THE SECOND MAIN THEOREM FOR SMALL FUNCTIONS AND RELATED PROBLEMS

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ABSTRACT. We shall establish the following three results in more general forms.

(1) *The second main theorem for small functions.* Let f be a meromorphic function on the complex plane $\mathbb{C}$. Let $a_1, \dots, a_q$ be distinct meromorphic functions on $\mathbb{C}$. Assume that a_i are small with respect to f; i.e., $T(r,a) < o(T(r,f))$ $||$. Then the inequality

$$(q-2-\varepsilon)T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r,a_i,f) + o(T(r,f)) \ ||$$

holds for all $\varepsilon > 0$ (Corollary 1). Here as usual in Nevanlinna theory, the terms $T(r,f)$ and $\overline{N}(r,a_i,f)$ denote for the characteristic function and the truncated counting function, respectively.

(2) *Application to functional equations.* Let $\mathfrak{K}_{\mathbb{C}}$ be the field of meromorphic functions on $\mathbb{C}$. For a function $\psi : \mathbb{R}_{>0} \to \mathbb{R}$, put $\mathfrak{K}_{\mathbb{C}}^{\psi} = \{a \in \mathfrak{K}_{\mathbb{C}};\ T(r,a) \leq O(\psi(r)) \ ||\}$, which is a subfield of $\mathfrak{K}_{\mathbb{C}}$. Then the following holds: Let $F(x,y) \in \mathfrak{K}_{\mathbb{C}}^{\psi}[x,y]$ be a polynomial in two variables over $\mathfrak{K}_{\mathbb{C}}^{\psi}$. Assume that the curve $F(x,y) = 0$ over $\mathfrak{K}_{\mathbb{C}}^{\psi}$ has genus greater than one. If $\zeta_1, \zeta_2 \in \mathfrak{K}_{\mathbb{C}}$ satisfy the functional equation $F(\zeta_1, \zeta_2) = 0$, then both ζ_1 and ζ_2 are contained in $\mathfrak{K}_{\mathbb{C}}^{\psi}$ (Corollary 2).

(3) *Height inequality for curves over function fields.* Let k be a function field of one variable over $\mathbb{C}$. Let X be a smooth projective curve over k, let $D \subset X$ be a reduced divisor, let L be an ample line bundle on X and let $\varepsilon > 0$. Then we have

$$h_{k,K_X(D)}(P) \leq N_{k,S}^{(1)}(D,P) + d_k(P) + \varepsilon h_{k,L}(P) + O_{\varepsilon}(1)$$

for all $P \in X(\overline{k}) \backslash D$ (Theorem 5). Here the notations are introduced in [V1], [V3] (see also section 9).

Our proof uses Ahlfors' theory of covering surfaces and the geometry of the moduli space of q-pointed stable curves of genus 0.

1. INTRODUCTION

1.1. *Results.* One of the most interesting results in Value distribution theory is the Defect Relation obtained by R. Nevanlinna: If f is a non-constant meromorphic function on $\mathbb{C}$, then for arbitrary collection of distinct $a_1, \cdots, a_q \in \mathbb{P}^1$, the following defect relation holds

$$\sum_{i=1}^{q} (\delta(a_i,f) + \theta(a_i,f)) \leq 2. \tag{1.1.1}$$

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Here, as usual in Nevanlinna theory, the terms $\delta(a_i, f)$, $\theta(a_i, f)$ are defined by

$$\delta(a_i, f) = \liminf_{r\to\infty} \left\{ 1 - \frac{N(r, a_i, f)}{T(r, f)} \right\},$$

$$\theta(a_i, f) = \liminf_{r\to\infty} \left\{ \frac{N(r, a_i, f) - \overline{N}(r, a_i, f)}{T(r, f)} \right\},$$

hence satisfy $0 \leq \delta(a_i, f),\ \theta(a_i, f) \leq 1$ (for the definitions of the terms $T(r, f)$, $N(r, a_i, f)$ and $\overline{N}(r, a_i, f)$, see [N2], [H] and the following subsections).

Nevanlinna asked whether inequality (1.1.1) is still true when we replace constants a_i to arbitrary collection of distinct *small functions* a_i with respect to f (cf. [N1]). Here we say a meromorphic function a on $\mathbb{C}$ is a small function with respect to f if a satisfies the condition $T(r, a) < o(T(r, f))\ ||$, that is,

$$T(r, a) < o(T(r, f)) \quad \text{when } r \to \infty \text{ and } r \notin E$$

for an exceptional set $E \subset \mathbb{R}_{>0}$ with $\int_E d\log\log r < \infty$. Nevanlinna pointed out that the case $q = 3$ for this question is valid, because we may reduce the problem to the case that a_1, a_2, a_3 are all constants by using Möbius transform. But for the case $q > 3$, this method doesn't work.

Later, N. Steinmetz [St] and C. Osgood [O] proved

$$\sum_{i=1}^{q} \delta(a_i, f) \leq 2$$

for distinct small functions a_i. They used differential polynomials for f and a_i ($1 \leq i \leq q$), so it may be regarded as a generalization of Nevanlinna's original proof of (1.1.1). Though Nevanlinna used only the first order derivative of f, Steinmetz and Osgood used higher order derivatives of f. Hence the truncation level of the counting function is greater than one in general. See also C. Chuang [C] and G. Frank-G. Weissenborn [FW].

But it is hopped that the generalization of (1.1.1) for small functions is true with the form including the term $\theta(a_i, f)$ (cf. [D]).

In this paper, we give a solution for this problem by the following theorem.

Theorem 1. *Let Y and B be two Riemann surfaces with proper, surjective holomorphic maps $\pi_Y : Y \to \mathbb{C}$ and $\pi_B : B \to \mathbb{C}$. Assume that π_Y factors through π_B, i.e., there exists a proper, surjective holomorphic map $\pi : Y \to B$ such that $\pi_Y = \pi_B \circ \pi$. Let f be a meromorphic function on Y. Let $a_1, \cdots, a_q$ be distinct meromorphic functions on B. Assume that $f \neq a_i \circ \pi$ for $i = 1, \ldots, q$. Then for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that*

$$(1.1.2)\quad (q - 2 - \varepsilon)T(r, f) \leq \sum_{i=1}^{q} \overline{N}(r, a_i \circ \pi, f) + N_{\mathrm{ram}\,\pi_Y}(r) + C(\varepsilon)\left(\sum_{i=1}^{q} T(r, a_i) + N_{\mathrm{ram}\,\pi_B}(r)\right) + o\left(T(r, f)\right)\ ||.$$

Remarks 1.1.3. (1) The term $N_{\mathrm{ram}\,\pi_Y}(r)$ is called the ramification counting function for π_Y. In the case $Y = \mathbb{C}$ and $\pi_Y = \mathrm{id}_{\mathbb{C}}$, we have $N_{\mathrm{ram}\,\pi_Y}(r) = 0$. Similarly for $N_{\mathrm{ram}\,\pi_B}(r)$.

(2) We can also define the terms $T(r, f)$, $T(r, a_i)$ and $\overline{N}(r, a_i \circ \pi, f)$ for algebroid functions f, $a_1, \dots a_q$ by the similar way for meromorphic functions on $\mathbb{C}$. See the following subsections.

As an immediate corollary, applying the theorem to the case that $Y = B = \mathbb{C}$, $\pi_Y = \pi_B = \mathrm{id}_{\mathbb{C}}$ and all a_i are small functions with respect to f, we have the following.

Corollary 1. *Let f be a meromorphic function on $\mathbb{C}$ and let $a_1, \cdots, a_q$ be distinct meromorphic functions on $\mathbb{C}$. Assume that a_i are small functions with respect to f for $i = 1, \dots, q$. Then we have the second main theorem:*

$$(q - 2 - \varepsilon)T(r, f) \le \sum_{i=1}^{q} \overline{N}(r, a_i, f) + o\left(T(r, f)\right) \ || \quad \textit{for all} \quad \varepsilon > 0,$$

and the defect relation:

$$\sum_{i=1}^{q} \left(\delta(a_i, f) + \theta(a_i, f)\right) \le 2.$$

A special case of this corollary that f is a transcendental meromorphic function and a_i are rational functions was proved in [Y2]. The present paper is a development of the previous one.

We shall prove two other results. The first one is a corollary of the above theorem. This is suggested by A. Eremenko [E]. Let $\mathfrak{K}_Y$ and $\mathfrak{K}_B$ are the fields of meromorphic functions on Y and B, respectively. For a function $\psi : \mathbb{R}_{>0} \to \mathbb{R}$, we define the subset $\mathfrak{K}_B^{\psi}$ of $\mathfrak{K}_B$ by

$$\mathfrak{K}_B^{\psi} = \{a \in \mathfrak{K}_B;\ T(r, a) \le O(\psi(r)) \ || \ \}.$$

Here, as before, the symbol $||$ means that the inequality holds when $r \to \infty$ and $r \notin E$ for some exceptional set $E \subset \mathbb{R}_{>0}$ with $\int_E d \log \log r < \infty$. Then this $\mathfrak{K}_B^{\psi}$ is a subfield of $\mathfrak{K}_B$. For instance, if ψ is a bounded function, then $\mathfrak{K}_B^{\psi}$ is the field of constant functions, i.e., $\mathfrak{K}_B^{\psi} = \mathbb{C}$.

Let $F(x, y) \in \mathfrak{K}_B^{\psi}[x, y]$ be a polynomial in two variables with coefficients in $\mathfrak{K}_B^{\psi}$. For general $z \in B$, we denote by $F_z(x, y) \in \mathbb{C}[x, y]$ the polynomial obtained by taking the values at z of the meromorphic functions in the coefficients of $F(x, y)$.

Corollary 2. *Let Y, B and π be the same as Theorem 1 and let $\psi : \mathbb{R}_{>0} \to \mathbb{R}$. Let $F(x, y) \in \mathfrak{K}_B^{\psi}[x, y]$ be a polynomial such that the equation $F_z(x, y) = 0$ defines a curve of genus greater than one for general $z \in B$. Assume that $\zeta_1, \zeta_2 \in \mathfrak{K}_Y$ satisfy the functional equation $F(\zeta_1, \zeta_2) = 0$, where we consider $\mathfrak{K}_B^{\psi}$ as a subfield of $\mathfrak{K}_Y$ by the natural inclusion defined by π. Then we have*

$$T(r, \zeta_i) \le O\left(\psi(r) + N_{\mathrm{ram}\,\pi_Y}(r) + N_{\mathrm{ram}\,\pi_B}(r)\right) \ ||$$

for $i = 1, 2$.

If we apply this corollary to the case that $Y = B = \mathbb{C}$, $\pi_Y = \pi_B = \mathrm{id}_{\mathbb{C}}$ and ψ is a bounded function, then we conclude that $T(r, \zeta_i) < O(1) \ ||$ for $i = 1, 2$. Hence, both ζ_1 and ζ_2 are constant functions. This is equivalent to a result of E. Picard:

A holomorphic map $f : \mathbb{C} \to X$, where X is a curve of genus greater than one, is a constant map.

The next result is an algebraic analogue of the above theorem.

Theorem 2. *Let $q \geq 3$ be a positive integer. For all $\varepsilon > 0$, there exists a positive constant $C(q,\varepsilon) > 0$ with the following property: Let Y and B be compact Riemann surfaces with a proper, surjective holomorphic map $\pi : Y \to B$. Let f be a rational function on Y. Let $a_1, \cdots, a_q$ be distinct rational functions on B. Assume that $f \neq a_i \circ \pi$ for all $i = 1, \ldots, q$. Then we have*

$$(1.1.4) \quad (q-2-\varepsilon)\deg f \leq \sum_{1\leq i\leq q} \overline{n}(a_i \circ \pi, f, Y) + 2g(Y) + C(q,\varepsilon)\deg \pi \left(\max_{1\leq i\leq q} (\deg a_i) + g(B) + 1 \right).$$

Here we put $\overline{n}(a_i \circ \pi, f, Y) = \mathrm{card}\{z \in Y;\ f(z) = a_i \circ \pi(z)\}$ and denote by $g(\cdot)$ genus of curves. Using this theorem, we can prove the Height inequality for curves over function fields, which is a geometric analogue of a conjectural Diophantine inequality in Number theory ([V1], [V3]). Since the formulation of this Height inequality requires some notations, we postpone stating it until section 9 (cf. Theorem 5). A proof of Theorem 2 is similar to that of Theorem 1. But we don't need Nevanlinna theory in this case. The following scheme for the proof of Theorem 1 also works for that of Theorem 2, if we replace "$B(R)$" by "B". We also note that the inequality (1.1.4) is an analogue of unintegrated version of (1.1.2).

1.2. *Rough outline of proof of Theorem 1.* We use Ahlfors' theory of covering surfaces (cf. [A], [N2], [H]) and the geometry of the moduli space of q-pointed stable curves of genus 0 (cf. [Kn]), especially properties around the degenerate locus whose point corresponds to a degenerate, nodal curve.

We first divide $\mathbb{P}^1$ by a non-simple curve γ such that $\mathbb{P}^1 \backslash \gamma$ is finite disjoint union of sufficiently small Jordan domains D_k ($1 \leq k \leq K$), i.e., $\mathbb{P}^1\backslash\gamma = \cup_{1\leq k\leq K} D_k$. This division of $\mathbb{P}^1$ gives the division of $(\mathbb{P}^1)^q$ in the form of open subsets

$$D_{k_1} \times \cdots \times D_{k_q}, \quad 1 \leq k_i \leq K \quad \text{for} \quad 1 \leq i \leq q.$$

Then this division and the holomorphic map

$$\underline{a} = a_1 \times \cdots \times a_q : B \to (\mathbb{P}^1)^q$$

give the division of the open set

$$B(R) = \pi_B^{-1}(\{z \in \mathbb{C};\ |z| < R\})$$

by the open subsets

$$F(\underline{k}) = F(k_1, \cdots, k_q) = B(R) \cap \underline{a}^{-1}(D_{k_1} \times \cdots \times D_{k_q}).$$

Note that on each $F(\underline{k})$, the move of a_i is bounded in $\mathbb{P}^1$. Hence the situation becomes closer to the case that a_i are all constants. We apply Ahlfors' theory of covering surfaces to the subcovering $f : \pi^{-1}(F(\underline{k})) \to \mathbb{P}^1$ and Jordan domains D_{k_i} ($1 \leq i \leq q$) on $\mathbb{P}^1$. Then we obtain unintegrated version of (1.1.2) for each domain $F(\underline{k})$. By adding over all $\underline{k}$, we get unintegrated version of (1.1.2) for $B(R)$. Using the

Schwarz inequality, we conclude the inequality (1.1.2). This is a very rough plan of our proof (we use the moduli space of q-pointed stable curves of genus 0 instead of the above space $(\mathbb{P}^1)^q$). There are some problems to work out above process correctly. The major problem comes from the *degenerating points* $z \in B$ where two distinct functions a_i and a_j degenerate into the same value $a_i(z) = a_j(z)$; the problem is how to *separate* the functions a_i and a_j at the degenerating points z in relevant way. To motivate the rest of this introduction, we only remark the following two points, which are closely related.

(1) If $z \in F(k_1, \cdots, k_q)$ is a degenerating point such that $a_i(z) = a_j(z)$, then we have $D_{k_i} = D_{k_j}$. Hence we can't apply usual method of Ahlfors' theory; we need to modify it. The idea of the modification is roughly as follows. We use Ahlfors' theory in two steps (in several steps in general). First, we apply Ahlfors' theory to the subcovering

$$f : \pi^{-1}\left(F(\underline{k})\right) \to \mathbb{P}^1.$$

Secondly, we apply Ahlfors' theory to the covering

$$\frac{f - a_i}{a_j - a_i} : f^{-1}(D_{k_i}) \cap \pi^{-1}\left(F(\underline{k})\right) \to \mathbb{P}^1.$$

Note that we choose the function $\lambda(w) = \frac{w - a_i}{a_j - a_i}$ so as to separate the functions a_i and a_j, i.e., $\lambda(a_i) \equiv 0$ and $\lambda(a_j) \equiv 1$. Combining these two steps, we get rid of the above degenerating point z. Hence, we can say that our idea is the systematic use of the functions of the form $\lambda(f)$ to reduce the problem of degenerate cases to that of non-degenerate cases. In this paper, we use a system of *contraction maps* (cf. subsection 1.5) instead of the functions of the form λ.

(2) Let $\mathfrak{K} = \mathbb{C}(a_1, \cdots, a_q)$ be the subfield of $\mathfrak{K}_B$ generated over $\mathbb{C}$ by the meromorphic functions $a_1, \ldots, a_q$. In general, the transcendental degree of the field extension $\mathfrak{K}/\mathbb{C}$ has high dimension, which requires us to use higher dimensional algebraic geometry. The most natural way to control the degeneration such as $a_i(z) = a_j(z)$ in relevant way is to consider the moduli space of q-pointed stable curves of genus 0, denoted by $\overline{\mathscr{M}}_{0,q}$. Roughly speaking, this space is a quotient of $(\mathbb{P}^1)^q$ by the diagonal action of $\mathrm{Aut}(\mathbb{P}^1)$. For generic $z \in B$, the points $a_1(z), \cdots, a_q(z) \in \mathbb{P}^1$ are distinct. We consider these points as q-marked points of $\mathbb{P}^1$. Since the space $\overline{\mathscr{M}}_{0,q}$ is the classification space of q-marked points of stable curves of genus 0, we have the classification map

$$\mathrm{cl}_a : B \to \overline{\mathscr{M}}_{0,q}.$$

This map is a modification of the above map $\underline{a}$. When we consider the degenerating point $z \in B$, then the image $\mathrm{cl}_a(z)$ is contained in the degenerate locus $\mathscr{Z}_q \subset \overline{\mathscr{M}}_{0,q}$. And what is important is that we may consider the points $a_1(z), \cdots, a_q(z)$ as distinct marked points of degenerate, nodal curve instead of considering as non-distinct points of $\mathbb{P}^1$. Hence in this sense, we can say that the values $a_1(z), \ldots, a_q(z)$ are also separated at the degenerating points z. This is one reason for why we employ the space $\overline{\mathscr{M}}_{0,q}$.

Next we prepare some notations and formulate Theorem 4 from which we derive both Theorem 1 and Theorem 2. And we shall discuss farther details of the proofs of our theorems.

Remark 1.2.1. When we consider the special case that f is a transcendental meromorphic function on $\mathbb{C}$ and a_i are distinct rational functions on $\mathbb{C}$, the proof becomes simpler than that of the general case. One reason for this is that the field $\mathfrak{K}$ is contained in the field of rational functions on $\mathbb{C}$, hence the transcendental degree of the field extension $\mathfrak{K}/\mathbb{C}$ is equal to or less than one. Especially, we don't need Algebraic geometry nor the moduli space of stable curves. This case was treated in [Y2]. In the present paper, we freely use the language of Algebraic geometry.

1.3. *Notations.* In this paper, we assume that all domains on Riemann surfaces have piecewise analytic (or empty) boundaries. We also assume that all curves on a Riemann surface are piecewise analytic.

Let $\mathscr{F}$ be a Riemann surface. We say that F is a *finite domain* of $\mathscr{F}$ when F is a compactly contained, connected domain of $\mathscr{F}$ and F is bordered by a finite disjoint union of Jordan curves. Then F is compact if and only if $\mathscr{F}$ is compact and $F = \mathscr{F}$. We denote by $\overline{F}$ the closure of F and by ∂F the boundary of F.

Take a triangulation of $\overline{F}$ by a finite number of triangles, where $\overline{F}$ may be a bordered surface. We define the *characteristic* $\rho(F)$ of F by

$$-[\text{number of interior vertices}] + [\text{number of interior edges}] - [\text{number of triangles}].$$

Then it is well known that this definition is independent of the choice of the triangulation. This characteristic is normalized such that $\rho(\text{disc}) = -1$ as usual in Ahlfors' theory. We also put $\rho^+(F) = \max\{0, \rho(F)\}$.

Let Ω be an open subset of $\mathscr{F}$. We denote by $\mathcal{C}(\Omega)$ the set of connected components of Ω.

Let f and a be meromorphic functions on Ω. Assume that $f \neq a$. Put

$$\overline{n}(a, f, \Omega) = \text{card}\big(\{z \in \Omega;\ f(z) = a(z)\}\big).$$

Let M be a smooth complex algebraic variety and let ω be a smooth (1,1) form on M. Let $g : \mathscr{F} \to M$ be a holomorphic map. We put

$$A(g, \Omega, \omega) = \int_\Omega g^*\omega.$$

Let γ be a Jordan arc on $\mathscr{F}$ and let ω_M be a Kähler form on M. We denote by

$$\ell(g, \gamma, \omega_M)$$

the length of the curve $g|_\gamma : \gamma \to M$ with respect to the associated Kähler metric of ω_M.

Let $Z \subset M$ be a Zariski closed subset such that $g(\mathscr{F}) \not\subset \text{supp}\, Z$. We put

$$n(g, Z, \Omega) = \sum_{x \in \Omega} \text{ord}_x\, g^* Z,$$

and

$$\overline{n}(g, Z, \Omega) = \sum_{x \in \Omega} \min\{1, \text{ord}_x\, g^* Z\} = \text{card}\left(\Omega \cap \text{supp}\, g^{-1}(Z)\right).$$

Let $\mathscr{F}'$ be a Riemann surface and let $\pi : \mathscr{F}' \to \mathscr{F}$ be a proper, surjective holomorphic map. We denote by $\text{ram}\,\pi$ the ramification divisor of π, which is a divisor on $\mathscr{F}'$. Put

$$\operatorname{disc}(\pi, \Omega) = \sum_{x \in \pi^{-1}(\Omega)} \operatorname{ord}_x(\operatorname{ram} \pi).$$

1.4. *Nevanlinna theory.* Let Y be a Riemann surface with a proper, surjective holomorphic map $\pi : Y \to \mathbb{C}$. Let M be a smooth projective variety. Let $g : Y \to M$ be a holomorphic map. Let $Z \subset M$ be a Zariski closed subset such that $g(Y) \not\subset \operatorname{supp} Z$ and let ω be a smooth (1,1)-form on M. For $r > 1$, we put

$$N(r, g, Z) = \frac{1}{\deg \pi} \int_1^r \frac{n(g, Z, Y(t))}{t} dt,$$

$$\overline{N}(r, g, Z) = \frac{1}{\deg \pi} \int_1^r \frac{\overline{n}(g, Z, Y(t))}{t} dt,$$

$$T(r, g, \omega) = \frac{1}{\deg \pi} \int_1^r \frac{A(g, Y(t), \omega)}{t} dt$$

and

$$N_{\operatorname{ram} \pi}(r) = \frac{1}{\deg \pi} \int_1^r \frac{\operatorname{disc}(\pi, \mathbb{C}(t))}{t} dt.$$

Here $\mathbb{C}(t) = \{z \in \mathbb{C};\ |z| < t\}$ and $Y(t) = \pi^{-1}(\mathbb{C}(t))$.

Let E be a line bundle on M. Let $||\cdot||_1$ and $||\cdot||_2$ be two Hermitian metrics on E. Let ω_1 and ω_2 be the curvature forms of $||\cdot||_1$ and $||\cdot||_2$, respectively. Then we have

$$T(r, g, \omega_1) = T(r, g, \omega_2) + O(1) \quad \text{when } r \to \infty \qquad \text{(cf. [NO, p.180])}.$$

So we define $T(r, g, E)$ up to bounded function by

$$T(r, g, E) = T(r, g, \omega_1) + O(1).$$

Let f and a be meromorphic functions on Y such that $f \neq a$. Then we put

$$\overline{N}(r, a, f) = \frac{1}{\deg \pi} \int_1^r \frac{\overline{n}(a, f, Y(t))}{t} dt.$$

We denote by $\omega_{\mathbb{P}^1}$ the Fubini-Study form on the projective line $\mathbb{P}^1$; i.e.,

$$\omega_{\mathbb{P}^1} = \frac{1}{(1+|w|^2)^2} \frac{\sqrt{-1}}{2\pi} dw \wedge d\overline{w}.$$

We define the spherical characteristic function by

$$T(r, f) = T(r, f, \omega_{\mathbb{P}^1}) = \frac{1}{\deg \pi} \int_1^r \frac{dt}{t} \int_{Y(t)} f^* \omega_{\mathbb{P}^1}.$$

Then it is well known that this function $T(r, f)$ is equal to the usual characteristic function up to bounded term in r (cf. Shimizu-Ahlfors theorem).

1.5. *Moduli space of stable curves.* Our basic references are [Kn], [Ke], [FP] and [M].

Definition 1.5.1. A *q-pointed stable curve of genus* 0 (or simply q-pointed stable curve) is a connected reduced curve C of genus 0 with distinct q marked points $(s_1, \dots, s_q)$ provided:

- Each irreducible component of C is isomorphic to the projective line $\mathbb{P}^1$.
- C is a tree of $\mathbb{P}^1$ with at worst ordinary double points.
- s_i is a smooth point of C for $i = 1, \cdots, q$.
- Each irreducible component of C has at least three special points, which are either the marked points or the nodes where the component meets the other components.

Let $C = (C, s_1, \cdots, s_q)$ and $C' = (C', s'_1, \cdots, s'_q)$ be two q-pointed stable curves. We say that C and C' are isomorphic if there exists an isomorphism $\tau : C \to C'$ such that $\tau(s_i) = s'_i$ for all $i = 1, \dots, q$.

We use the following notations.

$\overline{\mathscr{M}}_{0,q}$ **:** the moduli space of q-pointed stable curves of genus 0 ($\overline{\mathscr{M}}_{0,q}$ is a smooth projective variety).

$\overline{\mathscr{U}}_{0,q} \overset{\varpi_q}{\to} \overline{\mathscr{M}}_{0,q}$ **:** the universal curve, where $\overline{\mathscr{U}}_{0,q}$ is a smooth projective variety and ϖ_q is a flat morphism.

$\sigma_1, \cdots, \sigma_q$ **:** the universal sections of ϖ_q, where $\sigma_i(\overline{\mathscr{M}}_{0,q}) \cap \sigma_j(\overline{\mathscr{M}}_{0,q}) = \emptyset$ for $i \neq j$.

$\mathscr{D}_q$ **:** the divisor on $\overline{\mathscr{U}}_{0,q}$ determined by $\sum_{i=1}^q \sigma_i(\overline{\mathscr{M}}_{0,q})$.

$\mathscr{C}_x$ **:** a fiber $\varpi_q^{-1}(x)$ over $x \in \overline{\mathscr{M}}_{0,q}$.

$K_{\overline{\mathscr{U}}_{0,q}/\overline{\mathscr{M}}_{0,q}}$ **:** the line bundle on $\overline{\mathscr{U}}_{0,q}$ associated to the relative dualizing sheaf of the morphism $\varpi_q : \overline{\mathscr{U}}_{0,q} \to \overline{\mathscr{M}}_{0,q}$.

K_q **:** the line bundle $K_{\overline{\mathscr{U}}_{0,q}/\overline{\mathscr{M}}_{0,q}}(\mathscr{D}_q)$. $\quad (\deg K_q|_{\mathscr{C}_x} = q - 2)$

ω_q **:** a fixed Kähler form on $\overline{\mathscr{U}}_{0,q}$.

η_q **:** a fixed Kähler form on $\overline{\mathscr{M}}_{0,q}$.

κ_q**:** the curvature form of a fixed Hermitian metric on K_q.

(q) **:** the set $\{1, \cdots, q\}$.

$\mathscr{I} = \mathscr{I}^q$ **:** the set $\{(i,j,k,l); 1 \le i < j < k < l \le q\}$.

$\mathscr{J} = \mathscr{J}^q$ **:** the set $\{(i,j,k); 1 \le i < j < k \le q\}$.

Remark 1.5.2. By definition, the family $\varpi_q : \overline{\mathscr{U}}_{0,q} \to \overline{\mathscr{M}}_{0,q}$ with the distinct q-sections $\sigma_1, \dots, \sigma_q$ has the following properties:
(1) For a point $x \in \overline{\mathscr{M}}_{0,q}$, the q-pointed fiber $C_x = (\mathscr{C}_x, \sigma_1(x), \cdots, \sigma_q(x))$ is a q-pointed stable curve.
(2) Let $C = (C, s_1, \cdots, s_q)$ be a q-pointed stable curve. Then there exists the unique point $x \in \overline{\mathscr{M}}_{0,q}$ such that C and C_x are isomorphic.

The complex structure of $\overline{\mathscr{M}}_{0,q}$ is naturally defined by using a similar statement for families of q-pointed stable curves. But in this paper, we only describe the complex structure of $\mathscr{M}_{0,q}$, which is a Zariski open subset of $\overline{\mathscr{M}}_{0,q}$ (see below).

Space $\mathscr{M}_{0,q}$: Two pairs $s = (s_1, \cdots, s_q)$ and $s' = (s'_1, \cdots, s'_q)$ of q-points on $\mathbb{P}^1$ are said to be isomorphic if and only if there exists an isomorphism τ of $\mathbb{P}^1$ such that $s'_i = \tau(s_i)$ for all $i = 1, \dots, q$. We denote by $\mathscr{M}_{0,q}$ the space of *q-distinct* points on $\mathbb{P}^1$

modulo isomorphism. Then $\mathscr{M}_{0,q}$ is isomorphic to

$$\mathscr{P}_q = \underbrace{\mathbb{P}^1\backslash\{0,1,\infty\} \times \cdots \times \mathbb{P}^1\backslash\{0,1,\infty\}}_{q-3 \text{ factors}} \backslash[\text{diagonals}].$$

Here note that an isomorphism of $\mathbb{P}^1$ is determined by its action on three distinct points. Then $\overline{\mathscr{M}}_{0,q}$ gives a compactification of $\mathscr{M}_{0,q}$ by the natural inclusion $\mathscr{M}_{0,q} \subset \overline{\mathscr{M}}_{0,q}$ because q-distinct points on $\mathbb{P}^1$ naturally determine a q-pointed stable curve whose underlying space is non-singular. Put $\mathscr{Z}_q = \overline{\mathscr{M}}_{0,q}\backslash\mathscr{M}_{0,q}$, which is a divisor on $\overline{\mathscr{M}}_{0,q}$ and called the degenerate locus.

Remarks 1.5.3. (1) We have $\mathscr{M}_{0,q} = \{x \in \overline{\mathscr{M}}_{0,q};\ \mathscr{C}_x \simeq \mathbb{P}^1\}$.

(2) For $i = 1, \ldots, q$, we define the holomorphic maps $p_i : \mathscr{P}_q \to \mathbb{P}^1$ as follows. For $i = 1, \ldots, q-3$, let p_i be the obvious map coming from the projection to the i-th factor. Put $p_{q-2} \equiv 0$, $p_{q-1} \equiv 1$ and $p_q \equiv \infty$. Put

$$\overline{p}_i = \mathrm{id}_{\mathscr{P}_q} \times p_i : \mathscr{P}_q \to \mathscr{P}_q \times \mathbb{P}^1.$$

Then $\overline{p}_i$ is a section of the first projection $\mathscr{P}_q \times \mathbb{P}^1 \to \mathscr{P}_q$. Put $\mathscr{U}_{0,q} = \varpi_q^{-1}(\mathscr{M}_{0,q})$. For $i = 1, \ldots, q$, let $\sigma_i' : \mathscr{M}_{0,q} \to \mathscr{U}_{0,q}$ be the restriction of σ_i. Then there exist isomorphisms $\psi : \mathscr{M}_{0,q} \to \mathscr{P}_q$ and $\psi' : \mathscr{U}_{0,q} \to \mathscr{P}_q \times \mathbb{P}^1$ fit into the following commutative diagram.

(1.5.4)
$$\begin{array}{ccc} \mathscr{U}_{0,q} & \xrightarrow{\psi'} & \mathscr{P}_q \times \mathbb{P}^1 \\ {\scriptstyle \varpi_q}\downarrow & & \downarrow{\scriptstyle \text{1st.proj}} \\ \mathscr{M}_{0,q} & \xrightarrow[\psi]{} & \mathscr{P}_q \end{array}$$

Here $\psi' \circ \sigma_i' = \overline{p}_i \circ \psi$ for $i = 1, \ldots, q$.

Dual graph Γ_x*:* Let $x \in \overline{\mathscr{M}}_{0,q}$ be a point. Then $(\mathscr{C}_x, \sigma_1(x), \cdots, \sigma_q(x))$ is a q-pointed stable curve. Let Γ_x be the associated graph, that is, each element v of the set of vertices $\mathrm{vert}(\Gamma_x)$ corresponds to the irreducible component C_v of $\mathscr{C}_x$ and two vertices v and v' are adjacent if and only if C_v and $C_{v'}$ meet transversally at the node $\nu(v, v') \in \mathscr{C}_x$. Then Γ_x is a tree.

Classification maps cl_a *and* $\mathrm{cl}_{(f,a)}$*:* Let $\pi : \mathscr{F}' \to \mathscr{F}$ be a proper, surjective holomorphic map of Riemann surfaces $\mathscr{F}'$ and $\mathscr{F}$. Let f be a meromorphic function on $\mathscr{F}'$ and $a_1, \ldots, a_q$ be distinct meromorphic functions on $\mathscr{F}$. Then we have the classification maps

$$\mathscr{F} \xrightarrow{\mathrm{cl}_a} \overline{\mathscr{M}}_{0,q}, \quad \mathscr{F}' \xrightarrow{\mathrm{cl}_{(f,a)}} \overline{\mathscr{U}}_{0,q}$$

fit into the following commutative diagram of holomorphic maps.

(1.5.5)
$$\begin{array}{ccc} \mathscr{F}' & \xrightarrow{\mathrm{cl}_{(f,a)}} & \overline{\mathscr{U}}_{0,q} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \varpi_q} \\ \mathscr{F} & \xrightarrow[\mathrm{cl}_a]{} & \overline{\mathscr{M}}_{0,q} \end{array}$$

These classification maps are defined by the following. Put

$$U = \{z \in \mathscr{F};\ a_1(z), \cdots, a_q(z) \text{ are all distinct}\} \subset \mathscr{F}.$$

We first define the restrictions

$$cl_a|_U : U \to \mathscr{M}_{0,q}, \quad \mathrm{cl}_{(f,a)}|_{\pi^{-1}(U)} : \pi^{-1}(U) \to \mathscr{U}_{0,q}.$$

For $z \in U$, let $\mathrm{cl}_a(z) \in \mathscr{M}_{0,q}$ be the unique point such that two q-pointed stable curves

$$(\mathbb{P}^1, a_1(z), \dots, a_q(z)), \quad (\mathscr{C}_{\mathrm{cl}_a(z)}, \sigma_1(\mathrm{cl}_a(z)), \dots, \sigma_q(\mathrm{cl}_a(z)))$$

are isomorphic (cf. Remark 1.5.2). Then there exists an isomorphism $\tau : \mathbb{P}^1 \to \mathscr{C}_{\mathrm{cl}_a(z)}$ such that

$$\tau(a_i(z)) = \sigma_i(\mathrm{cl}_a(z)) \quad \text{for all} \quad i = 1, \dots, q. \tag{1.5.6}$$

For $y \in \pi^{-1}(U)$, put

$$\tau(f(y)) = \mathrm{cl}_{(f,a)}(y) \in \mathscr{C}_{\mathrm{cl}_a(z)}. \tag{1.5.7}$$

Next, we define the holomorphic maps

$$\mathrm{cl}_a : \mathscr{F} \to \overline{\mathscr{M}}_{0,q}, \quad \mathrm{cl}_{(f,a)} : \mathscr{F}' \to \overline{\mathscr{U}}_{0,q}$$

by the unique extension of $\mathrm{cl}_a|_U$ and $\mathrm{cl}_{(f,a)}|_{\pi^{-1}(U)}$, respectively.

Remark 1.5.8. In view of (1.5.4), we may write

$$p_i \circ \psi \circ \mathrm{cl}_a(z) = \frac{a_i(z) - a_{q-2}(z)}{a_i(z) - a_q(z)} \frac{a_{q-1}(z) - a_q(z)}{a_{q-1}(z) - a_{q-2}(z)} \quad (i = 1, \dots, q-3) \tag{1.5.9}$$

for $z \in U$ and

$$s \circ \psi' \circ \mathrm{cl}_{(f,a)}(y) = \frac{f(y) - a_{q-2}(z)}{f(y) - a_q(z)} \frac{a_{q-1}(z) - a_q(z)}{a_{q-1}(z) - a_{q-2}(z)} \tag{1.5.10}$$

for $y \in \pi^{-1}(U)$. Here $s : \mathscr{P}_q \times \mathbb{P}^1 \to \mathbb{P}^1$ is the second projection. These equations (1.5.9) and (1.5.10) easily follow from the fact that two pairs of $(q+1)$-points on $\mathbb{P}^1$

$$(f(y), a_1(z), \dots, a_q(z))$$

and

$$(s \circ \psi' \circ \mathrm{cl}_{(f,a)}(y), p_1 \circ \psi \circ \mathrm{cl}_a(z), \dots, p_{q-3} \circ \psi \circ \mathrm{cl}_a(z), 0, 1, \infty)$$

are isomorphic for $z \in U$ and $y \in \pi^{-1}(U)$.

Contraction map φ_α*:* For $\alpha = (i,j,k) \in \mathscr{J}$, we denote by $\varphi_\alpha = \varphi_\alpha^{(q)}$ the morphism

$$\varphi_\alpha : \overline{\mathscr{U}}_{0,q} \to \mathbb{P}^1$$

uniquely characterized by the following:

- $\varphi_\alpha \circ \sigma_i \equiv 0$, $\varphi_\alpha \circ \sigma_j \equiv 1$ and $\varphi_\alpha \circ \sigma_k \equiv \infty$ (on $\overline{\mathscr{M}}_{0,q}$),
- the restriction $\varphi_\alpha|_{\mathscr{C}_x} : \mathscr{C}_x \to \mathbb{P}^1$ is an isomorphism for all $x \in \mathscr{M}_{0,q}$.

To obtain this φ_α, observe the following. By forgetting all markings except i, j, k, we get the following commutative diagram of holomorphic maps (cf. [M, p.93]).

$$\begin{array}{ccc} \overline{\mathscr{U}}_{0,q} & \xrightarrow{s'} & \overline{\mathscr{U}}_{0,3} \\ {\scriptstyle \varpi_q}\downarrow & & \downarrow{\scriptstyle \varpi_3} \\ \overline{\mathscr{M}}_{0,q} & \xrightarrow{s} & \overline{\mathscr{M}}_{0,3} \end{array}$$

Note that $\overline{\mathscr{M}}_{0,3} \simeq \mathrm{pt}$ and $\overline{\mathscr{U}}_{0,3} \simeq \mathbb{P}^1$. We normalize the three universal sections of ϖ_3 as 0, 1 and ∞. Then $s' \circ \sigma_i \equiv 0$, $s' \circ \sigma_j \equiv 1$ and $s' \circ \sigma_k \equiv \infty$. Put $\varphi_\alpha = s'$.

Contraction map ϕ_β: By forgetting the marking σ_q, we have the morphism $\tau_q : \overline{\mathscr{M}}_{0,q} \to \overline{\mathscr{M}}_{0,q-1}$. There is an isomorphism $\iota_q : \overline{\mathscr{M}}_{0,q} \to \overline{\mathscr{U}}_{0,q-1}$ fits into the following commutative diagram of holomorphic maps (cf. [M, p.93]).

$$\begin{array}{ccc} \overline{\mathscr{M}}_{0,q} & \xrightarrow{\iota_q} & \overline{\mathscr{U}}_{0,q-1} \\ {\scriptstyle \tau_q}\downarrow & & \downarrow{\scriptstyle \varpi_{q-1}} \\ \overline{\mathscr{M}}_{0,q-1} & = & \overline{\mathscr{M}}_{0,q-1} \end{array} \tag{1.5.11}$$

For $l < q$, put $\tau_{q,l} = \tau_{l+1} \circ \cdots \circ \tau_q : \overline{\mathscr{M}}_{0,q} \to \overline{\mathscr{M}}_{0,l}$. Put $\tau_{q,q} = \mathrm{id}_{\overline{\mathscr{M}}_{0,q}}$. For $\beta = (i, j, k, l) \in \mathscr{I}$, we define $\phi_\beta : \overline{\mathscr{M}}_{0,q} \to \mathbb{P}^1$ by the composition of the following morphisms

$$\overline{\mathscr{M}}_{0,q} \overset{\tau_{q,l}}{\to} \overline{\mathscr{M}}_{0,l} \overset{\iota_l}{\to} \overline{\mathscr{U}}_{0,l-1} \overset{\varphi^{(l-1)}_{(i,j,k)}}{\longrightarrow} \mathbb{P}^1.$$

1.6. *Outline of proofs.* The proof of Theorem 2 is similar to that of Theorem 1 (actually easier). So we only consider the case of Theorem 1. We first formulate the following.

Theorem 3. *Let Y, B and π be the same as Theorem 1. Consider the following commutative diagram of holomorphic maps.*

$$\begin{array}{ccc} Y & \xrightarrow{g} & \overline{\mathscr{U}}_{0,q} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \varpi_q} \\ B & \xrightarrow[b]{} & \overline{\mathscr{M}}_{0,q} \end{array} \tag{1.6.1}$$

Assume the non-degeneracy condition that $g(Y) \not\subset \operatorname{supp} \mathscr{D}_q \cup \varpi_q^{-1}(\operatorname{supp} \mathscr{Z}_q)$. Then for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} T(r, g, \kappa_q) \leq \overline{N}(r, g, \mathscr{D}_q) + N_{\mathrm{ram}\,\pi_Y}(r) + \varepsilon T(r, g, \omega_q) \\ + C(\varepsilon)\left(T(r, b, \eta_q) + N_{\mathrm{ram}\,\pi_B}(r)\right) + o(T(r, g, \omega_q)) \ ||. \end{aligned} \tag{1.6.2}$$

Remark 1.6.3. Consider the case $B = \mathbb{C}$ and $\pi_B = \mathrm{id}_{\mathbb{C}}$. A consequence of the general second fundamental conjecture is that the inequality

$$T(r, g, K_{\overline{\mathscr{U}}_{0,q}}(\mathscr{D}_q)) \leq \overline{N}(r, g, \mathscr{D}_q) + N_{\mathrm{ram}\,\pi_Y}(r) + \varepsilon T(r, g, \omega_q) + o(T(r, g, \omega_q)) \ || \tag{1.6.4}$$

holds for all $\varepsilon > 0$ and for all suitably non-degenerate g. Here $K_{\overline{\mathscr{U}}_{0,q}}$ is the canonical line bundle on $\overline{\mathscr{U}}_{0,q}$. Since we have

$$T(r, g, K_{\overline{\mathscr{U}}_{0,q}}(\mathscr{D}_q)) = T(r, g, \kappa_q) + T(r, b, K_{\overline{\mathscr{M}}_{0,q}}) + O(1),$$

the inequality (1.6.2) is a weak form of (1.6.4).

In Section 2, we derive Theorem 1 from Theorem 3, applying to the case that $g = \mathrm{cl}_{(f,a)}$ and $b = \mathrm{cl}_a$. Using the Schwarz inequality, we prove Theorem 3 from the following Theorem 4 in the same section.

Definition 1.6.5. (1) A *q-hol-quintet* is an object $(\mathscr{F}, \mathscr{R}, \pi, g, b)$ where $\mathscr{F}$ and $\mathscr{R}$ are Riemann surfaces with proper, surjective holomorphic map $\pi : \mathscr{F} \to \mathscr{R}$, and g and b are holomorphic maps fit into the following commutative diagram.

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\ g\ } & \overline{\mathscr{U}}_{0,q} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \varpi_q} \\ \mathscr{R} & \xrightarrow[\ b\ ]{} & \overline{\mathscr{M}}_{0,q} \end{array}$$

We say that a q-hol-quintet $(\mathscr{F}, \mathscr{R}, \pi, g, b)$ is *non-degenerate* if $b(\mathscr{R}) \not\subset \operatorname{supp} \mathscr{Z}_q$ and if the meromorphic functions $\varphi_\alpha \circ g$ on $\mathscr{F}$ are non-constant for all $\alpha \in \mathscr{J}$.

(2) A *specified q-hol-quintet* is an object $(\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ where $(\mathscr{F}, \mathscr{R}, \pi, g, b)$ is a q-hol-quintet, $R \subset \mathscr{R}$ is a finite domain and $F = \pi^{-1}(R)$. We say that a specified q-hol-quintet is *non-degenerate* if the q-hol-quintet $(\mathscr{F}, \mathscr{R}, \pi, g, b)$ is non-degenerate.

Theorem 4. *Let $q \geq 3$ be a positive integer. For all $\varepsilon > 0$, there exists a positive constant $C(q, \varepsilon) > 0$ with the following property: Let $(\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ be a non-degenerate specified q-hol-quintet. Then we have*

(1.6.6)

$$\begin{aligned} A(g, F, \kappa_q) \leq & \overline{n}(g, \mathscr{D}_q, F) + \operatorname{disc}(\pi, R) + \varepsilon A(g, F, \omega_q) \\ & + C(q, \varepsilon) \deg \pi \left(A(b, R, \eta_q) + \overline{n}(b, \mathscr{Z}_q, R) + \rho^+(R) + \ell(g, \partial F, \omega_q) \right). \end{aligned}$$

The most important part of this paper is a proof of Theorem 4. The proof naturally divides into the following three steps.

Step 1: We prove the local version of our theorem, which roughly says as follows: For each point $x \in \overline{\mathscr{M}}_{0,q}$, there exists an open neighborhood V_x of x such that if a non-degenerate specified q-hol-quintet satisfies the condition $b(R) \subset V_x$, then our theorem is valid. For the precise statement, see Lemma 6. To prove this, we use one lemma from [Y2], which is an application of Ahlfors' theory (cf. Lemma 3). For each vertex $v \in \Gamma_x$, we attach a contraction morphism $\varphi_{\langle v \rangle} : \overline{\mathscr{U}}_{0,q} \to \mathbb{P}^1$ ($\langle v \rangle \in \mathscr{J}$). This contraction map $\varphi_{\langle v \rangle}$ has the properties that the restriction to the component C_v is an isomorphism and that the restrictions to the other components $C_{v'}$ are constant maps. Applying Lemma 3 to $\upsilon = \varphi_{\langle v \rangle} \circ g$ and $\zeta = \varphi_{\langle v' \rangle} \circ g$, where v and v' are adjacent vertices, we obtain some sort of "difference" of usual Ahlfors' second main theorem. Adding these "difference"s over all the edges of Γ_x, we obtain (a modification of) usual Ahlfors' second main theorem. Applying Rouché's theorem (Lemma 4), we get the local version of our theorem. This method is similar to that of [Y2]. Major differences are that instead of the tree constructed in [Y2, Section 8], we use the tree Γ_x, and instead of the combinatorial lemma [Y2, Lemma 4], we use a geometric lemma (cf. Lemma 5).

Step 2: By a non-simple curve γ, we divide $\mathbb{P}^1$ into a finite number of Jordan domains D_k $(1 \leq k \leq K)$. This division of $\mathbb{P}^1$ gives the division of $(\mathbb{P}^1)^{\mathscr{I}}$ in the form of the open subsets

$$\prod_{i \in \mathscr{I}} D_{k_i}, \quad 1 \leq k_i \leq K. \tag{1.6.7}$$

Put $\Phi = \prod_{i \in \mathscr{I}} \phi_i : \overline{\mathscr{M}}_{0,q} \to (\mathbb{P}^1)^{\mathscr{I}}$. We consider the connected components of the pull-back of the open subsets (1.6.7) by the composition of the morphisms

$$R \xrightarrow{b} \overline{\mathscr{M}}_{0,q} \xrightarrow{\Phi} (\mathbb{P}^1)^{\mathscr{I}}$$

to get the finite domains $R' \subset R$, which divide R into finite set $\{R'\}$ of disjoint finite domains. Using the facts that $\overline{\mathscr{M}}_{0,q}$ is compact and that Φ is an injection (cf. Lemma 7), we conclude that if the Jordan domains D_k are small enough, then for all $R' \in \{R'\}$ there exists a point $x \in \overline{\mathscr{M}}_{0,q}$ such that $b(R') \subset V_x$.

Step 3: Applying the local version of the theorem for each finite domain R' and adding over all these finite domains, we get our theorem. Here we need to estimate extra error terms coming from

- the lengths $\ell(g, \partial' \pi^{-1}(R'), \omega_q)$, where $\partial' \pi^{-1}(R')$ are the parts of the boundaries of $\pi^{-1}(R')$ which lie in the interior of F,
- the sum of $\rho^+(R')$ over $R' \in \{R'\}$.

See Lemma 8 for these estimates. Here we only remark the idea of a method of the first estimate. Take a slightly small Jordan domain $D'_k \subset D_k$ for each k. We define finite domains $R'' \subset R'$ by the same manner for R' from the Jordan domains D'_k. Then using so-called length-area principle, if the areas $A(g, \pi^{-1}(R''), \omega_q)$ are sufficiently large, we can find finite domains $\tilde{R}$ with $R'' \subset \tilde{R} \subset R'$ such that the lengths $\ell(g, \partial' \pi^{-1}(\tilde{R}), \omega_q)$ are small enough. We replace $\{R'\}$ by $\{\hat{R}\}$. This is the idea of the estimate.

The paper is organized as follows. In section 2, after some algebraic preparation, we derive Theorem 1 from Theorem 3 and Theorem 3 from Theorem 4. The proof of Theorem 4 begins from section 3. The section 3 is a preliminary including some lemmas from [Y2]. In this section, we also review Ahlfors' theory, which will be used in the proof. In section 4 and 5, we prove Lemma 6 and 8, respectively. The proof of Theorem 4 ends at section 6. In section 7, we prove Corollary 2 from rather sharp estimate. In section 8, we prove Theorem 2 from Theorem 4. This proof is similar to that of Theorem 1. In section 9, we introduce some notations from [V1] and [V3], and prove the height inequality for curves over function fields.

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This paper is an expanded and largely rewritten version of [Y1].

2. Derivations of Theorem 1 from Theorem 3 and Theorem 3 from Theorem 4

2.1. *Algebraic lemma.* We denote by $\mathscr{L}$ the hyperplane section bundle on $\mathbb{P}^1$, which is the unique line bundle of degree one.

Lemma 1. *There exist a line bundle M on $\overline{\mathscr{M}}_{0,q}$ and a divisor Ξ on $\overline{\mathscr{U}}_{0,q}$ such that $\varpi_q(\mathrm{supp}\,\Xi) \subset \mathrm{supp}\,\mathscr{Z}_q$ and*

$$(q-2)\varphi_{(1,2,3)}^*\mathscr{L} = K_q + \varpi_q^* M + [\Xi].$$

Here $[\Xi]$ is the associated line bundle for the divisor Ξ.

Proof. Put $P = (q-2)\varphi_{(1,2,3)}^*\mathscr{L} - K_q$. For $x \in \mathscr{M}_{0,q}$, the restriction $\varphi_{(1,2,3)}|_{\mathscr{C}_x} : \mathscr{C}_x \to \mathbb{P}^1$ is an isomorphism. Since $\deg(K_q|_{\mathscr{C}_x}) = q-2$, we know that the restriction $P|_{\mathscr{C}_x}$ is the trivial line bundle. Since $\varpi_q^{-1}(\mathscr{M}_{0,q}) \to \mathscr{M}_{0,q}$ is a $\mathbb{P}^1$-bundle, we conclude that there exists a line bundle M_0 on $\mathscr{M}_{0,q}$ such that the restriction $P|_{\varpi_q^{-1}(\mathscr{M}_{0,q})}$ is isomorphic to $\varpi_q^* M_0$. Let M be an extension of M_0 to $\overline{\mathscr{M}}_{0,q}$. Put $P' = P - \varpi_q^* M$. Then $P'|_{\varpi_q^{-1}(\mathscr{M}_{0,q})}$ is the trivial line bundle. Hence there exists a divisor Ξ on $\overline{\mathscr{U}}_{0,q}$ such that $\varpi_q(\mathrm{supp}\,\Xi) \subset \mathrm{supp}\,\mathscr{Z}_q$ and $P' = [\Xi]$. This proves our lemma. □

2.2. *Theorem 3 implies Theorem 1.* Let f, $a_1, \dots, a_q$ be the functions in Theorem 1. We apply Theorem 3 to the case $g = \mathrm{cl}_{(f,a)}$ and $b = \mathrm{cl}_a$. The non-degeneracy condition of Theorem 3 easily follows from the assumption that $f \not\equiv a_i \circ \pi$ for $i = 1, \dots, q$. Then we get

$$\text{(2.2.1)}\quad \begin{aligned} T(r, \mathrm{cl}_{(f,a)}, \kappa_q) \leq \overline{N}(r, \mathrm{cl}_{(f,a)}, \mathscr{D}_q) + N_{\mathrm{ram}\,\pi_Y}(r) + \varepsilon T(r, \mathrm{cl}_{(f,a)}, \omega_q) \\ + O_\varepsilon\left(T(r, \mathrm{cl}_a, \eta_q) + N_{\mathrm{ram}\,\pi_B}(r)\right) + o\left(T(r, \mathrm{cl}_{(f,a)}, \omega_q)\right) \ || \end{aligned}$$

for all $\varepsilon > 0$.

Let $\mathfrak{K}_B$ be the field of meromorphic functions on B. Let $W \subset \overline{\mathscr{M}}_{0,q}$ be the Zariski closure of the image $\mathrm{cl}_a(B)$ and let $\mathbb{C}(W)$ be the rational function field of W. Then cl_a defines the natural injection $\iota : \mathbb{C}(W) \to \mathfrak{K}_B$ by the pullback of the rational functions on W. Let $\mathbb{C}(a_1, \cdots, a_q) \subset \mathfrak{K}_B$ be the subfield generated by the meromorphic functions $a_1, \dots, a_q$. Then by the definition of cl_a, we have $\iota(\mathbb{C}(W)) \subset \mathbb{C}(a_1, \cdots, a_q)$ (cf. (1.5.9)). Hence we have

$$\text{(2.2.2)}\qquad T(r, \mathrm{cl}_a, \eta_q) \leq O\left(\sum_{1 \leq i \leq q} T(r, a_i)\right).$$

Similarly, using the field $\mathfrak{K}_Y$ of meromorphic functions on Y, we have

$$\text{(2.2.3)}\qquad T(r, \mathrm{cl}_{(f,a)}, \omega_q) \leq O\left(T(r, f) + \sum_{1 \leq i \leq q} T(r, a_i)\right) \quad \text{(cf. (1.5.10))}.$$

Claim. The following inequalities hold

$$\text{(2.2.4)}\qquad \overline{N}(r, \mathrm{cl}_{(f,a)}, \mathscr{D}_q) \leq \sum_{1 \leq i \leq q} \overline{N}(r, a_i \circ \pi, f) + O\left(\sum_{1 \leq i \leq q} T(r, a_i)\right),$$

$$(2.2.5)\qquad (q-2)T(r,f)\le T(r,\mathrm{cl}_{(f,a)},\kappa_q)+O\left(\sum_{1\le i\le q}T(r,a_i)\right).$$

Proof. We first prove (2.2.4). Put

$$U=\{z\in B;\ a_1(z),\dots,a_q(z)\text{ are all distinct}\}.$$

Then by the definition of the classification maps, we have $\mathrm{cl}_a(U)\subset\mathscr{M}_{0,q}$. For $z\in U$ and $y\in\pi^{-1}(z)$, we have $\mathrm{cl}_{(f,a)}(y)\in\mathscr{D}_q$ if and only if $f(y)=a_i(z)$ for some $i\in(q)$ (cf. (1.5.6), (1.5.7)). Hence we have

$$\{y\in Y;\ \mathrm{cl}_{(f,a)}(y)\in\mathscr{D}_q\}\subset\{y\in Y;\ f(y)=a_i\circ\pi(y)\text{ for some }i\in(q)\}\cup\pi^{-1}(B\backslash U).$$

This implies that

$$\overline{n}(\mathrm{cl}_{(f,a)},\mathscr{D}_q,Y(r))\le\sum_{1\le i\le q}\overline{n}(a_i\circ\pi,f,Y(r))+\deg\pi\sum_{1\le i\ne j\le q}\overline{n}(a_i,a_j,B(r))$$

and

$$\overline{N}(r,\mathrm{cl}_{(f,a)},\mathscr{D}_q)\le\sum_{1\le i\le q}\overline{N}(r,a_i\circ\pi,f)+\sum_{1\le i\ne j\le q}\overline{N}(r,a_i,a_j).$$

Since we have

$$\sum_{1\le i\ne j\le q}\overline{N}(r,a_i,a_j)\le O\left(\sum_{1\le i\le q}T(r,a_i)\right),$$

we get (2.2.4).

Next we prove (2.2.5). Since $\omega_{\mathbb{P}^1}$ is the curvature form of the Fubini-Study metric on $\mathscr{L}$, Lemma 1 implies the inequality

$$(2.2.6)\quad (q-2)T(r,\varphi_{(1,2,3)}\circ\mathrm{cl}_{(f,a)})=T(r,\mathrm{cl}_{(f,a)},\kappa_q)+T(r,\mathrm{cl}_a,M)+T(r,\mathrm{cl}_{(f,a)},[\Xi])+O(1).$$

Since for $z\in\pi^{-1}(U)$, two pairs of 4-points on $\mathbb{P}^1$

$$(f(z),a_1\circ\pi(z),a_2\circ\pi(z),a_3\circ\pi(z)),\quad(\varphi_{(1,2,3)}\circ\mathrm{cl}_{(f,a)}(z),0,1,\infty)$$

are isomorphic (cf. (1.5.6), (1.5.7)), we have

$$\varphi_{(1,2,3)}\circ\mathrm{cl}_{(f,a)}(z)=\frac{f(z)-a_1\circ\pi(z)}{f(z)-a_3\circ\pi(z)}\frac{a_2\circ\pi(z)-a_3\circ\pi(z)}{a_2\circ\pi(z)-a_1\circ\pi(z)}.$$

Hence we get

$$(2.2.7)\qquad T(r,\varphi_{(1,2,3)}\circ\mathrm{cl}_{(f,a)})=T(r,f)+O\left(\sum_{1\le i\le q}T(r,a_i)\right).$$

By $\varpi_q(\mathrm{supp}\,\Xi)\subset\mathrm{supp}\,\mathscr{Z}_q$ and $\mathrm{cl}_a(B)\not\subset\mathrm{supp}\,\mathscr{Z}_q$, we have

$$T(r,\mathrm{cl}_{(f,a)},[\Xi])\le O(T(r,\mathrm{cl}_a,[\mathscr{Z}_q])\le O\left(\sum_{1\le i\le q}T(r,a_i)\right)\quad\text{(cf. (2.2.2))}.$$

Hence using (2.2.6), (2.2.7) and the inequality

$$T(r, \mathrm{cl}_a, M) \le O\left(\sum_{1\le i\le q} T(r, a_i)\right),$$

we get our inequality (2.2.5) and conclude the proof of our claim. □

Using (2.2.1), (2.2.2), (2.2.3) and the above claim, we get our Theorem 1.

2.3. *Theorem 4 implies Theorem 3.* We shall apply Theorem 4 to the specified q-hol-quintet $\lambda_i = (Y, B, \pi, g, b, Y_i(r), B_i(r))$ for $i = 1, \dots, u_r$, where $\{B_i(r)\}_{i=1}^{u_r} = \mathcal{C}(B(r))$ is the set of connected components of $B(r)$.

First, consider the case that λ_i are degenerate, i.e., there exists $\alpha \in \mathscr{J}$ such that $\varphi_\alpha \circ g \equiv c$ is constant. Then the image $g(Y)$ is contained in the divisor $E = \varphi_\alpha^*(c) \subset \overline{\mathscr{U}}_{0,q}$. Put $E_0 = E \cap \varpi_q^{-1}(\mathscr{M}_{0,q})$. Then the restriction of ϖ_q on E_0 gives an isomorphism $E_0 \to \mathscr{M}_{0,q}$. Since we are assuming that $g(Y) \not\subset \varpi_q^{-1}(\mathscr{Z}_q)$, we obtain $T(r, g, \kappa_q) < O(T(r, b, \eta_q))$. This proves Theorem 3 in the case λ_i are degenerate.

Next we consider the case that λ_i are non-degenerate. First, apply Theorem 4 to each λ_i, next take the summation over $i = 1, \dots, u_r$ and finally integrate the inequality. Then putting

$$L(r) = \frac{1}{\deg \pi_Y}\int_1^r \frac{\ell(g, \partial Y(t), \omega_q)}{t}dt, \quad J(r) = \frac{1}{\deg \pi_B}\int_1^r \frac{\sum_{i=1}^{u_t}\rho^+(B_i(t))}{t}dt,$$

we get

$$T(r, g, \kappa_q) \le \overline{N}(r, g, \mathscr{D}_q) + N_{\mathrm{ram}\,\pi_Y}(r) - N_{\mathrm{ram}_{\pi_B}}(r) + \varepsilon T(r, g, \omega_q) + O_\varepsilon\left(T(r, b, \eta_q) + \overline{N}(r, b, \mathscr{Z}_q) + J(r) + \deg \pi L(r)\right) \tag{2.3.1}$$

for all $\varepsilon > 0$. Here we note that $\mathrm{ram}\,\pi_Y = \pi^*(\mathrm{ram}\,\pi_B) + \mathrm{ram}\,\pi$, hence we have

$$\mathrm{disc}(\pi_Y, \mathbb{C}(r)) = \deg \pi\,\mathrm{disc}(\pi_B, \mathbb{C}(r)) + \mathrm{disc}(\pi, B(r)) \tag{2.3.2}$$

and

$$N_{\mathrm{ram}\,\pi_Y}(r) - N_{\mathrm{ram}_{\pi_B}}(r) = \frac{1}{\deg \pi_Y}\int_1^r \frac{\mathrm{disc}(\pi, B(t))}{t}dt.$$

Claim: The following inequalities hold

$$J(r) \le N_{\mathrm{ram}\,\pi_B}(r), \tag{2.3.3}$$

$$L(r) < o(T(r, g, \omega_q))\ ||. \tag{2.3.4}$$

Proof of Claim. We first prove (2.3.3). By Hurwitz's formula, we have

$$\rho(B_i(r)) = \deg(\pi_B|_{B_i(r)})\rho(\mathbb{C}(r)) + \mathrm{disc}(\pi_B|_{B_i(r)}, \mathbb{C}(r)).$$

Since $\rho(\mathbb{C}(r)) = -1$ and $\rho(B_i(r)) \ge -1$, we have

$$\rho^+(B_i(r)) \le \mathrm{disc}(\pi_B|_{B_i(r)}, \mathbb{C}(r)).$$

Hence we have $\sum_{i=1}^{u_r}\rho^+(B_i(r)) \le \mathrm{disc}(\pi_B, \mathbb{C}(r))$ and (2.3.3).

Next we prove (2.3.4). In this proof, we denote the covering map $\pi_Y : Y \to \mathbb{C}$ by p to avoid the confusion with the ratio of the circumference π. Put $g^*\omega_q =$

$\frac{\sqrt{-1}}{2}G^2 dp \wedge d\overline{p}$, where G is a C^∞-function on $Y\backslash\{z \in Y;\ p'(z) = 0\}$ with $G \geq 0$. Then we have

$$\ell(r) := \ell(g, \partial Y(r), \omega_q) = \int_{\partial Y(r)} Gr\ d\arg p$$

and

$$A(r) := A(g, Y(r), \omega_q) = \int_0^r dt \int_{\partial Y(t)} G^2 t\ d\arg p.$$

Put $e = \deg p$. Using the Schwarz inequality, for $r > 1$ we have

$$\begin{aligned}
eL(r) &= \int_1^r \ell(t)\frac{dt}{t} = \int_1^r \int_{\partial Y(t)} Gt\ d\arg p \frac{dt}{t} \\
&\leq \left(\int_1^r \int_{\partial Y(t)} d\arg p \frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_1^r \int_{\partial Y(t)} G^2 t^2\ d\arg p \frac{dt}{t}\right)^{\frac{1}{2}} \\
&= (2\pi e \log r)^{\frac{1}{2}} \left(A(r) - A(1)\right)^{\frac{1}{2}} \\
&\leq (2\pi r e^2 \log r)^{\frac{1}{2}} \left(\frac{d}{dr} T(r)\right)^{\frac{1}{2}},
\end{aligned}$$

where we put $T(r) = T(r, g, \omega_q)$. Take $r_0 > 1$ such that $T(r_0) > 1$. Let E be the subset of $[r_0, \infty)$ such that

$$L(r) \geq T(r)^{\frac{1}{2}} \log T(r).$$

Then we have

$$\begin{aligned}
\int_E d\log\log r = \int_E \frac{1}{r\log r} dr &\leq 2\pi \int_E \frac{\frac{d}{dr}T(r)}{L(r)^2} dr \\
&\leq 2\pi \int_{r_0}^\infty \frac{\frac{d}{dr}T(r)}{T(r)(\log T(r))^2} dr = \frac{2\pi}{\log T(r_0)}.
\end{aligned}$$

Hence outside the set E such that $\int_E d\log\log r < \infty$, we have

$$L(r) \leq T(r)^{\frac{1}{2}} \log T(r) = o(T(r)),$$

which proves our claim. □

Since we have

$$\overline{N}(r, b, \mathscr{X}_q) \leq O\left(T(r, b, \eta_q)\right),$$

the equation (2.3.1) and the above claim imply Theorem 3.

3. Preliminary for the proof of Theorem 4

Proofs of lemmas in this section can be found in [Y2].

3.1. *Topology.* Let $\mathscr{F}$ be a Riemann surface. Let Ω and G be two open subsets in $\mathscr{F}$. We define two subsets $\mathcal{I}(G,\Omega)$, $\mathcal{P}(G,\Omega)$ of the set of connected components of $G\cap\Omega$ by the following manner. Let G' be a connected component of $G\cap\Omega$, then G' is contained in $\mathcal{I}(G,\Omega)$ if and only if G' is compactly contained in Ω, otherwise G' is contained in $\mathcal{P}(G,\Omega)$. Then a connected component G' in $\mathcal{I}(G,\Omega)$ is also a connected component of G.

Let ζ be a non-constant meromorphic function on $\overline{\Omega}\subset\mathscr{F}$, where Ω is a domain of $\mathscr{F}$. Let E be a domain in $\mathbb{P}^1$. We consider the following condition for $\zeta:\overline{\Omega}\to\mathbb{P}^1$ and E;

(3.1.1) Let $a\in\overline{\Omega}$ be a branch point of ζ. Then $\zeta(a)\notin\partial E$.

Lemma 2 ([Y2, Lemma 1])**.** *Assume that a finite number of disjoint simple closed curves γ_i $(i=1,\cdots p)$ divide $\mathbb{P}^1$ into connected domains $D_1,\cdots,D_{p+1}$. Let ζ be a non-constant meromorphic function on $\overline{\Omega}$, where Ω is a finite domain of a Riemann surface $\mathscr{F}$. Assume that the condition* (3.1.1) *is satisfied for ζ and D_i $(1\le i\le p+1)$. Put $\mathcal{A}=\bigcup_{i=1}^{p+1}\mathcal{I}(\zeta^{-1}(D_i),\Omega)$, $\mathcal{B}=\bigcup_{i=1}^{p+1}\mathcal{P}(\zeta^{-1}(D_i),\Omega)$. Then we have*

$$\rho^+(\Omega)\ge\sum_{A\in\mathcal{A}}\rho(A)+\sum_{B\in\mathcal{B}}\rho^+(B).$$

3.2. *Review of Ahlfors' theory.* Recall that we denote by $\omega_{\mathbb{P}^1}$ the Fubini-Study form on the projective line $\mathbb{P}^1$. Let Ω_0 be a finite domain of $\mathbb{P}^1$. Let $\mathscr{F}$ be a Riemann surface, let $\Omega\subset\mathscr{F}$ be a finite domain and let ζ be a non-constant meromorphic function on $\overline{\Omega}$. Assume that $\zeta(\overline{\Omega})\subset\overline{\Omega_0}$. Then we may consider $\zeta:\Omega\to\Omega_0$ as a covering surface in the sense of [N2, p.323].

We call $\zeta^{-1}(\Omega_0)\cap\partial\Omega$ the relative boundary and $\ell(\zeta,\zeta^{-1}(\Omega_0)\cap\partial\Omega,\omega_{\mathbb{P}^1})$ the length of the relative boundary.

Let $D\subset\Omega_0$ be a domain which is bounded by a finite number of Jordan curves. We call

$$S_D=\frac{A(\zeta,\zeta^{-1}(D)\cap\Omega,\omega_{\mathbb{P}^1})}{\int_D\omega_{\mathbb{P}^1}}$$

the mean sheet number of ζ over D. We call S_{Ω_0} the mean sheet number of ζ. In the following two theorems, we denote by S and L the mean sheet number and the length of the relative boundary of the covering $\zeta:\Omega\to\Omega_0$, respectively.

Covering Theorem 1. ([N2, p.328]) There exists a positive constant $h=h(\Omega_0)>0$ which is independent of D, Ω and ζ such that

$$|S-S_D|\le\frac{h}{\int_D\omega_{\mathbb{P}^1}}L. \tag{3.2.1}$$

Consider ζ as the covering map of the closed surfaces $\zeta:\overline{\Omega}\to\overline{\Omega_0}$. Put

$$S(\partial\Omega_0)=\frac{\ell(\zeta,\zeta^{-1}(\partial\Omega_0),\omega_{\mathbb{P}^1})}{\text{length of }\partial\Omega_0\text{ with respect to the Fubini-Study metric}}.$$

Covering Theorem 2'. ([N2, p.331, Remark]) Assume that $\partial\Omega_0$ consists of analytic Jordan curves. Then there exists a positive constant $h=h(\Omega_0)>0$ which is independent of Ω and ζ such that

$$|S-S(\partial\Omega_0)|\le hL. \tag{3.2.2}$$

Note that an analytic Jordan curve is *regular* in the sense of [N2, p.326] (cf. [H, Lemma 5.1]). The Main Theorem ([N2, p.332]) of Ahlfors' theory was used to prove the following.

Lemma 3 ([Y2, Lemma 2]). *Let $E^\dagger$ be a Jordan domain in $\mathbb{P}^1$ or $\mathbb{P}^1$ itself. Let $E_1, \dots, E_p$, E_∞ be Jordan domains in $\mathbb{P}^1$. Assume that the closures $\overline{E_j}$ of E_j's ($j = 1, \cdots, p, \infty$) are mutually disjoint. Then there is a positive constant $h > 0$ which only depends on $E_1, \cdots, E_p, E_\infty$ with the following property: Let Ω be a finite domain of a Riemann surface $\mathscr{F}$ and υ, ζ be two non-constant meromorphic functions on $\overline{\Omega}$. Assume that*

$$\zeta\left(\upsilon^{-1}(\mathbb{P}^1 \backslash E^\dagger) \cap \overline{\Omega}\right) \subset E_\infty \tag{3.2.3}$$

and that ζ and E_j satisfy the condition (3.1.1) *for $j = 1, \cdots, p, \infty$.*
Put

$$\mathcal{G}^I = \mathcal{I}(\upsilon^{-1}(E^\dagger), \Omega),\ \mathcal{G}^P = \mathcal{P}(\upsilon^{-1}(E^\dagger), \Omega),$$

$$\mathcal{G}_j^I = \mathcal{I}(\zeta^{-1}(E_j), \Omega),\ \mathcal{G}_j^P = \mathcal{P}(\zeta^{-1}(E_j), \Omega) \text{ for } j = 1, \cdots, p,$$

and $\mathcal{G}_\infty^I = \mathcal{I}(\zeta^{-1}(E_\infty), \Omega \cap \upsilon^{-1}(E^\dagger))$. Then we have the following inequality.

$$\begin{aligned} \vartheta(\zeta, \upsilon) + \sum_{G \in \mathcal{G}^I} \rho(G) + \sum_{G \in \mathcal{G}^P} \rho^+(G) - \sum_{j=1}^{p} \sum_{G \in \mathcal{G}_j^I} \rho(G) \\ - \sum_{j=1}^{p} \sum_{G \in \mathcal{G}_j^P} \rho^+(G) - \sum_{G \in \mathcal{G}_\infty^I} \rho(G) \geq (p-1) A(\zeta, \Omega, \omega_{\mathbb{P}^1}) - h\ell(\zeta, \partial\Omega, \omega_{\mathbb{P}^1}), \end{aligned} \tag{3.2.4}$$

where $\vartheta(\zeta, \upsilon)$ is the number of connected components G in $\mathcal{G}^I$ such that $\zeta(G) \subset E_\infty$.

Remarks 3.2.5. (1) Since we have $\int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = 1$, the term $A(\zeta, \Omega, \omega_{\mathbb{P}^1})$ is equal to the mean sheet number of the covering $\zeta : \Omega \to \mathbb{P}^1$. Also, since $\mathbb{P}^1$ is compact, the term $\ell(\zeta, \partial\Omega, \omega_{\mathbb{P}^1})$ is equal to the length of the relative boundary of the covering ζ.

(2) Consider the case $E^\dagger = \mathbb{P}^1$. Then the condition (3.2.3) is satisfied automatically. If Ω is non-compact, then $\mathcal{G}^I = \emptyset$ and $\mathcal{G}^P = \{\Omega\}$, hence $\vartheta(\zeta, \upsilon) = 0$. On the other hand, if Ω is compact, then $\mathcal{G}^I = \{\Omega\}$ and $\mathcal{G}^P = \emptyset$. Since ζ is non-constant, we have $\zeta(\Omega) \not\subset E_\infty$ and $\vartheta(\zeta, \upsilon) = 0$. Hence we have $\vartheta(\zeta, \upsilon) = 0$, in both cases. Since we have $\rho(\Omega) \leq \rho^+(\Omega)$, we get

$$\begin{aligned} \rho^+(\Omega) - \sum_{j=1}^{p} \sum_{G \in \mathcal{G}_j^I} \rho(G) - \sum_{j=1}^{p} \sum_{G \in \mathcal{G}_j^P} \rho^+(G) - \sum_{G \in \mathcal{G}_\infty^I} \rho(G) \\ \geq (p-1) A(\zeta, \Omega, \omega_{\mathbb{P}^1}) - h\ell(\zeta, \partial\Omega, \omega_{\mathbb{P}^1}). \end{aligned} \tag{3.2.6}$$

Here we can write $\mathcal{G}_\infty^I$ as $\mathcal{I}(\zeta^{-1}(E_\infty), \Omega)$.

3.3. *Rouché's theorem.* We denote by $\mathrm{dist}(x, y)$ the distance of $x, y \in \mathbb{P}^1$ with respect to the Fubini-Study metric on $\mathbb{P}^1$.

Lemma 4 ([Y2, Lemma 3]). *Let $E \subset \mathbb{P}^1$ be a Jordan domain and let b be a point in E. Then there is a positive constant $C = C(E, b) > 0$ with the following property: Let Ω be a finite domain in a Riemann surface $\mathscr{F}$ and let ζ be a meromorphic function on $\mathscr{F}$ such that $\zeta(\Omega) = E$ and $\zeta(\partial\Omega) = \partial E$. Then for a meromorphic function α on $\mathscr{F}$ such that* $\mathrm{dist}(\alpha(z), b) < C$ *for $z \in \overline{\Omega}$, there exists a point $z \in \Omega$ with $\zeta(z) = \alpha(z)$.*

4. Local value distribution

4.1. *Notations.* In this section, we work around a neighborhood of a point $x \in \overline{\mathscr{M}}_{0,q}$. This point x will be fixed in this section. We denote by $\mathrm{edge}(\Gamma_x)$ the set of edges of Γ_x, i.e.,

$$\mathrm{edge}(\Gamma_x) = \big\{\{v, v'\};\ v \text{ and } v' \text{ are adjacent vertices of } \Gamma_x\big\}.$$

Then $\mathrm{edge}(\Gamma_x)$ is an empty set if and only if $x \in \mathscr{M}_{0,q}$. Let v and v' be distinct vertices of Γ_x. A *path* joining two vertices v and v' is a sequence of disjoint vertices

$$v = v_0, v_1, \cdots, v_r = v',$$

where v_{i-1} and v_i are adjacent for $i = 1, \dots, r$. Since Γ_x is a tree, for every distinct vertices v and v', there exists the unique path which join v and v'.

4.1.1. Take a vertex $v \in \mathrm{vert}(\Gamma_x)$. Recall that C_v is the irreducible component of $\mathscr{C}_x$ corresponding to $v \in \mathrm{vert}(\Gamma_x)$. Put

$$P_v^m = \{i \in (q);\ \sigma_i(x) \in C_v\} \qquad \text{("marked points" is abbreviated to } m\text{)},$$

$$P_v^n = \{v' \in \mathrm{vert}(\Gamma_x); v' \text{ is adjacent with } v\} \qquad \text{("nodes" is abbreviated to } n\text{)}.$$

Note that we have $\cup_{v\in \mathrm{vert}(\Gamma_x)} P_v^m = (q)$ and $P_v^m \cap P_{v'}^m = \emptyset$ for $v \neq v'$ because marked points are smooth points of $\mathscr{C}_x$. Hence for each $i \in (q)$, there exists the unique vertex $v \in \mathrm{vert}(\Gamma_x)$ such that $\sigma_i(x) \in C_v$. Put $P = (q) \coprod \mathrm{vert}(\Gamma_x)$, $P_v = P_v^m \coprod P_v^n \subset P$ and $d_v = \mathrm{card}\, P_v$.

4.1.2. Define $\varsigma : P_v \to C_v$ by the following rule. If $\tau \in P_v^m$, then $\varsigma(\tau) = \sigma_\tau(x)$; on the other hand, if $\tau \in P_v^n$, then $\varsigma(\tau) = C_v \cap C_\tau$. Then ς is an injection, and the image $\varsigma(P_v)$ is the set of the special points of C_v, which are either marked points or nodes. Hence P_v can be identified with the special points of C_v by ς, so $d_v \geq 3$ (cf. Definition 1.5.1).

4.1.3. *Definition of $\varphi_{\langle v\rangle}$.* For $v \in \mathrm{vert}(\Gamma_x)$, there exists $\langle v\rangle \in \mathscr{J}$ with the following property: The restriction $\varphi_{\langle v\rangle}|_{C_v} : C_v \to \mathbb{P}^1$ is an isomorphism and the restrictions $\varphi_{\langle v\rangle}|_{C_{v'}} : C_{v'} \to \mathbb{P}^1$ are constant maps for all $v' \in \mathrm{vert}(\Gamma_x)\backslash\{v\}$. To see this, observe the following. When $q = 3$, our assertion is trivial because $\overline{\mathscr{M}}_{0,3} \simeq \mathrm{pt}$ and $\overline{\mathscr{U}}_{0,3} \simeq \mathbb{P}^1$. Hence in the following, we consider the case $q \geq 4$. By forgetting a marking σ_j ($j \in (q)$), we have the following commutative diagram of holomorphic maps.

$$\begin{array}{ccc} \overline{\mathscr{U}}_{0,q} & \xrightarrow{c'_j} & \overline{\mathscr{U}}_{0,q-1} \\ {\scriptstyle \varpi_q}\downarrow & & \downarrow{\scriptstyle \varpi_{q-1}} \\ \overline{\mathscr{M}}_{0,q} & \xrightarrow{c_j} & \overline{\mathscr{M}}_{0,q-1} \end{array}$$

Here $c'_j \circ \sigma_i = \sigma'_i \circ c_j$ for $i \in (q)\backslash\{j\}$, where σ'_i are the universal sections of ϖ_{q-1} which are assumed to be labeled by the set $(q)\backslash\{j\}$. Let

$$\overline{c}_j = c'_j|_{\mathscr{C}_x} : \mathscr{C}_x \to \mathscr{C}_{c_j(x)} = \varpi_{q-1}^{-1}(c_j(x))$$

be the restriction on the fiber $\mathscr{C}_x$. Put

$$Q = \{v' \in \mathrm{vert}(\Gamma_x);\ \overline{c}_j|_{C_{v'}} : C_{v'} \to \mathbb{P}^1 \text{ is not constant}\}$$

and let $C = \cup_{v' \in Q} C_{v'}$ be the curve obtained by collapsing the components $C_{v'}$ for $v' \notin Q$. Then we know (cf. [Ke, p.547]) that

$$(4.1.1) \qquad Q = \left\{v' \in \mathrm{vert}(\Gamma_x);\ \mathrm{card}\,(P_{v'} \cap (P\backslash\{j\})) \geq 3\right\}$$

and that

(4.1.2) the induced map $C \to \mathscr{C}_{c_j(x)}$ from $\overline{c}_j$ is an isomorphism.

Now we may take $j \in (q)$ such that the number of special points on C_v other than $\sigma_j(x)$ is at least three. (If there exists $j' \in (q)$ with $\sigma_{j'}(x) \notin C_v$, then put $j = j'$. Otherwise, take arbitrary $j \in (q)$, where note that $q \geq 4$.) Then by (4.1.1), we have $v \in Q$. Hence by (4.1.2), the restriction $\overline{c}_j|_{C_v} : C_v \to \mathscr{C}_{c_j(x)}$ is an injection. Taking such j inductively, we may take $\alpha \in \mathscr{J}$ such that the restriction $\varphi_\alpha : C_v \to \mathbb{P}^1$ is an injection, hence an isomorphism. Here note that φ_α is the map which forgets all the markings except those elements of α. Using (4.1.2) inductively, we conclude that $\varphi_\alpha|_{C_{v'}}$ are constant maps for all $v' \in \mathrm{vert}(\Gamma_x)\backslash\{v\}$. Put $\langle v\rangle = \alpha$, which will be fixed for each $v \in \mathrm{vert}(\Gamma_x)$.

4.1.4. For $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$, put $w_v(\tau) = \varphi_{\langle v\rangle} \circ \varsigma(\tau) \in \mathbb{P}^1$. Then $w_v : P_v \to \mathbb{P}^1$ is an injection.

4.1.5. *Definitions of* $\hat{\tau}_v$ *and* ι_v. For $v \in \mathrm{vert}(\Gamma_x)$, we define the map $\hat{\tau}_v : (q) \to P_v$ by the following rule. Take $i \in (q)$. If $i \in P_v^m$, then put $\hat{\tau}_v(i) = i \in P_v$. Otherwise, take the vertex $v' \in \mathrm{vert}(\Gamma_x)\backslash\{v\}$ with $i \in P_{v'}^m$ and the unique path

$$v = v_0, v_1, \cdots, v_r = v'$$

joining v and v'. Put $\hat{\tau}_v(i) = v_1 \in P_v$. Then we have

$$(4.1.3) \qquad w_v(\hat{\tau}_v(i)) = \varphi_{\langle v\rangle} \circ \sigma_i(x) \quad \text{for all } i \in (q) \text{ and } v \in \mathrm{vert}(\Gamma_x).$$

There exists a section $\iota_v : P_v \to (q)$ of $\hat{\tau}_v : (q) \to P_v$. This ι_v is defined by the following rule. For $i \in P_v^m$, put $\iota_v(i) = i \in (q)$. For a vertex $v' \in P_v^n$, take a maximal path

$$(4.1.4) \qquad v, v', v_1, \cdots, v_r$$

starting from the edge $\{v, v'\}$, i.e., there exists no path extending (4.1.4). Then we have $\mathrm{card}\, P_{v_r}^n = 1$ (otherwise we can extend the path). By $d_{v_r} \geq 3$, there exists $i \in P_{v_r}^m$. Put $\iota_v(v') = i$. Then this ι_v is a section of $\hat{\tau}_v$, which will be fixed for each $v \in \mathrm{vert}(\Gamma_x)$.

If v and v' are adjacent vertices of Γ_x, we have

$$(4.1.5) \qquad \hat{\tau}_{v'}(\iota_v(v')) \neq v \quad (\text{as elements of } P_{v'}),$$

which easily follows from the geometric meaning of the above objects.

4.1.6. For $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$, put $\beta_{v,\tau} = \varphi_{\langle v \rangle} \circ \sigma_{\iota_v(\tau)} : \overline{\mathscr{M}}_{0,q} \to \mathbb{P}^1$. Then we have $\beta_{v,\tau}(x) = w_v(\tau) \in \mathbb{P}^1$, which follows from (4.1.3) and the fact that ι_v is a section of $\hat{\tau}_v$.

4.2. *Geometric lemma.* Recall that $\mathscr{L}$ is the hyper-plane section bundle on $\mathbb{P}^1$.

Lemma 5. *There exists a Zariski open neighborhood $U_x \subset \overline{\mathscr{M}}_{0,q}$ of x such that*

$$\sum_{v \in \mathrm{vert}(\Gamma_x)} (d_v - 2)\varphi^*_{\langle v \rangle}\mathscr{L} = K_q \quad \text{on } \varpi_q^{-1}(U_x). \tag{4.2.1}$$

Proof. Put $M = \sum_{v \in \mathrm{vert}(\Gamma_x)}(d_v - 2)\varphi^*_{\langle v \rangle}\mathscr{L} - K_q$. For $y \in \overline{\mathscr{M}}_{0,q}$, let M_y be the restriction of M on $\mathscr{C}_y$. Note that C_v are isomorphic to $\mathbb{P}^1$ for all $v \in \mathrm{vert}(\Gamma_x)$ and that the degrees of the restrictions $K_q|_{C_v}$ and $((d_v - 2)\varphi^*_{\langle v \rangle}\mathscr{L})|_{C_v}$ are both equal to $d_v - 2$ (cf. [M, p.202]). Hence $M_x|_{C_v}$ are the trivial line bundles on C_v for all $v \in \mathrm{vert}(\Gamma_x)$. Hence M_x is the trivial line bundle on $\mathscr{C}_x$, which follows from the fact that Γ_x is a tree.

Since ϖ_q is a flat morphism, by the theorem of semi-continuity [Ha], there exists a non-empty affine open neighborhood U_x of x such that

$$\dim H^0(\mathscr{C}_y, M_y) \le 1, \quad \dim H^0(\mathscr{C}_y, M_y^{-1}) \le 1 \tag{4.2.2}$$

for all $y \in U_x$. Put

$$Z = \{y \in U_x;\ \dim H^0(\mathscr{C}_y, M_y) = 1\}.$$

Again by the theorem of semi-continuity, we know that Z is a Zariski closed subset of U_x. Take a point y from $U_x \backslash \mathscr{Z}_q$, which is a non-empty Zariski open subset of U_x. Then $\mathscr{C}_y$ is isomorphic to $\mathbb{P}^1$, hence the condition (4.2.2) implies that M_y is the trivial line bundle on $\mathscr{C}_y$. Hence $U_x \backslash \mathscr{Z}_q \subset Z$. This implies that $Z = U_x$.

Now by the theorem of Grauert [Ha], we have a section $s \in H^0(\varpi_q^{-1}(U_x), M)$ such that the restriction $s|_{\mathscr{C}_x}$ is equal to the section 1 of the trivial line bundle M_x, where we note that U_x is affine. Let D be the divisor on $\varpi_q^{-1}(U_x)$ defined by $s = 0$. Since ϖ_q is a projective morphism, $\varpi_q(\mathrm{supp}\, D)$ is a Zariski closed subset of U_x, which does not contain x. Hence by replacing U_x by $U_x \backslash \varpi_q(\mathrm{supp}\, D)$, we may assume that s is a nowhere vanishing section on $\varpi_q^{-1}(U_x)$. This implies that the restriction $M|_{\varpi_q^{-1}(U_x)}$ is the trivial line bundle, which proves our lemma. □

4.3. *Local version of the theorem.*

Lemma 6. *Let Λ be a countable set of non-degenerate q-hol-quintets. Then for all $x \in \overline{\mathscr{M}}_{0,q}$, there exist an open neighborhood $V_x = V_x(\Lambda)$ of x and a positive constant $h_x = h_x(\Lambda) > 0$ with the following property: Let $(\mathscr{F}, \mathscr{R}, \pi, g, b) \in \Lambda$ be a q-hol-quintet contained in Λ. Let $R \subset \mathscr{R}$ be a finite domain such that $b(R) \subset V_x$. Put $F = \pi^{-1}(R)$. Then we have the following inequality*

$$\begin{aligned} A(g, F, \kappa_q) \le \overline{n}(g, \mathscr{D}_q, F) + \mathrm{disc}(\pi, R) + \deg \pi \rho^+(R) \\ + h_x \ell(g, \partial F, \omega_q) + h_x \deg \pi\ \overline{n}(b, \mathscr{Z}_q, R). \end{aligned} \tag{4.3.1}$$

Proof. For $(\mathscr{F}, \mathscr{R}, \pi, g, b) \in \Lambda$ and $\alpha \in \mathscr{J}$, put

$$g_\alpha = \varphi_\alpha \circ g,$$

which is a non-constant meromorphic function on $\mathscr{F}$. For $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$, since we have $w_v(\tau) \neq w_v(\tau')$ for $\tau \neq \tau'$, we may take a Jordan domain $E^v_\tau \subset \mathbb{P}^1$ such that

- $w_v(\tau) \in E^v_\tau$,
- $\overline{E^v_\tau} \cap \overline{E^v_{\tau'}} = \emptyset$ for $\tau \neq \tau'$,
- $g_\alpha : \mathscr{F} \to \mathbb{P}^1$ and E^v_τ satisfy the condition (3.1.1) for all $v \in \mathrm{vert}(\Gamma_x)$, $\tau \in P_v$, $\alpha \in \mathscr{J}$ and $(\mathscr{F}, \mathscr{R}, \pi, g, b) \in \Lambda$, i.e., if $a \in \mathscr{F}$ is a branch point of g_α for some $(\mathscr{F}, \mathscr{R}, \pi, g, b) \in \Lambda$ and $\alpha \in \mathscr{J}$, then $g_\alpha(a) \notin \partial E^v_\tau$ for all $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$.

Here in the third condition, we note that the ramification points of the coverings

$$\{g_\alpha : \mathscr{F} \to \mathbb{P}^1\}_{\alpha \in \mathscr{J}, (\mathscr{F}, \mathscr{R}, \pi, g, b) \in \Lambda}$$

are countable, because Λ is countable.

For each $\{v, v'\} \in \mathrm{edge}(\Gamma_x)$, put

$$\hat{D} = \varphi^{-1}_{\langle v \rangle}(\mathbb{P}^1 \backslash E^v_{v'}) \cap \varphi^{-1}_{\langle v' \rangle}(\mathbb{P}^1 \backslash E^{v'}_v),$$

which is a compact subset of $\overline{\mathscr{U}}_{0,q}$. Then we have $\varpi_q^{-1}(x) \cap \hat{D} = \emptyset$. Hence the image $\varpi_q(\hat{D}) \subset \overline{\mathscr{M}}_{0,q}$ is a compact subset which does not contain the point x. Hence, we conclude that there exists an open neighborhood $V_{v,v'}$ of x such that $\varpi_q^{-1}(V_{v,v'}) \cap \hat{D} = \emptyset$, that is,

$$\varphi_{\langle v \rangle}(\varphi^{-1}_{\langle v' \rangle}(\mathbb{P}^1 \backslash E^{v'}_v) \cap \varpi_q^{-1}(V_{v,v'})) \subset E^v_{v'}. \tag{4.3.2}$$

Let $V_x \subset \overline{\mathscr{M}}_{0,q}$ be an open neighborhood of x such that

- $\overline{V_x} \subset U_x$ (cf. Lemma 5),
- $\overline{V_x} \subset V_{v,v'}$ for all $\{v, v'\} \in \mathrm{edge}(\Gamma_x)$,
- $\mathrm{dist}(w_v(\tau), \beta_{v,\tau}(y)) < C(E^v_\tau, w_v(\tau))$ for all $y \in \overline{V_x}$, $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$ (the constant C is defined in Lemma 4), where we note that $\beta_{v,\tau}(x) = w_v(\tau)$,
- $\varphi_{\langle v \rangle} \circ \sigma_i(\overline{V_x}) \subset E^v_{\hat{\tau}_v(i)}$ for all $v \in \mathrm{vert}(\Gamma_x)$ and $i \in (q)$, where we note that $\varphi_{\langle v \rangle} \circ \sigma_i(x) = w_v(\hat{\tau}_v(i)) \in E^v_{\hat{\tau}_v(i)}$ (cf. (4.1.3)).

We denote by v_o the unique vertex of Γ_x such that $\sigma_1(x) \in C_{v_o}$. For each vertex $v \in \mathrm{vert}\,\Gamma_x \backslash \{v_o\}$, take the unique path joining v_o and v

$$v_o = v_0, v_1, \cdots, v_{r-1}, v_r = v.$$

We denote this vertex v_{r-1} by v^- which is uniquely determined from the vertex v.

Take $\lambda = (\mathscr{F}, \mathscr{R}, \pi, g, b) \in \Lambda$ and a finite domain $R \subset \mathscr{R}$ such that $b(R) \subset V_x$. Put $F = \pi^{-1}(R)$. For a vertex $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$, put

$$\mathcal{G}^I_{v,\tau} = \mathcal{I}(g^{-1}_{\langle v \rangle}(E^v_\tau), F), \quad \mathcal{G}^P_{v,\tau} = \mathcal{P}(g^{-1}_{\langle v \rangle}(E^v_\tau), F).$$

For the vertex v_o, we apply Lemma 3 (cf. (3.2.6)) to the case that

$$\mathscr{F} = \mathscr{F}, \ \Omega = H \in \mathcal{C}(F), \ \zeta = v = g_{\langle v_o \rangle}|_H,$$

$$E^\dagger = \mathbb{P}^1, \ \{E_j\}_{j=1,\dots,p} = \{E^{v_o}_{v'}\}_{v' \in P^n_{v_o}} \cup \{E^{v_o}_i\}_{i \in P^m_{v_o} \backslash \{1\}}, \ E_\infty = E^{v_o}_1.$$

Adding over all $H \in \mathcal{C}(F)$ and using the fact $\sum_{i \in P^m_{v_o} \setminus \{1\}} \sum_{G \in \mathcal{G}^P_{v_o,i}} \rho^+(G) \geq 0$, we obtain the following: There exists a positive constant $h_{v_o} > 0$ which does not depend on the choices of $\lambda \in \Lambda$ and R such that

$$\begin{aligned}\text{IE}(v_o)\text{:} \ &\sum_{H \in \mathcal{C}(F)} \rho^+(H) \\ &- \sum_{v \in P^n_{v_o}} \left(\sum_{G \in \mathcal{G}^I_{v_o,v}} \rho(G) + \sum_{G \in \mathcal{G}^P_{v_o,v}} \rho^+(G) \right) - \sum_{i \in P^m_{v_o}} \sum_{G \in \mathcal{G}^I_{v_o,i}} \rho(G) \\ &\qquad \geq (d_{v_o} - 2) A(g_{\langle v_o \rangle}, F, \omega_{\mathbb{P}^1}) - h_{v_o} \ell(g_{\langle v_o \rangle}, \partial F, \omega_{\mathbb{P}^1}).\end{aligned}$$

For a vertex $v \in \operatorname{vert} \Gamma_x \backslash \{v_o\}$, we put

$$\mathcal{G}^I_v = \mathcal{I}(g^{-1}_{\langle v \rangle}(E^v_{v^-}), F \cap g^{-1}_{\langle v^- \rangle}(E^{v^-}_v)).$$

By (4.3.2), we may apply Lemma 3 to the case that

$$\mathscr{F} = \mathscr{F}, \ \Omega = H \in \mathcal{C}(F), \ \zeta = g_{\langle v \rangle}|_H, \ \upsilon = g_{\langle v^- \rangle}|_H, \ E^\dagger = E^{v^-}_v,$$

$$\{E_j\}_{j=1,\dots,p} = \{E^v_{v'}\}_{v' \in P^n_v \setminus \{v^-\}} \cup \{E^v_i\}_{i \in P^m_v}, \ E_\infty = E^v_{v^-}.$$

Adding over all $H \in \mathcal{C}(F)$ and using the fact $\sum_{i \in P^m_v} \sum_{G \in \mathcal{G}^P_{v,i}} \rho^+(G) \geq 0$, we obtain the following: There exists a positive constant $h_v > 0$ which does not depend on the choices of $\lambda \in \Lambda$ and R such that

$$\begin{aligned}\text{IE}(v)\text{:} \ &\sum_{H \in \mathcal{C}(F)} \vartheta(g_{\langle v \rangle}|_H, g_{\langle v^- \rangle}|_H) + \sum_{G \in \mathcal{G}^I_{v^-,v}} \rho(G) + \sum_{G \in \mathcal{G}^P_{v^-,v}} \rho^+(G) \\ &- \sum_{v' \in P^n_v \setminus \{v^-\}} \left(\sum_{G \in \mathcal{G}^I_{v,v'}} \rho(G) + \sum_{G \in \mathcal{G}^P_{v,v'}} \rho^+(G) \right) - \sum_{i \in P^m_v} \sum_{G \in \mathcal{G}^I_{v,i}} \rho(G) - \sum_{G \in \mathcal{G}^I_v} \rho(G) \\ &\qquad \geq (d_v - 2) A(g_{\langle v \rangle}, F, \omega_{\mathbb{P}^1}) - h_v \ell(g_{\langle v \rangle}, \partial F, \omega_{\mathbb{P}^1}).\end{aligned}$$

Now, using the inequality IE(v_o) for the vertex v_o and the inequalities IE(v) for vertices $v \neq v_o$, we add the inequalities IE(v) over all $v \in \operatorname{vert}(\Gamma_x)$. Then we obtain

$$\begin{aligned}(4.3.3) \quad &\sum_{H \in \mathcal{C}(F)} \rho^+(H) - \sum_{v \in \operatorname{vert}(\Gamma_x)} \sum_{i \in P^m_v} \sum_{G \in \mathcal{G}^I_{v,i}} \rho(G) \\ &+ \sum_{v \in \operatorname{vert}(\Gamma_x) \backslash \{v_o\}} \left(\sum_{H \in \mathcal{C}(F)} \vartheta(g_{\langle v \rangle}|_H, g_{\langle v^- \rangle}|_H) - \sum_{G \in \mathcal{G}^I_v} \rho(G) \right) \\ &\qquad \geq \sum_{v \in \operatorname{vert} \Gamma_x} (d_v - 2) A(g_{\langle v \rangle}, F, \omega_{\mathbb{P}^1}) - h' \ell(g, \partial F, \omega_q).\end{aligned}$$

Here we note the following two facts:

- There exists a positive constant $h' > 0$ which does not depend on the choices of $\lambda \in \Lambda$ and R such that

$$\sum_{v \in \mathrm{vert}(\Gamma_x)} h_v \ell(g_{\langle v \rangle}, \partial F, \omega_{\mathbb{P}^1}) \le h' \ell(g, \partial F, \omega_q).$$

- For a vertex $v \neq v_o$, the term

$$\sum_{G \in \mathcal{G}^I_{v^-,v}} \rho(G) + \sum_{G \in \mathcal{G}^P_{v^-,v}} \rho^+(G)$$

  appears in the left hand side of IE(v), while the term

$$-\sum_{G \in \mathcal{G}^I_{v^-,v}} \rho(G) - \sum_{G \in \mathcal{G}^P_{v^-,v}} \rho^+(G)$$

  appears in the left hand side of IE(v^-) because $v \in P^n_{v^-}$ and $v \neq (v^-)^-$. Hence these terms are canceled when we add inequalities over all $v \in \mathrm{vert}(\Gamma_x)$.

Claim. The following inequalities hold

$$\sum_{H \in \mathcal{C}(F)} \vartheta(g_{\langle v \rangle}|_H, g_{\langle v^- \rangle}|_H) - \sum_{G \in \mathcal{G}^I_v} \rho(G) \le 2 \deg \pi \overline{n}(b, \mathscr{X}_q, R), \tag{4.3.4}$$

$$-\sum_{v \in \mathrm{vert}(\Gamma_x)} \sum_{i \in P^m_v} \sum_{G \in \mathcal{G}^I_{v,i}} \rho(G) \le \overline{n}(g, \mathscr{D}_q, F) + q \deg \pi \overline{n}(b, \mathscr{X}_q, R), \tag{4.3.5}$$

$$\sum_{H \in \mathcal{C}(F)} \rho^+(H) \le \mathrm{disc}(\pi, R) + \deg \pi \rho^+(R). \tag{4.3.6}$$

Proof of (4.3.4). For $H \in \mathcal{C}(F)$ and for $\{v, v'\} \in \mathrm{edge}(\Gamma_x)$, let $\vartheta'(v', v, H)$ denote the number of connected components G in $\mathcal{I}(g^{-1}_{\langle v \rangle}(E^v_{v'}), H)$ such that $g_{\langle v' \rangle}(G) \subset E^{v'}_v$. Then we have

$$\vartheta'(v, v^-, H) = \vartheta(g_{\langle v \rangle}|_H, g_{\langle v^- \rangle}|_H).$$

Note that

$$-\sum_{G \in \mathcal{G}^I_v} \rho(G) \le \mathrm{card}\, \mathcal{G}^I_v \le \sum_{H \in \mathcal{C}(F)} \vartheta'(v^-, v, H).$$

Hence to prove (4.3.4), it suffices to prove

$$\sum_{H \in \mathcal{C}(F)} \vartheta'(v', v, H) \le \deg \pi \overline{n}(b, \mathscr{X}_q, R) \tag{4.3.7}$$

for $\{v, v'\} \in \mathrm{edge}(\Gamma_x)$. Take $G \in \mathcal{I}(g^{-1}_{\langle v \rangle}(E^v_{v'}), H)$ such that $g_{\langle v' \rangle}(G) \subset E^{v'}_v$. Then by the definition of V_x, we may apply Lemma 4 to the case that

$$\mathscr{F} = H,\ E = E^v_{v'},\ \Omega = G,\ \zeta = g_{\langle v \rangle} (= \varphi_{\langle v \rangle} \circ g),\ \alpha = \beta_{v,v'} \circ b \circ \pi.$$

We conclude that there exists $z \in G$ such that

$$\varphi_{\langle v \rangle} \circ g(z) = \varphi_{\langle v \rangle} \circ \sigma_{\iota_v(v')} \circ b \circ \pi(z) \quad (\text{note that } \beta_{v,v'} = \varphi_{\langle v \rangle} \circ \sigma_{\iota_v(v')}). \tag{4.3.8}$$

We shall prove $b \circ \pi(z) \in \operatorname{supp} \mathscr{Z}_q$ by a contradiction. Suppose $b \circ \pi(z) \notin \operatorname{supp} \mathscr{Z}_q$. Then (4.3.8) implies

$$\varphi_{\langle v' \rangle} \circ g(z) = \varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')} \circ b \circ \pi(z), \tag{4.3.9}$$

which follows from the fact that the restrictions $\varphi_{\langle v \rangle}|_{\mathscr{C}_y}$ and $\varphi_{\langle v' \rangle}|_{\mathscr{C}_y}$ give isomorphisms $\mathscr{C}_y \to \mathbb{P}^1$ for $y \in \overline{\mathscr{M}}_{0,q} \backslash \mathscr{Z}_q$. By the assumption $g_{\langle v' \rangle}(G) \subset E_v^{v'}$, we have

$$\varphi_{\langle v' \rangle} \circ g(z) \in E_v^{v'}. \tag{4.3.10}$$

On the other hand, we have

$$\varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')} \circ b \circ \pi(z) \notin E_v^{v'}. \tag{4.3.11}$$

To see this, we note that $\varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')}(x) = w_{v'}(\hat{\tau}_{v'}(\iota_v(v')))$ (cf. (4.1.3)). By the definition of V_x, we have $\varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')}(y) \in E^{v'}_{\hat{\tau}_{v'}(\iota_v(v'))}$, hence $\varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')}(y) \notin E_v^{v'}$ on $y \in \overline{V_x}$ (cf. (4.1.5)). Since $b \circ \pi(z) \in V_x$, we get (4.3.11). These (4.3.9), (4.3.10) and (4.3.11) give a contradiction. Hence we have $b \circ \pi(z) \in \operatorname{supp} \mathscr{Z}_q$. This proves (4.3.7) and (4.3.4). □

Proof of (4.3.5). We have $-\rho(G) \leq 1$ for $G \in \mathcal{G}^I_{v,i}$, hence

$$-\sum_{v \in \operatorname{vert}(\Gamma_x)} \sum_{i \in P_v^m} \sum_{G \in \mathcal{G}^I_{v,i}} \rho(G) \leq \sum_{v \in \operatorname{vert}(\Gamma_x)} \sum_{i \in P_v^m} \operatorname{card} \mathcal{G}^I_{v,i}.$$

For $G \in \mathcal{G}^I_{v,i}$, by the definition of V_x, we may apply Lemma 4 to the case that

$$E = E_i^v,\ \Omega = G,\ \zeta = \varphi_{\langle v \rangle} \circ g,\ \alpha = \varphi_{\langle v \rangle} \circ \sigma_i \circ b \circ \pi$$

to conclude that there exists $z \in G$ such that

$$\varphi_{\langle v \rangle} \circ g(z) = \varphi_{\langle v \rangle} \circ \sigma_i \circ b \circ \pi(z).$$

This implies that either $g(z) = \sigma_i \circ b \circ \pi(z)$ or $b \circ \pi(z) \in \operatorname{supp} \mathscr{Z}_q$ is valid. (Note that $\varphi_{\langle v \rangle}|_{\mathscr{C}_y}$ is an isomorphism for $y \in \overline{\mathscr{M}}_{0,q} \backslash \mathscr{Z}_q$.) Hence for $i = P_v^m$, we have

$$\operatorname{card} \mathcal{G}^I_{v,i} \leq \overline{n}(g, \mathscr{D}_{q,i}, F) + \deg \pi \overline{n}(b, \mathscr{Z}_q, R),$$

where we put $\mathscr{D}_{q,i} = \sigma_i(\overline{\mathscr{M}}_{0,q}) \subset \overline{\mathscr{U}}_{0,q}$. Since $\mathscr{D}_q = \sum_{1 \leq i \leq q} \mathscr{D}_{q,i}$, $\mathscr{D}_{q,i} \cap \mathscr{D}_{q,i'} = \emptyset$ for $i \neq i'$, $\cup_{v \in \operatorname{vert}(\Gamma_x)} P_v^m = (q)$ and $P_v^m \cap P_{v'}^m = \emptyset$ for $v \neq v'$, we obtain

$$\sum_{v \in \operatorname{vert}(\Gamma_x)} \sum_{i \in P_v^m} \operatorname{card} \mathcal{G}^I_{v,i} \leq \overline{n}(g, \mathscr{D}_q, F) + q \deg \pi \overline{n}(b, \mathscr{Z}_q, R).$$

This proves (4.3.5). □

Proof of (4.3.6). For $H \in \mathcal{C}(F)$, the restriction $\pi|_H : H \to R$ is a proper map. Hence, by Hurwitz's formula, we have

$$\rho(H) = (\deg \pi|_H)\rho(R) + \operatorname{disc}(\pi|_H, R).$$

We also have $\rho(R) \leq \rho(H)$. Hence we get

$$\rho^+(H) \leq (\deg \pi|_H)\rho^+(R) + \operatorname{disc}(\pi|_H, R)$$

and

$$\sum_{H\in\mathcal{C}(F)}\rho^+(H)\leq\rho^+(R)\sum_{H\in\mathcal{C}(F)}\deg\pi|_H+\sum_{H\in\mathcal{C}(F)}\mathrm{disc}(\pi|_H,R)$$
$$=\rho^+(R)\deg\pi+\mathrm{disc}(\pi,R).$$

This proves (4.3.6) and conclude our proof of claim. $\square$

Now note that the Fubini-Study form $\omega_{\mathbb{P}^1}$ is the curvature form of the Fubini-Study metric on the hyper-plane section bundle $\mathscr{L}$. Hence by Lemma 5, the restriction of the (1,1)-form

$$\sum_{v\in\mathrm{vert}(\Gamma_x)}(d_v-2)\varphi^*_{\langle v\rangle}\omega_{\mathbb{P}^1}-\kappa_q$$

on $\varpi_q^{-1}(U_x)$ is a curvature form of the trivial line bundle. Hence, there exists a C^∞-function ψ on $\varpi_q^{-1}(U_x)$ such that

$$\sum_{v\in\mathrm{vert}(\Gamma_x)}(d_v-2)\varphi^*_{\langle v\rangle}\omega_{\mathbb{P}^1}-\kappa_q=dd^c\psi\quad\text{on }\varpi_q^{-1}(U_x).$$

Note that there exists a positive constant $h''>0$ which does not depend on the choices of $\lambda\in\Lambda$ and R such that

$$|A(g,F,dd^c\psi)|=\left|\int_F g^*dd^c\psi\right|=\left|\int_{\partial F}g^*d^c\psi\right|\leq h''\ell(g,\partial F,\omega_q),$$

because the image $g(\overline{F})$ is contained in the compact set $\varpi_q^{-1}(\overline{V_x})$. Hence we get

$$\sum_{v\in\mathrm{vert}\,\Gamma_x}(d_v-2)A(g_{\langle v\rangle},F,\omega_{\mathbb{P}^1})\geq A(g,F,\kappa_q)-h''\ell(g,\partial F,\omega_q).\tag{4.3.12}$$

Put $h_x=\max\{h'+h'',2\,\mathrm{card}(\mathrm{vert}(\Gamma_x))+q-2\}$, which is a positive constant independent of the choices of $\lambda\in\Lambda$ and R. Using (4.3.3), (4.3.4), (4.3.5), (4.3.6) and (4.3.12), we get Lemma 6. $\square$

5. Lemmas for division and summation

5.1. *Algebraic lemma.* Put $\Phi=\prod_{i\in\mathscr{I}}\phi_i:\overline{\mathscr{M}}_{0,q}\to(\mathbb{P}^1)^{\mathscr{I}}$.

Lemma 7. *Φ gives an injection.*

Proof. We prove by induction on q.
For $q=3$, our lemma is trivial because $\overline{\mathscr{M}}_{0,3}\simeq\mathrm{pt}$.
Suppose our lemma is valid for all q' with $q'\leq q$, where $q\geq 3$. We shall prove our lemma for $q+1$. Our lemma is equivalent to saying that for distinct points $x,y\in\overline{\mathscr{M}}_{0,q+1}$, there exists $i\in\mathscr{I}^{q+1}$ such that $\phi_i(x)\neq\phi_i(y)$. In the case that $\tau_{q+1}(x)$ and $\tau_{q+1}(y)$ are distinct points in $\overline{\mathscr{M}}_{0,q}$, our lemma follows from the induction hypothesis. Here $\tau_{q+1}:\overline{\mathscr{M}}_{0,q+1}\to\overline{\mathscr{M}}_{0,q}$ is the morphism obtained by forgetting the marking σ_{q+1}.

On the other case, put $z=\tau_{q+1}(x)$. Using the isomorphism $\iota_{q+1}:\overline{\mathscr{M}}_{0,q+1}\to\overline{\mathscr{U}}_{0,q}$, the fiber $\tau_{q+1}^{-1}(z)$ is isomorphic to $\mathscr{C}_z$ (cf. (1.5.11)).

We first consider the case that $\iota_{q+1}(x)$ is a smooth point of $\mathscr{C}_z$. Let $v\in\mathrm{vert}(\Gamma_z)$ be the unique vertex such that $\iota_{q+1}(x)\in C_v$. Then since $\varphi_{\langle v\rangle}|_{C_v}:C_v\to\mathbb{P}^1$ is an isomorphism and $\varphi_{\langle v\rangle}|_{C_{v'}}$ is constant for $v' \in \mathrm{vert}(\Gamma_x)\backslash\{v\}$, we have $\varphi_{\langle v\rangle}(\iota_{q+1}(x)) \neq \varphi_{\langle v\rangle}(\iota_{q+1}(y))$ as desired. (By definition, we may take $i \in \mathscr{I}^{q+1}$ with $\phi_i = \varphi_{\langle v\rangle} \circ \iota_{q+1}$.)

Next we consider the case that $\iota_{q+1}(x)$ is not a smooth point of $\mathscr{C}_z$. Then $\iota_{q+1}(x)$ is a node. And there are adjacent vertices v and v' such that $\iota_{q+1}(x) = C_v \cap C_{v'}$. If $\varphi_{\langle v\rangle}(\iota_{q+1}(x)) \neq \varphi_{\langle v\rangle}(\iota_{q+1}(y))$, the proof is done. If $\varphi_{\langle v\rangle}(\iota_{q+1}(x)) = \varphi_{\langle v\rangle}(\iota_{q+1}(y))$, then we can easily see that $\varphi_{\langle v'\rangle}(\iota_{q+1}(x)) \neq \varphi_{\langle v'\rangle}(\iota_{q+1}(y))$, which proves our lemma for $q+1$. □

5.2. *Estimates for summation.* Let $\lambda = (\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ be a specified q-hol-quintet. For $i \in \mathscr{I}$, put

$$b_i = \phi_i \circ b : \mathscr{R} \to \mathbb{P}^1$$

and

$$\mathscr{I}_\lambda = \{i \in \mathscr{I}; b_i \text{ is non-constant}\}.$$

Definition 5.2.1. We call $\mathscr{I}_\lambda$ the *type* of the specified q-hol-quintet λ.

Let $\hat{\mathscr{I}} \subset \mathscr{I}^q$ be a subset. Let $\mathfrak{D} = \{D_i\}_{i\in\hat{\mathscr{I}}}$ be an $\hat{\mathscr{I}}$-tuple of Jordan domains $D_i \subset \mathbb{P}^1$. Let $\mathfrak{D}' = \{D'_i\}_{i\in\hat{\mathscr{I}}}$ be another such tuple. We say that $\mathfrak{D}'$ is compactly contained in $\mathfrak{D}$ if all D'_i are compactly contained in D_i. We also write $\mathfrak{D}' \subset \mathfrak{D}$ if $D'_i \subset D_i$ for all $i \in \hat{\mathscr{I}}$. Let $\lambda = (\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ be a specified q-hol-quintet of type $\hat{\mathscr{I}}$. We consider the following condition for $\{b_i\}_{i\in\hat{\mathscr{I}}}$ and $\{D_i\}_{i\in\hat{\mathscr{I}}}$

(5.2.2) $\qquad b_i|_{\overline{R}} : \overline{R} \to \mathbb{P}^1$ and D_i satisfy the condition (3.1.1) for all $i \in \hat{\mathscr{I}}$.

Put $R_{\mathfrak{D}} = R \cap \big(\cap_{i\in\hat{\mathscr{I}}} b_i^{-1}(D_i)\big)$ and $F_{\mathfrak{D}} = \pi^{-1}(R_{\mathfrak{D}})$.

Lemma 8. *(1) Let $\hat{\mathscr{I}} \subset \mathscr{I}^q$ be a subset. Suppose $\mathfrak{D}' = \{D'_i\}_{i\in\hat{\mathscr{I}}}$ is compactly contained in $\mathfrak{D} = \{D_i\}_{i\in\hat{\mathscr{I}}}$. Then for all $\varepsilon > 0$, there exists a positive constant $\mu_1 = \mu_1(\varepsilon, \hat{\mathscr{I}}, \mathfrak{D}, \mathfrak{D}')$ with the following property: Let $(\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ be a specified q-hol-quintet of type $\hat{\mathscr{I}}$ such that the inequality*

$$A(g, F_{\mathfrak{D}'}, \omega_q) > \mu_1 \deg \pi \left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right) \tag{5.2.3}$$

holds. Then there exists an $\hat{\mathscr{I}}$-tuple of Jordan domains $\mathfrak{D}'' = \{D''_i\}_{i\in\hat{\mathscr{I}}}$ such that $\mathfrak{D}' \subset \mathfrak{D}'' \subset \mathfrak{D}$ and that the following inequality holds

$$\ell(g, \partial F_{\mathfrak{D}''}, \omega_q) \leq \varepsilon A(g, F_{\mathfrak{D}''}, \omega_q) + \ell(g, \partial F, \omega_q).$$

Moreover, we may take $\mathfrak{D}''$ such that $(b_i)_{i\in\hat{\mathscr{I}}}$ and $\mathfrak{D}''$ satisfy the condition (5.2.2).

(2) Let $\hat{\mathscr{I}}$, $\mathfrak{D}'$ and $\mathfrak{D}$ be the same as (1). Then there exists a positive constant $\mu_2 = \mu_2(\hat{\mathscr{I}}, \mathfrak{D}, \mathfrak{D}') > 0$ with the following property: Let $(\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ be a specified q-hol-quintet of type $\hat{\mathscr{I}}$. Let $\mathfrak{D}''$ be an $\hat{\mathscr{I}}$-tuple of Jordan domains such that

$$\mathfrak{D}' \subset \mathfrak{D}'' \subset \mathfrak{D}. \tag{5.2.4}$$

Suppose that $(b_i)_{i\in\hat{\mathscr{I}}}$ and $\mathfrak{D}''$ satisfy the condition (5.2.2). *Then we have*

$$\sum_{G\in\mathcal{C}(R_{\mathfrak{D}''})} \rho^+(G) \leq \rho^+(R) + \mu_2 \left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right).$$

Proof of (1). For each $i \in \hat{\mathscr{I}}$, we fix a biholomorphic identification $\chi_i : D_i \xrightarrow{\sim} \Delta$. Put $D_i(r) = \chi_i^{-1}(\Delta(r))$ for $0 \le r \le 1$. Here we put $\Delta(r) = \{z \in \mathbb{C};\ |z| < r\}$ and $\Delta = \Delta(1)$. Let $r_0 < 1$ be a constant such that $D_i' \subset D_i(r_0)$ for all $i \in \hat{\mathscr{I}}$.

By replacing D_i by $D_i(s)$ and Δ by $\Delta(s)$ for $r_0 < s < 1$, we may assume that χ_i gives a biholomorphic map between neighborhoods of $\overline{D_i}$ and $\overline{\Delta}$. In particular, we may assume that ∂D_i is analytic for all $i \in \hat{\mathscr{I}}$.

Let $\lambda = (\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ be a specified q-hol-quintet of type $\hat{\mathscr{I}}$. For $i \in \hat{\mathscr{I}}$, put

$$\xi_i = b_i \circ \pi|_{\overline{F}} : \overline{F} \to \mathbb{P}^1,$$

$$F^i = \xi_i^{-1}(D_i) \cap F$$

and

$$\zeta_i = \chi_i \circ \xi_i|_{\overline{F^i}} : \overline{F^i} \to \overline{\Delta}.$$

For $0 < r \le 1$, put

$$\gamma_i(r) = \xi_i^{-1}(\partial D_i(r)) \cap F.$$

Let ω_E be the Euclidean form on $\Delta \subset \mathbb{C}$, which is a Kähler form. Put $S_i = A(\xi_i, F, \omega_{\mathbb{P}^1})$ and $L_i = \ell(\xi_i, \partial F, \omega_{\mathbb{P}^1})$, which are the mean sheet number and the length of the relative boundary of $\xi_i : F \to \mathbb{P}^1$, respectively.

Claim 1: There exists a positive constant $\mathcal{Q}_1$ which does not depend on the choice of λ such that

$$\ell(\zeta_i, \gamma_i(r), \omega_E) \le \mathcal{Q}_1(S_i + L_i) \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1]. \tag{5.2.5}$$

Proof of Claim 1. In this proof, we denote by $\mathcal{Q}$ any positive constant which is independent of $i \in \hat{\mathscr{I}}$, $r \in [r_0, 1]$ and the choice of λ.

For $0 < r \le 1$, put $F^i(r) = \xi_i^{-1}(D_i(r)) \cap F$ and

$$\xi_i(r) = \xi_i|_{\overline{F^i(r)}} : \overline{F^i(r)} \to \overline{D_i(r)}.$$

Define the map $\psi_r : \overline{D_i(r)} \to \overline{D_i}$ by

$$\overline{D_i(r)} \ni z \mapsto \chi_i^{-1}\left(\frac{\chi_i(z)}{r}\right) \in \overline{D_i}.$$

Let $S_{i,r}$ be the mean sheet number and $L_{i,r}$ be the length of the relative boundary of the covering $\xi_i(r) : \overline{F^i(r)} \to \overline{D_i(r)}$. Let $S_{i,r}'$ be the mean sheet number and $L_{i,r}'$ be the length of the relative boundary of the covering $\psi_r \circ \xi_i(r) : \overline{F^i(r)} \to \overline{D_i}$. Since we have

$$\frac{1}{\mathcal{Q}}\psi_r^*(\omega_{\mathbb{P}^1}|_{\overline{D_i}}) < \omega_{\mathbb{P}^1}|_{\overline{D_i(r)}} < \mathcal{Q}\psi_r^*(\omega_{\mathbb{P}^1}|_{\overline{D_i}}) \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1], \tag{5.2.6}$$

we have

$$\ell(\xi_i(r), \gamma_i(r), \omega_{\mathbb{P}^1}) \le \mathcal{Q}\ell(\psi_r \circ \xi_i(r), \gamma_i(r), \omega_{\mathbb{P}^1}) \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1]. \tag{5.2.7}$$

Here we note that $\gamma_i(r) \subset \partial F^i(r)$. Since $\psi_r \circ \xi_i(r)(\gamma_i(r)) \subset \partial D_i$, using Covering theorem 2' (cf. (3.2.2)), we get

$$\ell(\psi_r \circ \xi_i(r), \gamma_i(r), \omega_{\mathbb{P}^1}) \le \mathcal{Q}(S_{i,r}' + L_{i,r}') \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1]. \tag{5.2.8}$$

Here we note that ∂D_i is analytic for $i \in \hat{\mathscr{I}}$ by the assumption made in the beginning of the proof of this lemma. By (5.2.6), we have

$$S'_{i,r} \leq \mathcal{Q} S_{i,r}, \quad L'_{i,r} \leq \mathcal{Q} L_{i,r} \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1],$$

hence combining with (5.2.7) and (5.2.8), we have

$$\ell(\xi_i(r), \gamma_i(r), \omega_{\mathbb{P}^1}) \leq \mathcal{Q}(S_{i,r} + L_{i,r}) \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1].$$

Since we have $\chi_i^* \omega_E \leq \mathcal{Q} \omega_{\mathbb{P}^1}|_{\overline{D_i}}$ and $\chi_i \circ \xi_i(r) = \zeta_i|_{\overline{F^i(r)}}$, we have

$$\ell(\zeta_i, \gamma_i(r), \omega_E) \leq \mathcal{Q} \ell(\xi_i(r), \gamma_i(r), \omega_{\mathbb{P}^1}),$$

hence

$$\ell(\zeta_i, \gamma_i(r), \omega_E) \leq \mathcal{Q}(S_{i,r} + L_{i,r}) \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1].$$

We have $S_{i,r} \leq \mathcal{Q}(S_i + L_i)$ for $r_0 \leq r \leq 1$ by Covering theorem 1 (cf. (3.2.1)). Using $L_{i,r} \leq L_i$, we obtain

$$\ell(\zeta_i, \gamma_i(r), \omega_E) \leq \mathcal{Q}(S_i + L_i) \quad \text{for} \quad i \in \hat{\mathscr{I}},\ r \in [r_0, 1].$$

This proves our claim. $\square$

We take a positive constant $\mathcal{Q}_2$ which does not depend on the choice of λ and satisfies the following estimates

$$\sum_{i \in \hat{\mathscr{I}}} S_i = \deg \pi \sum_{i \in \hat{\mathscr{I}}} A(b_i, R, \omega_{\mathbb{P}^1}) \leq \mathcal{Q}_2 \deg \pi A(b, R, \eta_q), \tag{5.2.9}$$

and

$$\sum_{i \in \hat{\mathscr{I}}} L_i = \sum_{i \in \hat{\mathscr{I}}} \ell(\xi_i, \partial F, \omega_{\mathbb{P}^1}) \leq \mathcal{Q}_2 \ell(g, \partial F, \omega_q) \leq \mathcal{Q}_2 \deg \pi \ell(g, \partial F, \omega_q). \tag{5.2.10}$$

(We note the trivial estimate $1 \leq \deg \pi$.)

Take a positive constant $\varepsilon > 0$ and put

$$\mu_1 = \frac{2\mathcal{Q}_1 \mathcal{Q}_2}{\varepsilon^2 (1 - r_0)},$$

which is a positive constant independent of the choice of λ.

To state the second claim, we introduce some notations. We shall also denote the restriction $\zeta_i|_{F_{\mathfrak{D}}}$ by ζ_i. Take a subset $I \subset \hat{\mathscr{I}}$ with the following properties:

- If $i \in I$ and $i' \in I$ are distinct, then $|\zeta_i| \neq |\zeta_{i'}|$ on $F_{\mathfrak{D}}$.
- For all $i \in \hat{\mathscr{I}}$ there exists $i' \in I$ such that $|\zeta_i| = |\zeta_{i'}|$ on $F_{\mathfrak{D}}$.

For $i \in I$ and $r \in [0, 1]$, put

$$\Omega_i = \{z \in F_{\mathfrak{D}};\ |\zeta_i(z)| > |\zeta_{i'}(z)| \quad \text{for all } i' \in I \backslash \{i\}\},$$
$$\Omega_i(r) = \{z \in \Omega_i;\ |\zeta_i(z)| < r\},$$
$$\hat{\gamma}_i(r) = \overline{\Omega_i} \cap \gamma_i(r)$$

and

$$\ell_i(r) = \ell(g, \hat{\gamma}_i(r), \omega_q), \quad A_i(r) = A(g, \Omega_i(r), \omega_q).$$

For $r \in [0, 1]$, put

$$\mathfrak{D}(r) = \{D_i(r)\}_{i \in \hat{\mathscr{I}}},$$

and

$$\ell(r)=\sum_{i\in I}\ell_i(r),\quad A(r)=A(g,F_{\mathfrak{D}(r)},\omega_q).$$

Then by the above definitions, we have

$$A(r)=\sum_{i\in I}A_i(r),\quad \ell(g,\partial F_{\mathfrak{D}(r)},\omega_q)\le \ell(g,\partial F,\omega_q)+\ell(r). \tag{5.2.11}$$

Define the subset $E(\varepsilon)\subset[r_0,1]$ by

$$r\in E(\varepsilon)\Longleftrightarrow \ell(r)>\varepsilon A(r).$$

Claim 2: Suppose that the inequality (5.2.3) holds for λ. Then the set $[r_0,1]\backslash E(\varepsilon)$ is not a null set.

Proof of Claim 2. For $i\in I$, put

$$g^*(\omega_q)|_{\overline{\Omega_i}}=\frac{\sqrt{-1}}{2}G_i d\zeta_i\wedge d\overline{\zeta_i},$$

where G_i is a C^∞ function on $\overline{\Omega_i}\backslash\{z\in\overline{\Omega_i};\zeta_i'(z)=0\}$ with $G_i\ge 0$. Then for $r\in(0,1]$, we have

$$\ell_i(r)=\int_{\hat\gamma_i(r)}\sqrt{G_i}\,rd\arg\zeta_i$$

and

$$A_i(r)=\int_0^r\left(\int_{\hat\gamma_i(t)}G_i t d\arg\zeta_i\right)dt.$$

Using (5.2.5), (5.2.11) and the Schwarz inequality, we have

$$\begin{aligned}
\ell(r)^2&=\left(\sum_{i\in I}\ell_i(r)\right)^2\\
&=\left(\sum_{i\in I}\int_{\hat\gamma_i(r)}\sqrt{G_i}\,rd\arg\zeta_i\right)^2\\
&\le\left(\sum_{i\in I}\int_{\hat\gamma_i(r)}rd\arg\zeta_i\right)\left(\sum_{i\in I}\int_{\hat\gamma_i(r)}G_i rd\arg\zeta_i\right)\\
&=\left(\sum_{i\in I}\ell(\zeta_i,\hat\gamma_i(r),\omega_E)\right)\left(\sum_{i\in I}\frac{d}{dr}A_i(r)\right)\quad(\text{for a.e. } r\in[r_0,1])\\
&\le \mathcal{Q}_1\sum_{i\in I}(S_i+L_i)\frac{d}{dr}A(r)
\end{aligned}$$

for a.e. $r \in [r_0, 1]$. Now, suppose that the set $[r_0, 1]\backslash E(\varepsilon)$ is a null set. Then using (5.2.3), (5.2.9) and (5.2.10), we have

$$\begin{aligned}
1 - r_0 &= \int_{E(\varepsilon)} dr \\
&\leq \mathcal{Q}_1 \sum_{i\in I}(S_i + L_i) \int_{E(\varepsilon)} \left(\frac{d}{dr}A(r)\right) \frac{1}{\ell(r)^2} dr \\
&\leq \frac{\mathcal{Q}_1 \sum_{i\in I}(S_i + L_i)}{\varepsilon^2} \int_{r_0}^{1} \left(\frac{d}{dr}A(r)\right) \frac{1}{A(r)^2} dr \\
&\leq \frac{\mathcal{Q}_1 \sum_{i\in \hat{\mathscr{I}}}(S_i + L_i)}{\varepsilon^2 A(r_0)} \\
&\leq \frac{\mathcal{Q}_1 \mathcal{Q}_2}{\varepsilon^2 A(r_0)} \deg \pi \left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right) \\
&\leq \frac{\mathcal{Q}_1 \mathcal{Q}_2}{\varepsilon^2 \mu_1} \frac{A(g, F_{\mathfrak{D}'}, \omega_q)}{A(r_0)} \\
&\leq \frac{1 - r_0}{2},
\end{aligned}$$

which is a contradiction. This proves our claim. □

Note that the set

$$\{r \in [r_0, 1];\ (b_i)_{i\in\hat{\mathscr{I}}} \text{ and } \mathfrak{D}(r) \text{ do not satisfy the condition (5.2.2)}\}$$

is a finite set, so a null set. Hence by Claim 2, if (5.2.3) holds for λ, we may take $r \in [r_0, 1]$ such that $(b_i)_{i\in\hat{\mathscr{I}}}$ and $\mathfrak{D}(r)$ satisfy the condition (5.2.2), and that the following inequality holds

$$\ell(r) \leq \varepsilon A(r).$$

Using (5.2.11), we have

$$\begin{aligned}
\ell(g, \partial F_{\mathfrak{D}(r)}, \omega_q) &\leq \ell(r) + \ell(g, \partial F, \omega_q) \\
&\leq \varepsilon A(r) + \ell(g, \partial F, \omega_q) \\
&= \varepsilon A(g, F_{\mathfrak{D}(r)}, \omega_q) + \ell(g, \partial F, \omega_q).
\end{aligned}$$

Put $\mathfrak{D}'' = \mathfrak{D}(r)$, which proves (1) of our Lemma.

Proof of (2). Let $\lambda = (\mathscr{F}, \mathscr{R}, \pi, g, b, F, R)$ be a specified q-hol-quintet of type $\hat{\mathscr{I}}$, and let $\mathfrak{D}''$ be an $\hat{\mathscr{I}}$-tuple of Jordan domains which satisfies (5.2.4). We also assume the condition (5.2.2) for $(b_i)_{i\in\hat{\mathscr{I}}}$ and $\mathfrak{D}''$. In this proof, we denote by $\mathcal{Q}$ any positive constant which only depends on $\mathfrak{D}$, $\mathfrak{D}'$ and $\hat{\mathscr{I}}$, and does not depend on the choices of λ and $\mathfrak{D}''$. We shall prove

$$\sum_{G\in\mathcal{C}(R_{\mathfrak{D}''})} \rho^+(G) \leq \rho^+(R) + \mathcal{Q}\left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right), \tag{5.2.12}$$

which proves our lemma.

For a subset $I \subset \hat{\mathscr{I}}$, put

$$R^I = R \cap \bigcap_{i\in I} b_i^{-1}(D_i''), \quad F^I = \pi^{-1}(R^I).$$

If $I \neq \hat{\mathscr{I}}$, take $i \in \hat{\mathscr{I}}$ with $i \notin I$, and put

$$\mathcal{I}_{i,I} = \mathcal{I}(b_i^{-1}(D_i''), R^I), \quad \mathcal{I}'_{i,I} = \mathcal{I}(b_i^{-1}(\mathbb{P}^1 \backslash \overline{D_i''}), R^I), \quad \mathcal{P}_{i,I} = \mathcal{P}(b_i^{-1}(D_i''), R^I).$$

By Lemma 2 applying to $\Omega = H \in \mathcal{C}(R^I)$, $\zeta = b_i$ and $\gamma_1 = \partial D_i''$ (cf. (5.2.2)), we have

$$\sum_{H \in \mathcal{C}(R^I)} \rho^+(H) \geq \sum_{H \in \mathcal{I}_{i,I}} \rho(H) + \sum_{H \in \mathcal{I}'_{i,I}} \rho(H) + \sum_{H \in \mathcal{P}_{i,I}} \rho^+(H). \tag{5.2.13}$$

Let $S_{D_i'}$ be the mean sheet number of $b_i : R \to \mathbb{P}^1$ over $D_i' \subset \mathbb{P}^1$. Then we have

$$\sum_{H \in \mathcal{I}_{i,I}} \rho^+(H) - \sum_{H \in \mathcal{I}_{i,I}} \rho(H) \leq \operatorname{card}(\mathcal{I}_{i,I}) \leq S_{D_i'} \quad \text{(cf. (5.2.4))}.$$

Using Covering theorem 1 (cf. (3.2.1)), we get

$$\sum_{H \in \mathcal{I}_{i,I}} \rho^+(H) - \sum_{H \in \mathcal{I}_{i,I}} \rho(H) < \mathcal{Q}\left(A(b_i, R, \omega_{\mathbb{P}^1}) + \ell(b_i, \partial R, \omega_{\mathbb{P}^1})\right). \tag{5.2.14}$$

Similarly, we have

$$-\sum_{H \in \mathcal{I}'_{i,I}} \rho(H) \leq \operatorname{card}(\mathcal{I}'_{i,I}) \leq S_{\mathbb{P}^1 \backslash \overline{D_i}} \leq \mathcal{Q}\left(A(b_i, R, \omega_{\mathbb{P}^1}) + \ell(b_i, \partial R, \omega_{\mathbb{P}^1})\right) \tag{5.2.15}$$

where $S_{\mathbb{P}^1 \backslash \overline{D_i}}$ is the mean sheet number of $b_i : R \to \mathbb{P}^1$ over $\mathbb{P}^1 \backslash \overline{D_i} \subset \mathbb{P}^1$. Put $I' = I \cup \{i\}$. Since we have $\mathcal{I}_{i,I} \cup \mathcal{P}_{i,I} = \mathcal{C}(R^{I'})$, using (5.2.13), (5.2.14) and (5.2.15), we get

$$\sum_{H \in \mathcal{C}(R^{I'})} \rho^+(H) \leq \sum_{H \in \mathcal{C}(R^I)} \rho^+(H) + \mathcal{Q}\left(A(b_i, R, \omega_{\mathbb{P}^1}) + \ell(b_i, \partial R, \omega_{\mathbb{P}^1})\right).$$

Using this estimate inductively, we have

$$\sum_{H \in \mathcal{C}(R_{\mathfrak{D}''})} \rho^+(H) \leq \rho^+(R) + \mathcal{Q} \sum_{i \in \hat{\mathscr{I}}} \big(A(b_i, R, \omega_{\mathbb{P}^1}) + \ell(b_i, \partial R, \omega_{\mathbb{P}^1})\big),$$

where we note that $R^{\emptyset} = R$ and $R^{\hat{\mathscr{I}}} = R_{\mathfrak{D}''}$. Using the inequalities

$$\sum_{i \in \hat{\mathscr{I}}} \big(A(b_i, R, \omega_{\mathbb{P}^1}) + \ell(b_i, \partial R, \omega_{\mathbb{P}^1})\big) \leq \mathcal{Q}\left(A(b, R, \eta_q) + \ell(b, \partial R, \eta_q)\right)$$

and

$$\ell(b, \partial R, \eta_q) \leq \mathcal{Q}\ell(g, \partial F, \omega_q),$$

we obtain (5.2.12), which proves (2). □

6. Conclusion of the proof of Theorem 4

6.1. *Weak version of the theorem.* We first prove the following.

Claim: Let $\hat{\mathscr{I}} \subset \mathscr{I}^q$ be a subset. Let Λ be a countable set of non-degenerate specified q-hol-quintets of type $\hat{\mathscr{I}}$. Then for all $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon, \hat{\mathscr{I}}, \Lambda)$ such that

$$\begin{aligned} A(g, F, \kappa_q) \leq & \overline{n}(g, \mathscr{D}_q, F) + \operatorname{disc}(\pi, R) + \varepsilon A(g, F, \omega_q) \\ & + C \deg \pi \left(A(b, R, \eta_q) + \overline{n}(b, \mathscr{Z}_q, R) + \rho^+(R) + \ell(g, \partial F, \omega_q)\right) \end{aligned} \tag{6.1.1}$$

for all $(\mathscr{F},\mathscr{R},\pi,g,b,F,R)\in\Lambda$.

Proof of Claim. Recall that we denote by $\mathrm{dist}(x,y)$ the distance between $x,y\in\mathbb{P}^1$ with respect to the Kähler metric associated to the Kähler form $\omega_{\mathbb{P}^1}$. Put

$$\Lambda'=\{(\mathscr{F},\mathscr{R},\pi,g,b);\ (\mathscr{F},\mathscr{R},\pi,g,b,F,R)\in\Lambda\},$$

which is a countable set of non-degenerate q-hol-quintets. For a point $x\in\overline{\mathscr{M}}_{0,q}$ and for $r>0$, put

$$W_x(r)=\{y\in\overline{\mathscr{M}}_{0,q};\ \mathrm{dist}(\phi_i(x),\phi_i(y))<r\quad\text{for all } i\in\mathscr{I}\}.$$

By Lemma 7, we may take $r_x>0$ such that $W_x(r_x)\subset V_x(\Lambda')$ (cf. Lemma 6). Consider the open covering

$$\overline{\mathscr{M}}_{0,q}=\bigcup_{x\in\overline{\mathscr{M}}_{0,q}}W_x\left(\frac{r_x}{2}\right).$$

Since $\overline{\mathscr{M}}_{0,q}$ is compact, we may take a finite set $\mathcal{S}$ of points $x\in\overline{\mathscr{M}}_{0,q}$ such that the open sets $W_x\left(\frac{r_x}{2}\right)$ for these $x\in\mathcal{S}$ give a covering of $\overline{\mathscr{M}}_{0,q}$. Let r_0 be the minimum of $\frac{r_x}{2}$ for $x\in\mathcal{S}$. Then for all $y\in\overline{\mathscr{M}}_{0,q}$, there exists $x\in\mathcal{S}$ such that

$$W_y(r_0)\subset W_x(r_x)\subset V_x(\Lambda'). \tag{6.1.2}$$

Next, take a line γ on $\mathbb{P}^1$ which has the following property (P):

> $\mathbb{P}^1\backslash\gamma$ is a finite disjoint union of Jordan domains $D_\alpha(\gamma)$ $(1\leq\alpha\leq\daleth)$ such that $\sup_{x,y\in D_\alpha(\gamma)}\mathrm{dist}(x,y)<r_0$.

Let ε be an arbitrary positive constant. Take a positive integer J such that $J>\frac{1}{\varepsilon}$, and take small deformations $\gamma_1,\dots,\gamma_J$ of γ with the following properties:

- Each γ_j $(1\leq j\leq J)$ also satisfies the property (P),
- $\gamma_j\cap\gamma_k\cap\gamma_l=\emptyset$ for $1\leq j<k<l\leq J$.

Then for each integer j with $1\leq j\leq J$, we may take a small closed neighborhood δ_j of γ_j with the following property (P'):

> $\mathbb{P}^1\backslash\delta_j$ is a finite disjoint union of Jordan domains $D_1(\delta_j),\dots,D_\daleth(\delta_j)$ where each $D_\alpha(\delta_j)$ $(1\leq\alpha\leq\daleth)$ is compactly contained in $D_\alpha(\gamma_j)$.

We also assume that

$$\delta_j\cap\delta_k\cap\delta_l=\emptyset\quad\text{for } 1\leq j<k<l\leq J. \tag{6.1.3}$$

Put $\mathcal{T}=\{1,\cdots,\daleth\}^{\hat{\mathscr{I}}}$. For $\beta=(\beta_i)_{i\in\hat{\mathscr{I}}}\in\mathcal{T}$ and $1\leq j\leq J$, put $\mathfrak{D}_{\beta,j}=\{D_{\beta_i}(\gamma_j)\}_{i\in\hat{\mathscr{I}}}$ and $\mathfrak{D}'_{\beta,j}=\{D_{\beta_i}(\delta_j)\}_{i\in\hat{\mathscr{I}}}$, which are $\hat{\mathscr{I}}$-tuples of Jordan domains. Then $\mathfrak{D}'_{\beta,j}$ is compactly contained in $\mathfrak{D}_{\beta,j}$.

We take a positive constant h such that

$$\begin{aligned} &h_y(\Lambda')<h\quad\text{for all } y\in\mathcal{S}\quad\text{(cf. Lemma 6)},\\ &\kappa_q<h\omega_q\quad\text{on}\quad\overline{\mathscr{U}}_{0,q},\\ &1<h.\end{aligned}\tag{6.1.4}$$

Note that h is independent of the choice of ε. We also take a positive constant μ such that

$$\mu > \mu_1(\varepsilon, \hat{\mathscr{I}}, \mathfrak{D}_{\beta,j}, \mathfrak{D}'_{\beta,j}), \quad \mu > \mu_2(\hat{\mathscr{I}}, \mathfrak{D}_{\beta,j}, \mathfrak{D}'_{\beta,j}) \quad \text{(cf. Lemma 8)} \tag{6.1.5}$$

for all $\beta \in \mathcal{T}$ and $1 \leq j \leq J$.

Take $(\mathscr{F}, \mathscr{R}, \pi, g, b, F, R) \in \Lambda$. We consider the covering

$$\xi_i = b_i \circ \pi|_F : F \to \mathbb{P}^1 \quad \text{for } i \in \hat{\mathscr{I}}.$$

Since we have

$$\sum_{j=1}^{J} A(g, \xi_i^{-1}(\delta_j), \omega_q) \leq 2A(g, F, \omega_q) \quad \text{(by (6.1.3))}$$

for all $i \in \hat{\mathscr{I}}$, we have

$$\sum_{j=1}^{J} \sum_{i \in \hat{\mathscr{I}}} A(g, \xi_i^{-1}(\delta_j), \omega_q) \leq 2\varkappa A(g, F, \omega_q) \quad (\varkappa = \operatorname{card} \hat{\mathscr{I}}).$$

Hence there exists j $(1 \leq j \leq J)$ such that

$$\sum_{i \in \hat{\mathscr{I}}} A(g, \xi_i^{-1}(\delta_j), \omega_q) \leq \frac{2\varkappa}{J} A(g, F, \omega_q) \leq 2\varepsilon\varkappa A(g, F, \omega_q). \tag{6.1.6}$$

For the rest of this proof, we fix this j.

Subclaim: For $\beta \in \mathcal{T}$, there exists an $\hat{\mathscr{I}}$-tuple of Jordan domains $\mathfrak{D}''_\beta$ which satisfies $\mathfrak{D}'_{\beta,j} \subset \mathfrak{D}''_\beta \subset \mathfrak{D}_{\beta,j}$ and the following inequality

$$\begin{aligned} A(g, F_{\mathfrak{D}''_\beta}, \kappa_q) \leq{}& \overline{n}(g, \mathscr{D}_q, F_{\mathfrak{D}''_\beta}) + \operatorname{disc}(\pi, R_{\mathfrak{D}''_\beta}) \\ &+ \deg \pi \rho^+(R) + h\mu \deg \pi \left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right) \\ &+ \varepsilon h A(g, F_{\mathfrak{D}''_\beta}, \omega_q) + h\ell(g, \partial F, \omega_q) + h \deg \pi \overline{n}(b, \mathscr{Z}_q, R_{\mathfrak{D}''_\beta}). \end{aligned} \tag{6.1.7}$$

Proof of Subclaim. We first consider the case

$$A(g, F_{\mathfrak{D}'_{\beta,j}}, \omega_q) \leq \mu \deg \pi \left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right).$$

Put $\mathfrak{D}''_\beta = \mathfrak{D}'_{\beta,j}$. Then using (6.1.4), we have

$$A(g, F_{\mathfrak{D}''_\beta}, \kappa_q) \leq hA(g, F_{\mathfrak{D}''_\beta}, \omega_q) \leq h\mu \deg \pi \left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right).$$

Since all terms in the right hand side of (6.1.7) are not negative, we obtain our claim in this case.

Next we consider the case

$$A(g, F_{\mathfrak{D}'_{\beta,j}}, \omega_q) > \mu \deg \pi \left(A(b, R, \eta_q) + \ell(g, \partial F, \omega_q)\right).$$

Let $\mathfrak{D}''_\beta$ be the $\hat{\mathscr{I}}$-tuple of Jordan domains obtained in Lemma 8 (1) (cf. (6.1.5)). By the property (P) of γ_j, we see that $b(R_{\mathfrak{D}''_\beta}) \subset W_{b(z)}(r_0)$ for $z \in R_{\mathfrak{D}''_\beta}$. Hence by

(6.1.2), we have $b(R_{\mathfrak{D}''_\beta}) \subset V_x$ for some $x \in \mathcal{S}$. Hence we may apply Lemma 6 for each connected component $G \in \mathcal{C}(R_{\mathfrak{D}''_\beta})$ to get

$$A(g, \pi^{-1}(G), \kappa_q) \le \overline{n}(g, \mathscr{D}_q, \pi^{-1}(G)) + \mathrm{disc}(\pi, G) + \deg \pi \rho^+(G) \\ + h\ell(g, \partial\pi^{-1}(G), \omega_q) + h \deg \pi \overline{n}(b, \mathscr{Z}_q, G).$$

Adding over all $G \in \mathcal{C}(R_{\mathfrak{D}''_\beta})$ and using the estimates of Lemma 8 (1) and (2), we obtain our assertion. □

Since we have $F = \bigcup_{\beta \in T} F_{\mathfrak{D}''_\beta} \cup \bigcup_{i \in \hat{\mathscr{I}}} \xi_i^{-1}(\delta_j)$ and $F_{\mathfrak{D}''_\beta} \cap F_{\mathfrak{D}''_{\beta'}} = \emptyset$ for $\beta' \ne \beta$, we have

$$\begin{aligned} A(g, F, \kappa_q) &\le \sum_{\beta \in \mathcal{T}} A(g, F_{\mathfrak{D}''_\beta}, \kappa_q) + h \sum_{i \in \hat{\mathscr{I}}} A(g, \xi_i^{-1}(\delta_j), \omega_q) \\ &\le \sum_{\beta \in \mathcal{T}} A(g, F_{\mathfrak{D}''_\beta}, \kappa_q) + 2\varkappa h \varepsilon A(g, F, \omega_q) \quad (\text{cf. } (6.1.6)). \end{aligned} \tag{6.1.8}$$

Adding the inequalities (6.1.7) over all $\beta \in T$ and using the above inequality (6.1.8), we get

$$A(g, F, \kappa_q) \le \overline{n}(g, \mathscr{D}_q, F) + \mathrm{disc}(\pi, R) + (2\varkappa + 1)h\varepsilon A(g, F, \omega_q) \\ + \daleth^\varkappa \deg \pi \rho^+(R) + h\mu \daleth^\varkappa \deg \pi A(b, R, \eta_q) \\ + h \daleth^\varkappa (\mu \deg \pi + 1)\ell(g, \partial F, \omega_q) + h \deg \pi \overline{n}(b, \mathscr{Z}_q, R).$$

Here we used the fact $\mathrm{card}\, \mathcal{T} = \daleth^\varkappa$. Note that the constants h, μ, $\varkappa$ and $\daleth$ are independent of the choice of $\lambda \in \Lambda$. Using the facts that $\varepsilon > 0$ is arbitrary and that the constant $(2\varkappa + 1)h$ is independent of the choice of ε, we see that the term $(2\varkappa + 1)h\varepsilon$ is also arbitrary positive number. This proves our claim. □

6.2. *End of proof.* We prove our theorem by a contradiction. Suppose our theorem is not correct. Then there exist $q \ge 3$ and $\varepsilon > 0$ with the following property: For all positive integer k, there exists a non-degenerate specified q-hol-quintet

$$\lambda_k = (\mathscr{F}_k, \mathscr{R}_k, \pi_k, g_k, b_k, F_k, R_k)$$

such that

$$A(g_k, F_k, \kappa_q) > \overline{n}(g_k, \mathscr{D}_q, F_k) + \mathrm{disc}(\pi_k, R_k) + \varepsilon A(g_k, F_k, \omega_q) \\ + k \deg \pi_k \left(A(b_k, R_k, \eta_q) + \overline{n}(b_k, \mathscr{Z}_q, R_k) + \rho^+(R_k) + \ell(g_k, \partial F_k, \omega_q) \right). \tag{6.2.1}$$

Put $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$. Replacing Λ by its subset, we may assume that the types of λ_k are all the same $\hat{\mathscr{I}} \subset \mathscr{I}$. Using the above claim and (6.2.1), we conclude that

$$kQ_k < C(\varepsilon, \hat{\mathscr{I}}, \Lambda) Q_k$$

for all positive integer k, where we put

$$Q_k = \deg \pi_k \left(A(b_k, R_k, \eta_q) + \overline{n}(b_k, \mathscr{Z}_q, R_k) + \rho^+(R_k) + \ell(g_k, \partial F_k, \omega_q) \right).$$

But this is a contradiction, since we have $Q_k \ge 0$. Hence we obtain our theorem.

7. Proof of Corollary 2

7.1. *Generalization of Theorem 1.* Let Y, B, π and ψ be the same as Corollary 2. Then we may consider $\mathfrak{K}_B$ as a subfield of $\mathfrak{K}_Y$ by the natural inclusion defined by $\pi : Y \to B$.

Corollary 3. *Let $F(x) \in \mathfrak{K}_B^{\psi}[x]$ be a polynomial in one variable with coefficients in $\mathfrak{K}_B^{\psi}$. Assume that $F(x) = 0$ has no multiple solutions. Take $\zeta \in \mathfrak{K}_Y$ such that $F(\zeta) \neq 0$. Then for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that*

$$(\deg F - 2 - \varepsilon)T(r,\zeta) \leq \overline{N}(r, 0, F(\zeta)) + N_{\mathrm{ram}\,\pi_Y}(r) \\ + C(\varepsilon)(N_{\mathrm{ram}\,\pi_B}(r) + \psi(r)) + o(T(r,\zeta)) \ ||,$$

where we consider $F(\zeta)$ as a meromorphic function on Y.

Remark 7.1.1. If we put $F(x) = (x - a_1) \cdots (x - a_q)$ for distinct $a_1, \ldots, a_q \in \mathfrak{K}_B^{\psi}$, then the above corollary implies Theorem 1. This is because we have

$$\overline{N}(r, 0, F(\zeta)) = \sum_{i=1}^{q} \overline{N}(r, a_i \circ \pi, \zeta) + O(\psi(r)) \ ||. \tag{7.1.2}$$

Note that the condition $F(\zeta) \neq 0$ is equivalent to $\zeta \neq a_i \circ \pi$ for all $i = 1, \ldots, q$.

Proof of Corollary 3. Let $\overline{\mathfrak{K}_{\mathbb{C}}}$ be an algebraic closure of $\mathfrak{K}_{\mathbb{C}}$. We consider the fields $\mathfrak{K}_B^{\psi}$ and $\mathfrak{K}_Y$ as subfields of $\overline{\mathfrak{K}_{\mathbb{C}}}$. Let $\mathfrak{L} \subset \overline{\mathfrak{K}_{\mathbb{C}}}$ be the splitting field of $F(x)$ over $\mathfrak{K}_B^{\psi}$. Then $\mathfrak{L}$ is a finite separable extension of $\mathfrak{K}_B^{\psi}$. Hence there is a primitive element $\alpha \in \mathfrak{L}$, i.e., $\mathfrak{L} = \mathfrak{K}_B^{\psi}(\alpha)$. Let $B' \xrightarrow{\pi'} B$ be the Riemann surface defined by α, i.e., $\mathfrak{K}_{B'}^{\psi} = \mathfrak{L}$. Then there exist $\alpha_1, \ldots, \alpha_q, \beta \in \mathfrak{K}_{B'}^{\psi}$ such that

$$F(x) = \beta(x - \alpha_1) \cdots (x - \alpha_q) \tag{7.1.3}$$

where $q = \deg F(x)$. Let $G(x) \in \mathfrak{K}_B^{\psi}[x]$ be the irreducible polynomial such that $G(\alpha) = 0$. Since the ramification points of π' are either the poles of the coefficients of G or the zeros of the discriminant of G, we have

$$N_{\mathrm{ram}\,\pi_{B'}}(r) \leq N_{\mathrm{ram}\,\pi_B}(r) + O(\psi(r)) \ ||, \tag{7.1.4}$$

where $\pi_{B'} = \pi_B \circ \pi'$ (cf. (2.3.2)).

Next let $Y' \xrightarrow{\pi''} Y$ be the Riemann surface such that $\mathfrak{K}_{Y'} = \mathfrak{K}_Y(\alpha)$, where we consider $\mathfrak{K}_{Y'}$ as a subfield of $\overline{\mathfrak{K}_{\mathbb{C}}}$. By the similar reason for (7.1.4), we have

$$N_{\mathrm{ram}\,\pi_{Y'}}(r) \leq N_{\mathrm{ram}\,\pi_Y}(r) + O(\psi(r)) \ ||. \tag{7.1.5}$$

Since $\mathfrak{K}_{B'}^{\psi}$ is a subfield of $\mathfrak{K}_{Y'}$, there exists a proper, surjective holomorphic map $\hat{\pi} : Y' \to B'$. Apply Theorem 1 to the case that $Y = Y'$, $B = B'$, $a_i = \alpha_i$ and $f = \zeta \circ \pi''$. Then we get

$$(q - 2 - \varepsilon)T(r,\zeta) \leq \sum_{i=1}^{q} \overline{N}(r, \alpha_i \circ \hat{\pi}, \zeta \circ \pi'') + N_{\mathrm{ram}\,\pi_{Y'}}(r) \\ + O_{\varepsilon}(N_{\mathrm{ram}\,\pi_{B'}}(r) + \psi(r)) + o(T(r,\zeta)) \ ||$$

for all $\varepsilon > 0$. Here note that $T(r, \zeta) = T(r, \zeta \circ \pi'') + O(1)$ and that $\alpha_1, \dots, \alpha_q$ are distinct because $F(x) = 0$ has no multiple solutions. By (7.1.2) and (7.1.3), we have

$$\begin{aligned}\sum_{i=1}^{q} \overline{N}(r, \alpha_i \circ \hat{\pi}, \zeta \circ \pi'') &= \overline{N}(r, 0, F(\zeta \circ \pi'')) + O(\psi(r)) \ || \\ &= \overline{N}(r, 0, F(\zeta)) + O(\psi(r)) \ ||.\end{aligned}$$

Hence using (7.1.4) and (7.1.5), we conclude our proof. □

7.2. *Geometric version of the corollary.*

Corollary 4. *Let X and M be smooth projective varieties over $\mathbb{C}$. Let $p : X \to M$ be a surjective morphism such that a fiber $p^{-1}(x)$ is a smooth projective curve for a general $x \in M$. Let L_X and L_M be ample line bundles on X and M, respectively. Let $K_{X/M}$ be the relative canonical bundle on X. Let Y, B and π be the same as Theorem 1. Consider the following commutative diagram of holomorphic maps.*

$$\begin{array}{ccc} Y & \xrightarrow{\ \zeta\ } & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\ \beta\ ]{} & M \end{array}$$

Assume that the image $\beta(B)$ is Zariski dense in M. Then for all $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon) > 0$ such that

$$\begin{aligned}T(r, \zeta, K_{X/M}) \leq\ & N_{\mathrm{ram}\,\pi_Y}(r) + \varepsilon T(r, \zeta, L_X) \\ & + C(\varepsilon)\left(T(r, \beta, L_M) + N_{\mathrm{ram}\,\pi_B}(r)\right) + o(T(r, \zeta, L_X)) \ ||.\end{aligned}$$

Proof. Suppose that the Zariski closure W of the image $\zeta(Y)$ is not equal to X. Then the field extension $\mathbb{C}(W)/\mathbb{C}(M)$ defined by $p|_W : W \to M$ is a finite extension. Hence we have

$$T(r, v \circ \zeta) \leq O(T(r, \beta, L_M))$$

for all $v \in \mathbb{C}(W)$. Hence we get

$$T(r, \zeta, K_{X/M}) \leq O(T(r, \beta, L_M)).$$

This prove our corollary in the case $W \neq X$.

Next we consider the case $W = X$. By blowing-up, we can assume that there exists a generically finite map $\alpha : X \to \mathbb{P}^1 \times M$ over M. Let $M_0 \subset M$ be an affine open subset such that the restriction $\alpha_0 = \alpha|_{X_0} : X_0 \to \mathbb{P}^1 \times M_0$ is finite, where $X_0 = p^{-1}(M_0)$. Put $E_0 = \mathrm{ram}(\alpha_0) \subset X_0$, i.e., the ramification divisor of α_0. Let $H_0 \subset \mathbb{P}^1 \times M_0$ be the reduced divisor supported by $\alpha_0(\mathrm{supp}\, E_0)$, i.e., $H_0 = \alpha_0(\mathrm{supp}\, E_0)_{\mathrm{red}}$. Put $G_0 = \alpha_0^*(H_0)_{\mathrm{red}}$. By the ramification formula, we have

$$K_{X_0/M_0}(G_0) = \alpha_0^*\big(K_{\mathbb{P}^1 \times M_0/M_0}(H_0)\big). \tag{7.2.1}$$

Here $K_{\mathbb{P}^1 \times M_0/M_0}$, which is a line bundle on $\mathbb{P}^1 \times M_0$, is the relative canonical bundle of the second projection $\mathbb{P}^1 \times M_0 \to M_0$. Let $H \subset \mathbb{P}^1 \times M$ be the natural extension of H_0. Then by (7.2.1), we can extend the divisor G_0 to a divisor G on X such that

$$K_{X/M}(G) = \alpha^*\big(K_{\mathbb{P}^1\times M/M}(H)\big),$$

where $K_{\mathbb{P}^1\times M/M}$ is the relative canonical bundle of the second projection $\mathbb{P}^1 \times M \to M$. Hence we have

$$T(r,\zeta,K_{X/M}(G)) = T(r,\alpha\circ\zeta,K_{\mathbb{P}^1\times M/M}(H)) + O(1). \tag{7.2.2}$$

Put $\psi(r) = T(r,\beta,L_M) + O(1)$.

Claim: For all $\varepsilon > 0$, the following inequality holds

$$\begin{aligned} T(r,\alpha\circ\zeta,K_{\mathbb{P}^1\times M/M}(H)) \le{}& \overline{N}(r,\alpha\circ\zeta,H) + N_{\mathrm{ram}\,\pi_Y}(r) + \varepsilon T(r,\zeta,L_X) \\ &+ O_\varepsilon\big(N_{\mathrm{ram}\,\pi_B}(r) + \psi(r)\big) + o(T(r,\zeta,L_X))\ ||. \end{aligned}$$

Proof of Claim. Let e be the generic point of M in the sense of Scheme theory. Let $\mathbb{P}^1_e$ be the generic fiber of the second projection $p' : \mathbb{P}^1 \times M \to M$. Then $\mathbb{P}^1_e$ is the projective line over the function field $\mathbb{C}(M)$ of M. Let $H_e \subset \mathbb{P}^1_e$ be the restriction of H. By a coordinate change of the first factor of $\mathbb{P}^1 \times M$, if necessary, we may assume that the divisor $(\infty) \subset \mathbb{P}^1_e$ is not a component of H_e. Hence we may take a polynomial $F(x) \in \mathbb{C}(M)[x]$ such that H_e is defined by $F(x) = 0$.

First, we consider $F(x)$ as a rational function on $\mathbb{P}^1 \times M$. Let $(F)_0 \subset \mathbb{P}^1 \times M$ be the divisor of zeros of $F(x)$. Then we have

$$\overline{N}(r,0,F\circ\alpha\circ\zeta) \le \overline{N}(r,\alpha\circ\zeta,(F)_0),$$

where $F\circ\alpha\circ\zeta$ is a non-constant meromorphic function on Y because of the assumption $W = X$. Note that we have

$$\overline{N}(r,\alpha\circ\zeta,(F)_0) \le \overline{N}(r,\alpha\circ\zeta,H) + O(\psi(r))$$

because of $p'\big(\mathrm{supp}((F)_0 - H)\big) \ne M$. Hence we get

$$\overline{N}(r,0,F\circ\alpha\circ\zeta) \le \overline{N}(r,\alpha\circ\zeta,H) + O(\psi(r)). \tag{7.2.3}$$

Next, let $\hat{F}(x)$ be the polynomial over $\mathfrak{K}^\psi_B$ obtained from $F(x)$ by the natural inclusion $\mathbb{C}(M) \subset \mathfrak{K}^\psi_B$ defined by β. Let $\kappa : \mathbb{P}^1 \times M \to \mathbb{P}^1$ be the first projection, and put $\hat{\zeta} = \kappa\circ\alpha\circ\zeta : Y \to \mathbb{P}^1$. Then we have

$$F\circ\alpha\circ\zeta = \hat{F}(\hat{\zeta}).$$

Hence, using (7.2.3), we get

$$\overline{N}(r,0,\hat{F}(\hat{\zeta})) \le \overline{N}(r,\alpha\circ\zeta,H) + O(\psi(r)). \tag{7.2.4}$$

We apply Corollary 3 to obtain

$$\begin{aligned} (\deg\hat{F} - 2 - \varepsilon)T(r,\hat{\zeta}) \le{}& \overline{N}(r,0,\hat{F}(\hat{\zeta})) + N_{\mathrm{ram}\,\pi_Y}(r) \\ &+ O_\varepsilon(N_{\mathrm{ram}\,\pi_B}(r) + \psi(r)) + o(T(r,\hat{\zeta}))\ || \end{aligned} \tag{7.2.5}$$

for all $\varepsilon > 0$. Here we note that $\hat{F}(x)$ has no multiple solutions because H is a reduced divisor.

Now since we have $\big((\deg F - 2)\kappa^*\mathscr{L}\big)|_{\mathbb{P}^1_e} \simeq \big(K_{\mathbb{P}^1\times M/M}(H)\big)|_{\mathbb{P}^1_e}$, we get

$$(\deg\hat{F} - 2)T(r,\hat{\zeta},\mathscr{L}) = T(r,\alpha\circ\zeta,K_{\mathbb{P}^1\times M/M}(H)) + O(\psi(r)). \tag{7.2.6}$$

Note that we have $T(r,\hat{\zeta},\mathscr{L}) = T(r,\hat{\zeta}) + O(1)$, because the Fubini-Study form $\omega_{\mathbb{P}^1}$ is the curvature form of the Fubini-Study metric on $\mathscr{L}$. Hence combining (7.2.4), (7.2.5) and (7.2.6), we get

$$T(r,\alpha\circ\zeta,K_{\mathbb{P}^1\times M/M}(H)) \leq \overline{N}(r,\alpha\circ\zeta,H) + N_{\mathrm{ram}\,\pi_Y}(r) + \varepsilon T(r,\hat{\zeta}) \\ + O_\varepsilon(N_{\mathrm{ram}\,\pi_B}(r)+\psi(r)) + o(T(r,\hat{\zeta}))\ ||$$

for all $\varepsilon > 0$. Using

$$T(r,\hat{\zeta}) \leq O(T(r,\zeta,L_X)),$$

we obtain our claim. □

Since we have $p\big(\mathrm{supp}((\alpha^*H)_{\mathrm{red}} - G)\big) \neq M$, we obtain

$$\overline{N}(r,\alpha\circ\zeta,H) = \overline{N}(r,\zeta,(\alpha^*H)_{\mathrm{red}}) = \overline{N}(r,\zeta,G) + O(\psi(r)).$$

Here we also use the assumption $W = X$ to ensure $\zeta(Y) \not\subset \mathrm{supp}\, G$. Hence combining with (7.2.2) and the above claim, we get

$$T(r,\zeta,K_{X/M}(G)) \leq \overline{N}(r,\zeta,G) + N_{\mathrm{ram}\,\pi_Y}(r) + \varepsilon T(r,\zeta,L_X) \\ + O_\varepsilon(N_{\mathrm{ram}\,\pi_B}(r)+\psi(r)) + o(T(r,\zeta,L_X))\ ||$$

for all $\varepsilon > 0$. Using $\overline{N}(r,\zeta,G) \leq T(r,\zeta,[G]) + O(1)$ and

$$T(r,\zeta,K_{X/M}(G)) = T(r,\zeta,K_{X/M}) + T(r,\zeta,[G]) + O(1),$$

we get our corollary. (Recall that $[G]$ is the associated line bundle for G.) □

7.3. *Proof of Corollary 2.* We use the notations in Corollary 2. Let $\mathfrak{L} \subset \mathfrak{K}_B^\psi$ be the smallest subfield containing both $\mathbb{C}$ and all the coefficients of $F(x,y)$. Then $\mathfrak{L}$ is a finitely generated field over $\mathbb{C}$. Hence there exists a smooth projective variety M over $\mathbb{C}$ such that the rational function field $\mathbb{C}(M)$ of M is isomorphic to $\mathfrak{L}$. We denote by e the generic point of M in the sense of Scheme theory. In the following, we fix one isomorphism $\iota : \mathbb{C}(M) \xrightarrow{\sim} \mathfrak{L}$. Then we have the holomorphic map $\beta : B \to M$ such that $v\circ\beta = \iota(v)$ for all $v \in \mathbb{C}(M)$. Note that β has Zariski dense image and satisfies

$$T(r,\beta,L_M) \leq O(\psi(r))\ || \tag{7.3.1}$$

for an ample line bundle L_M on M. Let $\bar{F}(x,y) \in \mathbb{C}(M)[x,y]$ be the polynomial obtained by $F(x,y)$ and the isomorphism $\iota^{-1} : \mathfrak{L} \to \mathbb{C}(M)$. Let Q be the quotient field of the ring $\mathbb{C}(M)[x,y]/\bar{F}(x,y)$. We may take a smooth projective variety X and a surjective morphism $p : X \to M$ such that the rational function field $\mathbb{C}(M)(X_e)$ of the generic fiber X_e of p (in the sense of Scheme theory) is isomorphic to Q. Note that X_e is a smooth projective curve over the field $\mathbb{C}(M)$. Then the rational function field $\mathbb{C}(X)$ of X is also isomorphic to Q. Since the meromorphic functions ζ_1 and ζ_2 on Y satisfy the functional equation $F(\zeta_1,\zeta_2) = 0$, we get the holomorphic map $\zeta : Y \to X$ such that $x\circ\zeta = \zeta_1$ and $y\circ\zeta = \zeta_2$, where we consider x and y as rational functions on X. Then β and ζ fit into the commutative diagram in Corollary 4. By the assumption that $F_z(x,y) = 0$ defines a curve of genus > 1 for general $z \in B$, we see that the curve X_e has genus > 1. Hence the canonical bundle K_{X_e} is ample. Let L_X be an ample line bundle on X.

Claim: $T(r,\zeta,L_X) \leq O(T(r,\zeta,K_{X/M}) + \psi(r))\ ||$.

Proof of Claim. There exists a positive integer m such that $mK_{X_e} - L_X|_{X_e}$ is very ample. Hence we may take an effective divisor H on X such that $[H|_{X_e}] = mK_{X_e} - L_X|_{X_e}$ and $\zeta(Y) \not\subset \operatorname{supp} H$. Since the restriction $K_{X/M}|_{X_e}$ is isomorphic to K_{X_e}, we see that the restriction $(mK_{X/M} - L_X - [H])|_{X_e}$ is the trivial bundle on X_e. Hence there exists a divisor G on X such that $p(\operatorname{supp} G) \neq M$ and $mK_{X/M} - L_X - [H] = [G]$. Since we have

$$-T(r, \zeta, [H]) \leq O(1)$$

and

$$-T(r, \zeta, [G]) \leq O(\psi(r)) \ || \quad \text{(cf. (7.3.1))},$$

we get our claim. $\square$

Now, applying Corollary 4 and using the above claim, we get

$$T(r, \zeta, L_X) \leq O_\varepsilon(N_{\mathrm{ram}\,\pi_Y}(r) + N_{\mathrm{ram}\,\pi_B}(r) + \psi(r)) + \varepsilon T(r, \zeta, L_X) + o(T(r, \zeta, L_X)) \ ||$$

for all $\varepsilon > 0$. Letting $\varepsilon < 1$, we get

$$T(r, \zeta, L_X) \leq O(N_{\mathrm{ram}\,\pi_Y}(r) + N_{\mathrm{ram}\,\pi_B}(r) + \psi(r)) \ ||. \tag{7.3.2}$$

Using $x \circ \zeta = \zeta_1$ and $y \circ \zeta = \zeta_2$, we obtain

$$T(r, \zeta_1) \leq O(T(r, \zeta, L_X)), \quad T(r, \zeta_2) \leq O(T(r, \zeta, L_X)). \tag{7.3.3}$$

By (7.3.2) and (7.3.3), we get our corollary. $\square$

8. Proof of Theorem 2

In this section, we prove Theorem 2. We fix a positive integer $q \geq 3$. Let $\varepsilon > 0$ be a positive constant, and let

$$Y,\ B,\ \pi,\ f,\ a_1, \cdots, a_q \tag{8.0.4}$$

be the objects in Theorem 2, which will be also fixed in the following. Consider the specified q-hol-quintet $\lambda = (Y, B, \pi, \mathrm{cl}_{(f,a)}, \mathrm{cl}_a, Y, B)$ defined by (8.0.4).

Put $\delta = \max_{1 \leq i \leq q} \deg a_i$. Since for $(i, j, k) \in \mathscr{J}$ and for general $z \in Y$, two pairs of 4-points on $\mathbb{P}^1$

$$(f(z), a_i \circ \pi(z), a_j \circ \pi(z), a_k \circ \pi(z)), \quad (\varphi_{(i,j,k)} \circ \mathrm{cl}_{(f,a)}(z), 0, 1, \infty)$$

are isomorphic (cf. (1.5.6), (1.5.7)), we have

$$\varphi_{(i,j,k)} \circ \mathrm{cl}_{(f,a)}(z) = \frac{f(z) - a_i \circ \pi(z)}{f(z) - a_k \circ \pi(z)} \frac{a_j \circ \pi(z) - a_k \circ \pi(z)}{a_j \circ \pi(z) - a_i \circ \pi(z)}.$$

Hence we get

$$|\deg(\varphi_{(i,j,k)} \circ \mathrm{cl}_{(f,a)}) - \deg f| \leq 7\delta \deg \pi. \tag{8.0.5}$$

Also, since for $(i, j, k, l) \in \mathscr{I}$ and for general $z \in B$, two pairs of 4-points on $\mathbb{P}^1$

$$(a_l(z), a_i(z), a_j(z), a_k(z)), \quad (\phi_{(i,j,k,l)} \circ \mathrm{cl}_a(z), 0, 1, \infty)$$

are isomorphic, we have

$$\phi_{(i,j,k,l)} \circ \mathrm{cl}_a(z) = \frac{a_l(z) - a_i(z)}{a_l(z) - a_k(z)} \frac{a_j(z) - a_k(z)}{a_j(z) - a_i(z)}.$$

Hence we get

$$\deg(\phi_{(i,j,k,l)} \circ \mathrm{cl}_a) \le 8\delta. \tag{8.0.6}$$

First, we consider the case that λ is non-degenerate. By the assumption that a_i are distinct, we conclude that

$$\mathrm{cl}_a(B) \not\subset \mathrm{supp}\, \mathscr{X}_q. \tag{8.0.7}$$

Hence we may apply Theorem 4 for the non-degenerate specified q-hol-quintet λ. Denoting by $C_1(q, \varepsilon)$ the constant $C(q, \varepsilon)$ obtained in Theorem 4, we get

$$\begin{aligned}\deg(\mathrm{cl}_{(f,a)})^* K_q &\le \overline{n}(\mathrm{cl}_{(f,a)}, \mathscr{D}_q, Y) + \mathrm{disc}(\pi, B) + \varepsilon A(\mathrm{cl}_{(f,a)}, Y, \omega_q) \\ &\quad + C_1(q,\varepsilon) \deg \pi \big(A(\mathrm{cl}_a, B, \eta_q) + \overline{n}(\mathrm{cl}_a, \mathscr{X}_q, B) + \rho^+(B)\big).\end{aligned} \tag{8.0.8}$$

Here we note that $A(\mathrm{cl}_{(f,a)}, Y, \kappa_q) = \deg(\mathrm{cl}_{(f,a)})^* K_q$, and that $\ell(\mathrm{cl}_{(f,a)}, \partial Y, \omega_q) = 0$ because Y is compact. By the Riemann-Roch theorem and the Hurwitz theorem, we have

$$\rho(B) = 2g(B) - 2, \quad \mathrm{disc}(\pi, B) = (2g(Y) - 2) - \deg \pi (2g(B) - 2), \tag{8.0.9}$$

so

$$\rho^+(B) \le 2g(B), \quad \mathrm{disc}(\pi, B) \le 2g(Y) + 2\deg \pi.$$

Hence by (8.0.8), we get

$$\begin{aligned}\deg(\mathrm{cl}_{(f,a)})^* K_q &\le \overline{n}(\mathrm{cl}_{(f,a)}, \mathscr{D}_q, Y) + 2g(Y) + \varepsilon A(\mathrm{cl}_{(f,a)}, Y, \omega_q) \\ &\quad + C_2(q,\varepsilon) \deg \pi \big(A(\mathrm{cl}_a, B, \eta_q) + \overline{n}(\mathrm{cl}_a, \mathscr{X}_q, B) + g(B) + 1\big),\end{aligned} \tag{8.0.10}$$

where we put $C_2(q, \varepsilon) = 2\max\{C_1(q, \varepsilon), 2\}$.

Claim. There exist positive constants $Q_1, \ldots, Q_5$ which are independent of the choices of $\varepsilon > 0$ and of the objects in (8.0.4) such that

$$A(\mathrm{cl}_a, B, \eta_q) \le Q_1 \delta, \tag{8.0.11}$$

$$\overline{n}(\mathrm{cl}_a, \mathscr{X}_q, B) \le Q_2 \delta, \tag{8.0.12}$$

$$A(\mathrm{cl}_{(f,a)}, Y, \omega_q) \le Q_3(\deg f + \delta \deg \pi), \tag{8.0.13}$$

$$\overline{n}(\mathrm{cl}_{(f,a)}, \mathscr{D}_q, Y) \le \sum_{i=1}^{q} \overline{n}(a_i \circ \pi, f, Y) + Q_4 \delta \deg \pi, \tag{8.0.14}$$

$$(q-2)\deg f \le \deg(\mathrm{cl}_{(f,a)})^* K_q + Q_5 \delta \deg \pi. \tag{8.0.15}$$

Proof of (8.0.11). For $i \in \mathscr{I}$, let $\mathrm{pr}_i : (\mathbb{P}^1)^{\mathscr{I}} \to \mathbb{P}^1$ be the projection to the i-th factor. Put

$$\overline{\mathscr{L}} = \sum_{i \in \mathscr{I}} \mathrm{pr}_i^* \mathscr{L},$$

which is an ample line bundle on $(\mathbb{P}^1)^{\mathscr{I}}$. By Lemma 7, the line bundle $\Phi^* \overline{\mathscr{L}}$ is an ample line bundle on $\overline{\mathscr{M}}_{0,q}$. Hence there exists a curvature form ω' of $\Phi^* \overline{\mathscr{L}}$ that is a positive (1,1)-form. Hence there exists a positive constant Q_1' such that $\eta_q < Q_1'\omega'$. Using (8.0.6), we have

$$\begin{aligned} A(\mathrm{cl}_a, B, \eta_q) &\leq Q_1' A(\mathrm{cl}_a, B, \omega') = Q_1' \deg(\Phi \circ \mathrm{cl}_a)^* \overline{\mathscr{L}} \\ &= Q_1' \sum_{i \in \mathscr{I}} \deg(\phi_i \circ \mathrm{cl}_a) \leq 8Q_1'(\mathrm{card}\, \mathscr{I})\delta. \end{aligned}$$

Put $Q_1 = 8Q_1' \,\mathrm{card}\, \mathscr{I}$ to conclude the proof of (8.0.11).

Proof of (8.0.12). There exists a positive integer Q_2' such that $Q_2'\Phi^*\overline{\mathscr{L}} - [\mathscr{Z}_q]$ is an ample line bundle. Hence using (8.0.6), we get

$$\overline{n}(\mathrm{cl}_a, \mathscr{Z}_q, B) \leq \deg(\mathrm{cl}_a)^* \mathscr{Z}_q \leq Q_2' \deg(\Phi \circ \mathrm{cl}_a)^* \overline{\mathscr{L}} \leq 8Q_2'(\mathrm{card}\, \mathscr{I})\delta.$$

Put $Q_2 = 8Q_2' \,\mathrm{card}\, \mathscr{I}$ to conclude the proof of (8.0.12).

Proof of (8.0.13). Using the isomorphism $\iota_{q+1} : \overline{\mathscr{M}}_{0,q+1} \to \overline{\mathscr{U}}_{0,q}$ (cf. (1.5.11)) and Lemma 7 for $\overline{\mathscr{M}}_{0,q+1}$, we see that the line bundle

$$P = \sum_{\alpha \in \mathscr{J}^q} \varphi_\alpha^* \mathscr{L} + \sum_{i \in \mathscr{I}^q} (\phi_i \circ \varpi_q)^* \mathscr{L}$$

is an ample line bundle on $\overline{\mathscr{U}}_{0,q}$. Hence there exists a positive constant Q_3' such that $\omega_q < Q_3'\omega''$ where ω'' is a curvature form of P that is a positive (1,1)-form. Using (8.0.5) and (8.0.6), we get

$$\begin{aligned} A(\mathrm{cl}_{(f,a)}, Y, \omega_q) &\leq Q_3' A(\mathrm{cl}_{(f,a)}, Y, \omega'') = Q_3' \deg(\mathrm{cl}_{(f,a)})^* P \\ &= Q_3' \left(\sum_{\alpha \in \mathscr{J}^q} \deg(\varphi_\alpha \circ \mathrm{cl}_{(f,a)}) + \sum_{i \in \mathscr{I}^q} \deg(\phi_i \circ \mathrm{cl}_a \circ \pi) \right) \\ &\leq (Q_3' \,\mathrm{card}\, \mathscr{J}^q + 7Q_3' \,\mathrm{card}\, \mathscr{J}^q + 8Q_3' \,\mathrm{card}\, \mathscr{I}^q)(\deg f + \delta \deg \pi). \end{aligned}$$

Put $Q_3 = Q_3' \,\mathrm{card}\, \mathscr{J}^q + 7Q_3' \,\mathrm{card}\, \mathscr{J}^q + 8Q_3' \,\mathrm{card}\, \mathscr{I}^q$ to conclude the proof of (8.0.13).

Proof of (8.0.14). (cf. proof of (2.2.4)) Put

$$U = \{z \in B;\ a_1(z), \ldots, a_q(z) \text{ are all distinct}\}.$$

Then by the definition of the classification map, we have $\mathrm{cl}_a(U) \subset \mathscr{M}_{0,q}$. For $z \in U$ and $y \in \pi^{-1}(z)$, we have $\mathrm{cl}_{(f,a)}(y) \in \mathscr{D}_q$ if and only if $f(y) = a_i(z)$ for some $i \in (q)$ (cf. (1.5.6) and (1.5.7)) . Hence we have

$$\{y \in Y;\ \mathrm{cl}_{(f,a)}(y) \in \mathscr{D}_q\} \subset \{y \in Y;\ f(y) = a_i \circ \pi(y) \text{ for some } i \in (q)\} \cup \pi^{-1}(B \backslash U).$$

This implies that

$$\overline{n}(\mathrm{cl}_{(f,a)}, \mathscr{D}_q, Y) \leq \sum_{1 \leq i \leq q} \overline{n}(a_i \circ \pi, f, Y) + \deg \pi \sum_{1 \leq i \neq j \leq q} \overline{n}(a_i, a_j, B).$$

Since we have

$$\overline{n}(a_i, a_j, B) \leq 2\delta,$$

we get (8.0.14). (Put $Q_4 = 2q(q-1)$.)

Proof of (8.0.15). (cf. proof of (2.2.5)) By Lemma 1, we have

$$(8.0.16)\quad (q-2)\deg(\varphi_{(1,2,3)}\circ \mathrm{cl}_{(f,a)}) = \deg(\mathrm{cl}_{(f,a)})^*K_q + \deg\pi\deg(\mathrm{cl}_a)^*M + \deg(\mathrm{cl}_{(f,a)})^*(\Xi),$$

where M and Ξ are obtained in the lemma. By $\varpi_q(\operatorname{supp}\Xi)\subset \operatorname{supp}\mathscr{Z}_q$, there exists a positive integer Q_5' such that the divisor $Q_5'\varpi_q^*\mathscr{Z}_q - \Xi$ is effective. Hence by (8.0.7) and by the proof of (8.0.12), we have

$$(8.0.17)\qquad \deg(\mathrm{cl}_{(f,a)})^*(\Xi) \le Q_5'\deg\pi\deg(\mathrm{cl}_a)^*(\mathscr{Z}_q) \le Q_2Q_5'\delta\deg\pi.$$

Since $\varPhi^*\overline{\mathscr{L}}$ is ample, there exists a positive constant Q_5'' such that the line bundle $Q_5''\varPhi^*\overline{\mathscr{L}} - M$ is ample. Using (8.0.6), we get

$$(8.0.18)\qquad \deg(\mathrm{cl}_a)^*M \le Q_5''\deg(\varPhi\circ\mathrm{cl}_a)^*\overline{\mathscr{L}} \le 8Q_5''(\operatorname{card}\mathscr{I})\delta.$$

Using (8.0.5), (8.0.16), (8.0.17) and (8.0.18) and putting

$$Q_5 = Q_2Q_5' + 8Q_5''\operatorname{card}\mathscr{I} + 7(q-2),$$

we get our inequality (8.0.15) and conclude the proof of the claim. □

Now using (8.0.10) and the above claim, we get

$$\begin{aligned}(q-2)\deg f \le \sum_{i=1}^{q}\overline{n}(a_i\circ\pi, f, Y) + 2g(Y)\\ + \varepsilon Q_3\deg f + (\varepsilon Q_3 + Q_4 + Q_5)\delta\deg\pi\\ + C_2(q,\varepsilon)\deg\pi\big((Q_1+Q_2)\delta + g(B) + 1\big).\end{aligned}$$

Put

$$C_3(q,\varepsilon) = \max\{\varepsilon Q_3 + Q_4 + Q_5 + C_2(q,\varepsilon)(Q_1+Q_2), C_2(q,\varepsilon)\}.$$

Replacing ε by $\frac{\varepsilon}{Q_3}$ and putting $C(q,\varepsilon) = C_3(q,\frac{\varepsilon}{Q_3})$, we get our theorem in the case that λ is non-degenerate.

Next we consider the case that λ is degenerate, i.e., there exists some $\alpha\in\mathscr{J}$ such that $\varphi_\alpha\circ\mathrm{cl}_{(f,a)}$ is constant. Then by (8.0.5), we conclude that

$$\deg f \le 7\delta\deg\pi.$$

Hence replacing $C(q,\varepsilon)$ by $\max\{C(q,\varepsilon), 7(q-2)\}$, we also get the theorem in the case that λ is degenerate. Here note that all terms in the right hand side of (1.1.4) are non-negative. This conclude the proof of our theorem.

9. Height inequality for curves over function fields

9.1. *Notations.* General references for this section are [L], [V1] and [V3]. Let k be a function field, i.e., a rational function field of a compact Riemann surface B. This B is uniquely determined by k (up to isomorphism), and called the model of k. We consider B as a smooth projective curve over $\mathbb{C}$. Let $S\subset B$ be a finite set of points which will be fixed throughout.

Let X be a smooth projective curve over k, and let $D\subset X$ be an effective divisor. Let L be a line bundle on X. Following P. Vojta [V3], we define the functions

$$h_{L,k}(P),\ N^{(1)}_{k,S}(D,P),\ d_k(P)$$

for $P \in X(\overline{k})$ as follows.

First, take a model of X over B, i.e., a smooth variety $\mathfrak{X}$ projective over B such that the generic fiber (in the sense of Scheme theory) is isomorphic to X over k. Then for each $P \in X(\overline{k}) = \mathfrak{X}(\overline{k})$ by taking the normalization of the Zariski closure of P in $\mathfrak{X}$, we can associate the following commutative diagram.

$$\begin{array}{ccc} Y & \xrightarrow{f_P} & \mathfrak{X} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ B & = & B \end{array}$$

Here Y is the model of $k(P)$.

Let $\mathfrak{D} \subset \mathfrak{X}$ be an extension of $D \subset X$, and let $\mathfrak{L}$ be an extension of L to $\mathfrak{X}$. Put

$$h_{\mathfrak{L},k}(P) = \frac{1}{\deg \pi} \deg f_P^* \mathfrak{L}$$

and

$$N_{k,S}^{(1)}(\mathfrak{D}, P) = \frac{1}{\deg \pi} \sum_{x \in Y \setminus \pi^{-1}(S)} \min(1, \operatorname{ord}_x f_P^* \mathfrak{D}) \quad (P \in X(\overline{k}) \setminus D).$$

If we replace the models $\mathfrak{X}$, $\mathfrak{D}$ and $\mathfrak{L}$ by other models $\mathfrak{X}'$, $\mathfrak{D}'$ and $\mathfrak{L}'$, we have

$$h_{\mathfrak{L},k}(P) = h_{\mathfrak{L}',k}(P) + O(1), \; N_{k,S}^{(1)}(\mathfrak{D}, P) = N_{k,S}^{(1)}(\mathfrak{D}', P) + O(1),$$

where $O(1)$ are bounded terms independent of $P \in X(\overline{k})$. Then we define the functions $h_{L,k}(P)$ and $N_{k,S}^{(1)}(D,P)$ by

$$h_{L,k}(P) = h_{\mathfrak{L},k}(P) + O(1)$$

and

$$N_{k,S}^{(1)}(D, P) = N_{k,S}^{(1)}(\mathfrak{D}, P) + O(1) \quad (P \in X(\overline{k}) \setminus D),$$

which are functions modulo bounded terms $O(1)$. Finally, put

$$d_k(P) = \frac{1}{\deg \pi} \operatorname{disc}(\pi, B) = \frac{2g(Y)}{\deg \pi} + O(1) \quad \text{(cf. (8.0.9))}.$$

The following facts are easy consequences of the above definitions.

(i) $N_{k,S}^{(1)}(D, P) \leq h_{[D],k}(P) + O(1)$, where $[D]$ is the associated line bundle.

(ii) $\overline{n}(f_P, \mathfrak{D}, Y) \leq \deg \pi \big(N_{k,S}^{(1)}(\mathfrak{D}, P) + \operatorname{card} S \big)$.

(iii) Let $\mathbb{P}_k^1$ be the projective line over k. In the following, we always take $\mathbb{P}^1 \times B$ as a model of $\mathbb{P}_k^1$ over B. Then a point $P \in \mathbb{P}_k^1(\overline{k})$ corresponds to the rational function $\hat{f}_P$ on Y obtained by the composition

$$\hat{f}_P : Y \xrightarrow{f_P} \mathbb{P}^1 \times B \xrightarrow{\text{1st proj}} \mathbb{P}^1.$$

Let $\mathscr{L}_k$ be the hyper plane section bundle on $\mathbb{P}_k^1$. Then we have

$$h_{\mathscr{L}_k,k}(P) = \frac{\deg \hat{f}_P}{\deg \pi} + O(1).$$

(iv) Let $k' \subset \overline{k}$ be a finite extension of k. Put $e = [k' : k]$ and $X' = X \otimes_k k'$. Let B' be the model of k'. Let $b : B' \to B$ and $\hat{b} : X' \to X$ be the natural maps. Put $D' = \hat{b}^* D$, $L' = \hat{b}^* L$ and $S' = b^{-1}(S)$. Then using the natural identification $X'(\overline{k}) = X(\overline{k})$, we have

$$h_{L',k'}(P) = eh_{L,k}(P) + O(1),\ N^{(1)}_{k',S'}(D',P) = eN^{(1)}_{k,S}(D,P) + O(1),$$

and

$$d_{k'}(P) \leq ed_k(P) + O_{k,k'}(1).$$

Here $O_{k,k'}(1)$ is a bounded term depend on k and k', and independent of $P \in X(\overline{k})$.

By these properties and Theorem 2, we obtain the following.

Lemma 9. *Let $D \subset \mathbb{P}^1_k$ be a reduced divisor and let $\varepsilon > 0$. Then we have*

$$h_{K_{\mathbb{P}^1_k}(D),k}(P) \leq N^{(1)}_{k,S}(D,P) + d_k(P) + \varepsilon h_{\mathscr{L}_k,k}(P) + O_\varepsilon(1) \tag{9.1.1}$$

for all $P \in \mathbb{P}^1_k(\overline{k})\backslash D$. Here $O_\varepsilon(1)$ denotes a bounded term depend on ε, and independent of $P \in \mathbb{P}^1_k(\overline{k})$.

Proof. Let $k' \subset \overline{k}$ be a finite extension of k such that the divisor $D' \subset \mathbb{P}^1_{k'}$ has the form $D' = (P_1) + \cdots + (P_q)$ by k'-rational points $P_i \in \mathbb{P}^1_{k'}(k')$ for $i = 1, \ldots, q$. Here and the following, we use the notations in (iii) and (iv) above. Each P_i corresponds to the rational function $\hat{f}_{P_i}$ on B', where we note that $k'(P_i) = k'$. By the assumption that D is reduced, P_i are distinct, hence $\hat{f}_{P_i}$ are distinct. Take a point $P \in \mathbb{P}^1_{k'}(\overline{k})\backslash D' = \mathbb{P}^1_k(\overline{k})\backslash D$. Let Y' be the model of $k'(P)$ and $\pi' : Y' \to B'$ be the natural map. Then P corresponds to the rational function $\hat{f}_P$ on Y'. Because $P \notin \operatorname{supp} D'$, we have $\hat{f}_P \neq \hat{f}_{P_i} \circ \pi'$ for $i = 1, \ldots, q$. Apply Theorem 2 to get

$$(q - 2 - \varepsilon) \deg \hat{f}_P \leq \sum_{i=1}^{q} \overline{n}(\hat{f}_{P_i} \circ \pi, \hat{f}_P, Y') + 2g(Y') + O_\varepsilon(1) \deg \pi.$$

Here we note that the functions $\hat{f}_{P_i}$ and the Riemann surface B' are fixed because the divisor D and the Riemann surface B are fixed. Hence by the above (ii) and (iii), we get

$$h_{K_{\mathbb{P}^1_{k'}}(D'),k'}(P) \leq N^{(1)}_{k',S'}(D',P) + d_{k'}(P) + \varepsilon h_{\mathscr{L}_{k'},k'}(P) + O_\varepsilon(1).$$

Here we use the facts that $K_{\mathbb{P}^1_{k'}} = -2\mathscr{L}_{k'}$, $[D'] = q\mathscr{L}_{k'}$ and

$$\sum_{i=1}^{q} \overline{n}(\hat{f}_{P_i} \circ \pi, \hat{f}_P, Y') \leq \overline{n}(f_P, \mathfrak{D}', Y') + O(1) \deg \pi,$$

where $\mathfrak{D}' \subset \mathbb{P}^1 \times B'$ is the Zariski closure of $D' \subset \mathbb{P}^1_{k'}$ and $f_P : Y' \to \mathbb{P}^1 \times B'$ is the associated holomorphic map for P. Using the above (iv), we conclude our proof. □

9.2. *Height inequality.* The following theorem proves the conjecture [V3, Conjecture 2.3] for the case of curves over function fields.

Theorem 5. *Let k be a function field. Let X be a smooth projective curve over k, let D be a reduced divisor on X, let L be an ample line bundle on X and let $\varepsilon > 0$. Then we have*

$$h_{K_X(D),k}(P) \leq N^{(1)}_{k,S}(D,P) + d_k(P) + \varepsilon h_{L,k}(P) + O_\varepsilon(1) \tag{9.2.1}$$

for all $P \in X(\overline{k}) \backslash D$.

Proof. Let $\alpha : X \to \mathbb{P}^1_k$ be a finite surjective map over k. Put $E = \big(\mathrm{ram}(\alpha)\big)_{\mathrm{red}} \subset X$. Note that we may choose α such that $\operatorname{supp} D \cap \operatorname{supp} E = \emptyset$, hence we assume it. Let $H \subset \mathbb{P}^1_k$ be the reduced divisor supported by $\alpha(\operatorname{supp} D \cup \operatorname{supp} E)$. Then there exists an effective divisor $G \subset X$ such that $\big(\alpha^*(H)\big)_{\mathrm{red}} = D + E + G$. By the ramification formula, we have

$$K_X(D + E + G) = \alpha^*(K_{\mathbb{P}^1_k}(H)). \tag{9.2.2}$$

Then by Lemma 9 and the above property (i) of the previous subsection, we have

$$\begin{aligned} h_{K_X(D+E+G),k}(P) &= h_{K_{\mathbb{P}^1_k}(H),k}(\alpha(P)) \\ &\leq N^{(1)}_{k,S}(H, \alpha(P)) + d_k(\alpha(P)) + \varepsilon h_{\mathscr{L}_k,k}(\alpha(P)) + O_\varepsilon(1) \\ &= N^{(1)}_{k,S}(D + E + G, P) + d_k(\alpha(P)) + \varepsilon h_{\alpha^*\mathscr{L}_k,k}(P) + O_\varepsilon(1) \\ &\leq N^{(1)}_{k,S}(D + E + G, P) + d_k(P) + \varepsilon C h_{L,k}(P) + O_\varepsilon(1) \\ &\leq N^{(1)}_{k,S}(D, P) + h_{[E+G],k}(P) + d_k(P) + \varepsilon C h_{L,k}(P) + O_\varepsilon(1), \end{aligned}$$

for all $P \in X(\overline{k}) \backslash (D + E + G)$. Here C is a positive integer such that the line bundle $CL - \alpha^*\mathscr{L}_k$ is ample, hence C is independent of P and ε. For the points $P \in \operatorname{supp}(E + G)$, the values $h_{K_X(D)}(P)$ are bounded because $\operatorname{supp}(E + G)$ consists of finite points. Hence, replacing ε by $\frac{\varepsilon}{C}$, we get

$$h_{K_X(D),k}(P) \leq N^{(1)}_{k,S}(D,P) + d_k(P) + \varepsilon h_{L,k}(P) + O_\varepsilon(1)$$

for all $P \in X(\overline{k}) \backslash D$. This proves our theorem. □

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