Representations of the Cuntz algebra $\mathcal{O}_2$ arising from complex quadratic transformations —Annular basis of $L_2(\mathbf{C})$—

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We construct a representation of the Cuntz algebra $\mathcal{O}_2$ arising from a complex quadratic transformation $Q(z) \equiv z^2$. The characterization of this representation is shown by orbit analysis of Q on C. We show the irreducible decomposition of this representation and construct a complete system of orthonormal functions on C associated with the action of $\mathcal{O}_2$.

1. Introduction

We study representations of the Cuntz algebra $\mathcal{O}_N$ arising from dynamical systems with branching(or bifurcation) in [**11**, **12**, **13**, **14**]. So-called iteration function systems([**7**]) on dynamical systems are represented as families of isometries on Hilbert spaces so that their composition is corresponded to the product of isometries. In this paper, we treat a representation arising from a naive complex dynamical system.

On $\mathbf{C}$, we consider a transformation

$$Q(z) \equiv z^2. \tag{1.1}$$

Put a representation $(L_2(\mathbf{C}), \pi_0)$ of $\mathcal{O}_2$ arising from Q by

$$(\pi_0(s_i)\phi)(z) \equiv m_i(z)\phi(Q(z)) \tag{1.2}$$

for $\phi \in L_2(\mathbf{C})$ and $z \in \mathbf{C}$ where $m_i(z) \equiv 2|z| \cdot \chi_{E_i}(z)$, $i = 1, 2$, $E_1 \equiv \{z \in \mathbf{C} : \mathrm{Im}\, z \geq 0\}$, $E_2 \equiv \{z \in \mathbf{C} : \mathrm{Im}\, z < 0\}$, χ_Y is the characteristic function on $Y \subset \mathbf{C}$, $L_2(\mathbf{C})$ is taken by a measure $d\mu_{\mathbf{R}}(z) = dxdy$ on $\mathbf{C}$ for $z = x+\sqrt{-1}y$, and s_1, s_2 are generators of $\mathcal{O}_2$.

On the other hand, $(\mathcal{H}_B, \pi_B)$ is the *barycentric representation* of $\mathcal{O}_2$ if $(\mathcal{H}_B, \pi_B)$ is a cyclic representation of $\mathcal{O}_2$ such that there is an eigen vector of $\pi_B(s_1 + s_2)$ with eigen value $\sqrt{2}$.

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Theorem 1.1. (i) *There is the following direct integral decomposition of representation of* $\mathcal{O}_2$*:*

$$(L_2(\mathbf{C}),\pi_0)=\int_{U(1)}^{\oplus}(\mathcal{H}_{\bar{w}},\,\hat{\pi}_{B,\bar{w}})\,d\eta(w). \tag{1.3}$$

where $(\mathcal{H}_w,\hat{\pi}_B^{(w)})$ *is a representation of* $\mathcal{O}_2$ *defined by*

$$\mathcal{H}_w\equiv\mathcal{K}\otimes\mathcal{H}_B,\quad \hat{\pi}_{B,w}(s_i)\equiv I\otimes\pi_{B,w}(s_i),\quad \pi_{B,w}(s_i)\phi\equiv\pi_B(ws_i)\phi \tag{1.4}$$

for $\phi\in\mathcal{H}$, $i=1,2$, $w\in U(1)$, $\mathcal{K}$ *is a separable infinite dimensional Hilbert space, and* η *is the Haar measure of* $U(1)$*, the equality in (*1.3*) means unitary equivalence.*

(ii) *Any two elements in* $\{(\mathcal{H}_B,\pi_{B,w}):w\in U(1)\}$ *are mutually inequivalent and* $(\mathcal{H}_B,\pi_{B,w})$ *is irreducible for each* $w\in U(1)$*.*

(iii) *This decomposition is unique up to unitary equivalences.*

By Theorem 1.1, $(L_2(\mathbf{C}),\pi_0)$ is completely reducible and its characterization is given by a well-known representation $(\mathcal{H}_B,\pi_B)$ ([**13**]).

Next we show an explicit decomposition formula of $L_2(\mathbf{C})$ by using this representation and describe a complete system of orthonormal functions on $\mathbf{C}$. For this purpose, we prepare several multi-index sets.

Definition 1.2. *Put* $\{1,2\}^*\equiv\bigcup_{k\geq 0}\{1,2\}^k$, $\{1,2\}^0\equiv\{0\}$, $\{1,2\}^k\equiv\{(j_l)_{l=1}^k : j_l=1,2,\,l=1,\ldots,k\}$ *for* $k\geq 1$*, and* $\Lambda_2\equiv\bigcup_{k\geq 1}\Lambda_{2,k}$, $\Lambda_{2,1}\equiv\{1,2\}$, $\Lambda_{2,k}\equiv\{(j_l)_{l=1}^k\in\{1,2\}^k : j_k=2\}$ *for* $k\geq 2$*. For* $J=(j_1,\ldots,j_k)$ *and* $J'=(j_1',\ldots,j_k')\in\{1,2\}^k$, $k\geq 1$*, define* $(J|J')\equiv\sum_{l=1}^k(j_l-1)(j_l'-1)$*.*

Theorem 1.3. *Let* $(L_2(\mathbf{C}),\pi_0)$ *be the representation of* $\mathcal{O}_2$ *in (*1.2*).*

Then there are families $\{A_{n,J_1}^{(i)} : i=1,2,\,n\in\mathbf{Z},\,J_1\in\{1,2\}^*\}$ *and* $\{AB_{n,J_1,J_2}^{(i)} : i=1,2,\,n\in\mathbf{Z},\,J_1,J_2\in\{1,2\}^*\}$ *of regions in* $\mathbf{C}$ *which satisfy the followings:*

$$\mathbf{C}=D_1\cup D_2\cup\{z\in\mathbf{C}:|z|=0,1\}$$

$$D_i=\bigcup_{n\in\mathbf{Z}}\bigcup_{J_1\in\{1,2\}^k}A_{n,J_1}^{(i)},\quad A_{n,J_1}^{(i)}=\bigcup_{J_2\in\{1,2\}^k}AB_{n,J_1,J_2}^{(i)}$$

for each $k\geq 0$*,* $D_1\equiv\{z\in\mathbf{C}:0<|z|<1\}$*,* $D_2\equiv\{z\in\mathbf{C}:1<|z|\}$*, such that the followings hold:*

(i) *For* $i,j=1,2$*,* $n\in\mathbf{Z}$*,* $J_1,J_2\in\{1,2\}^*$*.*

$$Q(A_{n,J_1}^{(i)})=A_{n+1,J_1}^{(i)},\quad Q(AB_{n,J_1,\{j\}\cup J_2}^{(i)})=AB_{n+1,J_1,J_2}^{(i)}$$

where $\{j\}\cup J=\{j,j_1,\ldots,j_k\}$ *when* $J=(j_1,\ldots,j_k)\in\{1,2\}^*$*. Furthermore* $\mu_{\mathbf{R}}(AB_{n,J_1,J_2}^{(i)}\cap AB_{m,J_1',J_2'}^{(j)})=0$ *when* $(i,n,J_1,J_2)\neq(j,m,J_1,J_2)$*,* $J_1,J_1'\in\{1,2\}^{k_1}$ *and* $J_2,J_2'\in\{1,2\}^{k_2}$*,* $k_1,k_2\geq 1$*.*

(ii) *Put*

$$N^{(i)}_{n,J_1,J_2}(z) \equiv b_n(z) \sum_{J_1'\in\{1,2\}^{k_1}} \sum_{J_2'\in\{1,2\}^{k_2}} (-1)^{(J_1|J_1')+(J_2|J_2')} K^{(i)}_{n,J_1',J_2'}(z),$$

for $i = 1, 2$, $n \in \mathbf{Z}$, $J_1 \in \Lambda_{2,k_1}$, $J_2 \in \Lambda_{2,k_2}$, $k_1, k_2 \geq 1$, *and* $z \in \mathbf{C}$ *where* $b_n(z) \equiv \left(|z|\sqrt{2^n\pi \log 2}\right)^{-1}$ *and* $K^{(i)}_{n,J_1',J_2'}$ *is the characteristic function on* $AB^{(i)}_{n,J_1',J_2'}$. *Then* $\{N^{(i)}_{n,J_1,J_2} : i = 1, 2,\ n \in \mathbf{Z},\ J_1, J_2 \in \Lambda_2\}$ *is a complete orthonormal basis of* $L_2(\mathbf{C})$ *which satisfies*

$$\pi_0(T_j)N^{(i)}_{n,J_1,1} = N^{(i)}_{n-1,J_1,j}, \quad \pi_0(T_j)N^{(i)}_{n,J_1,J_2} = N^{(i)}_{n-1,J_1,\{j\}\cup J_2}$$

for $i, j = 1, 2$, $n \in \mathbf{Z}$, $J_1 \in \Lambda_2$, $J_2 \in \Lambda_2 \setminus \{1\}$ *where* $T_1 \equiv 2^{-1/2}(s_1 + s_2)$ *and* $T_2 \equiv 2^{-1/2}(s_1 - s_2)$.

(iii) *There are the following decompositions as representation of* $\mathcal{O}_2$*:*

$$L_2(\mathbf{C}) = L_2(D_1) \oplus L_2(D_2), \quad L_2(D_i) = \bigoplus_{J_1\in\Lambda_2} \mathcal{L}^{(i)}_{J_1},$$

$$\mathcal{L}^{(i)}_{J_1} = \int^{\oplus}_{U(1)} \mathcal{L}^{(i)}_{J_1,\bar{w}}\, d\eta(w)$$

where

$$\mathcal{L}^{(i)}_{J_1} \equiv \overline{\mathrm{Lin} < \{N^{(i)}_{n,J_1,J_2} : n \in \mathbf{Z},\ J_2 \in \Lambda_2\} >} \quad (i = 1, 2,\ J_1 \in \Lambda_2).$$

(iv) *For each* $i, j = 1, 2$ *and* $J_1, J_1' \in \Lambda_2$, $\mathcal{L}^{(i)}_{J_1}$ *and* $\mathcal{L}^{(j)}_{J_1'}$ *are equivalent as representation of* $\mathcal{O}_2$.

(v) *For each* $i, j = 1, 2$, $J_1, J_1' \in \Lambda_2$ *and* $w, w' \in U(1)$, $\mathcal{L}^{(i)}_{J_1,w}$ *and* $\mathcal{L}^{(j)}_{J_1',w'}$ *are equivalent if and only if* $w = w'$.

(vi) $\mathcal{L}^{(i)}_{J_1,w}$ *is irreducible and equivalent to* $\pi_{B,w}$ *in* (1.4) *for* $w \in U(1)$.

(vii) *Decomposition in (iii) is unique up to unitary equivalences.*

In § 2, we prepare representation theory of the Cuntz algebra. In § 3, we introduce representations arising from dynamical systems. In § 4, we decompose a dynamical system $(\mathbf{C}, Q)$ in (1.1) into the direct product of a shift on $\mathbf{Z}$ and a branching function system on an interval $[0, 1)$. It is shown that the branching of $\sqrt{z}$ which is the inverse map of Q is represented by representation of $\mathcal{O}_2$. Theorem 1.1 is shown here. In § 5, we show the complete system of orthonormal functions on $\mathbf{C}$ in Theorem 1.3 (ii) which is called the *annular basis* by using the representation of $\mathcal{O}_2$. Theorem 1.3 is proved here. In § 6, we generalize our results to $\mathcal{O}_N$ and other dynamical systems by conjugations.

2. P-cycles and P-chains

For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**4**]), that is, it is a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad (i,j = 1,\ldots,N), \quad \sum_{i=1}^{N} s_i s_i^* = I. \tag{2.1}$$

In this paper, any representation means a unital $*$-representation. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1, \ldots, S_N$ on an infinite dimensional Hilbert space which satisfy (2.1) in order to construct a representation of $\mathcal{O}_N$. Put α an action of a unitary group $U(N)$ on $\mathcal{O}_N$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^{N} g_{ji} s_j$ for $i = 1, \ldots, N$. Specially we denote $\gamma_w \equiv \alpha_{g(w)}$ when $g(w) = w \cdot I \subset U(N)$ for $w \in U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$.

Let Iso$\mathcal{O}_N$ be the set of all isometries in $\mathcal{O}_N$.

Definition 2.1. *Let $P \in$ Iso$\mathcal{O}_N$.*

(i) *$(\mathcal{H}, \pi, \Omega)$ is a P-cycle of $\mathcal{O}_N$ if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(P)\Omega = \Omega$.*

(ii) *$(\mathcal{H}, \pi, \Omega)$ is a P-chain of $\mathcal{O}_N$ if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with cyclic unit vector $\Omega \in \mathcal{H}$ such that $\{\pi((P^*)^n)\Omega : n \in \mathbf{N}\}$ is an orthonormal family in $\mathcal{H}$, that is, $< \pi((P^*)^n)\Omega | \pi((P^*)^m)\Omega >= \delta_{nm}$ for $n, m \in \mathbf{N}$ where $\mathbf{N} = \{1, 2, 3, \ldots\}$.*

Notions of P-cycle and P-chain are generalization of generalized permutative representation of $\mathcal{O}_N$ in [**8**, **9**, **10**].

Put isometries $P_S, P_{B,w}$, $w \in U(1)$, in $\mathcal{O}_2$ by

$$P_S \equiv s_1, \quad P_{B,w} \equiv 2^{-1/2} w(s_1 + s_2) \quad (w \in U(1)). \tag{2.2}$$

Example 2.2. (i) The *standard representation* $(l_2(\mathbf{N}), \pi_S)$ of $\mathcal{O}_2$ is defined by

$$\pi_S(s_1)e_n \equiv e_{2n-1}, \quad \pi_S(s_2)e_n \equiv e_{2n} \quad (n \in \mathbf{N}) \tag{2.3}$$

where $\{e_n\}_{n\in\mathbf{N}}$ is the canonical basis of $l_2(\mathbf{N})$([**1**, **12**]). Then $(l_2(\mathbf{N}), \pi_S, e_1)$ is a P_S-cycle.

(ii) The *barycentric representation* $(L_2[0,1], \pi_B)$ of $\mathcal{O}_2$ is defined by

$$(\pi_B(s_1)\phi)(x) \equiv \chi_{[0,1/2]}(x)\phi(2x), \quad (\pi_B(s_2)\phi)(x) \equiv \sqrt{2}\chi_{[1/2,1]}(x)\phi(2x-1)$$

for $\phi \in L_2[0,1]$ and $x \in [0,1]$ where χ_Y is the characteristic function of a subset Y of $[0,1]$([**13**]). Then $(L_2[0,1], \pi_B, \Omega)$ is a $P_{B,1}$-cycle where Ω is the constant function on $[0,1]$ with value 1.

(iii) In (ii), $(L_2[0,1], \pi_B \circ \gamma_{\bar{w}}, \Omega)$ is a $P_{B,w}$-cycle for $w \in U(1)$. In fact, $(\pi_B \circ \gamma_{\bar{w}})(P_{B,w})\Omega = (\pi_B \circ \gamma_{\bar{w}})(wP_{B,1})\Omega = \pi_B(P_{B,1})\Omega = \Omega$.

(iv) Put $R_i \equiv \mathbf{Z}\times\mathbf{N}_i$, $\mathbf{N}_i \equiv \{2(n-1)+i : n\in\mathbf{N}\}$ for $i=1,2$. Then we have a decomposition $\mathbf{Z}\times\mathbf{N} = R_1\sqcup R_2$. Consider a branching function system $f\equiv\{f_1,f_2\}$ on $\mathbf{Z}\times\mathbf{N}$ defined by

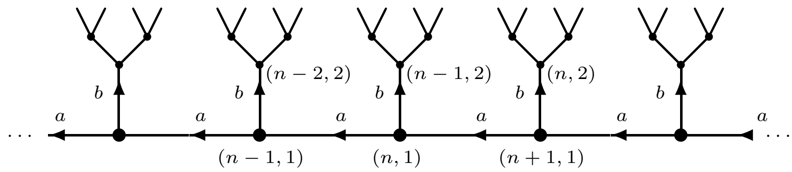
$$f_i : \mathbf{Z}\times\mathbf{N}\to R_i;\quad f_i(n,m)\equiv(n-1,\,2(m-1)+i) \tag{2.4}$$

for $i=1,2$. Then $f_1(n,1)=(n-1,1)$ for each $n\in\mathbf{Z}$. From this, we have $f_1^k(n,1)=(n-k,1)$ for $k\geq 1$ and $n\in\mathbf{Z}$. Put a representation $(l_2(\mathbf{Z}\times\mathbf{N}),\pi_f)$ of $\mathcal{O}_N$ by

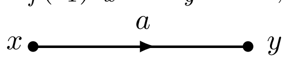
$$\pi_f(s_i)e_x \equiv e_{f_i(x)}\quad (x\in\mathbf{Z}\times\mathbf{N},\ i=1,2). \tag{2.5}$$

From this, we have $\pi_f(s_1^*)e_{(n,1)} = e_{(n+1,1)}$ for $n\in\mathbf{Z}$. Hence $\{\pi_f((s_1^*)^n)e_{(0,1)} : n\in\mathbf{N}\} = \{e_{(n,1)} : n\in\mathbf{N}\}$ is an orthonormal family.

The *tree* of the representation $(l_2(\mathbf{Z}\times\mathbf{N}),\pi_f)$ is following:



where vertices and edges mean the canonical basis $\{e_x\}_{x\in\mathbf{Z}\times\mathbf{N}}$ of $l_2(\mathbf{Z}\times\mathbf{N})$ and the action of operators $\pi_f(s_1),\pi_f(s_2)$ on $\{e_x\}_{x\in\mathbf{Z}\times\mathbf{N}}$, respectively. For example, if $\pi_f(s_1)e_x = e_y$ for $x,y\in\mathbf{Z}\times\mathbf{N}$, then it is represented as



where labels a,b of edges correspond to $\pi_f(s_1),\pi_f(s_2)$, respectively. $(l_2(\mathbf{Z}\times\mathbf{N}),\pi_f,e_{0,1})$ is a P_S-chain of $\mathcal{O}_2$.

It is easy to show that cyclicities and eigen equations in Example 2.2 follow from their definitions, respectively. π_S is a permutative representation in [**3**, **5**, **6**]. $\pi_{B,w}$ is not. π_S and $\pi_{B,w}$ are generalized permutative representations of $\mathcal{O}_2$ which correspond to those with parameters $(1,0),(2^{-1/2}w,2^{-1/2}w)$, respectively([**8**]).

Proposition 2.3. (i) *All of P_S-cycle, $P_{B,w}$-cycle and $P_{B,1}$-chain are unique up to unitary equivalences. We denote them by $(\mathcal{H}_S,\pi_S,\Omega_S)$, $(\mathcal{H}_{B,w},\pi_{B,w},\Omega_{B,w})$, $(\mathcal{H}_{B,1^\infty},\pi_{B,1^\infty},\Omega_{B,1^\infty})$ for $w\in U(1)$, respectively.*

(ii) *All of P_S-cycle, $P_{B,w}$-cycle, $w\in U(1)$, are irreducible.*

(iii) *$P_S, P_{B,w}$, $w\in U(1)$ are mutually inequivalent.*

Proof. See Appendix B. □

We often identify an equivalence class of representations and its representative when there is no ambiguity. Furthermore we often use a symbol

$\pi_S, \pi_{B,w}$ as $(\mathcal{H}_S, \pi_S, \Omega_S)$, $(\mathcal{H}_{B,w}, \pi_{B,w}, \Omega_{B,w})$. Notations of $\pi_S, \pi_{B,w}$ in Example 2.2 are justified by Proposition 2.3 (i). For $P \in \mathrm{Iso}\mathcal{O}_N$, a P-cycle and a P-chain are neither unique nor irreducible in general.

For a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ and a unitary operator U on a Hilbert space $\mathcal{K}$, we have a new representation $(\mathcal{K}\otimes\mathcal{H},\, U\boxtimes\pi)$ of $\mathcal{O}_N$ which is defined by

$$(U \boxtimes \pi)(s_i) \equiv U \otimes \pi(s_i) \quad (i = 1, \ldots, N). \tag{2.6}$$

Lemma 2.4. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$ and U a unitary operator on a Hilbert space $\mathcal{K}$. If there are $p \in \mathbf{Z}$ and a complete orthonormal basis $\{e_n : n \in \mathbf{Z}\}$ of $\mathcal{K}$ such that $Ue_n = e_{n+p}$ for each $n \in \mathbf{Z}$, then $(\mathcal{K}\otimes\mathcal{H},\, U\boxtimes\pi)$ in (2.6) is decomposed as*

$$\begin{cases} \displaystyle\int_{U(1)}^{\oplus} (\mathcal{H},\, \pi \circ \gamma_{w^p})\, d\eta(w) & (p \neq 0), \\ (\mathcal{H}, \pi)^{\oplus\infty} & (p = 0). \end{cases}$$

Proof. When $p = 0$, the assertion follows clearly. Assume $p \neq 0$. Put W a unitary from $\mathcal{K} \otimes \mathcal{H}$ to $L_2(U(1), \mathcal{H})$ by $(W(e_n \otimes \phi))(w) \equiv \phi \cdot \zeta_n(w)$ for $n \in \mathbf{Z}$, $w \in U(1)$ and $\phi \in \mathcal{H}$ where $\zeta_n(w) \equiv w^n$ for $n \in \mathbf{Z}$ and $w \in U(1)$. Then

$$W(U \boxtimes \pi)(s_i)W^*(\phi\zeta_n) = W((Ue_n) \otimes (\pi(s_i)\phi)) = (\pi(s_i)\phi)\zeta_{n+p}.$$

From this, $(W(U \boxtimes \pi)(s_i)W^*(\phi\zeta_n))\,(w) = (\pi(s_i)\phi)\zeta_{n+p}(w) = w^p\zeta_n(w)(\pi(s_i)\phi)$. Hence $(W(U\boxtimes\pi)(s_i)W^*(\phi\zeta_n))(w) = (\pi(w^p s_i)\phi)\zeta_n(w) = ((\pi\circ\gamma_{w^p})(s_i)\phi)\zeta_n(w)$ for each $n \in \mathbf{Z}$. Therefore we have $(W(U\boxtimes\pi)(s_i)W^*\psi)(w) = (\pi\circ\gamma_{w^p})(s_i)\psi(w)$ for $\psi \in L_2(U(1), \mathcal{H})$, $w \in U(1)$ and $i = 1, \ldots, N$. By definition of direct integral decomposition, we have the assertion. □

For shortness' sake, we often denote this type assertion by

$$U \boxtimes \pi \sim \begin{cases} \displaystyle\int_{U(1)}^{\oplus} \pi \circ \gamma_{w^p}\, d\eta(w) & (p \neq 0), \\ \pi^{\oplus\infty} & (p = 0) \end{cases}$$

where a symbol $\sim$ means the unitary equivalence of representations.

Proposition 2.5. (i) *Let $(\mathcal{H}, \pi, \Omega)$ be the P_S-chain. Then there is the following direct integral decomposition holds:*

$$(\mathcal{H}, \pi) \sim \int_{U(1)}^{\oplus} (\mathcal{H}_{S,\bar{w}}, \pi_{S,\bar{w}})\, d\eta(w)$$

where $\mathcal{H}_{S,w} \equiv l_2(\mathbf{N})$ and $\pi_{S,w} \equiv \pi_S \circ \gamma_w$ for $w \in U(1)$.

(ii) *Let $(\mathcal{H},\pi,\Omega)$ be the $P_{B,1}$-chain. Then there is the following direct integral decomposition holds:*

$$(\mathcal{H},\pi)\sim\int_{U(1)}^{\oplus}(\mathcal{H}_{B,\bar{w}},\pi_{B,\bar{w}})\,d\eta(w)$$

where $\mathcal{H}_{B,w}\equiv l_2(\mathbf{N})$ and $\pi_{B,w}\equiv\pi_B\circ\gamma_w$ for $w\in U(1)$.

Proof. By Lemma 2.4 for $p=-1$ and Example 2.2 (iv), $\pi=U\boxtimes\pi_S$ for $(\mathcal{H}_S,\pi_S)$, $\mathcal{K}\equiv l_2(\mathbf{Z})$ and $Ue_n\equiv e_{n-1}$. we have (i). In the same way, by taking $\pi=U\boxtimes\pi_B$ for $(\mathcal{H}_B,\pi_B)$, we have (ii). □

3. Dynamical systems and representations of $\mathcal{O}_N$

In order to analyze $(L_2(\mathbf{C}),\pi_0)$ in (1.2), we prepare a method of construction of isometries and representations of $\mathcal{O}_N$ on measure spaces([**11**, **12**, **13**, **14**]) here briefly.

3.1. Representations arising from branching function systems. Let (X,μ) and (Y,ν) be measure spaces and f a measurable map from X to Y which is injective and there exists the Radon-Nikodým derivative Φ_f of $\nu\circ f$ with respect to μ and Φ_f is non zero almost everywhere in X. We denote the set of such maps by $RN(X,Y)$. We simply denote $RN(X)\equiv RN(X,X)$. Note that $RN(X)$ is a semigroup with respect to composition of transformations on (X,μ). Denote $\mathrm{Iso}(L_2(X,\mu))$ the semigroup of isometries on $L_2(X,\mu)$.

Definition 3.1. *For $f\in RN(X,Y)$, define an operator $S(f)$ from $L_2(X,\mu)$ to $L_2(Y,\nu)$ by*

$$(S(f)\phi)(y)\equiv\begin{cases}\left\{\Phi_f\left(f^{-1}(y)\right)\right\}^{-1/2}\phi(f^{-1}(x)) & (\textit{when } y\in R(f)\,),\\ 0 & (\textit{otherwise})\end{cases}$$

for $\phi\in L_2(X,\mu)$ and $y\in Y$ where $R(f)$ is the image of f.

For measure spaces (X,μ) and (Y,ν), we denote $X\times Y$ and $X\cup Y$, the direct product and the direct sum of (X,μ) and (Y,ν) as measure space, respectively. For $f\in RN(X_1,Y_1)$ and $g\in RN(X_2,Y_2)$, $f\oplus g\in RN(X_1\cup X_2,Y_1\cup Y_2)$ is defined by $(f\oplus g)|_{X_1}\equiv f$, $(f\oplus g)|_{X_2}\equiv g$.

Lemma 3.2. *Let (X_i,μ_i) be measure spaces for $i=1,2,3,4$.*

(i) *For $f\in RN(X_1,X_2)$, $S(f)$ is an isometry.*

(ii) *For $f\in RN(X_1,X_2)$ and $g\in RN(X_2,X_3)$, $g\circ f\in RN(X_1,X_3)$ and*

$$S(g)S(f)=S(g\circ f)\tag{3.1}$$

Specially, a map S from $RN(X_1)$ to $\mathrm{Iso}(L_2(X_1,\mu_1))$ is a homomorphism between semigroups.

(iii) *If* $f \in RN(X_1, X_2)$ *is bijective and* $f^{-1} \in RN(X_2, X_1)$*, then*

$$S(f^{-1}) = S(f)^*.$$

Specially, $S(id_{X_1})$ *is the identity operator on* $L_2(X_1, \mu_1)$*.*

(iv) *For* $f \in RN(X_1, X_2)$ *and* $g \in RN(X_3, X_4)$*,*

$$S(f \times g) = S(f) \otimes S(g), \quad S(f \oplus g) = S(f) \oplus S(g)$$

where we identify $L_2(X_i \times X_j, \mu_i \times \mu_j)$ *and* $L_2(X_i, \mu_i) \otimes L_2(X_j, \mu_j)$*,* $L_2(X_i \cup X_j, \mu_i \cup \mu_j)$ *and* $L_2(X_i, \mu_i) \oplus L_2(X_j, \mu_j)$ *for* $i, j = 1, 2, 3, 4$*, respectively.*

Proof. About (i), (ii) and (iii), see [**11**].
(iv) Put a unitary U_{ij} from $L_2(X_i \times X_j, \mu_i \times \mu_j)$ to $L_2(X_i, \mu_i) \otimes L_2(X_j, \mu_j)$ by $U_{ij}(\phi_i \phi_j) \equiv \phi_i \otimes \phi_j$ for $\phi_i \in L_2(X_i, \mu_i)$ and $\phi_j \in L_2(X_i, \mu_j)$ where $(\phi_i \phi_j)(x, y) \equiv \phi_i(x)\phi_j(y)$ for $(x, y) \in X_i \times X_j$ for $i, j = 1, 2, 3, 4$. By Definition 3.1, $(S(f \times g)\phi_1\phi_3)(x, y) = (S(f)\phi_1)(x) \cdot (S(g)\phi_3)(y)$ for $\phi_1 \in L_2(X_1, \mu_1)$, $\phi_3 \in L_2(X_3, \mu_3)$ and $(x, y) \in X_2 \times X_4$. From this, $U_{24}S(f \times g)U_{13}^* = S(f) \otimes S(g)$. We can show $S(f \oplus g)|_{L_2(X_1,\mu_1)} = S(f)$ and $S(f \oplus g)|_{L_2(X_3,\mu_3)} = S(g)$ by direct computation. □

Remark that $g \circ f$ in rhs of (3.1) is only the composition of two transformations f and g but not special product of them. By Lemma 3.2, we see that the map S realizes the iteration of transformations on a measure space as the product of operators on a Hilbert space naturally.

Let $N \geq 2$.

Definition 3.3. *Let* (X, μ) *and* (Y, ν) *be measure spaces.*

(i) $f = \{f_i\}_{i=1}^N$ *is a branching function system on* (X, μ) *if* $f_i \in RN(X)$*,* $i = 1, \ldots, N$*, and* $f_i(X) \cap f_j(X)$*,* $1 \leq i < j \leq N$*,* $X \setminus \bigcup_{i=1}^N f_i(X)$ *are* μ*-null sets.*

(ii) F *is the coding map of a branching function system* $f = \{f_i\}_{i=1}^N$ *on* (X, μ) *if* F *is a map from* X *to* X *such that* $(F \circ f_i)(x) = x$ *for each* $x \in X$ *and* $i = 1, \ldots, N$*.*

(iii) *For branching function systems* $f = \{f_i\}_{i=1}^N$ *on* (X, μ) *and* $g = \{g_i\}_{i=1}^N$ *on* (Y, ν)*,* $f \sim g$ *if there is* $\varphi \in RN(X, Y)$ *such that* φ *is bijective,* $\varphi^{-1} \in RN(Y, X)$ *and* $\varphi \circ f_i \circ \varphi^{-1} = g_i$ *for* $i = 1, \ldots, N$*.*

(iv) *For* $f \in RN(X)$ *and a branching function system* $g = \{g_i\}_{i=1}^N$ *on* (Y, ν)*, we denote* $f \boxtimes g \equiv \{f \times g_i\}_{i=1}^N$*.*

(v) *For branching function systems* $f = \{f_i\}_{i=1}^N$ *on* (X, μ) *and* $g = \{g_i\}_{i=1}^N$ *on* (Y, ν)*, we denote* $f \oplus g \equiv \{f_i \oplus g_i\}_{i=1}^N$*.*

The notion of branching function system was introduced in [**3**] in order to construct a representation of $\mathcal{O}_N$ from a family of transformations.

Proposition 3.4. *Let* (X, μ) *and* (Y, ν) *be measure spaces.*

(i) *For a branching function system* $f = \{f_i\}_{i=1}^N$ *on* (X, μ),

$$\pi_f(s_i) \equiv S(f_i) \quad (i = 1, \dots, N),$$

defines a representation $(L_2(X,\mu), \pi_f)$ *of* $\mathcal{O}_N$. *We denote* $(L_2(X,\mu), \pi_f)$ *by* π_f *simply.*

(ii) *Let* $f = \{f_i\}_{i=1}^N$ *and* $g = \{g_i\}_{i=1}^N$ *be branching function systems on* (X, μ) *and* (Y, ν), *respectively. If* $f \sim g$, *then* $\pi_f \sim \pi_g$.

(iii) *If there are* $f \in RN(X)$ *and a branching function system* $g = \{g_i\}_{i=1}^N$ *on* (Y, ν) *such that* f *is bijective and* $f^{-1} \in RN(X)$, *then* $f \boxtimes g$ *is a branching function system on* $(X \times Y, \mu \times \nu)$ *and*

$$\pi_{f \boxtimes g} \sim S(f) \boxtimes \pi_g$$

where $S(f) \boxtimes \pi_g$ *is in* (2.6).

(iv) *If there are branching function systems* $f = \{f_i\}_{i=1}^N$ *and* $g = \{g_i\}_{i=1}^N$ *on* (X, μ) *and* (Y, ν), *respectively, then*

$$\pi_{f \oplus g} \sim \pi_f \oplus \pi_g.$$

Proof. (i) and (ii) follow from Lemma 3.2.
(iii) By Lemma 3.2 (iv), $\pi_{f\boxtimes g}(s_i) = S(f \times g_i) = S(f) \otimes S(g_i) = (S(f) \boxtimes \pi_g)(s_i)$ for $i = 1, \dots, N$. Therefore the statement holds.
(iv) By Lemma 3.2 (iv), $\pi_{f \oplus g}(s_i) = S(f_i \oplus g_i) = S(f_i) \oplus S(g_i) = \pi_f(s_i) \oplus \pi_g(s_i) = (\pi_f \oplus \pi_g)(s_i)$ for $i = 1, \dots, N$. Hence we have the statement. □

3.2. Representations arising from dynamical systems. In this paper, any dynamical system means a pair (X, F) of a measure space (X, μ) and a measurable transformation F on (X, μ). Any map between dynamical systems is assumed measurability.

Definition 3.5. *Let* (X_1, F_1) *and* (X_2, F_2) *be dynamical systems.*

(i) (X_1, F_1) *and* (X_2, F_2) *are conformal conjugate if there is* $\varphi \in RN(X_1, X_2)$ *such that* φ *is bijective,* $\varphi^{-1} \in RN(X_2, X_1)$ *and* $\varphi \circ F_1 \circ \varphi^{-1} = F_2$.

(ii) (X_1, F_1) *and* (X_2, F_2) *are weakly conformal conjugate if there are invariant subspaces* $Y_1 \subset X_1$ *and* $Y_2 \subset X_2$ *with respect to* F_1 *and* F_2, *respectively such that* $X_1 \setminus Y_1$ *and* $X_2 \setminus Y_2$ *are null sets, and* $(Y_1, F_1|_{Y_1})$ *and* $(Y_2, F_2|_{Y_2})$ *are conformal conjugate.*

(iii) $f = \{f_i\}_{i=1}^N$ *is the branching function system of* (X_1, F_1) *if* f *is a branching function system on* (X_1, μ_1) *such that* F_1 *is the coding map of* f.

Lemma 3.6. *Let* (X_i, F_i) *be a dynamical system on a measure space* (X_i, μ_i) *for* $i = 1, 2$. *Assume that there are branching function systems* $f = \{f_i\}_{i=1}^N$ *and* $f' = \{f_i'\}_{i=1}^N$ *of* F_1 *and* F_2, *respectively. If* (X_1, F_1) *and* (X_2, F_2) *are weakly conformal conjugate, then* π_f *and* $\pi_{f'}$ *are unitarily equivalent.*

Proof. Let φ be a weakly conformal map between X_1 and X_2. Put Y_i the invariant subspace of X_i under F_i such that $\mu_i(X_i \setminus Y_i) = 0$, $\varphi(Y_1) = Y_2$ for $i = 1, 2$. Since $\varphi \circ F_1 \circ \varphi^{-1} = F_2$, $\varphi \circ f_i \circ \varphi^{-1} = f_i^{'}$ for $i = 1, \ldots, N$. By Proposition 3.4, $S(\varphi)$ is a unitary which satisfies $(\mathrm{Ad}S(\varphi)) \circ \pi_f|_{L_2(Y_2,\mu_2)} = \pi_{f'}|_{L_2(Y_2,\mu_2)}$. This equation can be extended from the whole $L_2(X_1, \mu_1)$ to $L_2(X_2, \mu_2)$ by the assumption of Y_1 and Y_2. □

Definition 3.7. *Let (X_i, F_i) be a dynamical system on a measure space (X_i, μ_i) for $i = 1, 2$.*

(i) *(X, F) is the direct product of (X_1, F_1) and (X_2, F_2) if $(X, \mu) = (X_1, \mu_1) \times (X_2, \mu_2)$ is the direct product of measure spaces and $F = F_1 \times F_2$ on $X = X_1 \times X_2$. We simply denote $(X, F) = (X_1, F_1) \times (X_2, F_2)$.*

(ii) *(X, F) is the direct sum of (X_1, F_1) and (X_2, F_2) if $(X, \mu) = (X_1, \mu_1) \oplus (X_2, \mu_2)$ is the direct sum of measure spaces and $F|_{X_i} = F_i$ for $i = 1, 2$. We simply denote $(X, F) = (X_1, F_1) \oplus (X_2, F_2)$.*

Proposition 3.8. *Let (X_i, F_i) be a dynamical system on a measure space (X_i, μ_i) with the branching function system $\{f_j^{(i)}\}_{j=1}^N$ for $i = 1, 2$.*

(i) *Assume that $h \in RN(X_1)$ is bijective and $h^{-1} \in RN(X_1)$. The direct product $(X_1, h) \times (X_2, F_2)$ has the branching function system $h^{-1} \boxtimes f^{(2)}$ and*
$$\pi_{h^{-1} \boxtimes f^{(2)}} \sim S(h^{-1}) \boxtimes \pi_{f^{(2)}}.$$

(ii) *The direct sum $(X_1, F_1) \oplus (X_2, F_2)$ has the branching function system $f^{(1)} \oplus f^{(2)}$ and*
$$\pi_{f^{(1)} \oplus f^{(2)}} \sim \pi_{f^{(1)}} \oplus \pi_{f^{(2)}}.$$

Proof. By Proposition 3.4 (iii) and (iv), (i) and (ii) follow respectively. □

Proposition 3.9. *Let (X, F) be a dynamical system with the branching function system $f = \{f_i\}_{i=1}^N$ of F and σ_{-p}, $p \in \mathbf{Z}$, the shift on $\mathbf{Z}$ which is defined by $\sigma_{-p}(n) \equiv n - p$ for $n \in \mathbf{Z}$. Then the direct product $(\mathbf{Z} \times X, \sigma_{-p} \times F)$ has a branching function system $\sigma_p \boxtimes f$ and the following holds:*
$$\pi_{\sigma_p \boxtimes f} \sim \begin{cases} \displaystyle\int_{U(1)}^{\oplus} \pi_f \circ \gamma_{w^p} \, d\eta(w) & (p \neq 0), \\ (\pi_f)^{\oplus \infty} & (p = 0). \end{cases}$$

Proof. By checking Definition 3.3 (i), we see that $\sigma_p \boxtimes f$ is a branching function system on $(\mathbf{Z} \times X, \tilde{\mu})$ where $\tilde{\mu}(A \times Y) \equiv (\#A) \cdot \mu(Y)$ for $A \subset \mathbf{Z}$ and $Y \subset X$. By definition 3.1, $S(\sigma_p)e_n = e_{n+p}$ for $n \in \mathbf{Z}$ where $\{e_n\}_{n \in \mathbf{Z}}$ is the canonical basis of $l_2(\mathbf{Z})$. By Proposition 3.8 (i) and Lemma 2.4, it follows.

□

4. Proof of Theorem 1.1

Let Q be the transformation on $\mathbf{C}$ defined in (1.1). The behavior of Q on $\mathbf{C}$ is well-known([**7**]). By the action of Q, $\mathbf{C}$ is decomposed into three invariant parts $D_1 \equiv \{z \in \mathbf{C} : 0 < |z| < 1\}$, $\hat{S}^1 \equiv \{z \in \mathbf{C} : |z| = 0, 1\}$, $D_2 \equiv \{z \in \mathbf{C} : |z| > 1\}$. Therefore $(\mathbf{C}, Q)$ is decomposed into three dynamical systems $(D_1, Q|_{D_1})$, $(D_2, Q|_{D_2})$, $(\hat{S}^1, Q|_{S^1})$. For the aim to consider operators on $L_2(\mathbf{C})$, we neglect $(\hat{S}^1, Q|_{\hat{S}^1})$ since $\hat{S}^1$ is a null set in $\mathbf{C}$ with respect to the measure $d\mu_{\mathbf{R}}(z) = dxdy$ for $z = x + \sqrt{-1}y$. Q has the following branching function system $q = \{q_1, q_2\}$ on each parts:

$$q_1(z) \equiv \sqrt{z}, \quad q_2(z) \equiv -\sqrt{z} \quad (z \in \mathbf{C}) \tag{4.1}$$

where $\sqrt{z} \equiv \sqrt{r}e^{\pi\sqrt{-1}\theta}$ when $z = re^{2\pi\sqrt{-1}\theta}$, $0 \le \theta < 1$, $r \ge 0$.

By the polar coordinate on $\mathbf{C}$, $z = z(r, \theta) \equiv re^{2\pi\sqrt{-1}\theta}$, we can rewrite

$$Q(r, \theta) \equiv (Q_R(r), H(\theta)),$$

$$Q_R(r) \equiv r^2, \quad H(\theta) \equiv 2\theta \bmod 1 \quad ((r, \theta) \in [0, \infty) \times [0, 1)). \tag{4.2}$$

In this way, the action of Q on $\mathbf{C}$ is decomposed into the direct product of transformations on $[0, \infty)$ and $[0, 1)$, respectively.

Consider $([0, \infty), Q_R)$ in (4.2). Let $X \equiv [0, \infty)$ and a family $\{X_n^{(i)} : i = 1, 2,\ n \in \mathbf{Z}\}$ of an intervals in X by

$$X_n^{(1)} \equiv [2^{2^{n-1}}, 2^{2^n}), \quad X_n^{(2)} \equiv [2^{-2^n}, 2^{-2^{n-1}}) \quad (n \in \mathbf{Z}).$$

For example, $X_0^{(1)} = [\sqrt{2}, 2)$, $X_1^{(1)} = [2, 4)$, $X_{1,-1} = [2^{1/4}, \sqrt{2})$, $X_0^{(2)} = [1/2, 1/\sqrt{2})$, $X_1^{(2)} = [1/4, 1/2)$, $X_{-1}^{(2)} = [1/\sqrt{2}, 2^{-1/4})$. Hence we have the following decomposition:

$$X = X^{(0)} \cup \{0, 1\}, \quad X^{(0)} \equiv \coprod_{i=1,2} \coprod_{n \in \mathbf{Z}} X_n^{(i)} \tag{4.3}$$

Q satisfies $Q_R(X_n^{(i)}) = X_{n+1}^{(i)}$ for each $n \in \mathbf{Z}$ and $i = 1, 2$. Both points 0 and 1 in $[0, \infty)$ are fixed points with respect to Q_R.

Put a direct product $Y \equiv \mathbf{Z} \times [0, 1) \times \{1, 2\}$ of measure spaces $\mathbf{Z}$, $[0, 1)$ and $\{1, 2\}$ where $\mathbf{Z}$ and $\{1, 2\}$ are regarded as discrete measure space. Put maps $\psi_1 : [\sqrt{2}, 2) \to [0, 1)$; $\psi_1(x) \equiv (x - \sqrt{2})/(2 - \sqrt{2})$, $\psi_2 : [1/2, 1/\sqrt{2}) \to [0, 1)$; $\psi_2(x) \equiv -2(x - 2^{-1/2})/(\sqrt{2} - 1)$.

Lemma 4.1. *Define a map φ from $X^{(0)}$ to Y by*

$$\varphi(r) \equiv \left(n,\ \left(\psi_i \circ Q_R^{-n}\right)(r),\ i\right) \quad (\textit{when } r \in X_{i,n}) \tag{4.4}$$

where $Q_R^n \equiv \underbrace{Q_R \circ \cdots \circ Q_R}_{n \text{ times}}$, $Q_R^{-n} \equiv (Q_R^{-1})^n$ *for* $n \geq 1$, $Q_R^0 = id$. *Then* φ *is a measurable bijection in* $RN(X^{(0)}, Y)$. *Let* σ *be the shift on* $\mathbf{Z}$. *Then* $(X^{(0)}, Q_R)$ *and* $(Y, \sigma \times id \times id)$ *are conformal conjugate.*

Proof. It is easy to show $\varphi \circ Q_R \circ \varphi^{-1} = \sigma \times id \times id$. □

Consider $([0,1), H)$ in (4.2). Let h_1, h_2 be transformations on $[0,1)$ defined by

$$h_1(x) \equiv \frac{1}{2}x, \quad h_2(x) \equiv \frac{1}{2}x + \frac{1}{2}. \tag{4.5}$$

Then $h \equiv \{h_1, h_2\}$ is the branching function system of H.

Proposition 4.2. *Put* $\hat{Y} \equiv \mathbf{Z} \times [0,1) \times \{1,2\} \times [0,1)$ *and* $\hat{Q} \equiv \sigma \times id \times id \times H$. *Then* $(\mathbf{C}, Q)$ *and* $(\hat{Y}, \hat{Q})$ *are weakly conformal conjugate.*

Proof. Put $\hat{X} \equiv X^{(0)} \times [0,1)$. Then $\hat{X}$ is an invariant subspace of $\mathbf{C}$ and $\mathbf{C} \setminus \hat{X} = S^1 \cup \{0\}$ is a null set in $\mathbf{C}$ where we identify $\hat{X}$ and $\{re^{2\pi\sqrt{-1}\theta} \in \mathbf{C} : (r,\theta) \in \hat{X}\}$. Put $\hat{\varphi} \equiv \varphi \times id$ for φ in (4.4). Then we have $\hat{\varphi} \circ Q \circ \hat{\varphi}^{-1} = \hat{Q}$. From this, $(\hat{X}, Q)$ and $(\hat{Y}, \hat{Q})$ are conformal conjugate. Hence $(\mathbf{C}, Q)$ and $(\hat{Y}, \hat{Q})$ are weakly conformal conjugate by Lemma 4.1. □

Lemma 4.3. *Let* $\hat{q} \equiv \{\hat{q}_1, \hat{q}_2\}$ *be the branching function system of* $\hat{Q}$ *in Proposition* 4.2 *on* $\hat{Y}$ *given by*

$$\hat{q}_i(n, x, j, y) \equiv (n-1, x, j, h_i(y)) \quad (i = 1, 2). \tag{4.6}$$

Then the representation $(L_2(\mathbf{C}), \pi_0)$ *in (*1.2*) is unitarily equivalent to* $(l_2(\mathbf{Z}) \otimes L_2[0,1] \otimes \mathbf{C}^2 \otimes L_2[0,1],\ \pi_{\hat{q}})$.

Proof. The branching function system $q = \{q_1, q_2\}$ of Q in (4.1) is weakly conformal conjugate with $\hat{q} \equiv \{\hat{q}_1, \hat{q}_2\}$. The representation $(L_2(\mathbf{C}), \pi_q)$ of $\mathcal{O}_2$ by q is just $(L_2(\mathbf{C}), \pi_0)$ in (1.2) by Definition 3.1 and Proposition 3.4. By natural identification, $L_2(\hat{Y}) \sim l_2(\mathbf{Z}) \otimes L_2[0,1] \otimes \mathbf{C}^2 \otimes L_2[0,1]$. By Lemma 3.6, $(L_2(\mathbf{C}), \pi_q)$ is unitarily equivalent to $(l_2(\mathbf{Z}) \otimes L_2[0,1] \otimes \mathbf{C}^2 \otimes L_2[0,1],\ \pi_{\hat{q}})$. In consequence, the statement holds. □

Proof of Theorem 1.1*:* By (4.6), $\hat{q}_i = \sigma_{-1} \times \hat{h}_i$ where $\hat{h}_i \equiv id \times id \times h_i$ for $i = 1, 2$. That is, $\hat{q} = \sigma_{-1} \boxtimes \hat{h}$ in Proposition 3.9. By applying Proposition 3.9 (i) for the case $p = -1$, $\pi_{\hat{q}}$ is equivalent to

$$\pi_{\sigma_{-1} \boxtimes \hat{h}} \sim \int_{U(1)}^{\oplus} \pi_{\hat{h}} \circ \gamma_{w^{-1}}\ d\eta(w).$$

By Proposition 3.4 (iii), $\pi_{\hat{h}}\circ\gamma_{w^{-1}} \sim (I\otimes I)\boxtimes(\pi_h\circ\gamma_{\bar{w}})$ where $\mathcal{K} \equiv L_2[0,1]\otimes\mathbf{C}^2$. By Example 2.2 (ii), $\pi_h = \pi_B$. Hence (i) is proved. (ii) and (iii) follow from Proposition 2.3. □

5. Construction of annular basis of $L_2(\mathbf{C})$

We construct a basis of $L_2(\mathbf{C})$ by using π_0 in (1.2). In stead of considering the orbit of Q which consists of points in $\mathbf{C}$, we treat that of regions with non-zero surface volume.

For $X, Y \subset \mathbf{R}$, define subsets of $\mathbf{C}$ by

$$A(X) \equiv \{z \in \mathbf{C} : |z| \in X\}, \quad B(Y) \equiv \{z \in \mathbf{C} : (2\pi)^{-1}\arg(z) \in Y\},$$
$$AB(X,Y) \equiv A(X) \cap B(Y).$$

Note $AB(X,Y) \subset AB(X',Y')$ when $X \subset X'$ and $Y \subset Y'$. $A(X)$ is an annulus, and $AB(X,Y)$ is called a *chunk*([**7**]) (or a sector, a fan-shaped region) in $\mathbf{C}$. It is well known that Q maps a chunk to that.

Recall $\{1,2\}^*$ in Definition 1.2. For $J \in \{1,2\}^*$, the length $|J|$ of J is defined by $|J| \equiv k$ when $J \in \{1,2\}^k$, $k \geq 0$. For $J_1, J_2 \in \{1,2\}^*$, $J_1 \cup J_2 \equiv (j_1,\dots,j_k,j_1',\dots,j_l')$ when $J_1 = (j_1,\dots,j_k)$ and $J_2 = (j_1',\dots,j_l')$. Specially, we define $J \cup \{0\} = \{0\} \cup J = J$ for $J \in \{1,2\}^*$ for convention. For $J_1, J_2 \in \{1,2\}^*$, we denote $J_1 = * \cup J_2$ (*resp.* $J_1 = J_2 \cup *$) if there is $J_3 \in \{1,2\}^*$ such that $J_1 = J_3 \cup J_2$ (*resp.* $J_1 = J_2 \cup J_3$).

5.1. Annular decomposition of C. Put closed intervals

$$X_{n,0}^{(1)} \equiv [2^{-2^n}, 2^{-2^{n-1}}], \quad X_{n,0}^{(2)} \equiv [2^{2^{n-1}}, 2^{2^n}] \quad (n \in \mathbf{Z}).$$

Let $\mathcal{S}$ be the set of all bounded closed intervals of $[0,\infty)$. For $i = 0,1,2$, put Ξ_i the transformations of $\mathcal{S}$ by

$$\Xi_0 \equiv id, \quad \Xi_1([a,b]) \equiv [a, \sqrt{ab}], \quad \Xi_2([a,b]) \equiv [\sqrt{ab}, b] \quad ([a,b] \in \mathcal{S}).$$

Define

$$X_{n,J}^{(i)} \equiv \Xi_{\bar{J}}\left(X_{n,0}^{(i)}\right) \quad (i = 1,2,\ n \in \mathbf{Z},\ J \in \{1,2\}^*) \tag{5.1}$$

where $\Xi_{\bar{J}} \equiv \Xi_{j_k} \circ \cdots \circ \Xi_{j_1}$ for $J = (j_1,\dots,j_k)$, $k \geq 1$. For example, $X_{0,0}^{(1)} = [2^{-1}, 2^{-1/2}]$, $X_{0,1}^{(1)} = [2^{-1}, 2^{-3/4}]$, $X_{0,12}^{(1)} = [2^{-7/8}, 2^{-3/4}]$. Then $\{A(X_{n,J}^{(i)}) : i = 1,2,\ n \in \mathbf{Z},\ J \in \{1,2\}^*\}$ is a family of annuli in $\mathbf{C}$ with common center $0 \in \mathbf{C}$. Furthermore $A(X_{n,J}^{(i)}) \subset A(X_{n,J'}^{(i)})$ when $J = J' \cup *$. There is the following decomposition:

$$\mathbf{C} = \bigcup_{i=1,2} \bigcup_{n\in\mathbf{Z}} \bigcup_{J\in\{1,2\}^k} A(X_{n,J}^{(i)}) \cup \{z \in \mathbf{C} : |z| = 0,1\}$$

for each $k \geq 0$.

Lemma 5.1. *For $i = 1, 2$, $n \in \mathbf{Z}$ and $J \in \{1,2\}^*$,*

$$A(X^{(i)}_{n,J}) = A(X^{(i)}_{n,J\cup\{1\}}) \cup A(X^{(i)}_{n,J\cup\{2\}}), \quad Q\left(A(X^{(i)}_{n,J})\right) = A(X^{(i)}_{n+1,J})$$

Proof. By $X^{(i)}_{n,J} = X^{(i)}_{n,J\cup\{1\}} \cup X^{(i)}_{n,J\cup\{2\}}$, the first equality holds. We show the second by induction with respect to $J \in \{1,2\}^*$.

$$Q\left(A(X^{(i)}_{n,0})\right) = \{z^2 \in \mathbf{C} : |z| \in X^{(i)}_{n,0}\} = \{z \in \mathbf{C} : |z| \in X^{(i)}_{n+1,0}\}$$

for each $i = 1, 2$ and $n \in \mathbf{Z}$. Assume that the statement holds for each $J \in \{1,2\}^l$, $l = 0, \ldots, k$. Put $J \in \{1,2\}^{k+1}$. Then we can denote $J = J' \cup \{j\}$ for $J' \in \{1,2\}^k$. By definition, $Q\left(A(X^{(i)}_{n,J})\right) = \left\{z^2 \in \mathbf{C} : |z| \in \Xi_{\bar{J}}(X^{(i)}_{n,0})\right\}$. If $[a,b] = X^{(i)}_{n,J'}$, then $X^{(i)}_{n,J} = [a, \sqrt{ab}]$ or $[\sqrt{ab}, b]$. From this, $z \in Q\left(A(X^{(i)}_{n,J})\right)$ if and only if $\sqrt{|z|} \in [a, \sqrt{ab}]$ or $[\sqrt{ab}, b]$ if and only if $|z| \in [a^2, ab]$ or $[ab, b^2]$. Since $A([a^2, b^2]) = Q\left(A(X^{(i)}_{n,J'})\right) = A(X^{(i)}_{n+1,J'})$, $[a^2, b^2] = X^{(i)}_{n+1,J'}$ and $X^{(i)}_{n+1,J} = [a^2, ab]$ or $[ab, b^2]$ according to $j = 1, 2$. Hence $z \in Q\left(A(X^{(i)}_{n,J})\right)$ if and only if $|z| \in X^{(i)}_{n+1,J}$. From this, $Q\left(A(X^{(i)}_{n,J})\right) = A(X^{(i)}_{n+1,J})$. □

By Lemma 5.1, Q is the shift of a family $\left\{A(X^{(i)}_{n,J})\right\}_{n\in\mathbf{Z}}$ of annuli in $\mathbf{C}$ for each $i = 1, 2$ and $J \in \{1,2\}^*$.

For $\Omega \subset \mathbf{C}$, put

$$\mathcal{I}(\Omega) \equiv \int_\Omega \frac{1}{|z|^2}\, d\mu_{\mathbf{R}}(z). \tag{5.2}$$

Lemma 5.2.

$$\mathcal{I}\left(A(X^{(i)}_{n,J})\right) = 2^{n-|J|}\pi \log 2 \quad (i = 1, 2,\ n \in \mathbf{Z},\ J \in \{1,2\}^*).$$

Proof. By definition,

$$\mathcal{I}\left(A(X^{(i)}_{n,J})\right) = \int_{A(X^{(i)}_{n,J})} \frac{1}{|z|^2}\, d\mu_{\mathbf{R}}(z) = 2\pi \int_{a_{i,n,J}}^{b_{i,n,J}} \frac{1}{r} dr = 2\pi \log \frac{b_{i,n,J}}{a_{i,n,J}}$$

where we take polar coordinate $z = re^{2\pi\sqrt{-1}\theta}$ and $a_{i,n,J}, b_{i,n,J}$ are real numbers such that $[a_{i,n,J}, b_{i,n,J}] = \Xi_{\bar{J}}(X^{(i)}_{n,0})$ and $a_{1,n,0} = 2^{-2^n}$, $b_{1,n,0} = 2^{-2^{n-1}}$, $a_{2,n,0} = 2^{2^{n-1}}$, $b_{2,n,0} = 2^{2^n}$. Note

$$\frac{b_{i,n,0}}{a_{i,n,0}} = 2^{2^{n-1}}, \quad \frac{b_{i,n,j}}{a_{i,n,j}} = \sqrt{\frac{b_{i,n,0}}{a_{i,n,0}}}, \quad \frac{b_{i,n,J}}{a_{i,n,J}} = \sqrt{\frac{b_{i,n,J'}}{a_{i,n,J'}}}$$

for $i, j = 1, 2$, $n \in \mathbf{Z}$ and $J = (j_1, \ldots, j_k)$, $J' = (j_1, \ldots, j_{k-1})$, $k \geq 2$. Hence $\log(b_{i,n,J}/a_{i,n,J}) = 2^{n-1-|J|}\log 2$. Therefore

$$\mathcal{I}\left(A(X^{(i)}_{n,J})\right) = 2\pi \log(b_{i,n,J}/a_{i,n,J}) = 2\pi \cdot \left(2^{n-1-|J|}\log 2\right) = 2^{n-|J|}\pi \log 2.$$

□

Next, we decompose annuli into chunks in $\mathbf{C}$. For transformations h_1, h_2 in (4.5), put

$$Y_0 \equiv [0,1], \quad Y_J \equiv h_J([0,1]) \quad (J \in \{1,2\}^* \setminus \{0\}). \tag{5.3}$$

Then $Y_J \subset Y_{J'}$ when $J = * \cup J'$.

Lemma 5.3. (i) *For $i,j = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \{1,2\}^*$,*

$$A(X^{(i)}_{n,J_1}) = AB(X^{(i)}_{n,J_1}, Y_1) \cup AB(X^{(i)}_{n,J_1}, Y_2),$$

$$AB(X^{(i)}_{n,J_1}, Y_{J_2}) = AB(X^{(i)}_{n,J_1\cup\{1\}}, Y_{J_2}) \cup AB(X^{(i)}_{n,J_1\cup\{2\}}, Y_{J_2}).$$

(ii) *For $i,j = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \{1,2\}^*$, $|J_2| \geq 1$,*

$$Q\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right) = AB(X^{(i)}_{n+1,J_1}, Y_{J_2'}),$$

$$q_j\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right) = AB(X^{(i)}_{n-1,J_1}, Y_{\{j\}\cup J_2})$$

where q_j is in (4.1) and $J_2' = (j_2, \ldots, j_k)$ when $J_2 = (j_1, \ldots, j_k)$.

Proof. (i) The first follows by $Y_0 = Y_1 \cup Y_2$ and $AB(X^{(i)}_{n,J_1}, Y_0) = A(X^{(i)}_{n,J_1})$. The second follows by Lemma 5.1.
(ii) $z \in Q\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right)$ if and only if $z \in A(X^{(i)}_{n+1,J_1})$ by Lemma 5.1 and $h_1(\theta) \in Y_{J_2}$ or $h_2(\theta) \in Y_{J_2}$ where $\theta \equiv (2\pi)^{-1} \cdot \arg(z)$. Therefore the statement holds. □

Lemma 5.4. (i) *For $i = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \{1,2\}^*$, we have*

$$\mathcal{I}\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right) = 2^{n-|J_1|-|J_2|}\pi \log 2.$$

(ii) *For $i,j = 1,2$, $m,n \in \mathbf{Z}$ and $J_1, J_2, J_1', J_2' \in \{1,2\}^*$, $AB(X^{(i)}_{n,J_1}, Y_{J_2}) \cap AB(X^{(j)}_{m,J_1'}, Y_{J_2'})$ is a null set in $\mathbf{C}$ with respect to the measure $\mu_{\mathbf{R}}$ when $(i,n,J_1,J_2) \neq (j,m,J_1',J_2')$, $|J_1| = |J_1'|$ and $|J_2| = |J_2'|$.*

Proof. (i) Note that $\mathbf{C}$ is equally divided into images of h_1 and h_2. By Lemma 5.2, we have

$$\mathcal{I}\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right) = 2^{-|J_2|} \cdot \mathcal{I}\left(A(X^{(i)}_{n,J_1})\right) = 2^{n-|J_1|-|J_2|}\pi \log 2.$$

(ii) By definition of $AB(X^{(i)}_{n,J_1}, Y_{J_2})$, it follows. □

We have the following decomposition of an annulus into chunks:

$$A(X^{(i)}_{n,J_1}) = \bigcup_{J_2\in\{1,2\}^k} AB(X^{(i)}_{n,J_1}, Y_{J_2}) \tag{5.4}$$

for each $k \geq 1$. We see that multi-indices is used to decompose $\mathbf{C}$ into annuli and chunks with respect to the action of Q.

5.2. Annular decomposition of $L_2(\mathbf{C})$. We interpret the annular decomposition of $\mathbf{C}$ in § 5.1 to a decomposition of $L_2(\mathbf{C})$. Here annuli and chunks are interpreted as several functions on $\mathbf{C}$ which are related them.

Let $K^{(i)}_{n,J_1,J_2}$ be the characteristic function on $AB(X^{(i)}_{n,J_1}, Y_{J_2})$ for $i = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \{1,2\}^*$. For example, $K^{(1)}_{n,0,0} = \chi_{A([2^{-2^n}, 2^{-2^{n-1}}])}$ and $K^{(2)}_{n,0,0} = \chi_{A([2^{2^{n-1}}, 2^{2^n}])}$ for $n \in \mathbf{Z}$.

Lemma 5.5. *For each $i = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \{1,2\}^*$,*

$$K^{(i)}_{n,J_1,0} = K^{(i)}_{n,J_1,1} + K^{(i)}_{n,J_1,2}, \quad K^{(i)}_{n,J_1,J_2} = K^{(i)}_{n,J_1\cup\{1\},J_2} + K^{(i)}_{n,J_1\cup\{2\},J_2},$$

$$K^{(i)}_{n,J_1,J_2} \circ Q = K^{(i)}_{n-1,J_1,\{1\}\cup J_2} + K^{(i)}_{n-1,J_1,\{2\}\cup J_2}$$

where these equalities hold up to null sets in $\mathbf{C}$. Specially, $K^{(i)}_{n,J,0} \circ Q = K^{(i)}_{n-1,J,0}$.

Proof. By Lemma 5.3 (i), the first line follows. By Lemma 5.3 (ii), $Q(z) \in AB(X^{(i)}_{n,J_1}, Y_{J_2})$ if and only if $z \in AB(X^{(i)}_{n-1,J_1}, Y_{\{1\}\cup J_2})$ or $z \in AB(X^{(i)}_{n-1,J_1}, Y_{\{2\}\cup J_2})$. By Lemma 5.4 (ii), the statement holds. □

Put $L^{(i)}_{n,J_1,J_2}(z) \equiv \omega_{n,|J_1|,|J_2|}|z|^{-1}K_{n,J_1,J_2}(z)$ for $i = 1,2$, $n \in \mathbf{Z}$, $J_1, J_2 \in \{1,2\}^*$, $z \in \mathbf{C}$ where $\omega_{n,k,l} \equiv \left(2^{n-k-l}\pi \log 2\right)^{-1/2}$ for $n \in \mathbf{Z}$, $k, l \geq 0$. For $\Omega \subset \mathbf{C}$, denote $L_2(\Omega) \equiv \{\phi \in L_2(\mathbf{C}) : \int_{\mathbf{C}\setminus\Omega} |\phi(z)|^2 \, d\mu_{\mathbf{R}}(z) = 0\}$.

Lemma 5.6. (i) $\|L^{(i)}_{n,J_1,J_2}\| = 1$ *and* $L^{(i)}_{n,J_1,J_2} \in L_2\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right)$ *for* $i = 1,2,$ $n \in \mathbf{Z}$ *and* $J_1, J_2 \in \{1,2\}^*$.

(ii) $\{L^{(i)}_{n,J_1,J_2} : i = 1,2,\ n \in \mathbf{Z},\ J_1 \in \{1,2\}^k,\ J_2 \in \{1,2\}^l\}$ *is an orthonormal family in $L_2(\mathbf{C})$ for each $k, l \geq 0$.*

(iii) *For $i, j = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \{1,2\}^*$,*

$$\pi_0(s_j)L^{(i)}_{n,J_1,J_2} = L^{(i)}_{n-1,J_1,\{j\}\cup J_2} \tag{5.5}$$

where π_0 is in (1.2).

Proof. (i) For $i = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \{1,2\}^*$,

$$\|L^{(i)}_{n,J_1,J_2}\|^2 = (\omega_{n,|J_1|,|J_2|})^2 \cdot \mathcal{I}\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right).$$

By Lemma 5.4 (i), $\|L^{(i)}_{n,J_1,J_2}\| = 1$. $L^{(i)}_{n,J_1,J_2} \in L_2\left(AB(X^{(i)}_{n,J_1}, Y_{J_2})\right)$ follows by definition.
(ii) By (i) and Lemma 5.4 (ii), the assertion holds.
(iii) By definition,

$$\left(\pi_0(s_j)L^{(i)}_{n,J_1,J_2}\right)(z) = 2|z|\chi_{E_j}(z)\omega_{n,|J_1|,|J_2|}|z|^{-2}\left(K^{(i)}_{n,J_1,J_2}\circ Q\right)(z). \tag{5.6}$$

By Lemma 5.5 and $\omega_{n,|J_1|,|J_2|} = \omega_{n-1,|J_1|,|\{j\}\cup J_2|}/2$,

$$\left(\pi_0(s_j)L^{(i)}_{n,J_1,J_2}\right)(z) = \omega_{n-1,|J_1|,|\{j\}\cup J_2|}|z|^{-1}K^{(i)}_{n-1,J_1,\{j\}\cup J_2}(z) = L^{(i)}_{n-1,J_1,\{j\}\cup J_2}(z)$$

for $i,j = 1,2$, $n\in\mathbf{Z}$, $J_1, J_2 \in \{1,2\}^*$ and $z \in \mathbf{C}$. □

The definition of $L^{(i)}_{n,J_1,J_2}$ is natural in a sense of (5.5).

Lemma 5.7. (i) $2^{-1/2}\pi_0(s_1+s_2)L^{(i)}_{n,J,0} = L^{(i)}_{n-1,J,0}$ *for* $i = 1,2$, $n\in\mathbf{Z}$ *and* $J \in \{1,2\}^*$.

(ii) *For* $i,j = 1,2$, $n \in \mathbf{Z}$ *and* $J\in\{1,2\}^*\setminus\{0\}$,

$$\pi_0(s_j)^*L^{(i)}_{n,J_1,J_2} = \delta_{j,j_1}L^{(i)}_{n+1,J_1,J_2'},\quad \pi_0(s_j)^*L^{(i)}_{n,J_1,0} = 2^{-1/2}L^{(i)}_{n+1,J_1,0}$$

where $J_2' = (j_2,\dots,j_k)$ *when* $J_2 = (j_1,\dots,j_k)$.

(iii) *For* $i = 1,2$, $n\in\mathbf{Z}$ *and* $J_1, J_2 \in \{1,2\}^*$,

$$L^{(i)}_{n,J_1,J_2} = 2^{-1/2}(L^{(i)}_{n,J_1\cup\{1\},J_2} + L^{(i)}_{n,J_1\cup\{2\},J_2}).$$

Proof. (i) By Lemma 5.6 (iii), we have

$$\left(\pi_0(s_i)L^{(j)}_{n,J,0}\right)(z) = L^{(j)}_{n-1,J,i}(z) = \sqrt{2}\chi_{E_i}(z)L^{(j)}_{n-1,J,0}(z)$$

for $i,j = 1,2$, $n \in \mathbf{Z}$, $J\in\{1,2\}^*$ and $z\in\mathbf{C}$. Hence the statement holds by Lemma 5.6 (iii).
(ii) By (1.2), we have $(\pi_0(s_i)^*\phi)(z) = (2\sqrt{|z|})^{-1}\phi((-1)^{i-1}\sqrt{z})$ for $\phi \in L_2(\mathbf{C})$, $i = 1,2$, $z\in\mathbf{C}$. By (i) and Lemma 5.6 (iii) , it follows.
(iii) Assume $J_1 \in \{1,2\}^k$. By definition, $L^{(i)}_{n,J_1\cup\{1\},J_2} + L^{(i)}_{n,J_1\cup\{2\},J_2}$

$$= \left(2^{n-(k+1)-|J_2|}\pi\log 2\right)^{-1/2}\left(K^{(i)}_{n,J_1\cup\{1\},J_2} + K^{(i)}_{n,J_1\cup\{2\},J_2}\right).$$

By Lemma 5.5, we have the assertion. □

Lemma 5.8. *For* $i = 1,2$ *and* $J \in \{1,2\}^*$, *put* $\mathcal{H}^{(i)}_J \equiv \{\pi_0(x)L^{(i)}_{0,J,0} : x \in \mathcal{O}_2\}$. *Then*

$$\mathcal{H}^{(i)}_J = \overline{\mathrm{Lin} < \{L^{(i)}_{n,J,J'} : J' \in \{1,2\}^*,\, n\in\mathbf{Z}\} >}. \tag{5.7}$$

Furthermore $\mathcal{H}^{(i)}_J$ *and* $\mathcal{H}^{(j)}_{J'}$ *are orthogonal when* $(i,J)\neq(j,J')$ *for* $i,j = 1,2$ *and* $J, J' \in \{1,2\}^k$, $k\geq 1$.

Proof. Fix $i=1,2$ and $J\in\{1,2\}^*$. Denote the rhs in (5.7) by $\mathcal{H}'$. By Lemma 5.7 (i), we see $L^{(i)}_{n,J,0}\in\mathcal{H}^{(i)}_J$ for each $n\in\mathbf{Z}$. By Lemma 5.6 (iii), $L^{(i)}_{n,J,J_2}=\pi_0(s_{J_2})L^{(i)}_{n+|J_2|,J,0}$ for each $J_2\in\{1,2\}^*$. Hence $\mathcal{H}'\subset\mathcal{H}^{(i)}_J$. By Lemma 5.7 (ii), $\mathcal{H}'\supset\mathcal{H}^{(i)}_J$. Therefore $\mathcal{H}'=\mathcal{H}^{(i)}_J$. (5.7) is shown. By Lemma 5.6 (ii), the last statement holds. □

Note that $(\mathcal{H}^{(i)}_J,\pi_0)$ is a cyclic representation of $\mathcal{O}_2$ for each $i=1,2$ and $J\in\{1,2\}^*$.

Lemma 5.9. *Define*

$$M^{(i)}_{n,J_1,J_2}\equiv\pi_0(T_{J_2})L^{(i)}_{n+|J_2|,J_1,0} \tag{5.8}$$

for $i=1,2$, $n\in\mathbf{Z}$, $J_1\in\{1,2\}^*$, $J_2\in\Lambda_2$ *where* Λ_2 *is in Definition* 1.2 *and* $T_1\equiv 2^{-1/2}(s_1+s_2)$, $T_2\equiv 2^{-1/2}(s_1-s_2)$, $T_J\equiv T_{j_1}\cdots T_{j_k}$, $T_0\equiv I$ *when* $J=(j_1,\ldots,j_k)\in\{1,2\}^*\setminus\{0\}$. *Then the followings hold:*

(i) *For* $i=1,2$, $n\in\mathbf{Z}$, $J_1\in\{1,2\}^*$ *and* $J_2\in\Lambda_{2,l}$, $l\geq 1$, *we have*

$$M^{(i)}_{n,J_1,J_2}=2^{-l/2}\sum_{J_2'\in\{1,2\}^l}(-1)^{(J_2|J_2')}L^{(i)}_{n,J_1,J_2'}$$

where $(J_2|J_2')$ *is in Definition* 1.2.

(ii) *For* $i=1,2$, $n\in\mathbf{Z}$, $J_1\in\{1,2\}^*$ *and* $J_2\in\Lambda_2$, $M^{(i)}_{n,J_1,J_2}\in L_2\left(A(X^{(i)}_{n,J_1})\right)$.

Proof. (i) By (5.8), we have

$$M^{(i)}_{n,J_1,J_2}=\ 2^{-l/2}\sum_{J_2'\in\{1,2\}^l}(-1)^{(J_2|J_2')}\pi_0(s_{J_2'})L^{(i)}_{n+l,J_1,0}$$

where we denote $s_J\equiv s_{j_1}\cdots s_{j_k}$, $s_J^*\equiv s_{j_k}^*\cdots s_{j_1}^*$, $s_0\equiv I$ for $J=(j_1,\ldots,j_k)\in\{1,2\}^*$. By Lemma 5.6 (iii), the assertion holds.

(ii) Since $M^{(i)}_{n,J_1,J_2}$ is the image of the isometry $\pi_0(T_{J_2})$ from $L^{(i)}_{n,J_1,0}\in L_2(\mathbf{C})$, $M^{(i)}_{n,J_1,J_2}\in L_2(\mathbf{C})$. By Lemma 5.6 (i) and (5.4), we have

$$M^{(i)}_{n,J_1,J_2}\in\bigoplus_{J_2'\in\{1,2\}^k}L_2\left(AB(X^{(i)}_{n,J_1},Y_{J_2'})\right)\subset L_2\left(A(X^{(i)}_{n,J_1})\right)$$

when $J_2\in\Lambda_{2,k}$. □

Lemma 5.10. $\{M^{(i)}_{n,J_1,J_2}:n\in\mathbf{Z},\,J_2\in\Lambda_2\}$ *is a complete orthonormal basis of* $\mathcal{H}^{(i)}_{J_1}$ *for* $i=1,2$ *and* $J_1\in\{1,2\}^*$.

Proof. Fix $i=1,2$ and $J_1\in\{1,2\}^*$. By Lemma 5.4 (ii) and Lemma 5.9 (ii),

$$<M^{(i)}_{n,J_1,J_2}|M^{(i)}_{m,J_1,J_2'}>=\delta_{n,m}<\pi_0(T_{J_2})L^{(i)}_{n+|J_2|,J_1,0}|\pi_0(T_{J_2'})L^{(i)}_{n+|J_2'|,J_1,0}>.$$

If $|J_2| = |J_2'|$, then $< M^{(i)}_{n,J_1,J_2} | M^{(i)}_{n,J_1,J_2'} >= \delta_{J_2 J_2'}$ by Lemma 5.6 (ii). Assume $J_2 = (j_1, \ldots, j_{k+l})$ and $J_2' = (j_1', \ldots, j_k')$, $l \geq 1$. Then

$$< M^{(i)}_{n,J_1,J_2} | M^{(i)}_{n,J_1,J_2'} >= \delta_{J_{2,1} J_2'} < M^{(i)}_{n+k,J_1,J_{2,2}} | L^{(i)}_{n+k,J_1,0} >$$

where $J_{2,1} \equiv (j_1, \ldots, j_k)$ and $J_{2,2} \equiv (j_{k+1}, \ldots, j_{k+l})$. By Lemma 5.9 (i),

$$< M^{(i)}_{n+k,J_1,J_{2,2}} | L^{(i)}_{n+k,J_1,0} >= 2^{-l/2} \sum_{I_2 \in \{1,2\}^l} (-1)^{(J_{2,2}|I_2)} < L^{(i)}_{n+k,J_1,I_2} | L^{(i)}_{n+k,J_1,0} > .$$

By definition,

$$< L^{(i)}_{n+k,J_1,I_2} | L^{(i)}_{n+k,J_1,0} >= \omega_{n+k,k+l,k} \cdot \omega_{n+k,k+l,0} \cdot \mathcal{I}\left(AB(X^{(i)}_{n+k,J_1}, Y_{I_2}) \right).$$

By Lemma 5.4 (i),

$$< M^{(i)}_{n+k,J_1,J_{2,2}} | L^{(i)}_{n+k,J_1,0} >= \sum_{I_2 \in \{1,2\}^l} (-1)^{(J_{2,2}|I_2)}. \tag{5.9}$$

By choice of J_2 and Lemma A.1 (ii), the rhs in (5.9) equals 0 when $l \geq 1$. Hence $< M^{(i)}_{n,J_1,J_2} | M^{(i)}_{n,J_1,J_2'} >= 0$ when $|J_2| \neq |J_2'|$. From these considerations, we have

$$< M^{(i)}_{n,J_1,J_2} | M^{(i)}_{m,J_1,J_2'} >= \delta_{nm} \delta_{J_2 J_2'} \quad (n, m \in \mathbf{Z},\ J_2, J_2' \in \Lambda_2).$$

By Lemma 5.8 and (5.8),

$$\mathcal{H}^{(i)}_{J_1} \supset \{\pi_0(T_{J_2}) L^{(i)}_{n,J_1,0} : n \in \mathbf{Z},\ J_2 \in \Lambda_2\} = \{M^{(i)}_{n,J_1,J_2} : n \in \mathbf{Z},\ J_2 \in \Lambda_2\}.$$

Specially, $M^{(i)}_{n,J_1,1} = L^{(i)}_{n,J_1,0}$. Therefore

$$\pi_0(T_1) M^{(i)}_{n,J_1,1} = M^{(i)}_{n-1,J_1,1} \tag{5.10}$$

for each $n \in \mathbf{Z}$ by Lemma 5.7 (i). From these, $M^{(i)}_{0,J_1,1}$ is a cyclic unit vector of $\mathcal{H}^{(i)}_{J_1}$ by $\pi_0(\mathcal{O}_2)$. By Lemma 5.8, $(\mathcal{H}^{(i)}_{J_1}, \pi_0|_{\mathcal{H}^{(i)}_{J_1}}, M^{(i)}_{0,J_1,1})$ is the $P_{B,1}$-chain of $\mathcal{O}_2$ in Definition 2.1 and (2.2). By Proposition 2.3, $\{M^{(i)}_{n,J_1,J_2} : n \in \mathbf{Z},\ J_2 \in \Lambda_2\}$ is complete in $\mathcal{H}^{(i)}_{J_1}$. □

Lemma 5.11. (i) *For $i = 1, 2$, $n \in \mathbf{Z}$, $J_1 \in \{1,2\}^*$ and $J_2 \in \Lambda_2$,*

$$M^{(i)}_{n,J_1,J_2} = 2^{-1/2} (M^{(i)}_{n,J_1 \cup \{1\},J_2} + M^{(i)}_{n,J_1 \cup \{2\},J_2}). \tag{5.11}$$

(ii) *For $i, j = 1, 2$, $n, m \in \mathbf{Z}$, $J_1, J_1', J_3 \in \{1,2\}^*$, $|J_1| = |J_1'|$, and $J_2, J_2' \in \Lambda_2$, we have*

$$< M^{(i)}_{n,J_1,J_2} | M^{(j)}_{m,J_1' \cup J_3,J_2'} >= \delta_{ij} \delta_{nm} \delta_{J_2 J_2'} \delta_{J_1 J_1'} 2^{-|J_3|/2}.$$

Proof. (i) By (5.8), the rhs of (5.11) equals to

$$2^{-1/2}2^{-k/2}\sum_{J_2'\in\{1,2\}^k}(-1)^{(J_2|J_2')}\left(L^{(i)}_{n,J_1\cup\{1\},J_2'}+L^{(i)}_{n,J_1\cup\{2\},J_2'}\right)$$

when $J_2\in\Lambda_{2,k}, k\geq 1$. By Lemma 5.7 (iii), this equals to

$$2^{-k/2}\sum_{J_2'\in\{1,2\}^k}(-1)^{(J_2|J_2')}L^{(i)}_{n,J_1,J_2'}=M^{(i)}_{n,J_1,J_2}.$$

Hence the statement holds.
(ii) By Lemma 5.9 (ii),

$$< M^{(i)}_{n,J_1,J_2}|M^{(j)}_{m,J_1'\cup J_3,J_2'} >= \delta_{ij}\delta_{nm} < M^{(i)}_{n,J_1,J_2}|M^{(i)}_{n,J_1'\cup J_3,J_2'} > .$$

Fix $i=1,2$ and $n\in\mathbf{Z}$. Since $X^{(i)}_{n,J_1}\cap X^{(i)}_{n,J_1'\cup J_3}$ is a null set when $J_1'\neq J_1$, $< M^{(i)}_{n,J_1,J_2}|M^{(j)}_{m,J_1'\cup J_3,J_2'} >=0$ when $J_1'\neq J_1$.

Fix $J_1\in\{1,2\}^*$, too.

Assume $\underline{l\equiv |J_2'|-|J_2|\geq 1}$. Put $J_2'=J_{2,1}'\cup J_{2,2}'$, $J_{2,1}'=(j_1',\ldots,j_k')$. Then we have

$$< M^{(i)}_{n,J_1,J_2}|M^{(i)}_{n,J_1\cup J_3,J_2'} >= \delta_{J_2J_{2,1}'} < L^{(i)}_{n+k,J_1,0}|M^{(i)}_{n+k,J_1\cup J_3,J_{2,2}'} > .$$

By Lemma 5.9 (i),

$$< L^{(i)}_{n+k,J_1,0}|M^{(i)}_{n+k,J_1\cup J_3,J_{2,2}'} >= 2^{-l}\sum_{J_4\in\{1,2\}^l}(-1)^{(J_{2,2}'|J_4)}\cdot c_{k,k+l,J_3,J_4}$$

where $c_{k,k+l,J_3,J_4}\equiv < L^{(i)}_{n+k,J_1,0}|L^{(i)}_{n+k,J_1\cup J_3,J_4} >$. Then

$$c_{k,k+l,J_3,J_4}=\omega_{n+k,|J_1|,0}\cdot\omega_{n+k,|J_1\cup J_3|,l}\cdot\mathcal{I}\left(AB\left(X^{(i)}_{n+k,J_1\cup J_3},Y_{J_4}\right)\right).$$

By Lemma 5.4 (i), $\mathcal{I}\left(AB\left(X^{(i)}_{n+k,J_1\cup J_3},Y_{J_4}\right)\right)=2^{n+k-|J_1\cup J_3|-l}\pi\log 2$. From this, we can write

$$< L^{(i)}_{n+k,J_1,0}|M^{(i)}_{n+k,J_1\cup J_3,J_{2,2}'} >= W\cdot\sum_{J_4\in\{1,2\}^l}(-1)^{(J_{2,2}'|J_4)}$$

for some constant W. By choice of J_2', $J_{2,2}\neq(\underbrace{1,\ldots,1}_{l})$. By Lemma A.1 (ii), $< L^{(i)}_{n+k,J_1,0}|M^{(i)}_{n+k,J_1\cup J_3,J_{2,2}'} >=0$. Hence $< M^{(i)}_{n,J_1,J_2}|M^{(i)}_{n,J_1\cup J_3,J_2'} >=0$

Assume $\underline{k = |J_2| = |J_2'|}$. Then $< M^{(i)}_{n,J_1,J_2} | M^{(i)}_{n,J_1\cup J_3,J_2'} >= \delta_{J_2J_2'} < L^{(i)}_{n+k,J_1,0} | L^{(i)}_{n+k,J_1\cup J_3,0} >$ and

$$< L^{(i)}_{n+k,J_1,0} | L^{(i)}_{n+k,J_1\cup J_3,0} >= W' \cdot \mathcal{I}\left(A(X^{(i)}_{n+k,J_1\cup J_3}, Y_0)\right)$$

where $W' \equiv \omega_{n+k,|J_1|,0} \cdot \omega_{n+k,|J_1\cup J_3|,0} = 2^{-(n+k-|J_1|)+|J_3|/2}(\pi \log 2)^{-1}$. Hence $< L^{(i)}_{n+k,J_1,0} | L^{(i)}_{n+k,J_1\cup J_3,0} >= 2^{-|J_3|/2}$. Therefore $< M^{(i)}_{n,J_1,J_2} | M^{(i)}_{n,J_1\cup J_3,J_2'} >= 2^{-|J_3|/2}\delta_{J_2J_2'}$.

Regarding every case, we have the assertion. □

5.3. Commuting two representations of $\mathcal{O}_2$. Here we construct another representation of $\mathcal{O}_2$ which commutes π_0 in (1.2). By using this, we decompose $(L_2(\mathbf{C}), \pi_0)$.

Lemma 5.12. *Put a closed subspace of $L_2(\mathbf{C})$*
(5.12)

$$\mathcal{K}(M_*) \equiv \overline{\mathrm{Lin} < \{M^{(i)}_{n,J_1,J_2} : i = 1,2,\ n \in \mathbf{Z},\ J_1 \in \{1,2\}^*,\ J_2 \in \Lambda_2\} >}$$

and a function

$$N^{(i)}_{n,J_1,J_2} \equiv 2^{-k/2} \sum_{J_1' \in \{1,2\}^k} (-1)^{(J_1|J_1')} M^{(i)}_{n,J_1',J_2} \tag{5.13}$$

for $i = 1,2$, $n \in \mathbf{Z}$ and $J_1 \in \{1,2\}^k, J_2 \in \Lambda_2$, $k \geq 1$. Then the followings hold:

(i) $N^{(i)}_{n,J_1\cup\{1\},J_2} = N^{(i)}_{n,J_1,J_2}$ *for each $J_1 \in \{1,2\}^* \setminus \{0\}$.*
(ii) $\mathcal{K}(M_*) = L_2(\mathbf{C})$.

Proof. (i) For $i = 1,2$, $n \in \mathbf{Z}$, $J_1 \in \{1,2\}^k$, $J_2 \in \Lambda_2$, we have

$$\begin{aligned} N^{(i)}_{n,J_1\cup\{1\},J_2} &= 2^{-(k+1)/2} \sum_{I\in\{1,2\}^k} (-1)^{(J_1|I)}(M^{(i)}_{n,I\cup\{1\},J_2} + M^{(i)}_{n,I\cup\{2\},J_2}) \\ &= 2^{-(k+1)/2} \sum_{I\in\{1,2\}^k} (-1)^{(J_1|I)}\sqrt{2} M^{(i)}_{n,I,J_2} \\ &= N^{(i)}_{n,J_1,J_2} \end{aligned}$$

where we use Lemma 5.11 (i).
(ii) By Lemma 5.9 and (5.7), $\mathcal{K}(M_*) \supset \{L^{(i)}_{n,J_1,J_2} : i = 1,2,\ n \in \mathbf{Z},\ J_1, J_2 \in \{1,2\}^*\}$. By (5.4), $\mathrm{Lin} < \{L^{(i)}_{n,J_1,J_2} : i = 1,2,\ n \in \mathbf{Z},\ J_1, J_2 \in \{1,2\}^*\} >$ is dense in $L_2(\mathbf{C})$. Hence the assertion holds. □

Lemma 5.13. *$\{N^{(i)}_{n,J_1,J_2} : J_1, J_2 \in \Lambda_2\}$ is an orthonormal family of $L_2\left(A(X^{(i)}_{n,0})\right)$ for each $i = 1, 2$ and $n \in \mathbf{Z}$.*

Proof. Fix $i = 1, 2$ and $n \in \mathbf{Z}$. By Lemma 5.9 (ii), $N^{(i)}_{n,J_1,J_2} \in L_2(A(X^{(i)}_{n,0}))$. By Lemma 5.11 (ii), $< N^{(i)}_{n,J_1,J_2}|N^{(i)}_{n,J_1',J_2'} >= 0$ when $J_2 \neq J_2'$. Fix $J_2 \in \Lambda_2$, too. Assume $k_1 = |J_1|$ and $k_2 = |J_2|$. Then

$$C_{J_1,J_1'} \equiv < N^{(i)}_{n,J_1,J_2}|N^{(i)}_{n,J_1',J_2} >$$

$$= 2^{-(k_1+k_2)/2} \sum_{J_3\in\{1,2\}^{k_1}} \sum_{J_3'\in\{1,2\}^{k_2}} (-1)^{(J_1|J_3)+(J_1'|J_3')} < M^{(i)}_{n,J_3,J_2}|M^{(i)}_{n,J_3',J_2} > .$$

Assume $\underline{l \equiv k_1 - k_2 \geq 1}$. Then $< M^{(i)}_{n,J_3,J_2}|M^{(i)}_{n,J_3',J_2} >= 0$ when $J_3 \neq J_3' \cup *$. Assume that there is $J_4 \in \{1, 2\}^l$ such that $J_3 = J_3' \cup J_4$. By Lemma A.1 (i) and $< M^{(i)}_{n,J_3'\cup J_4,J_2}|M^{(i)}_{n,J_3',J_2} >= 2^{-l/2}$, we have

$$\begin{aligned} C_{J_1,J_1'} &= 2^{-(k_1+l)} \sum_{J_4\in\{1,2\}^l} \sum_{J_3'\in\{1,2\}^{k_2}} (-1)^{(J_{1,2}|J_4)+(J_{1,1}|J_3')+(J_1'|J_3')} \\ &= 2^{-l}\delta_{J_{1,1}J_1'} \sum_{J_4\in\{1,2\}^l} (-1)^{(J_{1,2}|J_4)} \end{aligned}$$

where $J_1 = J_{1,1} \cup J_{1,2}$, $|J_{1,1}| = k_2$, $|J_{1,2}| = l$. By choice of J_1, $J_{1,2} \neq (1^l)$. By Lemma A.1 (ii), $C_{J_1,J_1'} = 0$.

Assume $\underline{k \equiv |J_1| = |J_1'|}$. Then

$$C_{J_1,J_1'} = 2^{-k} \sum_{J_3,J_3'\in\{1,2\}^k} (-1)^{(J_1|J_3)+(J_1'|J_3')} < M^{(i)}_{n,J_3,J_2}|M^{(i)}_{n,J_3',J_2} > .$$

By Lemma 5.9 (ii) and Lemma 5.11 (ii),

$$C_{J_1,J_1'} = 2^{-|J_1|} \sum_{J_3\in\{1,2\}^k} (-1)^{(J_1|J_3)+(J_1'|J_3)}.$$

By Lemma A.1 (i), $C_{J_1,J_1'} = \delta_{J_1J_1'}$.

In consequence $< N^{(i)}_{n,J_1,J_2}|N^{(i)}_{n,J_1',J_2} >= \delta_{J_1J_1'}$ for each $J_1, J_1' \in \Lambda_2$. Therefore $\{N^{(i)}_{n,J_1,J_2} : J_1, J_2 \in \Lambda_2\}$ is an orthonormal family of $L_2(A(X^{(i)}_{n,0}))$. □

Corollary 5.14. *$\{N^{(i)}_{n,J_1,J_2} : i = 1, 2,\ n \in \mathbf{Z},\ J_1, J_2 \in \Lambda_2\}$ is an orthonormal family of $L_2(\mathbf{C})$.*

Proposition 5.15. *Put*

$$\mathcal{K}(N_*) \equiv \overline{\mathrm{Lin} < \{N^{(i)}_{n,J_1,J_2} : i = 1,2,\, n \in \mathbf{Z},\, J_1, J_2 \in \Lambda_2\} >},$$

$$\pi_1(s_j)N^{(i)}_{n,1,J_2} \equiv N^{(i)}_{n,j,J_2}, \quad \pi_1(s_j)N^{(i)}_{n,J_1,J_2} \equiv N^{(i)}_{n,\{j\}\cup J_1,J_2} \tag{5.14}$$

for $i, j = 1, 2$, $n \in \mathbf{Z}$, $J_1 \in \Lambda_2 \setminus \{1\}, J_2 \in \Lambda_2$. *Then* $(\mathcal{K}(N_*), \pi_1)$ *is a representation of* $\mathcal{O}_2$.

Proof. Because $\pi_1(s_i)$ is defined on the complete orthonormal basis $\{N^{(i)}_{n,J_1,J_2}\}$ of $\mathcal{K}(N_*)$, $\pi_1(s_i)$ is well defined and it is an isometry for $i = 1, 2$. By (5.14) and Corollary 5.14, $< \pi_1(s_j)N^{(i)}_{n,J_1,J_2} | \pi_1(s_{j'})N^{(i')}_{m,J_1',J_2'} >= \delta_{jj'}\delta_{(n,J_1,J_2,i),(m,J_1',J_2',i')}$. Therefore $\pi_1(s_i)^*\pi_1(s_j) = \delta_{ij}I$. By checking the image of $\pi_1(s_i)$, $\pi_1(s_1)\pi_1(s_1)^* + \pi_1(s_2)\pi_1(s_2)^* = I$ holds. Therefore $(\mathcal{K}(N_*), \pi_1)$ is a representation of $\mathcal{O}_2$. □

Lemma 5.16. *For* T_i, $i = 0, 1, 2$ *in Lemma* 5.9, *we have*

$$\pi_1(T_J)N^{(i)}_{n,1,J_2} = M^{(i)}_{n,J,J_2} \quad (i = 1, 2,\, n \in \mathbf{Z},\, J \in \{1,2\}^*,\, J_2 \in \Lambda_2).$$

Proof. Since $N^{(i)}_{n,1,J_2} = M^{(i)}_{n,0,J_2}$, the case $J = \{0\}$ holds. If $J \in \{1,2\}^k$, then

$$\begin{aligned}\pi_1(T_J)N^{(i)}_{n,1,J_2} &= 2^{-k/2} \sum_{I\in\{1,2\}^k} (-1)^{(I|J)} \pi_1(s_I) N^{(i)}_{n,1,J_2} \\ &= 2^{-k/2} \sum_{I\in\{1,2\}^k} (-1)^{(I|J)} N^{(i)}_{n,I,J_2}\end{aligned}$$

where we use $\pi_1(s_I)N^{(i)}_{n,1,J_2} = \pi_1(s_{i_1})\cdots\pi_1(s_{i_{k-1}})N^{(i)}_{n,i_k,J_2} = N^{(i)}_{n,I,J_2}$ when $I = (i_1, \ldots, i_k)$. By (5.13),

$$\begin{aligned}\pi_1(T_J)N^{(i)}_{n,1,J_2} &= 2^{-k} \sum_{I\in\{1,2\}^k} \sum_{I'\in\{1,2\}^k} (-1)^{(I|J)+(I|I')} M^{(i)}_{n,I',J_2} \\ &= 2^{-k} \sum_{I'\in\{1,2\}^k} \left(\sum_{I\in\{1,2\}^k} (-1)^{(I|J)+(I|I')} \right) M^{(i)}_{n,I',J_2} \\ &= \sum_{I'\in\{1,2\}^k} \delta_{I'J} M^{(i)}_{n,I',J_2} \\ &= M^{(i)}_{n,J,J_2}\end{aligned}$$

where we use Lemma A.1 (i). □

Proposition 5.17. *$\{N^{(i)}_{n,J_1,J_2} : i = 1,2,\, n \in \mathbf{Z},\, J_1, J_2 \in \Lambda_2\}$ is a complete orthonormal basis of $L_2(\mathbf{C})$*

Proof. By Lemma 5.16 and Proposition 5.15, $M^{(i)}_{n,J_1,J_2} \in \mathcal{K}(N_*)$ for each $i = 1,2$, $n \in \mathbf{Z}$, $J_1 \in \{1,2\}^*$ and $J_2 \in \Lambda_2$. Recall $\mathcal{K}(M_*)$ in (5.12). By Lemma 5.12, $L_2(\mathbf{C}) = \mathcal{K}(M_*) \subset \mathcal{K}(N_*) \subset L_2(\mathbf{C})$. Hence $L_2(\mathbf{C}) = \mathcal{K}(N_*)$. By Corollary 5.14, the statement holds. □

Lemma 5.18. *Let T_i, $i = 1,2$ be in Lemma 5.9. For $i, j = 1,2$, $n \in \mathbf{Z}$ and $J_1, J_2 \in \Lambda_2$, $J_2 \neq \{1\}$, we have*

$$\pi_0(T_j)N^{(i)}_{n,J_1,1} = N^{(i)}_{n-1,J_1,j}, \quad \pi_0(T_j)N^{(i)}_{n,J_1,J_2} = N^{(i)}_{n-1,J_1,\{j\}\cup J_2}.$$

Proof. By (5.10) and Definition of $M^{(i)}_{n,J_1,J_2}$, $\pi_0(T_j)M^{(i)}_{n,J_1',1} = M^{(i)}_{n-1,J_1',j}$. Hence

$$\pi_0(T_j)N^{(i)}_{n,J_1,1} = \; 2^{-k/2} \sum_{J_1' \in \{1,2\}^k} (-1)^{(J_1|J_1')} \pi_0(T_j) M^{(i)}_{n,J_1',1} = N^{(i)}_{n-1,J_1,j}.$$

In the same way,

$$\pi_0(T_j)N^{(i)}_{n,J_1,J_2} = 2^{-k/2} \sum_{J_1' \in \{1,2\}^k} (-1)^{(J_1|J_1')} M^{(i)}_{n-1,J_1',\{j\}\cup J_2} = N^{(i)}_{n-1,J_1,\{j\}\cup J_2}$$

for $J_2 \neq \{1\}$. □

Theorem 5.19. (i) *We have the following decomposition of invariant subspaces under the action π_1 of $\mathcal{O}_2$:*

$$L_2(\mathbf{C}) = \bigoplus_{i=1,2} \bigoplus_{n\in\mathbf{Z}} \bigoplus_{J_2\in\Lambda_2} \mathcal{K}^{(i)}_{n,J_2}, \quad \mathcal{K}^{(i)}_{n,J_2} \equiv \overline{\mathrm{Lin} < \{N^{(i)}_{n,J_1,J_2} : J_1 \in \Lambda_2\} >}.$$

Furthermore $(\mathcal{K}^{(i)}_{n,J_2}, \pi_1|_{\mathcal{K}^{(i)}_{n,J_2}})$ is the P_S-cycle of $\mathcal{O}_2$.

(ii) *We have the following decomposition of invariant subspaces under the action π_0 of $\mathcal{O}_2$:*

$$L_2(\mathbf{C}) = \bigoplus_{i=1,2} \bigoplus_{J_1\in\Lambda_2} \mathcal{L}^{(i)}_{J_1}, \quad \mathcal{L}^{(i)}_{J_1} \equiv \overline{\mathrm{Lin} < \{N^{(i)}_{n,J_1,J_2} : n \in \mathbf{Z},\, J_2 \in \Lambda_2\} >}.$$

Furthermore $(\mathcal{L}^{(i)}_{J_1}, \pi_0|_{\mathcal{L}^{(i)}_{J_1}})$ is the $P_{B,1}$-chain of $\mathcal{O}_2$.

(iii) $\pi_1(\mathcal{O}_2) \subset (\pi_0(\mathcal{O}_2))'$.

Proof. (i) By (5.14), $\pi_1(s_1)N^{(i)}_{n,1,J_2} = N^{(i)}_{n,1,J_2}$ for each $i = 1, 2$, $n \in \mathbf{Z}$ and $J_2 \in \Lambda_2$. By Proposition 5.17, we have the statement.
(ii) Note $T_1 = P_{B,1}$. By Lemma 5.18, we have $\pi_0(T_1)N_{n,J_1,1} = N_{n-1,J_1,1}$ for each $i = 1, 2$, $n \in \mathbf{Z}$ and $J_1 \in \Lambda_2$. By Proposition 5.17 and definition of $\mathcal{L}^{(i)}_{J_1}$, we have the statement.
(iii) Put $\pi_3 \equiv \pi_0 \circ \alpha_g$ for $g \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then $\pi_3(s_i) = \pi_0(T_i)$ for $i = 1, 2$. By Lemma 5.18 and (5.14), we have the followings:

$$\pi_3(s_i)\pi_1(s_j)N^{(k)}_{n,1,J_2} = N^{(k)}_{n-1,j,\{i\}\cup J_2} = \pi_1(s_j)\pi_3(s_i)N^{(k)}_{n,1,J_2},$$

$$\pi_3(s_i)\pi_1(s_j)N^{(k)}_{n,J_1,J_2} = N^{(k)}_{n-1,\{j\}\cup J_1,\{i\}\cup J_2} = \pi_1(s_j)\pi_3(s_i)N^{(k)}_{n,J_1,J_2},$$

$$\pi_3(s_i)\pi_1(s_j)^*N^{(k)}_{n,\{l\}\cup J_1,J_2} = \delta_{jl}N^{(k)}_{n-1,J_1,\{i\}\cup J_2} = \pi_1(s_j)^*\pi_3(s_i)N^{(k)}_{n,\{l\}\cup J_1,J_2},$$

$$\pi_3(s_i)\pi_1(s_j)^*N^{(k)}_{n,1,J_2} = \delta_{j1}N^{(k)}_{n-1,1,\{i\}\cup J_2} = \pi_1(s_j)^*\pi_3(s_i)N^{(k)}_{n,1,J_2},$$

$$\pi_3(s_i)\pi_1(s_j)^*N^{(k)}_{n,2,J_2} = \delta_{j2}N^{(k)}_{n-1,1,\{i\}\cup J_2} = \pi_1(s_j)^*\pi_3(s_i)N^{(k)}_{n,2,J_2}.$$

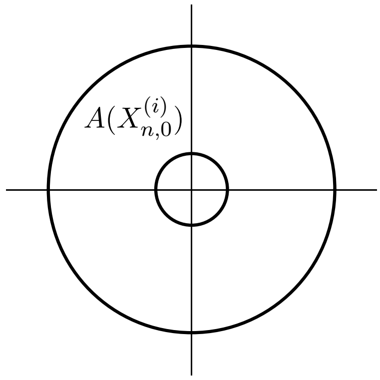
From these relations, we have $[\pi_3(s_i), \pi_1(s_j)] = 0$, $[\pi_3(s_i)^*, \pi_1(s_j)] = 0$ for $i, j = 1, 2$. Therefore $[\pi_0(s_i), \pi_1(s_j)] = 0$, $[\pi_0(s_i)^*, \pi_1(s_j)] = 0$ for $i, j = 1, 2$, too. Hence $[\pi_0(x), \pi_1(y)] = 0$ for each $x, y \in \mathcal{O}_2$. □

Proof of Theorem 1.3*:* Put $A^{(i)}_{n,J_1} \equiv A(X^{(i)}_{n,J_1})$ and $AB^{(i)}_{n,J_1,J_2} \equiv AB(X^{(i)}_{n,J_1}, Y_{J_2})$.
(i) By Lemma 5.3 (ii), it follows.
(ii) We see that $N^{(i)}_{n,J_1,J_2}$ in (5.13) is just that in Theorem 1.3 (ii). By Proposition 5.17 and Lemma 5.18, the assertion follows.
(iii) By the first paragraph in § 4 and Proposition 3.8 (ii), $L_2(\mathbf{C})$ is decomposed into $L_2(D_1)$ and $L_2(D_2)$ as representation of $\mathcal{O}_2$. By Theorem 5.19, π_1-action of $\mathcal{O}_2$ decomposes $L_2(D_i)$ with respect to the index set Λ_2. By Lemma 5.18, $(\mathcal{L}^{(i)}_{J_1}, \pi_0, N^{(i)}_{0,J_1,1})$ is the $P_{B,1}$-chain for each $i = 1, 2$ and $J_1 \in \Lambda_2$. Hence its direct integral decomposition follows by Proposition 2.5 (ii).
(iv) Because $\mathcal{L}^{(i)}_{J_1}$ is equivalent to the $P_{B,1}$-chain, it holds.
(v) By the proof of (iii) and decomposition of them, $\mathcal{L}^{(i)}_{J_1,w}$ is equivalent to the $P_{B,w}$-cycle. Hence the statement follows.
(vi) By the proof of (iv), it holds.
(vii) By Proposition 2.3, it follows. □

We call $\{N^{(i)}_{n,J_1,J_2} : i = 1, 2,\ n \in \mathbf{Z},\ J_1, J_2 \in \Lambda_2\}$ the *annular basis* of $L_2(\mathbf{C})$. By Lemma 4.3, $\pi_0 = \pi_q$ is naturally arising from the dynamical system $(\mathbf{C}, Q)$. In this sense and Lemma 5.18, we see that the annular basis of $L_2(\mathbf{C})$ is arising from $(\mathbf{C}, Q)$ naturally.

5.4. Illustration of annular basis. We illustrate the annular basis of $L_2(\mathbf{C})$ by figures.

Consider an annulus $A(X_{n,0}^{(i)})$ in § 5.1:



Then $\{N_{n,J_1,J_2}^{(i)} : J_1, J_2 \in \Lambda_2\}$ is a complete orthonormal basis of $L_2\left(A(X_{n,0}^{(i)})\right)$. For example,

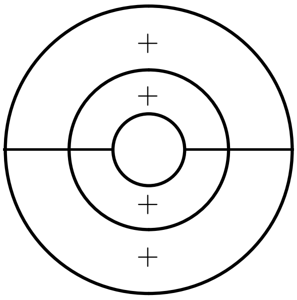
$$N_{n,1,1}^{(i)}(z) = b_n(z)\cdot\chi_{A(X_{n,0}^{(i)})}(z), \quad N_{n,2,1}^{(i)}(z) = b_n(z)\left(\chi_{A(X_{n,1}^{(i)})}(z) - \chi_{A(X_{n,2}^{(i)})}(z)\right),$$

$$N_{n,1,2}^{(i)}(z) = b_n(z)\left(\chi_{AB(X_{n,0}^{(i)},Y_1)}(z) - \chi_{AB(X_{n,0}^{(i)},Y_2)}(z)\right),$$

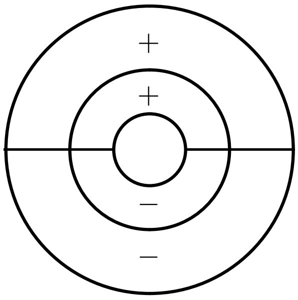
$$N_{n,2,2}^{(i)}(z) = b_n(z)\sum_{j_1,j_2=1,2}(-1)^{j_1+j_2-2}\chi_{AB(X_{n,j_1}^{(i)},Y_{j_2})}(z)$$

where $b_n(z) \equiv (|z|\sqrt{2^n\pi\log 2})^{-1}$ for $n \in \mathbf{Z}$, $z \in \mathbf{C}$. These are illustrated as follows:

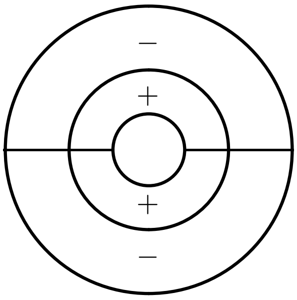
$N^{(i)}_{n,1,1}$



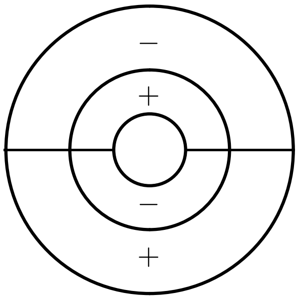
$N^{(i)}_{n,1,2}$



$N^{(i)}_{n,2,1}$

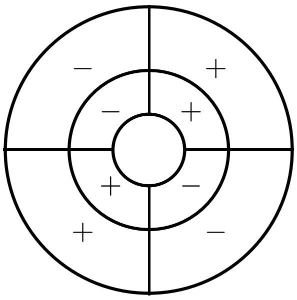


$N^{(i)}_{n,2,2}$

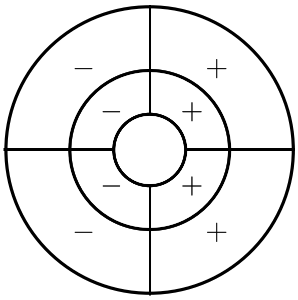


In the same way, we have the following illustration:

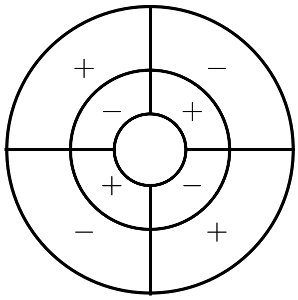
$N^{(i)}_{n,1,12}$



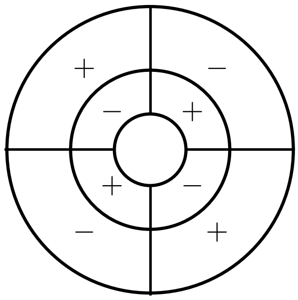
$N^{(i)}_{n,1,22}$



$N^{(i)}_{n,2,12}$



$N^{(i)}_{n,2,22}$



6. Generalization

6.1. $Q_c(z) \equiv z^2 + c$ **for general** $c \in \mathbf{C}$**.** We start from general form of quadratic transformations on $\mathbf{C}$. Consider a transformation $F_{a,b,c}$ over $\mathbf{C}$ defined by

$$F_{a,b,c}(z) \equiv az^2 + bz + c \tag{6.1}$$

for $a, b, c \in \mathbf{C}$, $a \neq 0$. $F_{a,b,c}$ is injective over subsets $X_i(a, b, c)$ of $\mathbf{C}$ which are defined by $X_i(a, b, c) \equiv \left\{ z - b/(2a) : (-1)^i \cdot \mathrm{Im} z \leq 0 \right\}$ for $i = 1, 2$. The representation $(L_2(\mathbf{C}), \pi)$ of $\mathcal{O}_2$ arising from $F_{a,b,c}$ is given by

$$(\pi(s_i)\phi)(z) \equiv m_i(z)\phi(F_{a,b,c}(z)) \quad (\phi \in L_2(\mathbf{C})) \tag{6.2}$$

where $m_i(z) \equiv \chi_{X_i(a,b,c)}(z)|2az+b|$ for $i = 1, 2$ and $z \in \mathbf{C}$. By conjugation of complex affine transformations, the following transformation are conjugate with $Q(z) = z^2$:

$$\frac{1}{a} z^2, \quad (z - b)^2 + b^2, \quad z^2 - 2z + 2, \quad z^2 + 2z, \quad 2(z - b)^2 + b, \quad 2z^2 + 2z$$

for $a, b \in \mathbf{C}$, $a \neq 0$. Since the transformation $F_{a,b,c}$ in (6.1) is always conjugate with $Q_c(z) \equiv z^2 + c$ by affine transformation for some $c \in \mathbf{C}$, the representation in (6.2) is equivalent to that from Q_c by Lemma 3.6. In the future, we wish try to treat the representation arising from Q_c for $c \neq 0$.

By § 18.5 in [**7**], we have representations of $\mathcal{O}_2$ from the following transformations on $\mathbf{C}$ except null sets:

$$z^2 - 2, \quad \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \frac{1}{2}\left(z - \frac{1}{z}\right), \quad z - \frac{(z - a)(z - b)}{2z - a - b}, \quad \bar{z}^2$$

for $a, b \in \mathbf{C}$. Every representations associated with these transformations are equivalent to the case $Q(z) \equiv z^2$.

6.2. $\mathcal{O}_N$ **case.** Let $P_N(z) \equiv z^N$, $z \in \mathbf{C}$, $N \geq 3$. Then the polar decomposition of P_N, $z = z(r,\theta) = re^{2\pi\sqrt{-1}\theta}$, is given by

$$P_N(r,\theta) = (P_{N,R}(r), H_N(\theta)),$$
$$P_{N,R}(r) \equiv r^N, \quad H_N(\theta) \equiv N\theta \pmod 1$$

for $0 \leq r$ and $0 \leq \theta < 1$. According to the similar argument in $N = 2$, we have a representation $(L_2(\mathbf{C}), \pi)$ of $\mathcal{O}_N$ from P_N and their decomposition holds:

$$(L_2(\mathbf{C}), \pi) \sim \int_{U(1)}^{\oplus} (\mathcal{H}_{\bar{w}}, \hat{\pi}_{B,\bar{w}})\, d\eta(w)$$

where

$$\mathcal{H}_w \equiv L_2[0,1] \otimes L_2[0,1], \quad \hat{\pi}_{B,w}(s_i) \equiv I \otimes \pi_B(ws_i) \quad (i = 1, \ldots, N)$$

for each $w \in U(1)$ and $(L_2[0,1], \pi_B)$ is the barycentric representation of $\mathcal{O}_N$([**13**]).

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Appendix A. Formulae about multi-indices

Recall notations $\{1,2\}^k$, $(J|J')$ in Definition 1.2.

Lemma A.1. (i) *For $J_1, J_2 \in \{1,2\}^k$, $k \geq 1$, we have*

$$\sum_{J \in \{1,2\}^k} (-1)^{(J_1|J)+(J_2|J)} = 2^k \delta_{J_1 J_2}.$$

(ii) *If $J \in \{1,2\}^k$, $k \geq 1$, then*

$$\sum_{J' \in \{1,2\}^k} (-1)^{(J|J')} = 2^k \delta_{J,(1^k)}$$

where $(1^k) \equiv (\underbrace{1, \ldots, 1}_{k})$.

Proof. (i) Denote $J_i = (j_{i,l})_{l=1}^k$ for $i = 1, 2$. By checking the following equation

$$\sum_{J \in \{1,2\}^k} (-1)^{(J_1|J)+(J_2|J)} = \prod_{l=1}^{k} \left\{1 + (-1)^{j_{1,l}+j_{2,l}-2}\right\} = \prod_{l=1}^{k} \left\{2\delta_{j_{1,l},j_{2,l}}\right\},$$

we have the assertion.

(ii) In (i), choose $J_1 = J$ and $J_2 = (1^k)$. Since $((1^k)|J^{'}) = 0$ for each $J^{'} \in \{1,2\}^k$,

$$\sum_{J^{'}\in\{1,2\}^k} (-1)^{(J|J^{'})} = \sum_{J^{'}\in\{1,2\}^k} (-1)^{(J|J^{'})+((1^k)|J^{'})} = 2^k \delta_{J,(1^k)}.$$

□

Appendix B. Proof of Proposition 2.3

The results in § 2 are obtained in [**8**, **9**, **10**]. We show several claims here for convenience. Specially, the uniqueness of P_S,$P_{B,w}$-cycles are shown in Appendix in [**13**, **14**].

Proposition B.1. *$P_S, P_{B,w}$-cycles, are irreducible and inequivalent each other.*

Proof. In § 2.1 and Appendix A in [**12**], $GP(1,0)$ and $GP(2^{-1/2}w, 2^{-1/2}w)$ are just P_S-cycle and $P_{B,w}$-cycle for each $w \in U(1)$, respectively. Hence statements hold. □

In order to show the uniqueness of $P_{B,1}$-chain, we construct the canonical basis of $P_{B,1}$-chain. Put $\Lambda(1^\infty) \equiv \{(J,n) \in \Lambda_2 \times \mathbf{Z}\}$. For $(J^{'}, n) \in \Lambda(1^\infty)$, put $e_{J^{'},n} \equiv \pi(s_{J^{'}} s_1^n) e_1$ where $s_1^n = \underbrace{s_1 \cdots s_1}_{n}$, $s_1^{-n} = (s_1^*)^n$ when $n \geq 1$, $s_1^0 = I$.

Lemma B.2. *Let $(\mathcal{H}, \pi, \Omega)$ be a P_S-chain of $\mathcal{O}_N$. Then $\{e_J : J \in \Lambda(1^\infty)\}$ is a complete orthonormal basis of $\mathcal{H}$.*

Proof. For $(J,n), (J^{'}, m) \in \Lambda(1^\infty)$, if $|J| = |J^{'}|$, then $< e_{J,n}|e_{J^{'},m} >= \delta_{JJ^{'}} < \pi(s_1^n)\Omega|\pi(s_1^m)\Omega >$. Assume that $J = J_1 \cup J_2$, $J_1 = (j_1, \ldots, j_k)$, $J_2 = (j_{k+1}, \ldots, j_{k+l})$, and $J^{'} = (j_1^{'}, \ldots, j_k^{'})$. Then $< e_J|e_{J^{'}} >= \delta_{J_1 J^{'}} < \pi(s_{J_2})\Omega|\Omega >$. Note $\Omega = \pi(s_1)^l e_{1+l}$. Hence $< \pi(s_{J_2})\Omega|\Omega >= \delta_{J_2 J_3} < \Omega|e_{l+1} >= 0$ where $J_3 = (\underbrace{1, \ldots, 1}_{l})$. Therefore $\{e_J : J \in \Lambda(1^\infty)\}$ is an orthonormal family of $\mathcal{H}$. By cyclicity of $\mathcal{H}$, $\{e_J : J \in \Lambda(1^\infty)\}$ is complete. □

Lemma B.3. *Let $f = \{f_i\}_{i=1}^N$ be in Example* 2.2 *(iv).*

(i) $(l_2(\mathbf{Z} \times \mathbf{N}), \pi_f, e_{0,1})$ *is a P_S-chain.*

(ii) *For* $g \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in U(2)$, $(l_2(\mathbf{Z} \times \mathbf{N}), \pi_f \circ \alpha_g, e_{0,1})$ *is a $P_{B,1}$-chain.*

Proof. (i) Note $\{f_J(n,1) : J \in \{1,2\}^*,\, n \in \mathbf{Z}\} = \mathbf{Z} \times \mathbf{N}$. Because $\pi_f(s_1^n)e_{0,1} = e_{n,1}$ for $n \in \mathbf{Z}$, $\{e_x : x \in \mathbf{Z} \times \mathbf{N}\} \subset \pi_f(\mathcal{O}_2)e_{0,1}$. Hence the cyclicity follows. By definition of f and (2.5), we have $\pi_f(s_1)e_{n,1} = e_{n-1,1}$ for $n \in \mathbf{Z}$. By putting $e_n^{'} \equiv e_{n,1}$ for $n \geq 1$, we have the statement. (ii) The cyclicity follows by (i), too. Note $2^{-1/2}\alpha_g(s_1+s_2) = s_1$. From this, we have $2^{-1/2}(\pi_f \circ \alpha_g)(s_1+s_2)e_{n,1} = \pi_f(s_1)e_{n,1} = e_{n-1,1}$ for $n \in \mathbf{Z}$. Hence $\pi_f \circ \alpha_g$ is a $P_{B,1}$-chain. □

Proposition B.4. *If $(\mathcal{H}_i, \pi_i, \Omega_i)$ is a P_S-chain for $i = 1, 2$. Then $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are unitarily equivalent.*

Proof. By Lemma B.2, any P_S-chain has the canonical basis. From this, both $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ have such canonical basis with the common index set $\Lambda(1^\infty)$. By corresponding their basis, define a unitary U between $\mathcal{H}_1$ and $\mathcal{H}_2$. Then U gives a unitary equivalence between $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$. □

Theorem B.5. *The P_S-chain and the $P_{B,1}$-chain are unique up to unitary equivalences.*

Proof. By Proposition B.4, the statements holds about a P_S-chain. By this result and Lemma B.3 (i), we can identify $(l_2(\mathbf{N}), \pi_f, e_{0,1})$ in (2.5) and any P_S-chain. By Lemma B.3 (ii) and the uniqueness of P_S-chain, $P_{B,1}$-chain is unique, too. □

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