ON SEMISTABLE MORI CONTRACTIONS

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Abstract. We study Fano-Mori contractions with fibers of dimension at most one satisfying the semistability assumption. As an application of our technique we give a new proof of the existence of semistable 3-fold flips.

1. Introduction

This paper is a continuation of our study of Mori contractions from threefolds to surfaces (see [12], [13], [14]). We refer to [10], [8] for the terminology of the minimal model theory.

Let X be a normal algebraic threefold over $\mathbb{C}$ (or three-dimensional normal complex space) with only terminal singularities. A proper surjective morphism $f\colon X \to Z$ is called a *Fano-Mori contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Z$ and the anticanonical divisor $-K_X$ is f-ample. Our interest is in the *local structure* of such contractions, so we shall always assume that Z is not a point and Z and X are sufficiently small (algebraic or analytic) neighborhoods of some point $o \in Z$ and the fiber $f^{-1}(o)$, respectively. Note that we do not assume that f is an *extremal Mori contraction* (i.e., X is $\mathbb{Q}$-factorial and $\rho(X/Z) = 1$). If the dimension of fibers of f (near $f^{-1}(o)$) is at most one we can distinguish the following cases:

- $\dim Z = 2$, then f is called a *Mori conic bundle*,
- $\dim Z = 3$ and f contracts a divisor to a curve, then f is called a 2-1-*type divisorial contraction*,
- $\dim Z = 3$ and the exceptional locus of f is one-dimensional, then f is called a *flipping contraction*.

In this paper we deal with *semistable* Fano-Mori contractions which appear in the semistable minimal model program, see [19], [4], [18], [5], [8], and references therein.

Definition 1.1. A Fano-Mori contraction $f\colon X \to Z \ni o$ is said to be *semistable* if there exists an effective Cartier divisor $o \in T \subset Z$ such that (X, f^*T) is divisorial log terminal (dlt).

It is clear that in the above definition we may replace T with a general hyperplane section through o. In particular, in the case $\dim Z = 3$ we may assume that T is irreducible, normal, and (X, f^*T) is purely log terminal (plt).

Example 1.2 ([12, Example 2.1])**.** The toric contraction $(\mathbb{P}^1 \times \mathbb{C}^2)/\boldsymbol{\mu}_n(0:a;1,-1) \to \mathbb{C}^2/\boldsymbol{\mu}_n(1,-1)$ is a semistable Mori conic bundle with $T = \{x_1x_2 = 0\}/\boldsymbol{\mu}_n$.

We shall show that the example above is very special: in "most" cases the surface $f^{-1}(T)$ is irreducible and normal.

Proposition 1.3. *Let $f\colon X \to Z \ni o$ be a semistable Mori conic bundle and let T be a general hyperplane section through o.*

- (i) *If T is reducible, then f is analytically isomorphic to the Mori conic bundle from Example* 1.2*. In particular, $Z \ni o$ is a Du Val point of type A_{m-1}.*
- (ii) *If T is irreducible, then $Z \ni o$ is smooth and the pair (X, f^*T) is purely log terminal (plt). In this case $S := f^*T$ is a normal surface with cyclic quotient singularities of type* T *or Du Val of type A_n (see* §3 *for the definition of* T*-singularities).*

In case (ii) the structure of X and f is completely determined by the structure of the surface S and the contraction $S \to T$. We study such contractions in §5.

In practice, it is very difficult to construct nontrivial examples of Mori conic bundles explicitly (cf. [12, §5]). Using deformation theory it is very easy to prove the existence (or non-existence) of semistable ones, see §4. In particular, we prove the following.

Theorem 1.4. *For any three-dimensional terminal semistable singularity $U \ni P$, there is a semistable Mori conic bundle $f\colon X \to Z \ni o$ with a unique singular point which is analytically isomorphic to $U \ni P$.*

The following result was inspired by M. Reid's "general elephant" conjecture (cf. [12, §4]) and provides an evidence for it.

Theorem 1.5 (cf. [3, Th. 0.4.5] [7, Th. 2.2], [18, Corr. 4.9])**.** *Let $f\colon X \to Z \ni o$ be a semistable Fano-Mori contraction such that the dimension of fibers is at most one and let T be a general hyperplane section through o. Then $K_X + f^*T$ is* 1*-complementary* [15]*, i.e., there exists an effective integral Weil divisor F such that $K_X + f^*T + F$ is log canonical* (*lc*) *and linearly trivial over Z. Moreover, $K_X + F$ is canonical and linearly trivial, the surface F is normal, has only Du Val singularities of type A_n, and in the Stein factorization $F \to \bar{F} \to f(F)$, the same holds for $\bar{F}$.*

Using this theorem we give a new proof of the existence of semistable flips in §7. Our proof is based on Theorem 1.5, Kawamata's double covering trick [4], and the existence of certain canonical flops.

Terminology. The semistable MMP is originated in semistable degenerations of surfaces. Namely, let $h\colon \mathfrak{X} \to \mathit{\Delta}$ be a projective surjective morphism from smooth threefold to a smooth curve such that the general fiber is a smooth surface and special fibers are reduced simple normal crossings divisors. In this situation $\mathfrak{X}/\mathit{\Delta}$ satisfies the following property:

$$(*) \qquad (\mathfrak{X}, h^*P) \text{ is dlt for every point } P \in \mathit{\Delta}.$$

In order to obtain either a minimal or relative Fano model we run the K-MMP over $\mathit{\Delta}$. Every step of the K-MMP is in the same time a step of the $K + h^*P$-MMP. In particular, the property (*) is preserved and all contractions and flips are semistable in our sense. This agrees more or less with definitions given in [19], [4], [1].

The semistability defined by Shokurov in [18] (cf. [11]) is also close to our one by [18, Lemma 1.4]. However the construction is inductive and given in terms of some (not necessarily projective) resolution.

Kollár and Mori [7, p. 541] defined semistable extremal neighborhoods $f\colon X \to Z \ni o$ in terms of general member $F_Z \ni |-K_Z|$: f is semistable if $F_Z \ni o$ is a Du Val singularity of type A_n. By Theorem 1.5 our definition 1.1 implies Mori-Kollár's one. Conversely, if $F_Z \ni o$ is a Du Val singularity of type A_n, then $K_X + F + f^*T$ is log canonical (but not necessarily dlt) for some effective Cartier divisor T on Z. Thus f is "almost" semistable in our sense.

Our technique uses the Kawamata-Viehweg vanishing theorem, so the proofs work only in characteristic zero. The positive and mixed characteristic case was treated in [5].

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2. Preliminary results

In this section we prove Proposition 1.3.

Let S be an algebraic surface (or two-dimensional complex space) having at worst quotient singularities and let $\varphi\colon S \to T$ be a contraction. We assume that T is not a point and T is a sufficiently small neighborhood of $o \in T$. We say that φ is a *log contraction* if $-K_S$ is φ-ample. If furthermore any singularity of S is of type T or Du Val, then we say that φ is a T-*contraction*.

Lemma 2.1 (see [14, Lemma 2.5])**.** *Let $f\colon X \to Z \ni o$ be a Mori conic bundle and let T be an effective $\mathbb{Q}$-Weil divisor on Z such that (X, f^*T) is lc (resp. plt) at some point $P \in f^{-1}(o)$. Then (Z, T) is lc (resp. plt).*

Lemma 2.2. *Let $f\colon X \to Z \ni o$ be a semistable Mori conic bundle and let T be an effective Cartier divisor such that (X, f^*T) is dlt. If T is irreducible, then $Z \ni o$ is smooth and for a general hyperplane section $o \in T_{\mathrm{gen}} \subset Z$ the pair (X, f^*T_{gen}) is plt.*

Proof. Follows by Lemma 2.1 and Bertini's theorem. □

Proposition 2.3. *Let $f\colon X \to Z \ni o$ be a Mori conic bundle. Assume that there is an effective Weil divisor T on Z such that (X, f^*T) is lc. If T is reducible, then f is analytically isomorphic to one of the following the Mori conic bundles*

(i) *f from Example* 1.2*, or*
(ii) *$X'/\boldsymbol{\mu}_2(1:0:0;1,1) \to \mathbb{C}^2/\boldsymbol{\mu}_2(1,1)$, where $X' = \{x_0^2 + x_1^2 + x_2^2\phi(u,v) = 0\} \subset \mathbb{P}^2_{x_0,x_1,x_2} \times \mathbb{C}^2_{u,v}$ and $\phi(u,v)$ is a $\boldsymbol{\mu}_2$-invariant without multiple factors, see* [12, Example 2.3]*.*

In particular, $Z \ni o$ is Du Val of type A_{m-1}. Moreover, the statement of Theorem 1.5 *holds for f.*

Proof. By Lemma 2.1, T has exactly two components T_1 and T_2. Consider a base change

$$\begin{array}{ccc} X' & \xrightarrow{\ \upsilon\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\ \pi\ } & Z \end{array}$$

where Z' is smooth, $Z = Z'/\boldsymbol{\mu}_n$, $X = X'/\boldsymbol{\mu}_n$, and $\boldsymbol{\mu}_n$ acts on Z' free in codimension one (see [12, (1.9)]). Put $S = f^*T$, $S' = \upsilon^{-1}(S)$, $T' = \pi^*(T)$, $T'_i = \pi^*(T_i)$, and $S'_i = f'^*T'_i$. Then (X', S') is lc. Replacing T'_1 and T'_2 with general hyperplane sections, we may assume that T'_1 is smooth and $(S'_1, S'_1|_{S'_2})$ is lc, where $S'_1|_{S'_2}$ is a Cartier divisor on S'_1. In this situation, S'_1 has at worst Du Val singularities. Hence X' is Gorenstein. Since $f'^{-1}(o')$ is reduced, by [12, §2] X/Z we have only two choices for the action of $\boldsymbol{\mu}_n$. Finally, the statement of Theorem 1.5 easily follows by the proposition below. □

Proposition 2.4 (see [15, Prop. 4.4.1])**.** *Let $f\colon X \to Z \ni o$ be a contraction and let S be a reduced divisor on X such that $S \cap f^{-1}(o) \neq \varnothing$, (X,S) is plt and $-(K_X+S)$ is f-nef and f-big. If $K_S + \mathrm{Diff}_S$ is n-complementary, then so is K_X+S. Here Diff_S is the* different, *a correcting term in the Adjunction Formula $K_S + \mathrm{Diff}_S = (K_X+S)|_S$, see* [10, Ch. 16].

From now on we consider semistable Fano-Mori contractions $f\colon X \to Z \ni o$ such that (X, f^*T) is plt (and the dimension of fibers is at most one).

3. Singularities of class T

Let $\boldsymbol{\mu}_n$ acts on $\mathbb{C}^2$ via $(x,y) \to (\eta^a x, \eta^b y)$, where η is a primitive nth root of unity and $\gcd(n,a) = \gcd(n,b) = 1$. In this case we say that the quotient $\mathbb{C}^2/\boldsymbol{\mu}_n$ is a singularity of type $\frac{1}{n}(a,b)$. This singularity can be written as $\frac{1}{n}(1,q)$, so it is determined by the fraction n/q. The minimal resolution of $\frac{1}{n}(1,q)$ can be described as follows. The dual graph of the exceptional divisor is a chain of smooth rational curves whose self-intersections $-b_1,\dots,-b_\varrho$ are determined by the continued fraction expansion

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots \cfrac{1}{b_\varrho}}}. \tag{3.1}$$

For typographical reasons we denote the fraction in (3.1) by $[b_1,\dots,b_\varrho]$. Put $\varrho(n/q) := \varrho$. Define also the following invariants:

- $\iota(n/q) = n/\gcd(n,q+1)$, the index of $\frac{1}{n}(1,q)$,
- $\beta(n/q) = \gcd(n,q+1)/\iota(n/q) = \gcd(n,q+1)^2/n$,
- $\gamma(n/q) = (q+1)/\gcd(n,q+1)$.

By definitions, $\iota,\gamma \in \mathbb{N}$, $\gamma \le \iota$, $\gcd(\iota,\gamma) = 1$. Thus we have the triple (ι,β,γ) which determines n/q:

$$n = \beta\iota^2, \quad q = \beta\iota\gamma - 1,$$

Note that presentation of a cyclic quotient singularity in the form $\frac{1}{n}(1,q)$ is not unique: $\frac{1}{n}(1,q')$ defines the same singularity if and only if either $q \equiv q'$ or $qq' \equiv 1 \mod n$. Clearly, $\varrho(n/q) = \varrho(n/q')$. Since $\gcd(n,q+1) = \gcd(n,q'+1)$, we have $\beta(n/q) = \beta(n/q')$ and $\iota(n/q) = \iota(n/q')$.

Definition 3.2. A cyclic quotient non-Du Val singularity $\frac{1}{n}(a,b)$ is said to be of type T (or simply T-singularity) if $(a+b)^2 \equiv 0 \mod n$. (It is easy to see that this definition does not depend on the representation in the form $\frac{1}{n}(a,b)$).

If $\frac{1}{n}(1,q)$ is a T- (resp. Du Val) singularity, then we say that n/q is a T- (resp. Du Val) fraction and $[b_1,\dots,b_\varrho]$ is T- (resp. Du Val) chain. Thus n/q is a T-fraction if and only if $\beta(n/q) \in \mathbb{Z}$.

Lemma 3.3. *Let $qq' \equiv 1 \mod n$. Then n/q is a* T*-fraction if and only if $q+q' = n-2$ if and only if $\gamma(n/q') + \gamma(n/q) = \iota(n/q)$.*

Remark 3.4. If $\iota(n/q) = 2$, then $\gamma(n/q) = 1$ and n/q is a T-fraction. It is easy to see that this implies either

$$n/q = [4], \qquad \text{or} \qquad n/q = [3,2,\dots,2,3]. \tag{3.5}$$

Moreover, $\beta(n/q) = \varrho(n/q)$. Conversely, any chain such as in (3.5) has $\iota = 2$.

The minimal resolutions of T-singularities are completely described.

Proposition 3.6 ([9])**.** (i) *If the chain $[b_1,\dots,b_\varrho]$ is of type* T*, then so are the chains*

a) $[2, b_1, \dots, b_\varrho + 1]$ *and b)* $[b_1+1, \dots, b_\varrho, 2]$

(ii) *Every* T*-chain can be obtained by starting with one of the chains* (3.5) *and iterating the steps described in* (i)*.*

For a chain $[b_1,\dots,b_\varrho]$, we denote corresponding log discrepancies by $\alpha_1,\dots,\alpha_\varrho$.

Lemma 3.7. *In the above notation one has $\alpha_1 = (q+1)/n = \gamma/\iota$. If n/q is a* T*-fraction, then $\alpha_\varrho = 1 - \gamma/\iota$.*

Proof. The $\frac{1}{n}(1,q)$-weighted blow-up gives us the first relation. The second one follows by Lemma 3.3. □

Corollary 3.8. *Let $[b_1,\dots,b_\varrho]$ be any chain. The following are equivalent:*

(i) $[b_1,\dots,b_\varrho]$ *is of type* T*;*
(ii) $\alpha_1 + \alpha_\varrho = 1$.

Theorem 3.9 ([2, Prop. 5.9], [9])**.** *Let $S \ni P$ be a germ of a two-dimensional quotient singularity. The following are equivalent:*

(i) *$S \ni P$ is either Du Val or of type* T*,*
(ii) *$S \ni P$ has a $\mathbb{Q}$-Gorenstein one-parameter smoothing,*

(iii) *there is a terminal three-dimensional singularity* $X \ni P$ *and an embedding* $P \in S \subset X$ *such that* $S \subset X$ *Cartier at* P *and* (X, S) *is plt.*

4. Constructing semistable Mori conic bundles via deformations

In this section all varieties are assumed to be analytic spaces.

Definition 4.1. A log (resp. T) contraction $\varphi\colon S \to T \ni o$ with $\dim T = 1$ is called a *log* (resp. T) *conic bundle.*

Theorem 4.2. *Let* $\varphi\colon S \to T \ni o_T$ *be a* T*-conic bundle. There exists a semistable Mori conic bundle* $f\colon X \to Z \ni o_Z$ *with smooth base and embeddings*

$$\begin{array}{ccc} S & \hookrightarrow & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} \\ T & \overset{\upsilon}{\hookrightarrow} & Z \end{array}$$

such that $\upsilon(o_T) = o_Z$ *and* (X, S) *is plt.*

We shall construct X as an one-parameter family of T-contractions.

Proof. We replace S and T with their compactifications so that S and T are projective, $T \simeq \mathbb{P}^1$, and $\varphi\colon S \to T$ is smooth outside of $\varphi^{-1}(o_T)$. Let P_i be singular points of S.

Denote $\mathrm{Def}(S)$ (resp. $\mathrm{Def}(S, P_i)$) the base space of the versal deformation of S (resp. of the singularity $S \ni P_i$). Let $s\colon \mathcal{S} \to \mathrm{Def}(S)$ be the versal family. Thus we may assume that $S = s^{-1}(o)$ for some $o \in \mathrm{Def}(S)$.

From [7, Proposition 11.4] we obtain the existence of the diagram of morphisms of complex analytic spaces.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\mathcal{F}} & \mathcal{T} = T \simeq \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathrm{Def}(S) & \longrightarrow & \mathrm{Def}(T) = \mathrm{pt} \end{array}$$

where $\mathcal{F}(S) = T$ and $\mathcal{F}|_S = \varphi$.

There is a natural (pull-back) morphism of germs of analytic spaces

$$\mathrm{Def}(S) \longrightarrow \prod_i \mathrm{Def}(S, P_i) \tag{4.3}$$

The obstruction to globalize deformations in $\prod_i \mathrm{Def}(S, P_i)$ lies in $R^2\varphi_*\Theta_S$, where $\Theta_S = (\Omega^1_S)^\vee$, the tangent sheaf of S. Since $R^2\varphi_*\Theta_S =$

0, the map (4.3) is smooth. In particular, every deformation of singularities $S \ni P_i$ may be globalized (cf. [7, 11.4.2]).

By Theorem 3.9 every singularity of S admits a $\mathbb{Q}$-Gorenstein one-parameter smoothing. Therefore there is a smoothing $g\colon X \to \mathit{\Delta} \ni 0$, where $g^{-1}(0) = S$, X is $\mathbb{Q}$-Gorenstein and $\mathit{\Delta} \subset \mathbb{C}$ is a small disc. By Inversion of Adjunction (X, S) is plt and since S is Cartier, X has at worst terminal singularities near S.

Put $Z = T \times \mathit{\Delta}$ and let $f\colon X \to Z$ be the projection. It is clear that $f|_S = \varphi\colon S \to T$. Therefore $-K_X$ is f-ample near S. Shrinking Z we get a Mori conic bundle $f\colon X \to Z \ni o = (o_T, 0)$. □

5. Two-dimensional log contractions

Notation 5.1. Let $\varphi\colon S \to T \ni o$ be a log contraction. We assume that S has at least one non-Du Val singularity. Let $\mu\colon \tilde{S} \to S$ be a minimal resolution and let $\phi\colon \tilde{S} \to T$ be the composition map. Take an effective Cartier divisor D on S such that $\operatorname{Supp}(D) = \varphi^{-1}(o)$ and $-D$ is φ-nef. For example, in the case $\dim T = 1$ we can put $D = \varphi^*(o)$ (the scheme-theoretical fiber) while in the case $\dim T = 2$ we can put $D := \varphi^*\varphi_* H - H$, where H is a very ample divisor on S such that $\varphi_* H$ is Cartier.

One can write the standard formula

$$\mu^* K_S = K_{\tilde{S}} + \Delta, \tag{5.2}$$

where Δ is an effective exceptional divisor, so-called, *codiscrepancy divisor.* Since the singularities of S are log terminal, $\lceil \Delta \rceil = 0$. We also write $\mu^* D = \sum l_i L_i + e_j E_j$, where the E_i (resp. L_i) are μ-exceptional (resp. non-μ-exceptional) components and $l_i, e_i \in \mathbb{N}$. Put $L = \sum l_i L_i$ and $E = e_j E_j$. Thus, $D = \mu_* L$ and $\operatorname{Supp}(\Delta) \subset \sum E_i$.

Lemma 5.3. *Notation as above.*

(i) $\operatorname{Supp}(L + E)$ *is a tree of smooth rational curves;*
(ii) *all the components of* L *are* (-1)*-curves;*
(iii) $\Delta \cdot L_i < 1$ *for all* i*;*
(iv) *if* $\dim T = 1$*, then* $L \cdot \Delta + 2 = \sum l_i$.

Proof. (i) is obvious because ϕ is flat in the case $\dim T = 1$ and because $Z \ni o$ is a rational singularity in case $\dim T = 2$. By (5.2) we have

$$0 > \mu^* K_S \cdot L_i = K_{\tilde{S}} \cdot L_i + \Delta \cdot L_i = \Delta \cdot L_i - 2 - L_i^2.$$

Since $L_i^2 < 0$ and $\Delta \cdot L_i \geq 0$, we have $L_i^2 = K_{\tilde{S}} \cdot L_i = -1$ and $\Delta \cdot L_i < 1$. This shows (ii) and (iii). For (iv) we note that $\mu^* K_S \cdot L = -2$. Thus,

$$\sum l_i = -K_{\tilde{S}} \cdot L = -\mu^* K_S \cdot L + \Delta \cdot L = \Delta \cdot L + 2.$$

□

Remark 5.4. (i) It is easy to see that the condition (iii) of 5.3 is also sufficient. Assume that conditions 5.1 hold except for the ampleness of $-K_S$. If $\Delta \cdot L_i < 1$ for all i, then φ is a log contraction, i.e., $-K_S$ is ample.

(ii) If S has a unique non-Du Val point, then (iii) holds.

To describe log contractions we make use the weighted graph language. By a *weighted* graph Γ we mean a graph where each vertex is given a weight $b_i \geq 1$. With a weighted graph $\Gamma = \{v_1, \ldots, v_\varrho\}$ we associate a *quadratic form* by setting $v_i^2 = -b_i$ and $v_i \cdot v_j$ for $i \neq j$ is equal to the number of edges joining v_i and v_j. We say that a weighted graph $\Gamma = \{v_1, \ldots, v_\varrho\}$ is *elliptic* (resp. *parabolic*) if the associated quadratic form has signature $(0, \varrho)$ (resp. $(0, \varrho - 1)$). Vertices with $b_i = 1$ will be referred to (and drawn) as *black* vertices those with $b_i \geq 2$ as *white.* Weights $b_i = 1$ and $b_i = 2$ will be omitted.

By the *blow-up of a vertex* v_i we mean the following transformation: the weight of the vertex v_i increases by one, that is, $b_i' = b_i + 1$ and a new black vertex is added to the graph, attached by an edge to the vertex v_i'. Similarly the *blow-up of an edge* $\{v_i, v_j\}$ is the following transformation: the weight of the vertices v_i and v_j increase by one, the number of edges joining v_i and v_j decreases by one, and a new black vertex is added to the graph, attached by edges to the vertices v_i' and v_j'. The inverse transformations are called *contractions.* One can easily see how the above transformations are related to blow-ups of curves on a smooth surface.

We denote by $[a_1, \ldots, a_r]$ a (weighted) chain and by $[p \mid a_1, \ldots, a_r \mid b_1, \ldots, b_s \mid c_1, \ldots, c_l]$ a fork Γ having the central vertex v_0 of weight p so that $\Gamma \setminus \{v_0\}$ is a disjoined union of chains $[a_1, \ldots, a_r]$, $[b_1, \ldots, b_s]$, and $[c_1, \ldots, c_l]$, where vertices corresponding a_1, b_1, and c_1 are adjacent to v_0.

Now in notation 5.1 we denote by $\Gamma(\varphi)$ the dual graph of the fiber $\phi^{-1}(o)$. By (i) of Lemma 5.3, $\Gamma(\varphi)$ is a tree. Moreover, the graph $\Gamma(\varphi)$ is parabolic whenever $\dim T = 1$ and elliptic whenever $\dim T = 2$.

Lemma 5.5. *The fork* $[1 \mid a \mid b \mid c]$ *is not elliptic for* $a, b, c \geq 1$. *The following graphs are parabolic* (*and not elliptic*):

(i) *chains* $[1, 1]$, $[1, 2, \ldots, 2, 1]$, $[2, 1, 2]$,

(ii) *the fork* $[2 \mid 2 \mid 2 \mid 2, \ldots, 2, 1]$

Corollary 5.6. *Let* D_i *be irreducible components of* D, *then*

(i) *intersection points* $D_i \cap D_j$ *are singular and not Du Val,*

(ii) *there are at most two singular points on every* D_i,

(iii) (S, D_i) *is plt near every Du Val point,*
(iv) S *has no Du Val singularities of type* D_n *and* E_n.

Configuration of singular points.

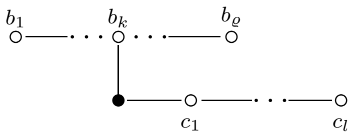
Lemma 5.7. *Let* $\varphi\colon S \to T \ni o$ *be a* T*-contraction and let* C *be a component of* D. *Assume that* C *contains exactly two singular points of* S *and they are of type* T *(not Du Val). Then* (S, C) *is plt.*

Proof. If $C \neq \operatorname{Supp}(D)$, then C is contractible over T, i.e., there is a decomposition $S \to T' \to T$ such that $\varphi'\colon S \to T'$ is birational and C is the only φ'-exceptional divisor. Replacing T with T' we may assume that $C = \operatorname{Supp}(D)$.

Let P_1, P_2 be singular points. Assume that (S, C) is not plt near P_1. We claim that (S, C) is plt near P_2. Indeed, take two general hyperplane sections H_1 and H_2 passing through P_1 and P_2. For some $0 < \varepsilon' \ll \varepsilon \ll 1$ the divisor $-(K_S + (1-\varepsilon')C + \varepsilon H_1 + \varepsilon H_2)$ is φ-ample and the pair $(S, (1-\varepsilon')C + \varepsilon H_1 + \varepsilon H_2)$ is not lc at P_1 and P_2. This contradicts Connectedness Lemma [17, 5.7].

Thus (S, C) is plt near P_2 and $\Gamma(\varphi)$ has the form

(5.8)



where $[b_1, \dots, b_\varrho]$ and $[c_1, \dots, c_l]$ are T-chains (i.e. $\Gamma(\varphi) = [b_k \mid b_{k-1}, \dots, b_1 \mid b_{k+1}, \dots, b_\varrho \mid c_1, \dots, c_l]$). Let α_1' and α_k be log discrepancies of the vertices corresponding to c_1 and b_k, respectively. By Lemma 5.3 we have $\alpha_1' + \alpha_k > 1$. Let

$$\Gamma(\varphi) = \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_r = \Gamma'$$

be the sequence of contractions of black vertices. If $b_k = 2$, then Γ_1 contains the fork $[1 \mid b_{k-1} \mid b_{k+1} \mid c_1 - 1]$. This contradicts Lemma 5.5. Therefore $b_k \geq 3$. The same arguments show that in each graph Γ_i the central vertex (corresponding to b_k) is not black. Since $[b_1, \dots, b_\varrho]$ is a chain of type T, we may assume that $b_\varrho = 2$ and $b_1 \geq 3$. Thus

$$\Gamma' = [b_k - s \mid b_{k-1}, \dots, b_1 \mid b_{k+1}, \dots, b_\varrho \mid c_s - 1, c_{s+1}, \dots, c_l],$$

where $s \geq 1$, $b_k - s \geq 2$, and $c_s - 1 \geq 2$. Clearly, Γ' is elliptic and log terminal (because $-(K_{\tilde{S}} + \Delta)$ is nef and big over T). Now we use the classification of two-dimensional log terminal singularities (see, e.g., [10, Ch. 3]). Since $b_1 > 2$, we have $[c_s - 1, \dots, c_l] = n/q$, where $1 \leq q < n$, $\gcd(n, q) = 1$ and $n = 2, 3, 4$, or 5. So, $[c_s - 1, \dots, c_l] = [n]$, $[2, \dots, 2]$, $[2, 3]$, or $[3, 2]$.

Assume that $s = 1$. Then $[c_1, \dots, c_l] = [4]$ or $[3, 3]$ (see Proposition 3.6), $k = \varrho - 1$, and $[b_1, \dots, b_{k-1}] = n'/q'$, where $1 \le q' < n' - 1$, $\gcd(n', q') = 1$ and $n' = 3, 4$, or 5. Further, $\alpha_1' = 1/2$. Easy computations (see [10, (3.1.3)]) show that $2/b_k > \alpha_k > 1/2$. Thus $b_k = 3$. We get only one possibility $[b_1, \dots, b_k, \dots, b_\varrho] = [4, 3, 2]$. But then $\alpha_k = 1/5$, a contradiction.

Assume that $s > 1$. Then $c_1 = \dots = c_{s-1} = 2$. Hence, $c_l \ge 3$ and $[c_1, \dots, c_l] = [2, \dots, 2, n + 1]$, or $[2, \dots, 2, 3, 3]$. Since $[c_1, \dots, c_l]$ is a T-chain, $n \ge 4$ and $[c_1, \dots, c_l]= [2, 5]$, or $[2, 2, 6]$. As above we get $\alpha_1' \le 1/3$, $2/b_k > \alpha_k > 2/3$, and $b_k = 2$, a contradiction. □

Lemma 5.9. *Notation as above. Let D_1 and D_2 be two components of D such that*

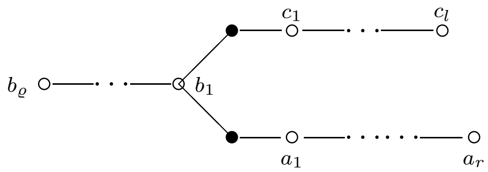
(i) $D_1 \cap D_2 \ne \varnothing$,

(ii) *there are* T*-points* $P_i \in D_i$, $P_i \notin D_1 \cap D_2$.

Then $K_S + D_1 + D_2$ is lc.

Proof. Assume the converse. By the previous lemma, $\Gamma(\varphi)$ contains a subgraph Γ of the form

(5.10)



where $\varrho \ge 2$ and $[a_1, \dots, a_r]$, $[b_1, \dots, b_\varrho]$, $[c_1, \dots, c_l]$ are T-chains.

Note that $b_1 \ge 3$. By Corollary 3.8 and Lemma 5.3 we have $a_1 = c_1 = 2$. Take $s, m \ge 1$ so that

$$a_1 = \dots = a_s = 2,\ a_{s+1} > 2,\ c_1 = \dots = c_m = 2,\ c_{m+1} > 2.$$

Contracting black vertices successively we get the following log terminal graph

$$\Gamma' = [b_1 - s - m - 2 \mid b_2, \dots, b_\varrho \mid a_{s+1} - 1, \dots, a_r \mid c_{m+1} - 1, \dots, c_l].$$

By Proposition 3.6 we have $\sum a_i = 2 - \beta + 3r$, where β is the number of vertices of the corresponding chain (3.5) with $\iota = 2$ (β coincides with $\beta(n/q)$ introduced in §3 but we do not need this fact). Since $\beta + s \le r$,

$$a_{s+1} + \dots + a_r = 2 - \beta + 3r - 2s \ge 2 + r + \beta \ge 5.$$

Similarly, $c_{m+1} + \dots + c_l \ge 5$. Therefore, $\varrho = b_\varrho = 2$ and we may assume that $[a_{s+1} - 1, \dots, a_r] = 3/q$. On the other hand, $a_r \ge 3$, so $[a_{s+1} - 1, \dots, a_r] = [3]$ and $[a_1, \dots, a_r] = [2, \dots, 2, 4]$, a contradiction. □

Corollary 5.11. *Let $\varphi \colon S \to T \ni o$ be a* T*-contraction.*

(i) *If S has exactly one non-Du Val point P, then all the components of D pass through P.*
(ii) *If S has more than one non-Du Val points, then (S, D_i) is plt for any component D_i of D containing two non-Du Val points.*

Now Theorem 1.5 is a consequence of Propositions 2.4 and 5.12 below.

Proposition 5.12. *Let $\varphi\colon S \to T \ni o$ be a* T*-contraction. Then K_S is* 1*-complementary.*

5.13. To begin with, assume that $\varphi\colon S \to T \ni o$ is an *arbitrary* log contraction. We apply the technique developed in [16]. Take δ so that $(S, \delta D)$ is maximally log canonical.

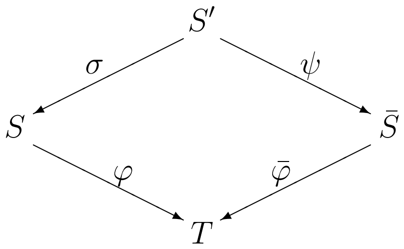
Note that on S the LMMP works with respect to any divisor G. Indeed, there is a boundary F such that $K_X + F$ is klt, numerically trivial, and the components of F generate $\mathrm{Pic}(S) \otimes \mathbb{Q}$. Then G-MMP is equivalent to $K_S + F + \varepsilon G'$-MMP for $0 < \varepsilon \ll 1$ and suitable $G' \sim_{\mathbb{Q}} G$.

Lemma 5.14. *Assume that $(S, \delta D)$ is plt. Then K_S is* 1*-complementary.*

Proof. Since $(S, \delta D)$ is maximally log canonical, $\lfloor \delta D \rfloor \neq 0$. Put $C = \lfloor \delta D \rfloor$ and $B = \delta D - C$. By Connectedness Lemma [17, 5.7], C is an irreducible curve. By Corollary 5.6, $\mathrm{Diff}_C(\delta B)$ is supported in two points, say P_1 and P_2. Run $-(K_S + C)$-MMP over T: $\psi\colon S \to \bar{S}$. Since $-K_S$ is ψ-ample, C is not contracted. Since $-(K_S + \delta D)$ is ψ-ample, the plt property of $(S, \delta D)$ is preserved. We get a plt model $(\bar{X}, \bar{C})$ such that $-(K_{\bar{S}} + \bar{C})$ is nef over T. By the above, $\mathrm{Diff}_{\bar{C}}(\delta \bar{B})$ is supported in two points. Hence $K_{\bar{S}} + \mathrm{Diff}_{\bar{C}}$ is 1-complementary (see [17, 5.2]). Since $-(K_{\bar{S}} + \bar{C})$ is nef and big over T, this complement can be extended to $\bar{S}$ (see [15, Prop. 4.4.1]). By [15, 4.3] this gives us an 1-complement of $K_S + C$. □

5.15. If $(S, \delta D)$ is not plt, there is an *inductive blowup* of $(S, \delta D)$. By definition it is a birational extraction $\sigma\colon S' \to S$ with irreducible exceptional divisor E satisfying the following properties:
(i) $K_{S'} + E + \delta D' = \sigma^*(K_S + \delta D)$ is log canonical, where D' is the proper transform of D,
(ii) (S', E) is plt.

Since the minimal resolution $\mu\colon \tilde{S} \to S$ is a log resolution of (S, D), we may assume that μ factors through σ (see [15, Proof of 3.1.4]). Then $K_{S'} + \alpha E = \sigma^* K_S$, where $\alpha \geq 0$ and $-(K_{S'} + \alpha E)$ is nef over T. As in the proof of Lemma 5.14 we run $-(K_{S'} + E)$-MMP over T. In this case again E cannot be contracted and we get a model $(\bar{S}, \bar{E})$ such that $-(K_{\bar{S}} + \bar{E})$ is nef over T:



Denote by N the exceptional divisor of ψ and let $V := \mathrm{Sing}(S') \cap E$. If $-(K_{S'}+E)$ is nef over T, we put $\psi = \mathrm{id}$ and $N = \varnothing$. Clearly all singular points $\bar{P}_1, \dots, \bar{P}_r$ of $\bar{S}$ lying on $\bar{E}$ are contained in $\psi(V) \cup \psi(N)$. If $r \leq 2$, then $K_{\bar{E}} + \mathrm{Diff}_{\bar{E}}$ is 1-complementary (see [17]). Since $-(K_{\bar{S}} + \bar{E})$ is nef over T, this complement can be extended to $\bar{S}$ (see [15, Prop. 4.4.1]). By [15, 4.3] this gives us an 1-complement of K_S.

Lemma 5.16. *Assume that S has a unique non-Du Val point and this point which is a cyclic quotient. Then K_S is 1-complementary.*

Proof. We may assume that $(S, \delta D)$ is not plt. Since $P := \sigma(E) \in S$ is a cyclic quotient singularity, V contains at most two points. Indeed, $-K_{S'}$ is ψ-ample and S' has at worst Du Val singularities outside of $\mathrm{Sing}(\bar{S}) \cap \bar{E}$. By [8, 3.38] discrepancies of all divisors of $\bar{S}$ over $\psi(E)$ are strictly positive. Hence $\bar{S}$ is smooth at points of $\psi(N)$, $\bar{P}_1, \dots, \bar{P}_r \in \psi(V)$ and $r \leq 2$. By the above K_S is 1-complementary. □

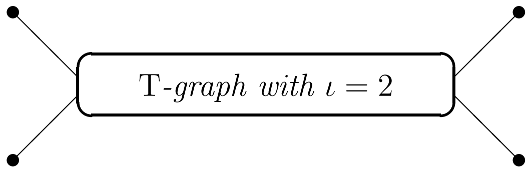
Proof of Proposition 5.12. Again V contains at most two points. If K_S is not 1-complementary, then $r \geq 3$. Take $\bar{P} \in \psi(N) \setminus \psi(V)$ and let $N_0 = \psi^{-1}(\bar{P})$. Then the point $N_0 \cap E \in S$ is smooth and N_0 contains at least one non-Du Val point of S'. If V is two points, then by Corollary 5.6 we get a subgraph (5.8), a contradiction.

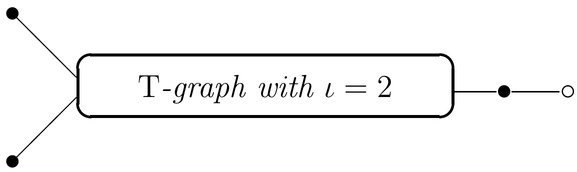
Assume that V is one point. Then there are two points $\bar{P}, \bar{P}_1 \in \psi(N) \setminus \psi(V)$ and similarly by Lemma 5.7 we get a subgraph (5.10), a contradiction.

Finally, assume that S' is smooth along E. Then E is a (-4)-curve. Hence $\Gamma(\varphi)$ contains a fork $[4 \mid 1, b_1 \mid 1, b_2 \mid 1, b_3]$. Such a graph cannot be elliptic. □

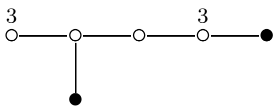
Examples.

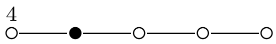
Proposition 5.17. *Let $\varphi \colon S \to T$ be a two-dimensional log conic bundle of index two. Then $\Gamma(\varphi)$ is one if the following:*

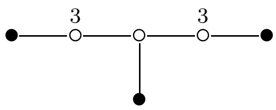
(I*)


(I**)


(I***)


(II*)


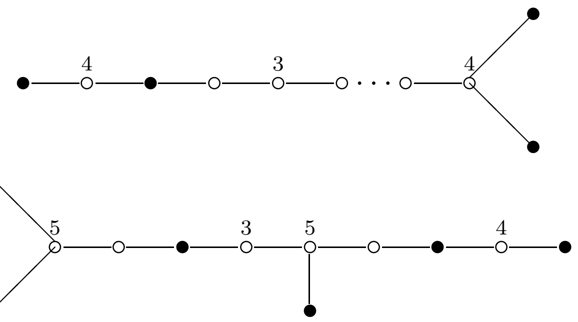
(III*)


(III**)


Our notation can be explained as follows. Consider a general member $B \in |-K_S|$ and let $S' \to S$ be a double covering branched along B. Then $K_{S'} = 0$, $S' \to T$ is an elliptic fibration, and S' has only Du Val singularities (cf. [12, §3]). If $\tilde{S}'$ is the minimal resolution, then the central fiber of the composition map $\tilde{S}' \to T$ is Kodaira's singular fiber.

Proof. For any index two log terminal point all log discrepancies of the minimal resolution are equal to $1/2$. By Lemma 5.3 there is at most one non-Du Val point on each component of D. By Corollary 5.6 there is exactly one non-Du Val point P on S and all the components of D pass through P. Again using Lemma 5.3 we have $\sum l_i = 4$, so the graph $\Gamma(\varphi)$ has at most four black vertices. Now the classification follows by Lemma 5.5. □

Example 5.18. The following graphs gives us examples of T-conic bundles with two and three non-Du Val points.

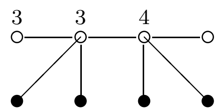


Proposition 5.19. *For any* T*-singularity* $\frac{1}{n}(1, q)$ *there is a* T*-conic bundle having exactly one singular point which is of type* $\frac{1}{n}(1, q)$.

Proof. One can start with graph (I*) and run the construction below. By Proposition 3.6 on each step we have only singularities of type T. □

Construction 5.20. Let φ be a log conic bundle with a unique singular point that is of type $[b_1, \dots, b_\varrho]$. Assume that in $\Gamma(\varphi)$ there is a black vertex adjacent to the end b_1, i.e., $\Gamma(\varphi)$ contains a string $[1, b_1, \dots, b_\varrho]$, where $[b_1, \dots, b_\varrho]$ corresponds to singular point. Blowing-up the ends 1 and b_ϱ, we get a graph Γ' containing a string $[1, 2, b_1, \dots, b_\varrho + 1, 2]$. By Remark 5.4, Γ' corresponds to a log conic bundle (i.e., the anticanonical divisor is ample).

Remark 5.21.
(i) Construction 5.20 provides only one series of T-conic bundles with a unique singular point. For example, the following T-conic bundle cannot be obtained by this way.



(ii) Similar to 5.20 one can obtain infinite series of T-conic bundles with two and three singular points starting with Example 5.18.

6. The case of irreducible central fiber

The following lemma shows that case of relative Picard number one is most important.

Lemma 6.1. *Let* $f \colon X \to Z \ni o$ *be a Fano-Mori contraction such that the dimension of fibers is at most one. There is a Fano-Mori contraction* $f' \colon X' \to Z$ *with the same property and a birational map* $h \colon X \dashrightarrow X'$ *over* Z *such that* h *is a morphism outside of* $f^{-1}(o)$, X'

is $\mathbb{Q}$-factorial and $\rho(X/Z) = 1$. In particular, $f'^{-1}(L)$ is irreducible for any irreducible divisor $L \subset Z$. If furthermore (X, f^*T) is dlt for some effective Cartier divisor T, then we can take X' so that $(X', f'^*(T))$ is dlt.

Proof. Let $q\colon X^{\mathrm{q}} \to X$ be a $\mathbb{Q}$-factorial modification. Thus X^{q} has only terminal $\mathbb{Q}$-factorial singularities, $K_{X^{\mathrm{q}}} = q^*K_X$, and q is a small birational contraction. Run MMP over Z. We get X' as above.

To show the second statement we construct $\mathbb{Q}$-factorialization $q\colon X^{\mathrm{q}} \to X$ of (X, S). Then $(X^{\mathrm{q}}, S^{\mathrm{q}})$ is dlt, where $S^{\mathrm{q}} = q^*S$. Then we just note that $K_{X^{\mathrm{q}}}$-MMP is the same as $K_{X^{\mathrm{q}}} + S^{\mathrm{q}}$-MMP. □

In analytic situation $\rho(X/Z) = 1$ implies that the fiber $f^{-1}(o)$ is irreducible. Now we classify semistable Mori conic bundles with irreducible central fiber.

Proposition 6.2 (cf. [14, Th. 2.5])**.** *Let $\varphi\colon S \to T \ni o$ be a* T*-conic bundle having at least one non-Du Val point. Assume that the fiber D is irreducible. Then φ is of type* (III*) *of* 5.17.

Corollary 6.3. *Let $f\colon X \to Z \ni o$ be a semistable Mori conic bundle such that $f^{-1}(o)$ is irreducible. Then X has exactly one non-Gorenstein point which is of index* 2*, see* [12, §3].

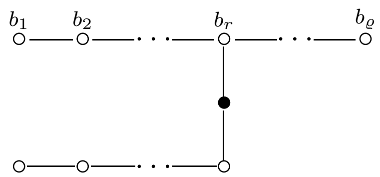
Proof. If $K_S + C$ is plt, then S has two singularities of types $\frac{1}{n}(1, q)$ and $\frac{1}{n}(1, n-q)$ (see [14, Th. 2.5]). If they are of type T, then

$$(q+1)^2 \equiv 0 \mod n, \qquad (n-q+1)^2 \equiv 0 \mod n.$$

This gives us $4 \equiv 0 \mod n$. Since the singularities of S are worse than Du Val, $n = 4$. We get case (III*).

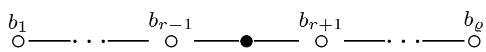
Now we consider the case when $K_S + C$ is not plt. By Lemma 5.7 and Corollary 5.6 the surface S has exactly one non-Du Val point and at most one Du Val point Q. Moreover, $S \ni Q$ is of type A_n and $K_S + C$ is plt near Q. Thus $\Gamma(\varphi)$ has the following form

(6.4)



where $r \neq 1$, $r \neq \varrho$. Contracting black vertices successively, on some step we get a subgraph

(6.5)



Lemma 6.6. *If the graph* (6.5) *is parabolic, then*

$$\sum_{i=1}^{r-1}(b_i-1)=\sum_{j=r+1}^{\varrho}(b_j-1)=\varrho-2. \tag{6.7}$$

In particular, $r\neq 1,\varrho$.

Apply the procedure described in Proposition 3.6 to $[b_1,\dots,b_r,\dots,b_\varrho]$. Each step preserves relation (6.7). At the end we get a chain $[b'_1,\dots,b'_{r'},\dots,b'_{\varrho'}]$ as in (3.5). Relation (6.7) give us $r'=\varrho'-r'+1=\varrho'-2$, i.e., $r'=3$ and $\varrho'=5$. Hence, in (6.5) we have $b_{r-1}=b_{r+2}=2$. This contradicts Lemma 5.5. □

7. The existence of semistable 3-fold flips

Theorem 7.1. *Let* $f\colon X\to Z$ *be a semistable three-dimensional flipping contraction. Assume that* f *is extremal* (*i.e.,* X *is* $\mathbb{Q}$*-factorial and* $\rho(X/Z)=1$). *Then the flip of* f *exists.*

Sketch of the proof (*see* [4, 8.5, 8.7]). The existence of the flip is equivalent to the finite generation of the $\mathcal{O}_Z$-algebra $\mathcal{R}_Z(K_Z):=\oplus_{m\geq 0}\mathcal{O}_Z(mK_Z)$, see [4, Lemma 3.1]. By Theorem 1.5 there is $L=2F\in|-2K_X|$ such that $K_X+f^*T+\frac{1}{2}L$ is lc. Since f is an extremal contraction and $K_X+f^*T+\frac{1}{2}L$ is numerically trivial, one can see that $K_Z+T+\frac{1}{2}L_Z$ is also lc, where $L_Z:=f_*L\in|-2K_Z|$. Therefore, the same holds for a general member $L_Z\in|-2K_Z|$ which is reduced and irreducible. As in [4, §8], consider a double covering $\pi\colon Z'\to Z$ ramified along L_Z. Then $K_{Z'}+\pi^*T=\pi^*(K_Z+T+\frac{1}{2}L_Z)$ is lc and Cartier. Since π^*T also is a Cartier divisor, Z' has at worst a canonical Gorenstein singularity at $o':=\pi^{-1}(o)$. Put $L_{Z'}:=\pi^*(L_Z)_{\mathrm{red}}$. By [4, 3.2] the finite generation of algebras $\mathcal{R}_Z(K_Z)$ and $\mathcal{R}_{Z'}(K'_Z-L'_Z)$ is equivalent. Finally, the last algebra is finitely generated by [4, 6.1] (see also [6], [10, §6], [8, §6]). □

Note that in our case the finite generation of $\mathcal{R}_{Z'}(K'_Z-L'_Z)$ is much easier because the presence of a Cartier divisor π^*T such that $K'_Z+\pi^*T$ is lc.

Corollary 7.2 ([7]). *Let* $f\colon X\to Z\ni o$ *be a semistable birational contraction with fibers of dimension at most one* (*either flipping or divisorial of type* 2-1) *and let* T *be a general hyperplane section through* o. *Then* $T\ni o$ *is a cyclic quotient singularity. If furthermore* f *is divisorial, then* $T\ni o$ *is of type* T.

Proof. By Theorem 1.5 the pair $(Z, T + f(F))$ is log canonical. By Adjunction so is $(T, \mathrm{Diff}_T(f(F)))$. Moreover, $K_T + \mathrm{Diff}_T(f(F)) \sim 0$. By the classification of log terminal singularities with a reduced boundary $T \ni o$ is a cyclic quotient singularity. □

References

[1] Corti A. *Semistable 3-fold flips*, e-print alg-geom/9505035

[2] Looijenga E. & Wahl J. *Quadratic functions and smoothing surface singularities*, Topology **25** (1986) 261–291

[3] Mori S. *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1** (1988) 117–253

[4] Kawamata Y. *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. Math. (2) **127** (1988) 93–163

[5] Kawamata Y. *Semistable minimal models of threefolds in positive or mixed characteristic*, J. Algebraic Geom. **3** (1994) 463–491

[6] Kollár J. *Flops*, Nagoya Math. J. **113** (1989) 15–36

[7] Kollár J. & Mori S. *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5** (1992) 533–703

[8] Kollár J. & Mori S. *Birational geometry of algebraic varieties*. Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998

[9] Kollár J. & Shepherd-Barron N. I. *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988) 299–338

[10] Kollár J. et al. *Flips and abundance for algebraic threefolds*, in "A summer seminar at the Univ. of Utah, Salt Lake City, 1991". Astérisque. **211** (1992)

[11] Kulikov Vik. S. *Degenerations of K3 surfaces and Enriques surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977) 1008–1042

[12] Prokhorov Yu. G. *On the existence of complements of the canonical divisor for Mori conic bundles*, Russian Acad. Sci. Sb. Math. **188** (1997) 1665–1685

[13] Prokhorov Yu. G. *On extremal contractions from threefolds to surfaces: The case of one nonsingular point*, Contemporary Math. AMS **207** (1997) 119–141

[14] Prokhorov Yu. G. *Mori conic bundles with a reduced log terminal boundary*, J. Math. Sci. (New York) **94** (1999) 1051–1059

[15] Prokhorov Yu. G. *Lectures on complements on log surfaces*. MSJ Memoirs. **10**. Tokyo: Mathematical Society of Japan. (2001)

[16] Prokhorov Yu. G. & Shokurov V. V. *The first main theorem on complements: from global to local*, Russian Acad. Sci. Izv. Math. **65** (2001) 1169–1196

[17] Shokurov V. V. *3-fold log flips*, Russ. Acad. Sci. Izv. Math. **40** (1993) 93–202

[18] Shokurov V. V. *Semistable 3-fold flips*, Russ. Acad. Sci. Izv., Math. **42** (1994) 371–425

[19] Tsunoda Sh. *Degenerations of surfaces*, Algebraic geometry, Sendai, 1985, 755–764, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987

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