Three representations of the Cuntz algebra $\mathcal{O}_2$ by a pair of operators arising from a $\mathbf{Z}_2$-graded dynamical system

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We construct three kinds of representations of the Cuntz algebra $\mathcal{O}_2$ by using a pair $\{T, M\}$ of operators from some dynamical systems with $\mathbf{Z}_2$-action, and show their relations. The characterization of these representations is shown by the eigen equation of T.

1. Introduction

Representation theory of the Cuntz algebra $\mathcal{O}_N$ is studied in [**1**, **2**, **4**, **5**, **6**, **8**, **9**, **10**]. Specially, we are interest in a representation of $\mathcal{O}_N$ arising from a dynamical system (X, F) which consists of a transformation F on a measure space (X, μ). A class of dynamical systems brings naive and computable examples of representations of $\mathcal{O}_N$. A system $\{f_i\}_{i=1}^N$ of transformations on X is a branching function system of F if $(F \circ f_i)(x) = x$ for $x \in X$, $i = 1, \ldots, N$, there are Radon-Nikodým derivatives of $\mu \circ f_1$, $\mu \circ f_2$ with respect to μ and they are non zero almost everywhere in X, and all of $f_i(X) \cap f_j(X)$, $1 \leq i < j \leq N$, $X \setminus f_1(X) \cup \cdots \cup f_N(X)$ are μ-null sets. If F has a branching function system $\{f_i\}_{i=1}^N$, then we can construct a family $\{S(f_i)\}_{i=1}^N$ of operators on $L_2(X, \mu)$ such that

$$S(f_i)S(f_j) = S(f_i \circ f_j) \quad (i, j = 1, \ldots, N) \tag{1.1}$$

where $(f_i \circ f_j)(x) \equiv f_i(f_j(x))$ for $x \in X$. In this way, such family represents iteration property of the dynamical system (X, F) faithfully. Furthermore $\{S(f_i)\}_{i=1}^N$ gives a representation of $\mathcal{O}_N$. For example, we construct representations of $\mathcal{O}_2$ and $\mathcal{O}_3$ arising from real quadratic and cubic transformations on a closed interval in [**8**, **10**], respectively. The properties of these are shown by using intertwining relations between several dynamical systems.

In this paper, we consider a relation between a dynamical system with $\mathbf{Z}_2$-symmetry and the property of representation of $\mathcal{O}_2$.

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We introduce our main theorem(Theorem 3.10) by using a case of interval dynamics here in brief. Let s_1, s_2 be generators of $\mathcal{O}_2$ which satisfies $s_i^* s_j = \delta_{ij} I$ and $s_1 s_1^* + s_2 s_2^* = I$.

Theorem 1.1. *For $a > 0$, let F be a transformation on a closed interval $[-a, a]$ such that F is monotone and C^1-class on $(0, a)$. Assume that F is a bijective from $[0, a]$ to $[-a, a]$ and $F(-x) = F(x)$ for $x \in [-a, a]$. Let a pair $\{T, M\}$ of operators on $L_2[-a, a]$ by*

$$(T\phi)(x)(x) \equiv \frac{1}{\sqrt{2}} \sqrt{|F^{'}(x)|}\, \phi(F(x)), \quad (M\phi)(x) \equiv \mathrm{sgn}(x) \cdot \phi(x)$$

for $\phi \in L_2[-a, a]$ and $x \in X$ where $F^{'}(x)$ at $x \in \{0, a\}$ is taken as the right limit of differential and $F^{'}(-a) \equiv F^{'}(a)$. Then the followings hold:

(i) *The following π_1, π_2, π_3 define representations of $\mathcal{O}_2$ on $L_2[-a, a]$:*

$$\pi_1(s_1) \equiv \frac{1}{\sqrt{2}}(I + M)T, \quad \pi_1(s_2) \equiv \frac{1}{\sqrt{2}}(I - M)T,$$

$$\pi_2(s_1) \equiv T, \quad \pi_2(s_2) \equiv MT, \quad \pi_3(s_1) \equiv TM, \quad \pi_3(s_2) \equiv MT.$$

(ii) *Ranges of $\pi_j(s_i)$, $j = 1, 2, 3$, $i = 1, 2$ are given by*

$$\pi_1(s_i s_i^*) L_2[-a, a] = \{\phi \in L_2[-a, a] : \phi|_{X_i} = 0\},$$

$$\pi_j(s_i s_i^*) L_2[-a, a] = \{\phi \in L_2[-a, a] : \phi(-x) = (-1)^{i-1}\phi(x),\, x \in [-a, a]\}$$

for $j = 2, 3$ and $i = 1, 2$ where $X_1 = [-a, 0]$ and $X_2 = [0, a]$.

(iii) *The following relations hold:*

$$\pi_3(s_1) = \pi_2(s_1 s_2 s_1^* + s_1 s_1 s_2^*), \quad \pi_3(s_2) = \pi_2(s_2),$$

$$\frac{1}{\sqrt{2}}\pi_1(s_1 + s_2) = \pi_2(s_1), \quad \frac{1}{\sqrt{2}}\pi_1(s_1 - s_2) = \pi_2(s_2).$$

(iv) *If T has an eigen function Ω with eigen value 1, then $\pi_i(\mathcal{O}_2)\Omega$ is an irreducible subspace of $L_2[-a, a]$ for each $i = 1, 2, 3$ and they are mutually inequivalent.*

In § 2, we introduce three representations of $\mathcal{O}_2$ which are defined by eigen equations of some operators and they are inequivalent each other. In § 3, we introduce dynamical systems with $\mathbf{Z}_2$-grading and construct representations of $\mathcal{O}_2$ from them. The family in (1.1) is explained in § 3.2. By these, Theorem 1.1 is proved as a corollary of more general claim. In § 4, we show examples. In § 5, we explain meaning of our results and discuss about remaining problems.

2. Three representations of $\mathcal{O}_2$

For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**3**]), that is, it is a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad (i,j = 1,\ldots,N), \quad \sum_{i=1}^{N} s_i s_i^* = I. \tag{2.1}$$

In this paper, any representation means a unital $*$-representation. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1, \ldots, S_N$ on an infinite dimensional Hilbert space which satisfy (2.1) in order to construct a representation of $\mathcal{O}_N$.

Let Iso$\mathcal{O}_N$ be the set of all isometries in $\mathcal{O}_N$.

Definition 2.1. *For $P \in \mathrm{Iso}\mathcal{O}_N$, a triplet $(\mathcal{H}, \pi, \Omega)$ is a P-representation of $\mathcal{O}_N$ if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with a cyclic unit vector Ω such that $\pi(P)\Omega = \Omega$.*

Note that the notion of P-representation is a generalization of generalized permutative representation of $\mathcal{O}_N$ with cycle in [**6**, **7**]. Permutative representations with chain([**2**, **4**, **5**]) is not included in the class of P-representations.

In this paper, we treat representations of only $\mathcal{O}_2$. We show examples of P-representation. Put isometries P_S, P_{12}, P_B in $\mathcal{O}_2$ by

$$P_S \equiv s_1, \quad P_{12} \equiv s_1 s_2, \quad P_B \equiv \frac{1}{\sqrt{2}}(s_1 + s_2). \tag{2.2}$$

(i) The *standard representation* $(l_2(\mathbf{N}), \pi_S)$ of $\mathcal{O}_2$ is defined by

$$\pi_S(s_1)e_n \equiv e_{2n-1}, \quad \pi_S(s_2)e_n \equiv e_{2n} \quad (n \in \mathbf{N}) \tag{2.3}$$

where $\{e_n\}_{n\in\mathbf{N}}$ is the canonical basis of $l_2(\mathbf{N})$ and $\mathbf{N} = \{1,2,3,\ldots\}$([**1**, **9**]). Then $(l_2(\mathbf{N}), \pi_S, e_1)$ is a P_S-representation.

(ii) Put a representation $(l_2(\mathbf{N}), \pi_{12})$ of $\mathcal{O}_2$ by

$$\begin{array}{c} \pi_{12}(s_1)e_{2n-1} \equiv e_{4n-1}, \quad \pi_{12}(s_1)e_{2n} \equiv e_{4n-3}, \\ \pi_{12}(s_2)e_n \equiv e_{2n} \end{array} \quad (n \in \mathbf{N}). \tag{2.4}$$

Then $(l_2(\mathbf{N}), \pi_{12}, e_1)$ is a P_{12}-representation.

(iii) The *barycentric representation* $(L_2[-1,1], \pi_B)$ of $\mathcal{O}_2$ is defined by

$$(\pi_B(s_1)\phi)(x) \equiv \sqrt{2}\chi_{[0,1]}(x)\phi(2|x|-1),$$

$$(\pi_B(s_2)\phi)(x) \equiv \sqrt{2}\chi_{[-1,0]}(x)\phi(2|x|-1)$$

for $\phi \in L_2[-1,1]$ and $x \in [-1,1]$ where χ_Y is the characteristic function of a subset Y of $[-1,1]$([**10**]). Then $(L_2[-1,1], \pi_B, \Omega)$ is a P_B-representation where Ω is the constant function on $[-1,1]$ with value $1/\sqrt{2}$. This example is treated in § 4.1 again.

It is easy to show that cyclicities and eigen equations in the above follow from their definitions, respectively. Both π_S and π_{12} are permutative representations in [**2**, **4**, **5**] (π_{12} is denoted by Rep(12) in [**1**]). π_B is not. π_S, π_{12} and π_B are generalized permutative representations of $\mathcal{O}_2$ which correspond to those with parameters $z = (1,0), (1,0) \otimes (0,1), \frac{1}{\sqrt{2}}(1,1)$, respectively([**6**]).

Proposition 2.2. *Recall P_S, P_{12}, P_B in (2.2).*

(i) *For $x \in \{S, 12, B\}$, P_x-representation is unique up to unitary equivalences. We denote them $(\mathcal{H}_x, \pi_x, \Omega_x)$ or $(\mathcal{H}_x, \pi_x)$ for $x \in \{S, 12, B\}$.*
(ii) *Any element in $\{(\mathcal{H}_x, \pi_x) : x \in \{S, 12, B\}\}$ is irreducible.*
(iii) *Any two elements in $\{(\mathcal{H}_x, \pi_x) : x \in \{S, 12, B\}\}$ are inequivalent.*

Proof. See Appendix A. □

We often identify an equivalence class of representations and its representative when there is no ambiguity. Furthermore we often use a symbol π_x as $(\mathcal{H}_x, \pi_x, \Omega_x)$ for $x \in \{S, 12, B\}$ simply. Notations of π_S, π_{12}, π_B in the above examples are justified by Proposition 2.2 (i). For $P \in \mathrm{Iso}\mathcal{O}_N$, a P-representation is neither unique nor irreducible in general.

Put α an action of a unitary group $U(2)$ on $\mathcal{O}_2$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^{2} g_{ji} s_j$ for $i = 1, 2$. A symbol $\sim$ means the unitary equivalence of representations.

Proposition 2.3. *Let π_x be the P_x-representation of $\mathcal{O}_2$ for $x \in \{S, 12, B\}$.*

(i) *If $g \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then $\pi_S \circ \alpha_g \sim \pi_B$.*
(ii) *For a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_2$, put $\pi^{(12)}$ by*

$$\pi^{(12)}(s_1) \equiv \pi(s_1 s_2 s_1^* + s_1 s_1 s_2^*), \quad \pi^{(12)}(s_2) \equiv \pi(s_2).$$

Then $(\mathcal{H}, \pi^{(12)})$ is a representation of $\mathcal{O}_2$, too. If $\pi \sim \pi_S$, then $\pi^{(12)} \sim \pi_{12}$.

Proof. (i) Since π_S is cyclic, $\pi_S \circ \alpha_g$ is cyclic, too. We can see that the eigen equation of the condition of π_B holds for $\pi_S \circ \alpha_g$ by direct computation. Hence $\pi_S \circ \alpha_g \sim \pi_B$ by Proposition 2.2 (i).
(ii) The first statement is directly shown by checking (2.1) for $\pi^{(12)}$. For π_S and π_{12} in (2.3) and (2.4), we have the following relations:

$$\pi_{12}(s_1) = \pi_S(s_1 s_2 s_1^* + s_1 s_1 s_2^*), \quad \pi_{12}(s_2) = \pi_S(s_2).$$

Assume that there is a unitary U from $\mathcal{H}$ to $l_2(\mathbf{N})$ such that $\mathrm{Ad}U \circ \pi = \pi_S$ where $(\mathrm{Ad}U \circ \pi)(x) \equiv U\pi(x)U^*$ for $x \in \mathcal{O}_2$. Then

$$\pi_{12}(s_1) = \pi_S(s_1 s_2 s_1^* + s_1 s_1 s_2^*) = U\pi(s_1 s_2 s_1^* + s_1 s_1 s_2^*)U^* = U\pi^{(12)}(s_1)U^*,$$

$$\pi_{12}(s_2) = \pi_S(s_2) = U\pi(s_2)U^* = U\pi^{(12)}(s_2)U^*.$$

Hence $\pi^{(12)}$ is unitarily equivalent to π_{12}. □

We show the property of π_{12} in more detail. Denote a set of multiindices $\{1,2\}^* \equiv \bigcup_{k\geq 0}\{1,2\}^k$ where $\{1,2\}^k \equiv \{(i_j)_{j=1}^k : i_j = 1 \text{ or } 2,\ j = 1,\ldots,k\}$, $\{1,2\}^0 \equiv \{\emptyset\}$. For $J = (j_1,\ldots,j_k), J' = (j_1',\ldots,j_l') \in \{1,2\}^*$, put

$$J \cup J' \equiv (j_1,\ldots,j_k,j_1',\ldots,j_l'), \quad J^k = \underbrace{J \cup \cdots \cup J}_{k},$$

$$s_J \equiv s_{j_1}\cdots s_{j_k}, \quad s_J^* \equiv s_{j_k}^*\cdots s_{j_1}^* \quad (k \geq 1), \quad s_\emptyset = I.$$

Proposition 2.4. (i) *If $(\mathcal{H},\pi)$ is π_{12} of $\mathcal{O}_2$, then the eigen value of $\pi(s_1 s_2)$ on $\mathcal{H}$ is only 1. The eigen vector of $\pi(s_1 s_2)$ is unique up to scalar multiplications.*

(ii) *If $(\mathcal{H},\pi)$ is a cyclic representation of $\mathcal{O}_2$ such that there is an eigen vector Ω of $\pi(s_2 s_1)$ with eigen value 1, then $(\mathcal{H},\pi)$ is π_{12}.*

(iii) *The GNS-representation of the following state ρ of $\mathcal{O}_2$*

$$\rho(s_J s_{J'}^*) = \begin{cases} 1 & (J \equiv J' \mod (12)) \\ 0 & (\textit{otherwise}) \end{cases} \quad (J, J' \in \{1,2\}^*), \tag{2.5}$$

is equivalent to π_{12} where $J \equiv J' \mod (12)$ if there are $k, l \in \mathbf{N}\cup\{0\}$ such that $J \cup (12)^k = J' \cup (12)^l$. Specially, ρ is pure.

Proof. See Appendix B. □

3. Construction of representations of $\mathcal{O}_2$ from $\mathbf{Z}_2$-graded dynamical systems

3.1. $\mathbf{Z}_2$-graded dynamical systems. We prepare notions of $\mathbf{Z}_2$-graded measure space, $\mathbf{Z}_2$-graded dynamical system, symmetric measure space and branching function system on a measure space ([**8**, **9**]). Here we treat branching function systems which consist of only two functions.

Definition 3.1. (i) *A measure space (X,μ) is $\mathbf{Z}_2$-graded if there are a measure preserving $\mathbf{Z}_2$-action γ on X and a subspace $Y \subset X$ such that $\mu(\gamma(Y)\cap Y) = 0$ and $\mu(X \setminus \gamma(Y) \cup Y) = 0$. We simply denote (X,μ,γ,Y) a $\mathbf{Z}_2$-graded measure space.*

(ii) *We call a triplet* (X, μ, γ) *a symmetric measure space if* (X, μ) *is a measure space and* γ *is a measure preserving map from* X *to* X *such that* $\gamma^2 = id_X$ *and* $\mu(X^\gamma) = 0$ *where* $X^\gamma = \{x \in X : \gamma(x) = x\}$.

(iii) (X, F) *is a* $\mathbf{Z}_2$*-graded dynamical system if* F *is a measurable transformation on a* $\mathbf{Z}_2$*-graded measure space* (X, μ, γ, Y) *and* F *is symmetric(with respect to* γ*), that is,* $F \circ \gamma = F$.

Definition 3.2. *Let* (X, μ) *be a measure space. Recall the definition of branching function system in* § 1.

(i) F *is the coding map of a branching function system* $f = \{f_1, f_2\}$ *on* (X, μ) *if* F *is a map from* X *to* X *such that* $(F \circ f_i)(x) = x$ *for* $x \in X$ *and* $i = 1, 2$.

(ii) *A branching function system* $f = \{f_1, f_2\}$ *on a symmetric measure space* (X, μ, γ) *is symmetric if* $\gamma \circ f_1 = f_2$.

If $f = \{f_1, f_2\}$ is a symmetric branching function system on a symmetric measure space (X, μ, γ), then $\mu(X \setminus X_1 \cup X_2) = 0$, $\gamma(X_1) = X_2$ for $X_i \equiv f_i(X)$, $i = 1, 2$. If F is the coding map of a symmetric branching function system, then $F \circ \gamma = F$.

Lemma 3.3. *Let* (X, F) *be a* $\mathbf{Z}_2$*-graded dynamical system with a* $\mathbf{Z}_2$*-graded measure space* (X, μ, γ, Y)*. Then the followings hold:*

(i) (X, μ, γ) *is a symmetric measure space.*

(ii) *Assume that* F *satisfies the following condition:*

(3.1) *the restriction* $F|_Y$ *of* F *to* Y *is a bijection from* Y *to* X *and there is the Radon-Nikodým derivative* Φ_F *of* $\mu \circ F$ *with respect to* μ *such that it is non zero almost everywhere in* Y.

Then there is a symmetric branching function system of F *on* X.

Proof. (i) Note that $X^\gamma = Z_1 \sqcup Z_2, Z_1 \equiv X^\gamma \setminus (Y \cup \gamma(Y)), Z_2 \equiv X^\gamma \cap (Y \cup \gamma(Y))$. Then $\mu(X^\gamma) = \mu(Z_1) + \mu(Z_2)$. Since $\mu(Z_1) \leq \mu(X \setminus Y \cup \gamma(Y)) = 0$, $\mu(X^\gamma) = \mu(Z_2)$. On the other hand, $\mu(Z_2) = \mu(X^\gamma \cap Y \cup X^\gamma \cap \gamma(Y))$. If $x \in X^\gamma \cap Y$, then $x = \gamma(x) \in X^\gamma \cap \gamma(Y)$. Hence $X^\gamma \cap Y = X^\gamma \cap \gamma(Y) \subset Y \cap \gamma(Y)$. Therefore $\mu(Z_2) \leq \mu(Y \cap \gamma(Y)) = 0$. Hence $\mu(X^\gamma) = 0$. From this, (X, μ, γ) is a symmetric measure space.
(ii) Assume that F is a symmetric transformation on a (X, μ, γ, Y) which satisfies the condition (3.1). Since $F|_Y : Y \to X$ is bijective, we obtain transformations f_1, f_2 on X by $f_1 \equiv (F|_Y)^{-1}, \quad f_2 \equiv \gamma \circ f_1$. Then $f_1(X) \cap f_2(X) = Y \cap \gamma(Y)$, $X \setminus (f_1(X) \cup f_2(X)) = X \setminus (Y \cup \gamma(Y))$ are μ-null sets. By property of the Radon-Nikodým derivative of F, those of f_1, f_2 exist and they are non zero almost everywhere in X. Hence $\{f_1, f_2\}$ is a branching function system of F on X. By definition of f_2, $\{f_1, f_2\}$ is symmetric. □

Proposition 3.4. *Let (X, μ, γ) be a symmetric measure space. Then the followings are equivalent:*

(i) *There are a symmetric transformation F on X and a subspace $Y \subset X$ such that (X, F) is a $\mathbf{Z}_2$-graded dynamical system with a $\mathbf{Z}_2$-graded measure space (X, μ, γ, Y) and F satisfies (3.1).*

(ii) *There is a symmetric branching function system $\{f_1, f_2\}$ on X.*

Proof. (i) to (ii) follows from Lemma 3.3 (ii). Assume (ii). Put $Y \equiv f_1(X)$. Then $Y \cap \gamma(Y) = f_1(X) \cap f_2(X)$, $X \setminus (Y \cup \gamma(Y)) = X \setminus (f_1(X) \cup f_2(X))$ are μ-null sets by definition of branching function system. Hence (X, μ, γ, Y) is a $\mathbf{Z}_2$-graded measure space. Put a transformation F on X by $F(x) \equiv f_1^{-1}(x)$ for $x \in Y$, $F(x) \equiv f_2^{-1}(x)$ for $x \in \gamma(Y)$, $F(x) \equiv x$ for $x \in X \setminus (Y \cup \gamma(Y))$. Then F is symmetric and a bijection from Y to X such that $\Phi_F(x) = 1/\Phi_{f_1}(f_1^{-1}(x)) > 0$ for $x \in Y$, $\Phi_F(x) = 1/\Phi_{f_2}(f_2^{-1}(x)) > 0$ for $x \in \gamma(Y)$. Hence Φ_F is non zero almost everywhere in X. Therefore F satisfies (3.1). (i) holds. □

3.2. Isometries arising from transformations on measure spaces.

We show the method of construction of isometries and representations of $\mathcal{O}_N$ on measure spaces([**9**, **10**]) here without proof.

Let (X, μ) be a measure space and f a transformation on X which is injective and there exists the Radon-Nikodým derivative Φ_f of $\mu \circ f$ with respect to μ and Φ_f is non zero almost everywhere in X. We denote the set of such transformations on X by $RN(X)$. Note that $RN(X)$ is a semigroup with respect to composition of transformations. Denote $\mathrm{Iso}(L_2(X, \mu))$ the semigroup of isometries on $L_2(X, \mu)$.

Definition 3.5. *Define an operator $S(f)$ on $L_2(X, \mu)$ by*

(3.2)
$$(S(f)\phi)(x) \equiv \begin{cases} \left\{\Phi_f\left(f^{-1}(x)\right)\right\}^{-1/2} \phi(f^{-1}(x)) & (\textit{when } x \in R(f)\,), \\ 0 & (\textit{otherwise}) \end{cases}$$

for $\phi \in L_2(X, \mu)$ and $x \in X$.

Lemma 3.6. *For $f \in RN(X)$, $S(f)$ in (3.2) is an isometry on $L_2(X, \mu)$. A map S from $RN(X)$ to $\mathrm{Iso}(L_2(X, \mu))$ is a semigroup homomorphism, that is,*

$$S(f)S(g) = S(f \circ g) \quad (f, g \in RN(X)). \tag{3.3}$$

Remark that $f \circ g$ in rhs of (3.3) is only the composition of two transformations f and g but not special product of them. By Lemma 3.6, we see that the map S realizes the iteration of transformations on a measure space as the product of operators on a Hilbert space naturally.

Recall the definition of branching function system in § 1. The notion of branching function system was introduced in [**2**] in order to construct a representation of $\mathcal{O}_N$ from a family of maps.

Let $N \geq 2$.

Proposition 3.7. *For a branching function system $f = \{f_i\}_{i=1}^N$ on (X, μ),*

$$\pi_f(s_i) \equiv S(f_i) \quad (i = 1, \ldots, N),$$

defines a representation $(L_2(X, \mu), \pi_f)$ of $\mathcal{O}_N$.

Proposition 3.8. *Let $f = \{f_i\}_{i=1}^N$ and $g = \{g_i\}_{i=1}^N$ be branching function systems over measure spaces (X, μ) and (Y, ν), respectively. Assume that there is a map φ from X to Y such that the Radon-Nikodým derivative of $\nu \circ \varphi$ with respect to μ exists and it is non zero almost everywhere in X, and map identities $g_i = \varphi \circ f_i \circ \varphi^{-1}$ for $i = 1, \ldots, N$ hold. Then $(L_2(X, \mu), \pi_f)$ and $(L_2(Y, \nu), \pi_g)$ are unitarily equivalent.*

3.3. Operators and representations of $\mathcal{O}_2$ arising from $\mathbf{Z}_2$-graded dynamical systems. Let (X, μ, γ, Y) be a $\mathbf{Z}_2$-graded measure space. Put an operator U on $L_2(X, \mu)$ by

$$(U\phi)(x) \equiv \phi(\gamma(x)) \quad (\phi \in L_2(X, \mu),\ x \in X). \tag{3.4}$$

Then U is a unitary and $U = U^*$. By U, we have the following decomposition:

$$L_2(X, \mu) = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{3.5}$$

where $\mathcal{H}_\pm = \{\phi \in L_2(X, \mu) : U\phi = \pm\phi\}$. $\mathcal{H}_+$ and $\mathcal{H}_-$ are called the *even* and *odd* subspace of $L_2(X, \mu)$ with respect to γ, respectively. Since a decomposition $\phi = \phi_+ + \phi_-$, $\phi_+ \equiv \frac{1}{2}(\phi + \phi \circ \gamma)$, $\phi_- \equiv \frac{1}{2}(\phi - \phi \circ \gamma)$ always holds for any $\phi \in L_2(X, \mu)$, (3.5) is shown.

On the other hand, we have another decomposition of $L_2(X, \mu)$ as follows:

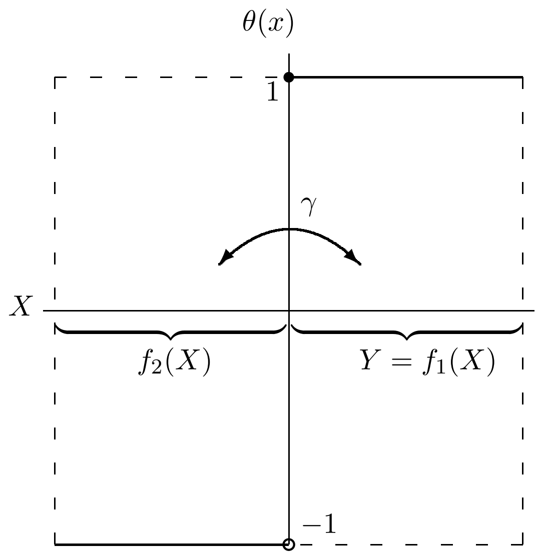
$$L_2(X, \mu) = L_2(Y, \mu) \oplus L_2(\gamma(Y), \mu)$$

where $L_2(Y, \mu)$ is a subspace of $L_2(X, \mu)$ which consists of $\phi|_{X \setminus Y} = 0$ and $L_2(\gamma(Y), \mu)$ is that which consists of $\phi|_{X \setminus \gamma(Y)} = 0$. If there is a symmetric branching function system $f = \{f_1, f_2\}$ on X, then $\pi_f(s_1 s_1^*) L_2(X, \mu) = L_2(Y, \mu)$ and $\pi_f(s_2 s_2^*) L_2(X, \mu) = L_2(\gamma(Y), \mu)$ by Proposition 3.7 and (3.2).

Let F be a transformation on X which satisfies the condition (3.1) with a branching function system $f = \{f_1, f_2\}$ in Lemma 3.3 (ii). Define a pair $\{T, M\}$ of operators on $L_2(X, \mu)$ by

$$(T\phi)(x) \equiv \frac{1}{\sqrt{2}} \{\Phi_F(x)\}^{1/2} \phi(F(x)), \quad (M\phi)(x) \equiv \theta(x)\phi(x) \tag{3.6}$$

for $\phi \in L_2(X, \mu)$ and $x \in X$ where θ is the *sign function* of X defined by $\theta(x) \equiv 1$ when $x \in f_1(X)$ and $\theta(x) \equiv -1$ when $x \in X \setminus f_1(X)$.



The adjoint T^* of T is given by

$$(T^*\phi)(x) = \frac{1}{\sqrt{2}}\left\{\{\Phi_{f_1}(x)\}^{1/2}\phi(f_1(x)) + \{\Phi_{f_2}(x)\}^{1/2}\phi(f_2(x))\right\}.$$

Lemma 3.9. *Let (X, F) be a $\mathbf{Z}_2$-graded dynamical system which satisfies (3.1), $\mathcal{H}_\pm$ in (3.5) and a pair $\{T, M\}$ in (3.6). Then the followings hold:*

(i) $M^2 = id$, $M^* = M$, *and* $M\mathcal{H}_+ = \mathcal{H}_-$.
(ii) *For U in (3.4),* $UT = T$.
(iii) *T is an isometry.*
(iv) $TT^*L_2(X,\mu) = \mathcal{H}_+$.

Proof. Let $\{f_1, f_2\}$ be the branching function system of F on X.
(i) follows by definition of M. By $F \circ \gamma = F$, (ii) holds.
(iii)

$$\begin{aligned}\|T\phi\|^2 = & \frac{1}{2}\int_X \Phi_F(x)|\phi(F(x))|^2\, d\mu(x) \\ = & \frac{1}{2}\left\{\int_{X_1} \Phi_F(x)|\phi(f_1^{-1}(x))|^2\, d\mu(x) + \int_{X_2} \Phi_F(x)|\phi(f_2^{-1}(x))|^2\, d\mu(x)\right\}.\end{aligned}$$

Since $\int_{X_i} \Phi_F(x)|\phi(f_i^{-1}(x))|^2\, d\mu(x) = \|\phi\|^2$ for $i = 1, 2$, we have $\|T\phi\| = \|\phi\|$. Hence (iii) is proved.
(iv) For $x \in X$, $(f_2 \circ F)(x) = (f_2 \circ f_2^{-1})(\gamma(|x|)) = \gamma(|x|)$ where

$$|x| \equiv \begin{cases} \gamma(x) & \text{(when } x \in f_2(X)), \\ x & \text{(otherwise).} \end{cases}$$

Hence

$$(TT^*\phi)(x) = \frac{1}{2}\{\phi(|x|) + \phi(\gamma(|x|))\} = \frac{1}{2}\{\phi(x) + \phi(\gamma(x))\}.$$

Therefore $TT^*L_2(X,\mu) = \mathcal{H}_+$. □

Recall Definition 2.1.

Theorem 3.10. *Let (X,F) be a $\mathbf{Z}_2$-graded dynamical system which satisfies (3.1) with the $\mathbf{Z}_2$-graded measure space (X,μ,γ,Y). Let $\{T,M\}$ be a pair in (3.6). Then the followings hold:*

(i) *The following π_1, π_2, π_3 define representations of $\mathcal{O}_2$ on $L_2(X,\mu)$:*

$$\pi_1(s_1) \equiv S(f_1), \quad \pi_1(s_2) \equiv S(f_2),$$

$$\pi_2(s_1) \equiv T, \quad \pi_2(s_2) \equiv MT, \quad \pi_3(s_1) \equiv TM, \quad \pi_3(s_2) \equiv MT$$

where $\{f_1, f_2\}$ is the branching function system of F and $S(f_i)$ is in (3.2).

(ii) *Ranges of $\pi_j(s_i)$, $j = 1,2,3$, $i = 1,2$ are given by*

$$\pi_1(s_i s_i^*)L_2(X,\mu) = \{\phi \in L_2(X,\mu) : \phi|_{X_i} = 0\},$$

$$\pi_j(s_i s_i^*)L_2(X,\mu) = \{\phi \in L_2(X,\mu) : \phi(-x) = (-1)^{i-1}\phi(x),\, x \in X\}$$

for $j = 2,3$ and $i = 1,2$ where $X_1 = \gamma(Y)$ and $X_2 = Y$.

(iii) *The following relations hold:*

$$\pi_3(s_1) = \pi_2(s_1 s_2 s_1^* + s_1 s_1 s_2^*), \quad \pi_3(s_2) = \pi_2(s_2),$$

$$\frac{1}{\sqrt{2}}\pi_1(s_1 + s_2) = \pi_2(s_1), \quad \frac{1}{\sqrt{2}}\pi_1(s_1 - s_2) = \pi_2(s_2).$$

(iv) *Assume that T has an eigen function Ω with eigen value 1, and put $\mathcal{H}_i \equiv \pi_i(\mathcal{O}_2)\Omega$ for $i = 1,2,3$ and $Q_1 \equiv \frac{1}{\sqrt{2}}(s_1+s_2)$, $Q_2 \equiv s_1$, $Q_3 \equiv s_1 s_2$. Then a subrepresentation $(\mathcal{H}_i, \pi_i|_{\mathcal{H}_i})$ of $(L_2(X,\mu),\pi_i)$ is equivalent to Q_i-representation for each $i = 1,2,3$. Specially, $(\mathcal{H}_i, \pi_i|_{\mathcal{H}_i})$ is irreducible for each $i = 1,2,3$ and they are mutually inequivalent.*

Proof. (i) holds by Proposition 3.7 and checking (2.1) directly. (ii) and (iii) follow by computation directly, too.
(iv) Let $\Omega \in L_2(X,\mu)$ be an eigen vector of T with eigen value 1. By Lemma 3.9 (i), $\pi_3(s_1 s_2)\Omega = TMMT\Omega = T^2\Omega = \Omega$. Then $(\mathcal{H}_3, \pi_3|_{\mathcal{H}_3}, \Omega)$ is Q_3-representation by Definition 2.1 where Ω is normalized. Cases of π_1, π_2 are proved in the same way. By Proposition 2.2, last statements hold. □

Proof of Theorem 1.1. A closed interval $[-a,a]$ is a $\mathbf{Z}_2$-graded measure space with respect to the Lebesgue measure and $\gamma(x) \equiv -x$, $x \in X$, $Y \equiv [0,a]$. Let F be a function on $[-a,a]$ which satisfies the condition in Theorem

1.1. Then F satisfies (3.1). From this, (i),(ii),(iii) hold by computation and Proposition 2.3. Applying Theorem 3.10 for F, we have the statement. $\square$

4. Examples

A closed interval $[-1,1]$ with Lebesgue measure is a $\mathbf{Z}_2$-graded measure space by the sign transformation $\gamma(x) \equiv -x$ for $x \in [-1,1]$ and $Y \equiv [0,1]$. We simply denote $[-1,1]$ as $([-1,1], dx, \gamma, [0,1])$. For a $\mathbf{Z}_2$-graded dynamical system $([-1,1], F)$, we use symbols T, M in (3.6), too.

4.1. A dynamical system of piecewise linear transformation. Let V be a transformation on $[-1,1]$ defined by

$$V(x) \equiv 2|x| - 1.$$

Note that V is a symmetric transformation on $[-1,1]$, and $\Phi_V(x) = 2$ for each $x \in [-1,1]$.

$$(T\phi)(x) = \phi(2|x|-1), \quad (M\phi)(x) = \theta(x)\phi(x) \quad (\phi \in L_2[-1,1])$$

where $\theta(x) = 1$ when $x \geq 0$, $\theta(x) = -1$ when $x < 0$. By Theorem 3.10, we have a representation $(L_2[-1,1], \pi_V)$ of $\mathcal{O}_2$ defined by

$$(4.1) \quad (\pi_V(s_1)\phi)(x) = \hat{\theta}(x)\phi(2|x|-1), \quad (\pi_V(s_2)\phi)(x) = \theta(x)\,\phi(2|x|-1)$$

for $\phi \in L_2[-1,1]$ and $x \in [-1,1]$ where $\hat{\theta}(x) \equiv 1$ when $x \in [-1,-1/2] \cup [1/2,1]$, $\hat{\theta}(x) \equiv -1$ when $x \in (-1/2,1/2)$, $\pi_V(s_1)$ and $\pi_V(s_2)$ are isometries on $L_2[-1,1]$ onto subspaces which consist of even and odd functions, respectively. Specially, meanings of even and odd coincide ordinary definition in this case. Adjoints of $\pi_V(s_1)$ and $\pi_V(s_2)$ are given as follows:

$$\begin{cases} (\pi_V(s_1)^*\phi)(x) = & \dfrac{1}{2}\theta(x)\,(\phi(v_1(x)) + \phi(v_2(x))), \\ \\ (\pi_V(s_2)^*\phi)(x) = & \dfrac{1}{2}(\phi(v_1(x)) - \phi(v_2(x))) \end{cases}$$

where $v_1(x) = \frac{1}{2}(x+1)$ and $v_2(x) = -\frac{1}{2}(x+1)$ on $[-1,1]$. Since T has an eigen function $\mathbf{1}$, $\pi_V(s_1s_2)\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is the constant function on $[-1,1]$ which takes value 1. Note that $\pi(s_1) \equiv T$ and $\pi(s_2) \equiv MT$ gives a representation $(L_2[-1,1], \pi)$ of $\mathcal{O}_2$ which is equivalent to π_S and $\pi^{(12)} = \pi_V$. By Proposition 2.3, $(L_2[-1,1], \pi_V, \Omega)$ is π_{12} where $\Omega \equiv \frac{1}{\sqrt{2}}\mathbf{1}$. Specially, $(L_2[-1,1], \pi_V)$ is irreducible.

This representation is a natural realization of π_{12} on $L_2[-1,1]$.

4.2. Quadratic case. Let F be a transformation on $[-1,1]$ defined by

$$F(x) \equiv 2x^2 - 1. \tag{4.2}$$

Then

$$(T\phi)(x) = \sqrt{2|x|}\phi(2x^2-1), \quad (M\phi)(x) = \theta(x)\phi(x) \quad (\phi \in L_2[-1,1]).$$

The representation $(L_2[-1,1],\pi)$ of $\mathcal{O}_2$ which is defined by T and M in Theorem 3.10 by (4.2) is

$$\begin{cases} (\pi(s_1)\phi)(x) = & \sqrt{2|x|}\theta(2x^2-1)\phi(2x^2-1), \\ \\ (\pi(s_2)\phi)(x) = & \sqrt{2|x|}\theta(x)\phi(2x^2-1) \end{cases}$$

for $\phi \in L_2[-1,1]$ and $x \in [-1,1]$. This representation is equivalent to that in § 4.1([**8**]). For example, a function

$$\Omega(x) \equiv \frac{1}{\sqrt{\pi}}\frac{1}{(1-x^2)^{1/4}} \quad (x \in [-1,1])$$

satisfies $\pi(s_1)\pi(s_2)\Omega = \Omega$ and $(L_2[-1,1],\pi)$ is cyclic. Hence $(L_2[-1,1],\pi)$ is π_{12}.

4.3. On R. Consider a $\mathbf{Z}_2$-graded measure space $(\mathbf{R},\mu,\gamma,[0,\infty))$ defined by $\gamma(x) \equiv -x$ and

$$d\mu(x) \equiv \frac{1}{2(1+|x|)^2}dx \quad (x \in \mathbf{R}).$$

A transformation F on $\mathbf{R}$ defined by

$$F(x) \equiv \begin{cases} \dfrac{1}{2} - \dfrac{1}{2|x|} & (x \in [-1,1]) \\ \\ \dfrac{|x|-1}{2} & (\text{otherwise}) \end{cases}$$

is symmetric and satisfies the condition in (3.1). We have

$$(T\phi)(x) = \phi(F(x)), \quad (M\phi)(x) = \theta(x)\phi(x),$$

$$(\pi(s_1)\phi)(x) = \theta\left(F(x)\right)\phi(F(x)), \quad (\pi(s_2)\phi)(x) = \theta\left(x\right)\phi(F(x))$$

for $\phi \in L_2(\mathbf{R},\mu)$. $(L_2(\mathbf{R},\mu),\pi)$ is π_{12} because there is a measure-space isomorphism R from $([-1,1],dx)$ to $(\mathbf{R},\mu)$ defined by $R(x) \equiv \mathrm{sgn}x \cdot (\frac{1}{1-|x|} - 1)$. R gives a unitary operator U from $L_2[-1,1]$ to $L_2(\mathbf{R},\mu)$ and $U\pi_V(x)U^* = \pi(x)$ for each $x \in \mathcal{O}_2$ where π_V is in (4.1) by Proposition 3.8.

5. Discussion

We summarize our results here.

For one thing, we give representations of $\mathcal{O}_2$ which are naturally defined on $L_2(X,\mu)$ by symmetric transformations on a measure space (X,μ) with $\mathbf{Z}_2$-grading. The $\mathbf{Z}_2$-symmetry of the transposition $s_1 \leftrightarrow s_2$ is realized as that of isometries with range space of even and odd functions(Proposition 2.4 (ii) and Theorem 3.10). For next thing, the representation π_{12} of $\mathcal{O}_2$ is characterized as an eigen equation of an operator which is naturally defined by a transformation of dynamical system(Theorem 3.10). Lastly, π_{12} is effectively realized as a system of isometries on $\mathbf{Z}_2$-graded measure space. The naive meaning of π_{12} is easily understood by this realization.

We show remaining problems about our subjects:

(i) For a $\mathbf{Z}_2$-graded dynamical system, when does T in Theorem 3.10 have an eigen function ? This is equivalent to the problem of the following functional equation:

$$\sqrt{\Phi_F(x)}\phi(F(x)) = \sqrt{2}\phi(x) \quad (x \in X). \tag{5.1}$$

For a given F, when does (5.1) have a solution ϕ in $L_2(X,\mu)$?

(ii) Generalize Theorem 3.10 for $\mathcal{O}_N$, $N \geq 3$. How can we treat $\mathbf{Z}_N$-symmetry as geometric symmetry ?

(iii) How can we realize other GP representation in [**6**] on $L_2(X,\mu)$ naturally by using geometric symmetric ?

(iv) What is a C*-algebra $\mathcal{A} \equiv C^* < \{T, M\} >$ for a pair $\{M, T\}$ of operators in (3.6) ? $\mathcal{A} \cong \mathcal{O}_2$? Can $\mathcal{A}$ be formulated without representation theory ?

(v) We have transformations between inequivalent representations in Proposition 2.3. We see that the transformation between π_S and π_B is invertible but not that between π_S and π_{12}. Can we construct a transformation from π_{12} to π_S concretely ?

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Appendix A. Proof of Proposition 2.2

Permutative representations of $\mathcal{O}_N$ are studied in [**2**, **4**, **5**] and they are generalized in [**6**, **7**]. Results in § 2 and this appendix are included in [**6**, **10**]. For convenience, we prove several properties here. Permutative representations are treated as GP representations here.

A.1. Construction of the canonical basis of the P_{12}-representation. We construct a complete orthonormal basis of a given P_{12}-representation of $\mathcal{O}_2$ in order to show uniqueness of P_{12}-representation.

Recall the notation of multiindices in § 2. For $J \in \{1,2\}^*$, $|J|$ is the length of J, that is, $|J| = k$ when $J = \{1,2\}^k$. Put a subset $\Lambda(12)$ of $\{1,2\}^*$ by

$$\Lambda(12) \equiv \coprod_{m\geq 0} \Lambda^{(m)}(12),$$

$$\Lambda^{(0)}(12) \equiv \{(12),(2)\}, \quad \Lambda^{(1)}(12) \equiv \{(112),(22)\},$$

$$\Lambda^{(m)}(12) \equiv \{J \cup J^{'} : J^{'} \in \Lambda^{(1)}(12),\ J \in \{1,2\}^{m-1}\} \quad (m \geq 2).$$

Lemma A.1. *(The canonical basis) Assume that $(\mathcal{H},\pi)$ is a representation of $\mathcal{O}_2$ such that there is an eigen vector Ω of $\pi(s_1s_2)$ in $\mathcal{H}$ with eigen value 1. Then the followings hold:*

(i) $\{\pi(s_Js^*_{J^{'}})\Omega : J,J^{'} \in \{1,2\}^*\} = \{\pi(s_J)\Omega : J \in \Lambda(12)\}$.

(ii) *If $(\mathcal{H},\pi)$ is cyclic and $\|\Omega\| = 1$, then $\{\pi(s_J)\Omega : J \in \Lambda(12)\}$ is a complete orthonormal basis of $\mathcal{H}$.*

Proof. Assume that $(\mathcal{H},\pi)$ is a representation of $\mathcal{O}_2$ with the eigen vector Ω of $\pi(s_1s_2)$ in $\mathcal{H}$ with eigen value 1.

Put a family $B \equiv \{\pi(s_J)\Omega : I \in \Lambda(12)\}$ of vectors in $\mathcal{H}$.

(i) Assume $J^{'} \in \{1,2\}^k$, $k \geq 1$. When $k = 2n$, $\pi(s_Js^*_{J^{'}})\Omega = \pi(s_Js^*_{J^{'}})(\pi(s_{12}))^n\Omega = \delta_{J^{'}J^{''}}\pi(s_J)\Omega$ for $J^{''} = (12)^n$. When $k = 2n-1$, $\pi(s_Js^*_{J^{'}})\Omega = \pi(s_Js^*_{J^{'}})(\pi(s_{12}))^n\Omega = \delta_{J^{'}J^{''}}\pi(s_{J\cup\{2\}})\Omega$ for $J^{''} = (12)^{n-1}\cup(1)$. From these, we see $\{\pi(s_Js^*_{J^{'}})\Omega : J,J^{'} \in \{1,2\}^*\} = \{\pi(s_J)\Omega : J \in \{1,2\}^*\}$. Furthermore, when $J = J^{'}\cup(12)^k$, $\pi(s_J)\Omega = \pi(s_{J^{'}})\Omega$. Hence $\{\pi(s_J)\Omega : J \in \{1,2\}^*\} = \{\pi(s_{J\cup\{1\}})\Omega, \pi(s_{J\cup\{12\}})\Omega : J \in \{1,2\}^*\} = B$ because $\pi(s_{J\cup\{1\}})\Omega = \pi(s_{J\cup\{112\}})\Omega$.

(ii) Because $\pi(s_J)$ for any $J \in \Lambda(12)$ is an isometry, B is a family of unit vectors in $\mathcal{H}$.

Assume $J,J^{'} \in \Lambda(12)$.

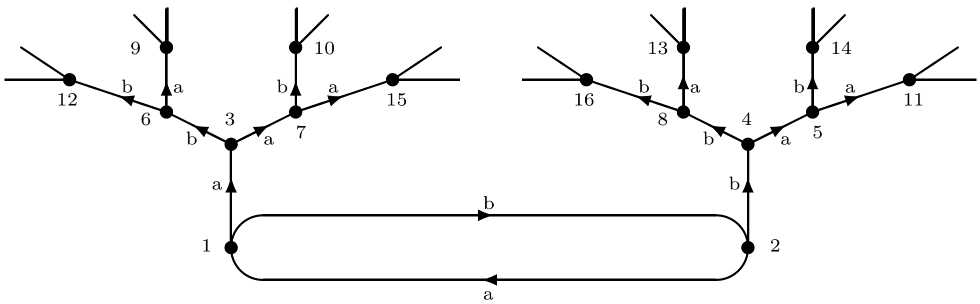
(a) If $|J| = |J^{'}|$, then $< \pi(s_J)\Omega|\pi(s_{J^{'}})\Omega >= \delta_{JJ^{'}} < \Omega|\Omega >= \delta_{JJ^{'}}$.

(b) If $|J| > |J^{'}|$ and $J = (j_1,\ldots,j_{k+l})$ and $J^{'} = (j^{'}_1,\ldots,j^{'}_k)$, $l \geq 1$, then $< \pi(s_J)\Omega|\pi(s_{J^{'}})\Omega >= \delta_{J_1J^{'}} < \pi(s_{J_2})\Omega|\Omega >$ where $J_1 = (j_1,\ldots,j_k)$ and $J_2 = (j_{k+1},\ldots,j_{k+l})$. If $l = 2n$ for some $n \in \mathbf{N}$, then $< \pi(s_{J_2})\Omega|\Omega >=< \pi(s_{J_2})\Omega|\pi(s_{12})^n\Omega >= \delta_{J_2J^{''}} < \Omega|\Omega >= \delta_{J_2J^{''}}$ where $J^{''} = (12)^n$. By definition of $\Lambda(12)$, $J_2 \neq J^{''}$. Hence $< \pi(s_{J_2})\Omega|\Omega >= 0$. If $l = 2n-1$ for some $n \in \mathbf{N}$, then $< \pi(s_{J_2})\Omega|\Omega >=< \pi(s_{J_2})\Omega|\pi(s_{12})^n\Omega >= \delta_{J_2J^{''}} < \Omega|\pi(s_2)\Omega >= \delta_{J_2J^{'''}}$ where $J^{'''} = (12)^{n-1}\cup(1)$. By definition of $\Lambda(12)$, $J_2 \neq J^{'''}$. Therefore $< \pi(s_J)\Omega|\pi(s_{J^{'}})\Omega >= 0$ when $|J| > |J^{'}|$.

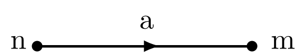
¿From these, $< \pi(s_J)\Omega|\pi(s_{J^{'}})\Omega >= \delta_{JJ^{'}}$ for $J,J^{'} \in \Lambda(12)$. Therefore B is an orthonormal family of $\mathcal{H}$. By assumption, $(\mathcal{H},\pi)$ is cyclic. Hence

Lin $< \{\pi(s_J s_{J'}^*)\Omega : J, J' \in \{1,2\}^*\} >$ is dense in $\mathcal{H}$. By (i), B is a complete orthonormal basis of $\mathcal{H}$. □

We illustrate the meaning of the basis of P_{12}-representation. The *tree* of $(l_2(\mathbf{N}), \pi_{12})$ in (2.4) is following:



where vertices and edges mean the canonical basis $\{e_n\}_{n\in\mathbf{N}}$ of $l_2(\mathbf{N})$ and the action of operators $\pi_{12}(s_1), \pi_{12}(s_2)$ on $\{e_n\}_{n\in\mathbf{N}}$, respectively. For example, if $\pi_{12}(s_1)e_n = e_m$ for $n, m \in \mathbf{N}$, then it is represented as



where labels a, b of edges correspond to s_1, s_2, respectively.

Note $\pi_{12}(s_1)\pi_{12}(s_2)e_1 = \pi_{12}(s_1)e_2 = e_1$. Hence $(l_2(\mathbf{N}), \pi_{12})$ satisfies the eigen equation of P_{12}-representation. The eigen equation $\pi_{12}(s_1 s_2)e_1 = e_1$ corresponds with the cycle in the tree of representation. On the other hand, the action of $\mathcal{O}_2$ on $l_2(\mathbf{N})$ generates every e_n, $n \in \mathbf{N}$ by (2.4). Therefore $(l_2(\mathbf{N}), \pi_{12})$ is cyclic. Hence $(l_2(\mathbf{N}), \pi)$ is a P_{12}-representation.

A.2. Uniqueness and irreducibility.

Proposition A.2. *Let $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ be P_{12}-representations of $\mathcal{O}_2$ with cyclic vectors Ω and Ω', respectively. Then $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ are equivalent.*

Proof. By Lemma A.1 (ii), $\{\pi(s_J)\Omega : J \in \Lambda(12)\}$ and $\{\pi'(s_J)\Omega' : J \in \Lambda(12)\}$ are complete orthonormal basis of $\mathcal{H}$ and $\mathcal{H}'$, respectively. Put $U\pi(s_J)\Omega \equiv \pi'(s_J)\Omega'$. Then U defines a unitary from $\mathcal{H}$ to $\mathcal{H}'$ and it satisfies $U\pi(s_i)U^* = \pi'(s_i)$ for $i = 1, 2$. □

By Proposition A.2, the uniqueness of P_{12}-representation is proved.

Proposition A.3. *The P_{12}-representation is irreducible.*

Proof. By Theorem 2.7 in [**2**], $(l_2(\mathbf{N}), \pi_{12})$ in (2.4) is irreducible. By Proposition A.2, every representation which is π_{12} is equivalent to $(l_2(\mathbf{N}), \pi_{12})$ in (2.4). Therefore the P_{12}-representation is irreducible. □

The uniqueness and irreducibility of P_S, P_B-representations are shown in [**9**, **10**] in detail.

A.3. Inequivalence.

Lemma A.4. *Let $(\mathcal{H}, \pi, \Omega)$ be the P_{12}-representation of $\mathcal{O}_2$. If there are a vector $\Omega^{'} \in \mathcal{H}$ and $(a_1, a_2) \in \mathbf{R}^2$ such that $a_1^2 + a_2^2 = 1$ and $\pi(a_1 s_1 + a_2 s_2)\Omega^{'} = \Omega^{'}$. Then $\Omega^{'} = 0$.*

Proof. By Lemma A.1 (ii), we can denote $\Omega^{'} = \sum_J c_J \pi(s_J)\Omega$ for $c_J \in \mathbf{C}$, $J \in \Lambda(12)$. Assume that $< \Omega^{'}|\pi(s_J)\Omega > \neq 0$. Put $T \equiv a_1 s_1 + a_2 s_2$. If $|J| = k$, then $< \Omega^{'}|\pi(s_J)\Omega > = < \pi(T)^k \Omega^{'}|\pi(s_J)\Omega > = \kappa_J < \Omega^{'}|\Omega >$ where $\kappa_J = \prod_{l=1}^{k} a_{j_l}$ when $J = (j_1, \ldots, j_k)$. Note $0 \leq |\kappa_J| \leq 1$.

If $|\kappa_J| < 1$, then $< \Omega^{'}|\pi(s_J)\Omega >$ must be 0.

Assume that $|\kappa_J| = 1$. Hence $|a| = 1$ or $|b| = 1$. If $|a| = 1$, then $J = (1)^k$. Since $(1)^k \notin \Lambda(12)$. $|a| \neq 0$. In the same way, $|b| \neq 1$. Therefore $|\kappa_J| \neq 1$.

Consequently, $< \Omega^{'}|\pi(s_J)\Omega > = 0$ for each $J \in \Lambda(12)$. Hence $\Omega^{'} = 0$ because $\{\pi(s_J)\Omega : J \in \Lambda(12)\}$ is a complete orthonormal basis of $\mathcal{H}$. □

Let $(\mathcal{H}_x, \pi_x, \Omega_x)$ be a P_x-representation of $\mathcal{O}_2$ for $x \in \{S, 12, B\}$. $(l_2(\mathbf{N}), \pi_S)$ and $(L_2[-1,1], \pi_B)$ in § 2. Assume that $\Omega, \Omega_S, \Omega_B$ are corresponded eigen vectors with respect to π, π_S, π_B, respectively.

Assume that $\pi \sim \pi_{12}$. Then we can identify $\pi = \pi_{12}$ on $l_2(\mathbf{N})$. Applying Lemma A.4 for $\Omega^{'} = \Omega_S$ for $a = 1$ and $b = 0$, we obtain $\Omega_S = 0$. This is contradiction. Hence $\pi \not\sim \pi_S$.

Assume that $\pi_B \sim \pi$ and identify $\pi_B = \pi$. Applying Lemma A.4 for $\Omega^{'} = \Omega_B$ for $a = b = \frac{1}{\sqrt{2}}$, we obtain $\Omega_B = 0$. This is contradiction. Hence $\pi \not\sim \pi_B$.

The inequivalence between π_S and π_B is shown in [**9**].

Appendix B. Proof of Proposition 2.4

B.1. Uniqueness of eigen value. Let $(\mathcal{H}, \pi, \Omega)$ be π_{12} of $\mathcal{O}_2$.

Assume that $\Omega^{'}$ is an eigen vector of $\pi(s_1 s_2)$ in $\mathcal{H}$ with eigen value c. Since $\pi(s_1 s_2)$ is an isometry, $|c| = 1$. Because $\pi(s_2^* s_1^*)\Omega^{'} = \bar{c}\Omega^{'}$,

$$< \Omega|\Omega^{'} > = < \pi(s_1 s_2)\Omega|\Omega^{'} > = < \Omega|\pi(s_1 s_2)^* \Omega^{'} > = \bar{c} < \Omega|\Omega^{'} > .$$

Hence $\bar{c} = 1$ or $< \Omega|\Omega^{'} > = 0$.

Assume that $\bar{c} \neq 1$. Then $< \Omega|\Omega^{'} > = 0$. By Lemma A.1, we can denote $\Omega^{'} = \sum_J a_J \pi(s_J)\Omega$. Hence there is $J \in \Lambda(12)$ such that $< \Omega^{'}|\pi(s_J)\Omega > \neq 0$.

By assumption, $J \neq (12)^n$ for any $n \geq 1$. If $|J| = 2n$, then $< \Omega^{'}|\pi(s_J)\Omega > = c^n < \pi(s_{12})^n \Omega^{'}|\pi(s_J)\Omega > = \delta_{J^{'}J} c^n < \Omega^{'}|\Omega > = 0$ where $J^{'} \equiv (12)^n$. This is contradiction. If $|J| = 2n - 1$, then $< \Omega^{'}|\pi(s_J)\Omega > =$

$c^n < \pi(s_{12})^n\Omega^{'}|\pi(s_J)\Omega >= \delta_{J^{'}J}c^n < \pi(s_2)\Omega^{'}|\Omega >= 0$ where $J^{'} \equiv (12)^{n-1}$. This is contradiction, too. Hence $c = 1$.

B.2. Uniqueness of eigen vector. Let $(\mathcal{H}, \pi, \Omega)$ be π_{12} of $\mathcal{O}_2$.

Assume that $\Omega^{'}$ is an eigen vector of $\pi(s_1s_2)$ in $\mathcal{H}$ with eigen value 1. By Lemma A.1, we can denote $\Omega^{'} = \sum_J a_J\pi(s_J)\Omega$. Hence there is $J \in \Lambda(12)$, such that $< \Omega^{'}|\pi(s_J)\Omega >\neq 0$.

Assume that $J \neq (12)$. If $|J| = 2n$ for $n \geq 1$, then $< \Omega^{'}|\pi(s_J)\Omega >=< \pi(s_{12})^n\Omega^{'}|\pi(s_J)\Omega >= \delta_{J^{'}J} < \Omega^{'}|\Omega >$ where $J^{'} \equiv (12)^n$. Since $J^{'} \neq J$, $< \Omega^{'}|\pi(s_J)\Omega >= 0$ and this is contradiction. If $|J| = 2n-1$, then $< \Omega^{'}|\pi(s_J)\Omega >=< \pi(s_{12})^n\Omega^{'}|\pi(s_J)\Omega >= \delta_{J^{'}J} < \pi(s_2)\Omega^{'}|\Omega >$ where $J^{'} \equiv (12)^{n-1}$. Since $J^{'} \neq J$, $< \Omega^{'}|\pi(s_J)\Omega >= 0$ and this is contradiction, too. Therefore $J = (12)$. Hence there is $c \in \mathbf{C}$, $c \neq 0$ such that $\Omega^{'} = c\Omega$. Proposition 2.4 (i) is shown.

B.3. Equivalence. Assume that $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_2$ such that there is an eigen vector Ω of $\pi(s_2s_1)$ with eigen value 1. Put $\Omega^{'} \equiv s_1\Omega$. Then $\pi(s_1s_2)\Omega^{'} = \pi(s_1s_2)\pi(s_1)\Omega = \pi(s_1)(\pi(s_2s_1)\Omega) = \pi(s_1)\Omega = \Omega^{'}$. Therefore we see that $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_2$ such that there is an eigen vector $\Omega^{'}$ of $\pi(s_1s_2)$ with eigen value 1. By definition of π_{12} and uniqueness, $(\mathcal{H}, \pi)$ is π_{12}. From this, Proposition 2.4 (ii) is shown.

More general results are shown in [**6**, **11**]. We show them without proof.

For $g \in U(2)$, the following holds:

$$\pi_{12}\circ\alpha_g \sim \pi_{12} \text{ if and only if } g \in \left\{ \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}, \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} \in U(2) : c \in U(1) \right\}.$$

Specially, a transposition $\beta \in \mathrm{Aut}\mathcal{O}_2$, $\beta(s_1) \equiv s_2$, $\beta(s_2) \equiv s_1$, preserves the equivalence class of π_{12}.

B.4. State. Let $(l_2(\mathbf{N}), \pi_{12})$ be the representation of $\mathcal{O}_2$ in (2.4). Put $\rho^{'}(x) \equiv< \Omega|\pi_{12}(x)\Omega >$ for $x \in \mathcal{O}_2$. Then $\rho^{'}$ defines a state of $\mathcal{O}_2$ and $\rho^{'} = \rho$ in Proposition 2.4 (iv). Since $(l_2(\mathbf{N}), \pi_{12})$ is irreducible and the GNS-representation of ρ is uniqueness up to unitary equivalences, Proposition 2.4 (iii) is proved.

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