omega_0 = A_2 B^*\mathbf{1} = B^* A_1\mathbf{1} = \sqrt{3}B^*\mathbf{1} = \sqrt{3}\Omega_0.$$

Hence Ω_0 is an eigen function of A_2 with eigen value $\sqrt{3}$.

Lemma 4.1 *The operator $S(t_1)+S(t_2)+S(t_3)$ has an eigen vector $(S(c_+)^*+S(c_-)^*)\mathbf{1}$ with eigen value $\sqrt{3}$.*

Let Ω be the normalization of Ω_0. Then

$$\Omega(x) = \frac{1}{\sqrt{\pi}}\frac{1}{(1-x^2)^{1/4}} \quad (x \in [-1,1]).$$

Theorem 4.2 *$(L_2[-1,1], \pi_t, \Omega)$ is the barycentric representation.*

Proof. By Lemma 4.1, it is sufficient to show the cyclicity of $(L_2[-1,1], \pi_t)$. By (1.2), $\pi_t(s_I)\Omega = M_{\chi_{D_I}} 3^{k/2}\Omega$. Hence $K \equiv \{\chi_{D_I} \cdot \Omega : I \in \{1,2,3\}^k,\ k \geq 1\} \subset \pi_t(\mathcal{O}_3)\Omega$. Since $\Omega(x) > 0$ for any $x \in (0,1)$, $L_2[-1,1] = \overline{\mathrm{Lin} < K >} \subset \pi_t(\mathcal{O}_3)\Omega$. Therefore $(L_2[-1,1], \pi_t)$ is cyclic. Hence $(L_2[-1,1], \pi_t, \Omega)$ is the barycentric representation. ∎

Proof of Theorem 1.1. By Proposition 2.3, $(l_2(\mathbf{N}), \pi^{'})$ in Theorem 1.1 is equivalent to the barycentric representation. By Proposition 2.2 (i) and Theorem 4.2, the statement in Theorem 1.1 follows. ∎

4.2 Generalization

We generalize our result slightly. Assume $a, b \in \mathbf{R}$, $a < b$. Let

$$\varphi_{(a,b)} : [-1,1] \to [a,b]; \quad \varphi_{(a,b)}(x) \equiv \frac{b-a}{2}x + \frac{a+b}{2}.$$

Put $T_3^{(a,b)} \equiv \varphi_{(a,b)} \circ T_3 \circ (\varphi_{(a,b)})^{-1}$, $\quad V_3^{(a,b)} \equiv \varphi_{(a,b)} \circ V_3 \circ (\varphi_{(a,b)})^{-1}$. Then

$$V_3^{(a,b)}(x) = 2\alpha\left|\frac{1}{\alpha}x - \beta\right| + \alpha(\beta-1),$$

$$T_3^{(a,b)}(x) = \frac{4}{\alpha^2}x^3 - \frac{12\beta}{\alpha}x^2 + 3(4\beta^2-1)x - 4\alpha\beta(\beta^2-1) \qquad (4.4)$$

where $\alpha = \frac{b-a}{2}$, $\beta = \frac{b+a}{b-a}$. For example, $T_3^{(-1,1)}(x) = T_3(x)$, $T_3^{(-2,2)}(x) = x^3 - 3x$ and $T_3^{(0,1)}(x) = x(4x-3)^2$.

By Proposition 2.12, for any $a, b \in \mathbf{R}$, $a < b$, the representation $(L_2[a,b], \pi_t^{(a,b)})$ of $\mathcal{O}_3$ arising from $T_3^{(a,b)}$ is given by

$$(\pi_t^{(a,b)}(s_i)\phi)(x) = m_i^{(a,b)}(x)\phi(T^{(a,b)}(x)) \quad (\phi \in L_2[a,b])$$

where

$$m_i^{(a,b)}(x) \equiv \chi_{D_i^{(a,b)}}(x) \cdot \sqrt{\left|\frac{12}{\alpha^2}x^2 - \frac{24\beta}{\alpha}x + 3(4\beta^2 - 1)\right|}$$

and $D_1^{(a,b)} = [a, \frac{3a+b}{4}]$, $D_2^{(a,b)} = [\frac{3a+b}{4}, \frac{a+3b}{4}]$, $D_3^{(a,b)} = [\frac{a+3b}{4}, b]$. Put

$$\Omega^{(a,b)}(x) \equiv \frac{1}{\sqrt{\pi}} \frac{1}{\{(3b - a - 2x)(b - 3a + 2x)\}^{1/4}}.$$

Then $(L_2[a,b], \pi_t^{(a,b)}, \Omega^{(a,b)})$ is the barycentric representation of $\mathcal{O}_3$.

5 Discussion

Our results in this paper is similar to those in [6]. Although, the intertwining relations (4.2) are more complicated than the former. Their abstractions are followings: Put $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$ the set of all unital $*$-representations of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$ for $N \geq 2$. By [6], there is a map

$$\zeta^{(2)} : \mathrm{Rep}(\mathcal{O}_2, \mathcal{H}) \times \mathrm{Rep}(\mathcal{O}_2, \mathcal{H}) \to \mathrm{Rep}(\mathcal{O}_2, \mathcal{H}); \quad (\pi', \pi) \mapsto \zeta_{\pi'}^{(2)}(\pi)$$

defined by

$$\begin{cases} \zeta_{\pi'}^{(2)}(\pi)(s_1) = & \pi'(s_1)\pi(s_1)\pi'(s_1)^* + \pi'(s_1)\pi(s_2)\pi'(s_2)^*, \\ \zeta_{\pi'}^{(2)}(\pi)(s_2) = & \pi'(s_2)\pi(s_1)\pi'(s_1)^* + \pi'(s_2)\pi(s_2)\pi'(s_2)^*. \end{cases}$$

On the other hand, by (4.2), there is a map

$$\zeta^{(3)} : \mathrm{Rep}(\mathcal{O}_3, \mathcal{H}) \times \mathrm{Rep}(\mathcal{O}_2, \mathcal{H}) \to \mathrm{Rep}(\mathcal{O}_3, \mathcal{H}); \quad (\pi', \pi) \mapsto \zeta_{\pi'}^{(3)}(\pi)$$

defined by

$$\begin{cases} \zeta_{\pi'}^{(3)}(\pi)(t_1) = & \pi'(s_1)\pi(t_1)\pi'(s_2)^* + \pi'(s_1)\pi(t_2)\pi'(s_1)^*, \\ \zeta_{\pi'}^{(3)}(\pi)(t_2) = & \pi'(s_2)\pi(t_3)\pi'(s_2)^* + \pi'(s_1)\pi(t_3)\pi'(s_1)^*, \\ \zeta_{\pi'}^{(3)}(\pi)(t_3) = & \pi'(s_2)\pi(t_2)\pi'(s_2)^* + \pi'(s_2)\pi(t_1)\pi'(s_1)^* \end{cases}$$

where $\{s_1, s_2\}$ and $\{t_1, t_2, t_3\}$ are generators of $\mathcal{O}_2$ and $\mathcal{O}_3$, respectively. Apparently, $\zeta^{(3)}$ is not a simple generalization of $\zeta^{(2)}$. In fact, we can not guess the case of $N = 4, 5, 6, \ldots$ for $\mathcal{O}_N$. Note that $\zeta^{(2)}$ and $\zeta^{(3)}$ are derived by intertwining relations of a quadratic transformation and a cubic transformation on a closed interval, respectively.

We can not yet make clear mechanism when "nice" intertwining relations occur. For example, we can not make representation theory from general cubic transformations $a_1x^3 + a_2x^2 + a_3x + a_4$ except $T_3^{(a,b)}$ in (4.4).

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Appendix A Proof of Proposition 2.2

Results in Proposition 2.2 are included in [5]. The barycentric representation is a kind of GP representation in [5]. Here we show the proof of Proposition 2.2 for convenience.

A.1 The standard representation of $\mathcal{O}_N$

We introduce the standard representation of $\mathcal{O}_N$ in order to prove Proposition 2.2. Let $l_2(\mathbf{N})$ be the Hilbert space with the canonical basis $\{e_n : n \in \mathbf{N}\}$, $\mathbf{N} = \{1, 2, 3, \ldots\}$, and make the following representation $(l_2(\mathbf{N}), \pi_S)$ of the Cuntz algebra $\mathcal{O}_N$ which is called *the standard representation of* $\mathcal{O}_N$([1]):

$$\pi_S(s_i)e_n \equiv e_{N(n-1)+i} \quad (i = 1, \ldots, N,\ n \in \mathbf{N}). \tag{A.1}$$

From this, we have $\pi_S(s_i)^* e_{N(n-1)+j} = \delta_{ij} e_n$ for $i, j = 1, \ldots, N$ and $n \in \mathbf{N}$. Note that this is a permutative representation of $\mathcal{O}_N$ by [2]. By (A.1),

$$\pi_S(s_1)e_1 = e_1. \tag{A.2}$$

Lemma A.1 $(l_2(\mathbf{N}), \pi_S)$ *is irreducible.*

Proof. By Theorem 2.7 in [2], this representation is irreducible. ∎

Proposition A.2 *For* $N \geq 2$*, put*

$$\Lambda_N \equiv \{1, \ldots, N\} \cup \bigcup_{k \geq 1} \left(\{1, \ldots, N\}^k \times \{2, \ldots, N\} \right).$$

Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{O}_N$. Assume that

$$\text{there is a unit cyclic vector } \Omega\in\mathcal{H} \text{ such that } \pi(s_1)\Omega=\Omega. \tag{A.3}$$

Then the followings hold:

(i) *$\{\pi(s_I)\Omega : I\in\Lambda_N\}$ is an orthonormal family in $\mathcal{H}$.*

(ii) *The representation of $\mathcal{O}_N$ which satisfies the condition (A.3) is unique up to unitary equivalences.*

Proof. (i) We denote $|I|$ the length of $I\in\Lambda_N$. Put $I,J\in\Lambda_N$.

If $|I|=|J|$, then $<\pi(s_I)\Omega|\pi(s_J)\Omega>=\delta_{I,J}$.

Assume that $|I|>|J|$. Put $I=(i_1,\ldots,i_{k+l})$ and $J=(j_1,\ldots,j_k)$, $k,l\geq 1$. Then

$$<\pi(s_I)\Omega|\pi(s_J)\Omega>= \delta_{I_1,J}<\pi(s_{I_2})\Omega|\Omega>=\delta_{I_1,J}<\pi(s_{I_2})\Omega|\pi(s_1^l)\Omega>$$

where we use $\pi(s_1^l)\Omega=\Omega$ and $I_1=(i_1,\ldots,i_k)$ and $I_2=(i_{k+1},\ldots,i_{k+l})$. By choice of I, $I_2\neq(1,\ldots,1)$. Hence $<\pi(s_{I_2})\Omega|\pi(s_1^l)\Omega>=\delta_{I_2,J_0}=0$ where $J_0\equiv(\underbrace{1,\ldots,1}_{l})$. Therefore $<\pi(s_I)\Omega|\pi(s_J)\Omega>=0$. From this, we obtain $<\pi(s_I)\Omega|\pi(s_J)\Omega>=\delta_{I,J}$ for each $I,J\in\Lambda_N$.

(ii) Put $\{1,\ldots,N\}^*\equiv\bigcup_{k\geq 0}\{1,\ldots,N\}^k$. Let $(\mathcal{H},\pi,\Omega)$ be a representation which satisfies (A.3). Because Ω is a cyclic vector, the linear span of $\{\pi(s_Is_J^*)\Omega : I,J\in\{1,\ldots,N\}^*\}$ is dense in $\mathcal{H}$. Because $\pi(s_1)\Omega=\Omega$, $\{\pi(s_Is_J^*)\Omega : I,J\in\{1,\ldots,N\}^*\}=\{\pi(s_I)\Omega : I\in\Lambda_N\}$. Hence $\{\pi(s_I)\Omega : I\in\Lambda_N\}$ is a complete orthonormal basis of $\mathcal{H}$ by (i). In this way, we know that any representation which satisfies the condition (A.3) always has such basis. Therefore we have a natural unitary between any two such representations. The uniqueness is proved. ∎

By this, we call the standard representation the representation which satisfies the condition (A.3), too.

A.2 The barycentric representation of $\mathcal{O}_N$

Recall Definition 2.1.

Lemma A.3 *Let $N\geq 2$.*

(i) *The barycentric representation of $\mathcal{O}_N$ exists.*

(ii) *The barycentric representation of $\mathcal{O}_N$ is unique up to unitary equivalences.*

(iii) *The barycentric representation of $\mathcal{O}_N$ is irreducible.*

(iv) *The barycentric representation of $\mathcal{O}_N$ is not equivalent to the standard representation.*

(v) *The GNS representation of $\mathcal{O}_N$ of the following state ρ*

$$\rho(s_J s_{J'}^*) = \frac{1}{N^{(|J|+|J'|)/2}} \quad (J, J' \in \{1, \ldots, N\}^k,\, k \geq 0) \tag{A.4}$$

is equivalent to the barycentric representation where we define $s_J = I$ when $J = \emptyset$. Specially, ρ is pure.

Proof. (i) Put $g = (g_{ij}) \in U(N)$ by

$$g_{ij} \equiv \frac{1}{\sqrt{N}} e^{2\pi\sqrt{-1}\tau_{ij}} \quad (i, j = 1, \ldots, N). \tag{A.5}$$

where $\tau_{ij} \equiv \frac{1}{N}(i-1)(j-1)$ for $i, j = 1, \ldots, N$. Then specially, $g_{1j} = \frac{1}{\sqrt{N}}$ for $i = 1, \ldots, N$. Let $(l_2(\mathbf{N}), \pi_S)$ be the standard representation of $\mathcal{O}_N$ in (A.1). Put $\pi' \equiv \pi_S \circ \alpha_g$ where α is an action of $U(N)$ on $\mathcal{O}_N$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^N g_{ji} s_j$ for $i = 1, \ldots, N$.

Since $(l_2(\mathbf{N}), \pi_S)$ is an irreducible representation of $\mathcal{O}_N$, $(l_2(\mathbf{N}), \pi')$ is too. Specially, $(l_2(\mathbf{N}), \pi')$ is cyclic. Because $\alpha_{g^*}(s_1) = \frac{1}{\sqrt{N}}(s_1 + \cdots + s_N)$,

$$\frac{1}{\sqrt{N}} \pi'(s_1 + \cdots + s_N) e_1 = (\pi \circ \alpha_g)(\alpha_{g^*}(s_1))\, e_1 = \pi_S(s_1) e_1 = e_1.$$

Hence $(l_2(\mathbf{N}), \pi', e_1)$ is a barycentric representation of $\mathcal{O}_N$.

(ii) Assume that $(\mathcal{H}_i, \pi_i, \Omega_i)$ is a barycentric representation of $\mathcal{O}_N$ for $i = 1, 2$. By the similar argument in (i), $(\mathcal{H}_i, \pi_i \circ \alpha_{g^*})$ is the standard representation of $\mathcal{O}_N$ with respect to $g \in U(N)$ in (i) for $i = 1, 2$. By uniqueness of the standard representation, $(\mathcal{H}_1, \pi_1 \circ \alpha_{g^*}) \sim (\mathcal{H}_2, \pi_2 \circ \alpha_{g^*})$. Hence $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$. Therefore the barycentric representation is unique up to unitary equivalences.

(iii) By (ii), the barycentric representation is equivalent to the action of the standard representation by suitable $g \in U(N)$. Since the standard representation is irreducible, the barycentric representation is too.

(iv) Let $(\mathcal{H}, \pi, \Omega)$ and $(\mathcal{H}', \pi', \Omega')$ be the standard representation and the barycentric representation of $\mathcal{O}_N$, respectively. Assume that $(\mathcal{H}, \pi)$ and

$(\mathcal{H}^{'}, \pi^{'})$ are unitarily equivalent. Then we can identify $(\mathcal{H}^{'}, \pi^{'}) = (\mathcal{H}, \pi)$. By assumption,

$$< \Omega|\Omega^{'} >=< \pi(s_1)\Omega|\pi(N^{-1/2}(s_1 + \cdots + s_N))\Omega^{'} >= N^{-1/2} < \Omega|\Omega^{'} > .$$

Hence $< \Omega|\Omega^{'} >= 0$. By cyclicity, there is $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $k \geq 1$, such that $< \Omega|\pi(s_J)\Omega^{'} >\neq 0$. However,

$$< \Omega|\pi(s_J)\Omega^{'} >=< \pi(s_1)^k\Omega|\pi(s_J)\Omega^{'} >= \left(\prod_{l=1}^{k} \delta_{1,j_l}\right) < \Omega|\Omega^{'} >= 0.$$

This is contradiction. Therefore $(\mathcal{H}, \pi)$ and $(\mathcal{H}^{'}, \pi^{'})$ are not unitarily equivalent.

(v) Let $(\mathcal{H}, \pi, \Omega)$ be the barycentric representation. Put $\rho(x) \equiv< \Omega|\pi(x)\Omega >$ for $x \in \mathcal{O}_N$. Then ρ satisfies the condition (A.4). Since the uniqueness of the GNS-representation and $(\mathcal{H}, \pi)$ is irreducible, the statement holds. ∎

Corollary A.4 *The representation* $(l_2(\mathbf{N}), \pi^{'}, e_1)$ *of* $\mathcal{O}_3$ *in Theorem 1.1 is the barycentric representation.*

Proof. By the proof of Lemma A.3 (i), it follows when $N = 3$. ∎

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