Generalized permutative representations of the Cuntz algebras. IV
—Gauge transformation of representations—

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We introduce a gauge transformation of representations of the Cuntz algebra $\mathcal{O}_N$ as a generalization of the canonical $U(N)$-action. We show their orbits and equivalence of representations. As application, we show properties of generalized permutative representations and automorphisms of $\mathcal{O}_N$.

1. Introduction

Nowadays, gauge theory is well established as the fundamental theories of physics([**18**]). The origin of gauge theory was introduced by H.Weyl in the 1920's and it is treated as fiber bundle and geometry of connection by C.N.Yang and K.L.Mills([**17**]). We introduce another aspect of gauge theory in a subject of representation theory of operator algebra. Since the original purpose of gauge theory in physics is the "recipe" of quantization of classical field(electromagnetic field, Higgs, Yang-Mills fields), it seems that the research of gauge theory in operator theory is natural.

It is well known that the canonical $U(1)$-action of the Cuntz algebra $\mathcal{O}_N$ is called the *gauge action*. In order to generalize permutative representations of $\mathcal{O}_N$([**2**, **4**, **5**]), we generalize the gauge action to transformations of representations which are not automorphisms in general.

Let $\mathcal{H}$ be a Hilbert space with a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ and $U_\Lambda(N)$ the group of all maps from Λ to $U(N)$ by pointwise product. For a unital $*$-representation π of $\mathcal{O}_N$ on $\mathcal{H}$ and $g \in U_\Lambda(N)$, we have a new representation π_g defined by

$$\pi_g(s_i)e_n \equiv \pi(\alpha_{g^*(n)}(s_i))e_n \quad (n \in \Lambda) \tag{1.1}$$

for $i = 1, \ldots, N$ where α is the canonical $U(N)$-action on $\mathcal{O}_N$. Denote $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$ the set of all unital $*$-representations of $\mathcal{O}_N$ on $\mathcal{H}$. For $g \in U_\Lambda(N)$ and $\pi \in \mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$, we see that

$$\Gamma_g(\pi) \equiv \pi_g \tag{1.2}$$

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is an action of $U_\Lambda(N)$ on $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$, that is,

$$\Gamma_g \Gamma_h = \Gamma_{gh} \quad (g, h \in U_\Lambda(N)).$$

We call Γ *the gauge action of* $U_\Lambda(N)$ *on* $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$ and $\Gamma_g(\pi)$ is the *gauge transformation of* π.

On the other hand, we introduced generalized permutative(=GP) representations of $\mathcal{O}_N$ in [**8**, **9**, **10**] which are generalization of permutative representation by [**2**, **4**, **5**]. There are two kinds of GP representations, one is "cycle" and other is "chain". They are kinds of limit set of Hilbert space by action of $\mathcal{O}_N$.

By gauge transformation, we show another characterization of GP representations.

Theorem 1.1. *Any GP representation is realized as a gauge transformation of permutative representation.*

A remarkable property of gauge transformation of representation is that gauge transformation does not transform irreducible representation to irreducible one in general. We can extends a class of representations of the Cuntz algebra by gauge transformation. The origin of GP representation is obtained by a gauge transformation of a permutative representation(§ 5.2).

In § 2, we review generalized permutative representations of $\mathcal{O}_N$ in [**8**, **9**, **10**] and the systematic method of construction of representation on a measure space from a branching function system. In § 3, we give the definition of gauge transformation on a measure space and their meaning in more general situation. As a special case, we derive (1.1) from this. The relation between GP representation and gauge transformation are shown. We show how gauge transformation transforms an equivalence class of representation of $\mathcal{O}_N$ on a concrete representation space. In § 4, we show a relation between ordinary $U(N)$-action of $\mathcal{O}_N$ and gauge transformation. In § 5, we show examples of gauge transformation.

2. Preliminaries

2.1. Multiindices and parameters. In order to define a generalized permutative representation, we prepare the parameter space of GP representations. Fix $N \geq 2$. Denote $S(\mathbf{C}^N) \equiv \{z \in \mathbf{C}^N : \|z\| = 1\}$ is the unit complex sphere in $\mathbf{C}^N$. Put a set of sequences $S(\mathbf{C}^N)^\infty \equiv \{(z^{(n)})_{n\in\mathbf{N}} : z^{(n)} \in S(\mathbf{C}^N),\, n \in \mathbf{N}\}$. Furthermore, put

$$S(\mathbf{C}^N)^{\otimes k} \equiv \{z^{(1)}\otimes\cdots\otimes z^{(k)} \in (\mathbf{C}^N)^{\otimes k} : z^{(j)} \in S(\mathbf{C}^N),\, j = 1,\ldots,k\} \quad (k \geq 1),$$

$$TS(\mathbf{C}^N) \equiv \coprod_{k\geq 1} S(\mathbf{C}^N)^{\otimes k}.$$

For $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$, denote

$$z[k] \equiv z^{(1)} \otimes \cdots \otimes z^{(k)} \quad (k \geq 1). \tag{2.1}$$

Let $\{\varepsilon_1, \ldots, \varepsilon_N\}$ be the canonical basis of $\mathbf{C}^N$.

Put T an action of $U(N)$ on a tensor space $T(\mathbf{C}^N) \equiv \bigoplus_{l\geq 0}(\mathbf{C}^N)^{\otimes l}$ by

$$T_g(v_1 \otimes \cdots \otimes v_l) = gv_1 \otimes \cdots \otimes gv_l$$

for $g \in U(N)$ and $v_1 \otimes \cdots \otimes v_l \in (\mathbf{C}^N)^{\otimes l}$, $l \geq 0$. Because $T_g(TS(\mathbf{C}^N)) \subset TS(\mathbf{C}^N) \subset T(\mathbf{C}^N)$, the restriction $T|_{TS(\mathbf{C}^N)}$ is an action of $U(N)$ on $TS(\mathbf{C}^N)$, too. We denote $T|_{TS(\mathbf{C}^N)}$ by T again. Denote W the shift on $TS(\mathbf{C}^N)$ defined by $W(z^{(1)} \otimes \cdots \otimes z^{(k)}) \equiv z^{(2)} \otimes \cdots \otimes z^{(k)} \otimes z^{(1)}$ for $k \geq 1$. For $p \in \mathbf{Z}$, put $W_p \equiv W^p$ and $W^{-p} \equiv (W_p)^{-1}$ when $p \geq 1$. Then W is an action of $\mathbf{Z}$ on $TS(\mathbf{C}^N)$. Clearly $W_pT_g = T_gW_p$ for each $p \in \mathbf{Z}$ and $g \in U(N)$.

On the other hand, for $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$ and $g \in U(N)$, define $T_gz \equiv (gz^{(n)})$. Then $T_gz \in S(\mathbf{C}^N)^\infty$ and T is an action of $U(N)$ on $S(\mathbf{C}^N)^\infty$. Furthermore, Wz is defined by $y \equiv (y^{(n)}) \in S(\mathbf{C}^N)^\infty$ $y^{(n)} \equiv z^{(n+1)}$ for $n \geq 1$. $W_p \equiv W^p$ for $p \geq 0$. Put $\mathbf{T}^\infty \equiv \{(w(n)) \in U_{\mathbf{N}}(N) : w_n \in U(1)\}$. τ is an action of $\mathbf{T}^\infty$ on $S(\mathbf{C}^N)^\infty$ by $\tau_wz = (w(n)z^{(n)})$ for $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$ and $w = (w(n)) \in \mathbf{T}^\infty$. Remark that W_p is not invertible, and W_p and τ_w do not commute in general on $S(\mathbf{C}^N)^\infty$.

Definition 2.1. (i) *For $k \geq 1$, $z \in S(\mathbf{C}^N)^{\otimes k}$ is periodic if there is $p \in \mathbf{Z} \setminus k\mathbf{Z}$ such that $W_pz = z$.*
(ii) *$z \in TS(\mathbf{C}^N)$ is non periodic if z is not periodic.*
(iii) *For $z, z' \in TS(\mathbf{C}^N)$, $z \sim z'$ if there are $k \geq 1$ and $p \in \mathbf{Z}$ such that $z, z' \in S(\mathbf{C}^N)^{\otimes k}$ and $W_pz = z'$.*

Definition 2.2. (i) *For $z \in S(\mathbf{C}^N)^\infty$ is eventually periodic if there are $p, M \geq 1$ and $w \in \mathbf{T}^\infty$ such that $W_Mz = \tau_wW_{p+M}z$.*
(ii) *$z \in S(\mathbf{C}^N)^\infty$ is non eventually periodic if z is not eventually periodic.*
(iii) *For $z, z' \in S(\mathbf{C}^N)^\infty$, $z \sim z'$ if there are $k, M \geq 0$ and $w \in \mathbf{T}^\infty$ such that $W_pz = \tau_wW_qz'$.*

Relations $\sim$ in Definition 2.2 (i) and Definition 2.1 (iii) are equivalence relations. When $z \sim y$, we call that z and y are equivalent.

2.2. Definition and properties of GP representations. For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**3**]), that is, it is a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ satisfying

$$s_i^*s_j = \delta_{ij}I \quad (i, j = 1, \ldots, N), \quad s_1s_1^* + \cdots + s_Ns_N^* = I. \tag{2.2}$$

In this paper, any representation means a unital $*$-representation. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1, \ldots, S_N$ on an infinite dimensional Hilbert space which satisfy (2.2) in order to construct a representation of $\mathcal{O}_N$. Put α an action of a unitary group $U(N)$ on

$\mathcal{O}_N$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^{N} g_{ji}s_j$ for $i = 1, \ldots, N$. Specially we denote $\gamma_w \equiv \alpha_{g(w)}$ when $g(w) \equiv w \cdot I \subset U(N)$ for $w \in U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$.

We review the definition and properties of generalized permutative representations([**8**, **9**, **10**]) here by using parameters in § 2.1.

For $z = (z_1, \ldots, z_N) \in S(\mathbf{C}^N)$, denote $s(z) \equiv z_1 s_1 + \cdots + z_N s_N$. For $z = z^{(1)} \otimes \cdots \otimes z^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$, $s(z) \equiv s(z^{(1)}) \cdots s(z^{(k)})$, $s(z)^* \equiv s(z^{(k)})^* \cdots s(z^{(1)})^*$.

Definition 2.3. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$.*

(i) *For $z \in TS(\mathbf{C}^N)$, a unit vector $\Omega \in \mathcal{H}$ satisfies the cycle condition with respect to z if $\pi(s(z))\Omega = \Omega$.*

(ii) *For $z \in TS(\mathbf{C}^N)$, a unit vector $\Omega \in \mathcal{H}$ satisfies the full cycle condition with respect to z if Ω satisfies the cycle condition with respect to z and $\dim \mathrm{Lin} < \{\pi(s(z^{(l)}) \otimes \cdots \otimes s(z^{(k)}))\Omega : l = 1, \ldots, k\} >= k$ when $z = z^{(1)} \otimes \cdots \otimes z^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$.*

(iii) *For $z \in S(\mathbf{C}^N)^\infty$, a unit vector $\Omega \in \mathcal{H}$ satisfies the chain condition with respect to z if $\{\pi(s(z[n]))^*\Omega : n \geq 1\}$ is an orthonormal family in $\mathcal{H}$.*

The condition (ii) in Definition 2.3 is stricter than (i). When $z \in TS(\mathbf{C}^N)$ is non periodic, we see that if Ω satisfies the cyclic condition with respect to z, then Ω always satisfies the full cyclic condition with respect to z(Lemma 4.5 in [**8**], Lemma B.1 in [**10**]). However, it is not in general when z is periodic([**9**]). In order to treat periodic case conveniently, we use the condition (ii) in Definition 2.3 for the definition of generalized permutative representation of $\mathcal{O}_N$ with cycle in stead of (i) in this paper.

Definition 2.4. (i) *For $z \in TS(\mathbf{C}^N)$, $(\mathcal{H}, \pi, \Omega)$ is a generalized permutative(=GP) representation of $\mathcal{O}_N$ with cycle by z if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with a unit cyclic vector $\Omega \in \mathcal{H}$ which satisfies the full cycle condition with respect to z.*

(ii) *For $z \in S(\mathbf{C}^N)^\infty$, $(\mathcal{H}, \pi, \Omega)$ is a GP representation of $\mathcal{O}_N$ with chain by z if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with a unit cyclic vector $\Omega \in \mathcal{H}$ which satisfies the chain condition with respect to z.*

We call Ω in Definition 2.4 both (i) and (ii) the GP vector of a GP representation and denote $GP(z) = (\mathcal{H}, \pi, \Omega)$ for (i), (ii) simply.

For two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathcal{O}_N$, $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$ means the unitary equivalence between $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$. Specially, $GP(z) \sim GP(z')$ means that two cyclic representations of $\mathcal{O}_N$ are unitarily equivalent.

We review results about GP representations. In order to unify statements of cycle and chain in [**8**, **9**, **10**], we put the unified parameter space

$$S(\mathbf{C}^N)^* \equiv TS(\mathbf{C}^N) \sqcup S(\mathbf{C}^N)^\infty.$$

$z \in S(\mathbf{C}^N)^*$ is periodic if z is eventually periodic when $z \in S(\mathbf{C}^N)^\infty$ and z is periodic when $z \in TS(\mathbf{C}^N)$. For $z, z' \in S(\mathbf{C}^N)^*$, $z \sim z'$ if $z, z' \in TS(\mathbf{C}^N)$ and $z \sim z'$, or $z, z' \in S(\mathbf{C}^N)^\infty$ and $z \sim z'$.

Theorem 2.5. (i) *(Existence and uniqueness) For any $z \in S(\mathbf{C}^N)^*$, there exists $GP(z)$ and it is unique up to unitary equivalences.*
(ii) *(Irreducibility) For $z \in S(\mathbf{C}^N)^*$, $GP(z)$ is irreducible if and only if z is non periodic.*
(iii) *(Equivalence) For $z, z' \in S(\mathbf{C}^N)^*$, $GP(z) \sim GP(z')$ if and only if $z \sim z'$.*

Proof. (i) When $z \in TS(\mathbf{C}^N)$, the existence is shown in Proposition 3.4 in [**8**]. If z is non periodic, then the uniqueness is shown in Proposition 5.4 in [**8**]. If z is periodic, the uniqueness is shown in Corollary 5.6 (v) in [**9**].

When $z \in S(\mathbf{C}^N)^\infty$, the existence is shown in Proposition 3.5 in [**10**]. The uniqueness is shown in Theorem 5.1 in [**10**].
(ii) When $z \in TS(\mathbf{C}^N)$, it is shown in Proposition 5.5 in [**8**]. When $z \in S(\mathbf{C}^N)^\infty$, it is shown in Theorem 5.14 in [**10**].
(iii) When $z, z' \in TS(\mathbf{C}^N)$, it is shown in Proposition 5.11 in [**8**]. When $z, z' \in S(\mathbf{C}^N)^\infty$, it is shown in Theorem 5.3 in [**10**]. When $z \in TS(\mathbf{C}^N)$ and $z' \in S(\mathbf{C}^N)^\infty$, $z \not\sim z'$ by definition. On the other hand, By Proposition 5.5 and Theorem 5.11 in [**10**], $GP(z) \not\sim GP(z')$. Hence the statement is proved. □

By Theorem 2.5 (i), we can regard a symbol $GP(z)$ as the representative element of an equivalence class of representations of $\mathcal{O}_N$.

We prepare a method of construction of isometries and representations of $\mathcal{O}_N$ on measure spaces([**11**, **12**, **13**, **14**, **15**]) here briefly.

Let (X, μ) be a measure space and f a measurable transformation on X which is injective and there exists the Radon-Nikodým derivative Φ_f of $\mu \circ f$ with respect to μ and Φ_f is non zero almost everywhere in X. We denote the set of such transformations by $RN(X)$.

Definition 2.6. *Let (X, μ) be a measure space.*
(i) *For $f \in RN(X)$, define an operator $S(f)$ on $L_2(X, \mu)$ by*

$$(S(f)\phi)(x) \equiv \begin{cases} \left\{\Phi_f\left(f^{-1}(x)\right)\right\}^{-1/2} \phi(f^{-1}(x)) & (\textit{when } x \in R(f)), \\ 0 & (\textit{otherwise}) \end{cases}$$

for $\phi \in L_2(X, \mu)$ and $x \in X$ where $R(f)$ is the image of f.
(ii) *For $N \geq 2$, $f = \{f_i\}_{i=1}^N$ is a branching function system on (X, μ) if $f_i \in RN(X)$, $i = 1, \ldots, N$ and the followings are μ-null sets: $f_i(X) \cap f_j(X)$, $1 \leq i < j \leq N$, $X \setminus \bigcup_{i=1}^N f_i(X)$.*

(iii) *For a branching function system* $f = \{f_i\}_{i=1}^N$ *on* (X,μ)*, a representation* $(L_2(X,\mu),\pi_f)$ *of* $\mathcal{O}_N$ *which is defined by* $\pi_f(s_i) \equiv S(f_i)$ *for* $i = 1,\ldots,N$ *is called the measure theoretical permutative representation by* f*. We denote* $(L_2(X,\mu),\pi_f)$ *by* π_f *simply.*

(iv) $(\mathcal{H},\pi)$ *is a permutative representation of* $\mathcal{O}_N$ *if there are a complete orthonormal basis* $\{e_n\}_{n\in\Lambda}$ *of* $\mathcal{H}$ *and a branching function system* f *on* Λ *such that* $\pi = \pi_f$ *in (iii) where* Λ *is regarded as a measure space with pointwise measure and* $\mathcal{H}$ *is identified with* L_2*-space on* Λ *with respect to this measure.*

About concrete examples of $(L_2(X,\mu),\pi_f)$, see [**11**, **12**, **13**, **14**, **15**].

For § 4, we prepare examples of GP representation here.

Example 2.7. (i) The *standard representation* $(l_2(\mathbf{N}),\pi_S)$ of $\mathcal{O}_N$ is defined by $\pi_S(s_i)e_n \equiv e_{N(n-1)+i}$ for $n \in \mathbf{N}$, $i = 1,\ldots,N$ where $\{e_n\}_{n\in\mathbf{N}}$ is the canonical basis of $l_2(\mathbf{N})$([**1**, **12**]). Then $(l_2(\mathbf{N}),\pi_S,e_1)$ satisfies the condition of $GP(\varepsilon_1)$. Because $\varepsilon_1 \in S(\mathbf{C}^N)$ is non periodic, $(l_2(\mathbf{N}),\pi_S)$ is irreducible by Theorem 2.5 (ii).

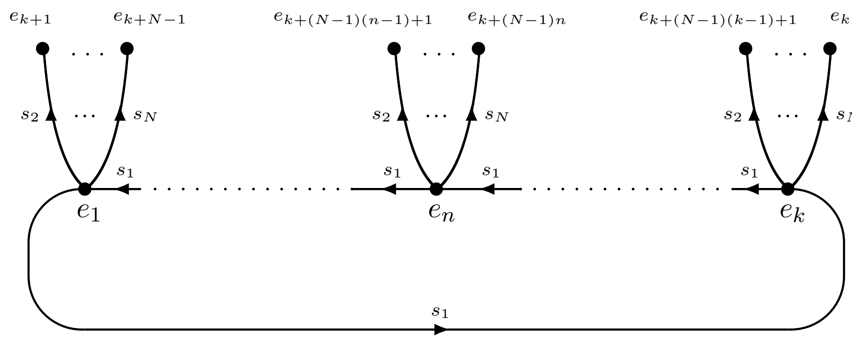
(ii) We generalize (i). Denote $(l_2(\mathbf{N}),\pi_S)$ by $(l_2(\mathbf{N}),\pi_1)$. For $k \geq 2$, let $(l_2(\mathbf{N}),\pi_k)$ be a representation of $\mathcal{O}_N$ defined by

$$\pi_k(s_1)e_1 \equiv e_n, \quad \pi_k(s_1)e_n \equiv e_{n-1} \quad (n = 2,\ldots,k),$$

$$\pi_k(s_i)e_n \equiv e_{k+i+(n-1)(N-1)} \quad (n = 1,\ldots,k,\ i = 2,\ldots,N),$$

$$\pi_k(s_i)e_n \equiv e_{N(n-1)+i} \quad (n \geq k+1,\ i = 1,\ldots,N).$$

Then $(l_2(\mathbf{N}),\pi_k,e_k)$ is $GP(\varepsilon_1^{\otimes k})$. The tree of representation is following:

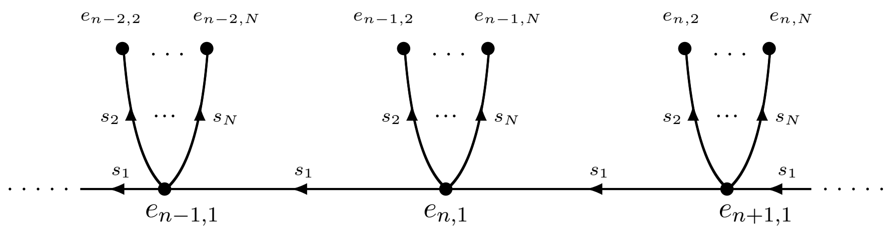


where vertices and edges mean $\{e_n\}_{n\in\mathbf{N}}$ and the action of operators $\pi_k(s_i)$, $i = 1,\ldots,N$ on $\{e_n\}_{n\in\mathbf{N}}$, respectively. Because $\varepsilon_1^{\otimes k} \in S(\mathbf{C}^N)^{\otimes k}$ is periodic when $k \geq 2$, $(l_2(\mathbf{N}),\pi_k)$ is not irreducible. By Theorem 5.3 in [**9**], we see the irreducible decomposition $(l_2(\mathbf{N}),\pi_k) \sim \bigoplus_{l=0}^{k-1} GP(\zeta_k^l\varepsilon_1)$ where $\zeta_k \equiv e^{2\pi\sqrt{-1}/k}$.

(iii) Put $R_i \equiv \mathbf{Z} \times \mathbf{N}_i$, $\mathbf{N}_i \equiv \{N(n-1)+i : n \in \mathbf{N}\}$ for $i = 1, \ldots, N$. Then we have a partition $\mathbf{Z}\times\mathbf{N} = R_1 \sqcup \cdots \sqcup R_N$. Consider a branching function system $f \equiv \{f_i\}_{i=1}^N$ on $\mathbf{Z} \times \mathbf{N}$ defined by

$$f_i : \mathbf{Z} \times \mathbf{N} \to R_i; \quad f_i(n,m) \equiv (n-1,\, N(m-1)+i) \tag{2.3}$$

for $i = 1, \ldots, N$. Then $f_1(n,1) = (n-1,1)$ for each $n \in \mathbf{Z}$. From this, we have $f_1^k(n,1) = (n-k,1)$ for $k \geq 1$ and $n \in \mathbf{Z}$. Put a representation $(l_2(\mathbf{Z}\times\mathbf{N}), \pi_f)$ of $\mathcal{O}_N$ by $\pi_f(s_i)e_x \equiv e_{f_i(x)}$ for $x \in \mathbf{Z}\times\mathbf{N}$ and $i = 1, \ldots, N$. From this, we have $\pi_f(s_1^*)e_{n,1} = e_{n+1,1}$ for $n \in \mathbf{Z}$. Hence $\{\pi_f((s_1^*)^n)e_{0,1} : n \in \mathbf{N}\} = \{e_{n,1} : n \in \mathbf{N}\}$ is an orthonormal family. The tree of representation is following:



In consequence, $(l_2(\mathbf{Z} \times \mathbf{N}), \pi_f, e_{0,1})$ is $GP(\varepsilon_1^\infty)$ of $\mathcal{O}_N$ where $\varepsilon_1^\infty \equiv (\varepsilon_1, \varepsilon_1, \varepsilon_1, \ldots) \in S(\mathbf{C}^N)^\infty$.

It is easy to check that cyclicities and eigen equations in Example 2.7 follow from their definitions, respectively. All of these are (cyclic)permutative representations in [**2**, **4**, **5**].

3. Gauge transformations of $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$

For a given representation of $\mathcal{O}_N$, we make a new representation by using a method of gauge transformation.

3.1. Mapping group as gauge group. Let $N \geq 1$ and (X, μ) a measure space. Define $U_X(N)$ the group of all measurable maps from X to a unitary group $U(N)$ with respect to pointwise product. For $g \in U_X(N)$, g^* is defined by $(g^*)(x) \equiv (g(x))^*$ for $x \in X$. We often identify $U(N)$ and the subgroup of $U_X(N)$ which consists of constant maps. For a set Λ without measure, we denote $U_\Lambda(N) = \{(g(n))_{n\in\Lambda} : g(n) \in U(N),\, n \in \Lambda\}$.

3.2. Gauge transformation. Let $N \geq 2$ and $\mathcal{B}$ a unital C*-algebra and $\mathcal{A}$ an abelian subalgebra in $\mathcal{B}$. Put $U(N, \mathcal{A})$ the group of all unitaries in the set $M_N(\mathcal{A})$ of all $\mathcal{A}$-valued $N \times N$-matrices and $\mathrm{Hom}(\mathcal{O}_N, \mathcal{B})$ the set of all unital $*$-homomorphisms from $\mathcal{O}_N$ to $\mathcal{B}$. For $f \in \mathrm{Hom}(\mathcal{O}_N, \mathcal{B})$ and $g = (g_{ij}) \in U(N, \mathcal{A})$, define $\Gamma_g(f) \in \mathrm{Hom}(\mathcal{O}_N, \mathcal{B})$ by

$$(\Gamma_g(f))(s_i) \equiv \sum_{j=1}^N f(s_j)g_{ji}^* \quad (i = 1, \ldots, N). \tag{3.1}$$

We see that Γ is an action of $U(N,\mathcal{A})$ on $\mathrm{Hom}(\mathcal{O}_N,\mathcal{B})$, that is, $\Gamma_g\circ\Gamma_h=\Gamma_{gh}$ for $g,h\in U(N,\mathcal{A})$ where we use the assumption that $\mathcal{A}$ is abelian. Remark g^*_{ji} and $f(s_j)$ in the rhs of (3.1) do not commute in general. Hence the order of them are important. We identify $U(N)$ and $\{(g_{ij}I)_{i,j=1}^N\in U(N,\mathcal{A}):g=(g_{ij})\in U(N)\}$. Note if $g\in U(N)\subset U(N,\mathcal{A})$, then $\Gamma_g(f)=f\circ\alpha_{g^*}$ for $f\in\mathrm{Hom}(\mathcal{O}_N,\mathcal{B})$ where α is the canonical $U(N)$-action of $\mathcal{O}_N$ in § 2.2. For a Hilbert space $\mathcal{H}$, $\mathrm{Rep}(\mathcal{O}_N,\mathcal{H})\equiv\mathrm{Hom}(\mathcal{O}_N,\mathcal{L}(\mathcal{H}))$ is the set of all representations of $\mathcal{O}_N$ on $\mathcal{H}$.

Definition 3.1. (i) Γ *in* (3.1) *is called the abstract gauge action of* $U(N,\mathcal{A})$ *on* $\mathrm{Hom}(\mathcal{O}_N,\mathcal{B})$.

(ii) *For a measure space* (X,μ) *which satisfies* $\mathrm{dim}L_2(X,\mu)=\infty$, Γ *is the gauge action of* $U_X(N)=U(N,L_\infty(X,\mu))$ *on* $\mathrm{Rep}(\mathcal{O}_N,L_2(X,\mu))$ *if* Γ *is the abstract gauge action of* $U_X(N)$ *on* $\mathrm{Hom}(\mathcal{O}_N,\mathcal{L}(L_2(X,\mu)))$ *where* $L_\infty(X,\mu)$ *is identified as an abelian subalgebra of* $\mathcal{L}(L_2(X,\mu))$. *For* $g\in U_X(N)$, Γ_g *is called the gauge transformation of* $\mathrm{Rep}(\mathcal{O}_N,L_2(X,\mu))$ *by* g.

(iii) *For a Hilbert space* $\mathcal{H}$ *with a complete orthonormal basis* $\{e_n\}_{n\in\Lambda}$, Γ *is the gauge action* $U_\Lambda(N)$ *on* $\mathrm{Rep}(\mathcal{O}_N,\mathcal{H})$ *with respect to* $\{e_n\}_{n\in\Lambda}$ *if*

$$(\Gamma_g(\pi))(s_i)e_n\equiv(\pi\circ\alpha_{g^*(n)})(s_i)e_n\quad(n\in\Lambda,\ i=1,\ldots,N)\tag{3.2}$$

for $g\in U_\Lambda(N)$ *and* $\pi\in\mathrm{Rep}(\mathcal{O}_N,\mathcal{H})$.

We see that (iii) in Definition 3.1 is a special case of (ii) with respect to a measure space Λ with one-point measure. For a measure space (X,μ), $\pi\in\mathrm{Rep}(\mathcal{O}_N,L_2(X,\mu))$ and $g\in U_X(N)$, (3.1) is written as following:

$$(\Gamma_g(\pi))(s_i)=\sum_{j=1}^N\pi(s_j)M_{g^*_{ji}}\quad(i=1,\ldots,N)\tag{3.3}$$

where $M_{g^*_{ji}}$ is the multiplication operator of $g^*_{ji}\in L_\infty(X,\mu)$ on $L_2(X,\mu)$.

Proposition 3.2. *If* $(L_2(X,\mu),\pi_f)$ *is in Definition* 2.6 *(iii) by a branching function system* $f=\{f_i\}_{i=1}^N$ *on* (X,μ) *and* $g\in U_X(N)$, *then*

$$\{(\Gamma_g(\pi_f))(s_i)\phi\}(x)=\sum_{j=1}^N g^*_{ji}(f_j^{-1}(x))(\pi_f(s_j)\phi)(x)\quad(i=1,\ldots,N)\tag{3.4}$$

for $x\in X$ *and* $\phi\in L_2(X,\mu)$. *If* $g_{ij}(x)=\delta_{ij}e^{\sqrt{-1}\theta_i(x)}$ *for* $i,j=1,\ldots,N$ *and* $x\in X$, *then* (3.4) *is written as*

$$\{(\Gamma_g(\pi_f))(s_i)\phi\}(x)=e^{-\sqrt{-1}\theta_i(f_i^{-1}(x))}(\pi_f(s_i)\phi)(x)\tag{3.5}$$

for $i=1,\ldots,N$.

Proof. By direct computation, we have statements. □

Remark that the rhs in (3.4) can not be written by the canonical $U(N)$-action of $\mathcal{O}_N$ in general. Therefore this is not transformation of $\mathcal{O}_N$ but that of the space of representations of $\mathcal{O}_N$.

We explain the notion of gauge transformation in the style of quantum field theory([**18**]). For convenience, we denote $\psi_i \equiv \pi_f(s_i)\phi$ for $\phi \in L_2(X,\mu)$ and $i = 1,\ldots,N$. Then (3.4) and (3.5) in Proposition 3.2 are rewritten as

$$\psi_i(x) \longmapsto \psi_i^{'}(x) = \sum_{j=1}^{N} g_{ji}^{*}(f_j^{-1}(x))\psi_j(x), \tag{3.6}$$

$$\psi_i(x) \longmapsto \psi_i^{'}(x) = e^{-\sqrt{-1}\theta_i(f_i^{-1}(x))}\psi_i(x) \tag{3.7}$$

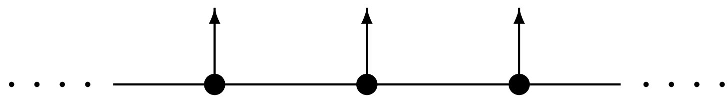
for $x \in X$ and $i = 1,\ldots,N$. We see that the former is just a non abelian gauge transformation of the second kind(or a local gauge transformation) and the later is abelian gauge transformation in physics([**18**]). In actuality, it may be better that Γ_g is called the *dual* gauge action.

3.3. GP representations of $\mathcal{O}_N$ and gauge transformations. Recall the permutative representation in Definition (2.6) (iv). Let $(l_2(\mathbf{N}),\pi_f)$ be a permutative representation by a branching function system f on $\mathbf{N}$. For $g = (g_{ij}) \in U_{\mathbf{N}}(N)$, the gauge transformation of π_f by g coincides the gauge transformation of $\mathrm{Rep}(\mathcal{O}_N, l_2(\mathbf{N}))$ with respect to $\{e_n\}_{n\in\mathbf{N}}$ in Definition 3.1 (iii). Specially, $(\Gamma_g(\pi_f))(s_i^*)e_{f_k(m)} = g_{ik}(m)e_m$.

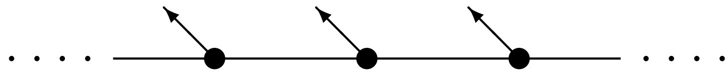
We illustrate the gauge transformation of permutative representation here. Consider the following one-dimensional half infinite chain indexed by $\mathbf{N}$. The vertex means the canonical basis of $l_2(\mathbf{N})$. Now $\mathcal{O}_N$ acts on $l_2(\mathbf{N})$ by some permutative representation(figure **(a)**). We show the action of $U_{\mathbf{N}}(N)$ by an arrow at each vertex.

If $g \in U(N)$, then a transformation $\pi \mapsto \pi \circ \alpha_g$ is illustrated by a *simultaneous transformation* at each vertex(figure **(b)**). This is called the global gauge transformation in quantum field theory([**18**]).

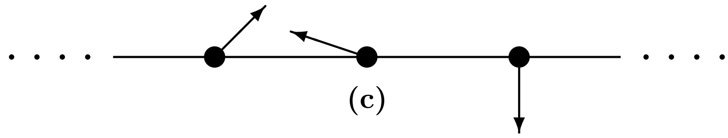
If $g \in U_{\mathbf{N}}(N)$, then a transformation $\pi \mapsto \Gamma_g(\pi)$ is illustrated by *asynchronous transformation*(figure **(c)**) by (3.2). This is called the local gauge transformation in quantum field theory.



(a)



(b)



(c)

Recall $GP(z)$ in Definition 2.4.

Lemma 3.3. *For $k \geq 1$, let $(l_2(\mathbf{N}), \pi_k)$ be in Example 2.7 (ii). Then for each $z \in S(\mathbf{C}^N)^{\otimes k}$, there is $g \in U_{\mathbf{N}}(N)$ such that $\Gamma_g(\pi_k)$ is $GP(z)$.*

Proof. Fix $z = z^{(1)} \otimes \cdots \otimes z^{(k)}$. Choose $g = (g(n))_{n \in \mathbf{N}} \in U_{\mathbf{N}}(N)$ as

$$g_{j1}(n) = z_j^{(n)} \quad (j = 1, \ldots, N,\ n = 1, \ldots, k), \quad g(n) = I \quad (n \geq k+1).$$

Put $\pi_g \equiv \Gamma_g(\pi_k)$. Then $\alpha_{g(n)}(s_1) = s(z^{(n)})$ and $\pi_g(s(z^{(n)}))e_n = (\pi \circ \alpha_{g(n)^*})(s(z^{(n)}))e_n = \pi_k(s_1)e_n$ for $n = 1, \ldots, k$ by (3.2). Hence $\pi_g(s(z))e_k = e_k$. Put $\mathcal{V} \equiv \pi_g(\mathcal{O}_N)e_k$. Then $(\mathcal{V}, \pi_g, e_k)$ is $GP(z)$. We see that $e_1, \ldots, e_k$ belong to $\mathcal{V}$ by action of $\pi_g(s(z^{(n)}))$. Furthermore $\pi_g(\alpha_{g(n)}(s_i))e_n = \pi_k(s_i)e_n$ for $n = 1, \ldots, k$ and $\pi_g(s_i)e_n = \pi_k(s_i)e_n$ for $n \geq k+1$. Hence $e_n \in \mathcal{V}$ for each $n \in \mathbf{N}$. Therefore $\mathcal{V} = l_2(\mathbf{N})$. In consequence, $(l_2(\mathbf{N}), \Gamma_g(\pi_k), e_k)$ is $GP(z)$. □

Lemma 3.4. *Let $(l_2(\mathbf{Z} \times \mathbf{N}), \pi_f)$ be in Example 2.7 (iii). Then for each $z \in S(\mathbf{C}^N)^{\infty}$, there is $g \in U_{\mathbf{Z}\times\mathbf{N}}(N)$ such that $\Gamma_g(\pi_f)$ is $GP(z)$.*

Proof. Fix $z = (z^{(n)}) \in S(\mathbf{C}^N)^{\infty}$. We extend z as $z = (z^{(n)})_{n \in \mathbf{Z}}$. $z^{(-n)} = \varepsilon_1$ when $n \geq 0$. Choose $g \in U_{\mathbf{Z}\times\mathbf{N}}(N)$ as

$$g_{j1}(n, 1) = z_j^{(n-1)} \quad (n \in \mathbf{Z}), \quad g(n, m) = I \quad (n \in \mathbf{Z},\ m \geq 2).$$

Put $\pi_g \equiv \Gamma_g(\pi_f)$. We see $\alpha_{g(n,1)}(s_1) = s(z^{(n-1)})$ for $n \in \mathbf{Z}$. From this, $\pi_g(s(z^{(n-1)}))e_{n,1} = \pi_f(s_1)e_{n,1} = e_{n-1,1}$ for $n \in \mathbf{Z}$. Put $\mathcal{V} \equiv \pi_g(\mathcal{O}_N)e_{0,1}$.

Then $(\mathcal{V}, \pi_g, e_{0,1})$ is $GP(z)$. We see $e_{n,1} \in \mathcal{V}$ for each $n \in \mathbf{Z}$. By choice of g, we have $\pi_g(\alpha_{g(n,m)}(s_i))e_{n,m} = \pi_f(s_i)e_{n,m}$ for each $(n,m) \in \mathbf{Z} \times \mathbf{N}$ and $i = 1, \ldots, N$. Hence $e_{n,m} \in \mathcal{V}$ for each $(n,m) \in \mathbf{Z}\times\mathbf{N}$. Hence $\mathcal{V} = l_2(\mathbf{Z}\times\mathbf{N})$. We see that $(l_2(\mathbf{Z} \times \mathbf{N}), \Gamma_g(\pi_f), e_{0,1})$ is $GP(z)$. □

Lemma 3.4 holds on $l_2(\mathbf{N})$ by using a suitable bijection $\mathbf{Z} \times \mathbf{N} \cong \mathbf{N}$.

We show another characterization of GP representations.

Definition 3.5. *$(\mathcal{H}, \pi)$ is a FP representation of $\mathcal{O}_N$ if there are a complete orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$, a branching function system $f = \{f_i\}_{i=1}^N$ on $\mathbf{N}$ and $g \in U_{\mathbf{N}}(N)$ such that $\pi(\alpha_{g(n)}(s_i))e_n = e_{f_i(n)}$ for $i = 1, \ldots, N$ and $n \in \mathbf{N}$.*

Proposition 3.6. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$. The followings are equivalent:*

(i) *$(\mathcal{H}, \pi)$ is a GP representation.*
(ii) *$(\mathcal{H}, \pi)$ is a cyclic FP representation.*
(iii) *$(\mathcal{H}, \pi)$ is equivalent to the gauge transformation of a cyclic permutative representation.*

Proof. For GP representations with chain, we treat on $l_2(\mathbf{N})$ instead of $l_2(\mathbf{Z} \times \mathbf{N})$ here.

(i) follows from (iii) by Lemma 3.3 and Lemma 3.4. (iii) follows from (i) by Lemma 4.1 and Lemma 4.4.

Assume (ii). Consider the gauge transformation Γ with respect to basis $\{e_n\}_{n\in\mathbf{N}}$. Put $\pi' \equiv \Gamma_{g^*}(\pi)$. Then $\pi'(s_i)e_n = e_{f_i(n)}$. Hence $(\mathcal{H}, \pi')$ is a permutative representation by f. Because $\pi = \Gamma_g(\pi')$, (iii) follows. (ii) follows from (iii). □

By Proposition 3.6, Theorem 1.1 is proved.

Remark that there is a difference between ordinary $U(N)$-action of $\mathcal{O}_N$ and gauge transformation. Let $(l_2(\mathbf{N}), \pi_k)$ be in Example 2.7 (ii). Put $k = 2$. Then $(l_2(\mathbf{N}), \pi_2)$ is not irreducible. On the other hand, by Lemma 3.3, we have $g \in U_{\mathbf{N}}(N)$ such that $\Gamma_g(\pi_2)$ is $GP(\varepsilon_1 \otimes \varepsilon_2)$ which is irreducible. In this way, some gauge transformation transform reducible representation transforms an irreducible representation. This is impossible by automorphism.

4. Equivalence and automorphisms

We consider the equivalence between GP representations by gauge transformations.

4.1. Equivalence of GP representations and gauge transformations(cycle).

Lemma 4.1. *For $k \geq 1$, let $(l_2(\mathbf{N}), \pi_k)$ in Example 2.7 (ii) and $g \in U_{\mathbf{N}}(N)$.*

(i) $(l_2(\mathbf{N}), \Gamma_g(\pi_k), e_k) = GP\left((g(1)\otimes\cdots\otimes g(k))\varepsilon_1^{\otimes k}\right)$.

(ii) $\Gamma_g(\pi_k) \sim \pi_k$ *if and only if* $g_{11}(1)\cdots g_{11}(k) = 1$.

Proof. Put $\pi_g \equiv \Gamma_g(\pi_k)$.
(i) In the proof of Lemma 3.3, we see $z^{(n)} = g(n)\varepsilon_1$ for $n = 1,\ldots,k$. Hence it is sufficient to show the cyclicity of $\mathcal{V} \equiv \pi_g(\mathcal{O}_N)e_k$. Note $\pi_g(\alpha_{g(n)}(s_i))e_n = \pi_k(s_i)e_n$ for each $n \in \mathbf{N}$ and $i = 1,\ldots,N$. Hence we see $e_n \in \mathcal{V}$ for each $n \in \mathbf{N}$. Therefore $\mathcal{V} = l_2(\mathbf{N})$.
(ii) $(l_2(\mathbf{N}), \pi_g, e_k)$ is $GP\left((g(1)\otimes\cdots\otimes g(k))\varepsilon_1^{\otimes k}\right)$ by (i) and $(l_2(\mathbf{N}), \pi_k, e_k)$ is $GP(\varepsilon_1^{\otimes k})$ by assumption. From these, $\pi_g \sim \pi_k$ if and only if $GP(z) \sim GP(\varepsilon_1^{\otimes k})$. This is equivalent to $z = z^{(1)}\otimes\cdots\otimes z^{(k)} \sim \varepsilon_1^{\otimes k}$. Therefore $z^{(n)} = (c_n, 0,\ldots,0)$ for $n = 1,\ldots,k$ and $c_1\cdots c_k = 1$. From this, $1 = c_1\cdots c_k = z_1^{(1)}\cdots z_1^{(k)} = g_{11}(1)\cdots g_{11}(k)$. In consequence, we have the assertion. □

Lemma 4.2. *For* $k \geq 1$, *let* $(l_2(\mathbf{N}), \pi_k)$ *in Example* 2.7 *(ii) and* $g \in U_{\mathbf{N}}(N)$. *For* $h \in U_{\mathbf{N}}(N)$, *the followings are equivalent:*

(i) $\Gamma_h(\Gamma_g(\pi_k)) \sim \Gamma_g(\pi_k)$.

(ii) *There is* $\sigma \in \mathbf{Z}_k$ *such that* $(g_\sigma^* hg)_{11}(1)\cdots(g_\sigma^* hg)_{11}(k) = 1$ *where* $g_\sigma(n) \equiv g(\sigma(n))$ *for* $n = 1,\ldots,k$.

Proof. By Lemma 4.1 (i), $\Gamma_g(\pi_k) = GP\left((g(1)\otimes\cdots\otimes g(k))\varepsilon_1^{\otimes k}\right)$ and $\Gamma_{hg}(\pi_k) = GP\left(((hg)(1)\otimes\cdots\otimes(hg)(k))\varepsilon_1^{\otimes k}\right)$. Hence $\Gamma_h(\Gamma_g(\pi_k)) \sim \Gamma_g(\pi_k)$ if and only if there is $\sigma \in \mathbf{Z}_k$ such that $(g_\sigma(1)\otimes\cdots\otimes g_\sigma(k))\varepsilon_1^{\otimes k} = ((hg)(1)\otimes\cdots\otimes(hg)(k))\varepsilon_1^{\otimes k}$. From this, we have the equivalent condition $((g_\sigma^* hg)(1)\otimes\cdots\otimes(g_\sigma^* hg)(k))\varepsilon_1^{\otimes k} = \varepsilon_1^{\otimes k}$. □

Proposition 4.3. *For* $k \geq 1$, *let* $(l_2(\mathbf{N}), \pi_k)$ *be in Example* 2.7 *(ii).*

(i) *Let* $H_1 \equiv \{g \in U_{\mathbf{N}}(N) : \Gamma_g(\pi_k) \sim \pi_k\}$. *Then* H_1 *is a subgroup of* $U_{\mathbf{N}}(N)$ *and* $H_1 = \{g \in U_{\mathbf{N}}(N) : g_{11}(1)\cdots g_{11}(k) = 1\}$.

(ii) *For* $g \in U_{\mathbf{N}}(N)$, *let* $H_g \equiv \{h \in U_{\mathbf{N}}(N) : \Gamma_h(\Gamma_g(\pi_k)) \sim \Gamma_g(\pi_k)\}$. *Then*

$$H_g = \left\{ h \in U_{\mathbf{N}}(N) : \begin{array}{c} \textit{there is } \sigma \in \mathbf{Z}_k \textit{ such that} \\ (g_\sigma^* hg)_{11}(1)\cdots(g_\sigma^* hg)_{11}(k) = 1 \end{array} \right\}.$$

Proof. (i) For $g, h \in H_1$, we can check $gh \in H_1$. By Lemma 4.1, it holds. (ii) By Lemma 4.2, it holds. □

Remark that H_g in Proposition 4.3 (ii) is not a subgroup of $U_{\mathbf{N}}(N)$ in general.

We show the case $k=1,2$ in Proposition 4.3. When $k=1$, $\Gamma_g(\pi_1) \sim \pi_1 \circ \alpha_{g^*(1)} \sim GP(g(1)\varepsilon_1)$. Because $\{h\varepsilon_1 : h \in U(N)\} = S(\mathbf{C}^N)$, $V_1 \equiv \{\Gamma_g(\pi_1) : g \in U_{\mathbf{N}}(N)\}/\sim = \{GP(z) : z \in S(\mathbf{C}^N)\}$ as $U(N)$-homogeneous space. Specially V_1 is a set of equivalence classes of irreducible representations of $\mathcal{O}_N$. Therefore $S(\mathbf{C}^N)$ is regarded as a subset of the spectrum of $\mathcal{O}_N$.

When $k=2$, then $\Gamma_g(\pi_2) \sim GP\left(g(1)\varepsilon_1 \otimes g(2)\varepsilon_1\right)$. Because $\{h_1\varepsilon_1 \otimes h_2\varepsilon_1 : h_1, h_2 \in U(N)\} = S(\mathbf{C}^N)^{\otimes 2}$, $V_2 \equiv \{\Gamma_g(\pi_2) : g \in U_{\mathbf{N}}(N)\}$. Then $V_2/\sim = \{GP(z) : z \in S(\mathbf{C}^N)^{\otimes 2}\}/\sim$ and V_2 decomposes into the irreducible part $V_{2,irr}$ and the reducible part $V_{2,red}$. We have $V_{2,red} = \{z^{\otimes 2} \in V_2 : z \in S(\mathbf{C}^N)\}$ and $V_{2,irr} = V_2 \setminus V_{2,red}$.

4.2. Equivalence of GP representations and gauge transformations(chain).

Lemma 4.4. *Let $(l_2(\mathbf{Z}\times\mathbf{N}), \pi_f)$ be in Example 2.7 (iii) and $g \in U_{\mathbf{Z}\times\mathbf{N}}(N)$.*

- (i) *$(l_2(\mathbf{Z}\times\mathbf{N}), \Gamma_g(\pi_f), e_{0,1})$ is $GP(z)$ for $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$, $z^{(n)} = g(n+1,1)\varepsilon_1$ for $n \geq 1$.*
- (ii) *$\Gamma_g(\pi_f) \sim \pi_f$ if and only if there is $M \geq 1$ such that $|g_{11}(n,1)| = 1$ for each $n \geq M$.*

Proof. (i) By Lemma 3.4, it follows. (ii) $\Gamma_g(\pi_f) \sim \pi_f$ if and only if $GP(z) \sim GP(\varepsilon_1^\infty)$ where $z \in S(\mathbf{C}^N)^\infty$ satisfies the condition in (i). This is equivalent to $z \sim \varepsilon_1^\infty$. Hence there is $M \geq 1$ such that $|z_1^{(n)}| = 1$ for each $n \geq M$. In consequence, the assertion holds. □

Lemma 4.5. *Let $(l_2(\mathbf{Z}\times\mathbf{N}), \pi_f)$ be in Example 2.7 (iii) and $g \in U_{\mathbf{Z}\times\mathbf{N}}(N)$. For $h \in U_{\mathbf{Z}\times\mathbf{N}}(N)$, the followings are equivalent:*

- (i) *$\Gamma_h(\Gamma_g(\pi_f)) \sim \Gamma_g(\pi_f)$.*
- (ii) *There are $L \in \mathbf{Z}$ and $M \in \mathbf{N}$ such that $|(g^*_{\sigma^L} hg)_{11}(n)| = 1$ for each $n \geq M$ where $g_{\sigma^L}(n) \equiv g(n-L)$.*

Proof. $\Gamma_h(\Gamma_g(\pi_f)) \sim \Gamma_g(\pi_f)$ if and only if $GP(y) \sim GP(z)$ where $z = (z^{(n)}), y = (y^{(n)}) \in S(\mathbf{C}^N)^\infty$ are defined by $z_j^{(n-1)} \equiv g_{j1}(n)$ and $y_j^{(n-1)} \equiv (hg)_{j1}(n)$ for $n \geq 2$ and $j = 1, \ldots, N$. This is equivalent to $y \sim z$. Hence there are $M, L \in \mathbf{N}\cup\{0\}$ and $\{c_n\}_{n\geq 1} \subset U(1)$ such that $y^{(n+L)} = c_n z^{(n)}$. for $n \geq M$. From this, $(hg)(n+L)\varepsilon_1 = c_n g(n)\varepsilon_1$ for $n \geq M$. Therefore $(g^*_{\sigma^L} hg)^*(n+L)\varepsilon_1 = c_n\varepsilon_1$. Hence $|(g^*_{\sigma^L} hg)_{11}(n+L)| = 1$. We have the assertion. □

Proposition 4.6. *Let $(l_2(\mathbf{Z}\times\mathbf{N}), \pi_f)$ be in Example 2.7 (iii).*

(i) *Let* $H_\infty \equiv \{g \in U_{\mathbf{Z}\times\mathbf{N}}(N) : \Gamma_g(\pi_f) \sim \pi_f\}$. *Then* H_∞ *is a subgroup of* $U_{\mathbf{Z}\times\mathbf{N}}(N)$ *and*

$$H_\infty = \{g \in U_{\mathbf{Z}\times\mathbf{N}}(N) : \text{ there is } M \geq 1 \text{ such that } |g_{11}(n)| = 1 \text{ for } n \geq M\}\,.$$

(ii) *For* $g \in U_{\mathbf{Z}\times\mathbf{N}}(N)$, *let* $H_g \equiv \{h \in U_{\mathbf{Z}\times\mathbf{N}}(N) : \Gamma_h(\Gamma_g(\pi_f)) \sim \Gamma_g(\pi_f)\}$. *Then*

$$H_g \equiv \left\{ h \in U_{\mathbf{Z}\times\mathbf{N}}(N) : \begin{array}{c} \text{there are } L \in \mathbf{Z} \text{ and } M \geq 1 \text{ such that} \\ |(g^*_{\sigma^L} hg)_{11}(n)| = 1 \text{ for each } n \geq M \end{array} \right\}.$$

Proof. (i) For $g, h \in H_\infty$, we can check $gh \in H_\infty$. By Lemma 4.4, it holds. (ii) By Lemma 4.5, it holds. □

4.3. Action of $U(N)$ on $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$.

Since the canonical $U(N)$-action on $\mathcal{O}_N$ can be regarded as the restriction of the gauge action $U_X(N)$ on $U(N) \subset U_X(N)$(which is called the *global action* in physics), properties of the action by $U(N)$ on $\mathcal{O}_N$ are obtained by Corollary of claims in § 4.1 and § 4.2. We give another proof of statements about $U(N)$ here.

We start simple statements about automorphisms and representations without proof.

Lemma 4.7. *Let* $\mathcal{A}$ *be a unital* C^**-algebra and* φ *an automorphism of* $\mathcal{A}$.

(i) *If* φ *is inner, then for any representation* π *of* $\mathcal{A}$, $\pi \circ \beta$ *and* π *are unitarily equivalent.*

(ii) *Let* π, π' *be representations of* $\mathcal{A}$. *If* $\pi \sim \pi'$, *then* $\pi \circ \varphi \sim \pi' \circ \varphi$.

(iii) *Let* π *be a representation of* $\mathcal{A}$. *If* π *is irreducible, then* $\pi \circ \varphi$ *is irreducible, too.*

Recall notations in § 2.1. For $(\mathcal{H}, \pi, \Omega) = GP(z)$ and $g \in U(N)$, we denote $(\mathcal{H}, \pi \circ \alpha_g, \Omega)$ by $GP(z) \circ \alpha_g$.

Lemma 4.8. *Let* $g = (g_{ij}) \in U(N)$.

(i) *If* $i_0 \in \{1, \ldots, N\}$ *and* $(\mathcal{H}, \pi, \Omega) = GP(\varepsilon_{i_0})$, *then* $\pi \circ \alpha_g \sim \pi$ *if and only if* $g_{i_0 i_0} = 1$.

(ii) *If* $\alpha_g \in \mathrm{Aut}\mathcal{O}_N$ *is inner, then* $g = I$.

Proof. (i) By assumption, $\pi(s_{i_0})\Omega = \Omega$. Put $z \in S(\mathbf{C}^N)$ by $z_j \equiv g^*_{j i_0}$ for $j = 1, \ldots, N$. Then $\alpha_{g^*}(s_{i_0}) = s(z)$. Hence $(\mathcal{H}, \pi \circ \alpha_g, u\Omega)$ is $GP(z)$. $\pi \circ \alpha_g \sim \pi$ if and only if $GP(z) \sim GP(\varepsilon_{i_0})$ if and only if $z \sim \varepsilon_{i_0}$. This is equivalent to $g^*_{i_0 i_0} = z_{i_0} = 1$.
(ii) If α_g is inner, then for each representation π, $\pi \circ \alpha_g \sim \pi$ by Lemma 4.7 (i). Therefore $GP(\varepsilon_i) \circ \alpha_g \sim GP(\varepsilon_i)$ for each $i = 1, \ldots, N$. From this, $g_{ii} = 1$ for each $i = 1, \ldots, N$ by (i). Therefore $g = I$ since g is unitary. □

The following claim was shown by [**6**, **16**]. We give other proof by using results about GP representation.

Proposition 4.9. *For each $g \in U(N) \setminus \{I\}$, α_g is outer.*

Proof. By Lemma 4.8 (ii), if $g \in U(N)$ and $g \neq I$, then α_g is not inner. Therefore α_g is outer. □

Theorem 4.10. *(*[**16**]*)*

(i) *If α is an automorphism on $\mathcal{O}_N$ satisfying $\alpha(s_1) = \lambda s_1$ for some complex number $\lambda \neq 1$ with modulus one, then α is outer.*

(ii) *The following automorphism σ of $\mathcal{O}_4$ is outer:*

$$\begin{aligned}\sigma(s_1) &\equiv s_1,\\ \sigma(s_2) &\equiv s_3,\\ \sigma(s_3) &\equiv s_2(s_1s_3^* + s_3s_1^* + s_2s_4^* + s_4s_2^*),\\ \sigma(s_4) &\equiv s_4(s_1s_3^* + s_3s_1^* + s_2s_4^* + s_4s_2^*).\end{aligned}$$

Proof. (i) If $\lambda \neq 1$, then $GP(\varepsilon_1) \circ \alpha = GP(\bar{\lambda}\varepsilon_1)$. Since $\varepsilon_1 \not\sim \bar{\lambda}\varepsilon_1$, $GP(\varepsilon_1) \circ \alpha \not\sim GP(\varepsilon_1)$. Hence α is outer by Lemma 4.7 (i).
(ii) By $\sigma(s_2) = s_3$, $GP(\varepsilon_3) \circ \sigma = GP(\varepsilon_2)$. Hence $GP(\varepsilon_3) \circ \sigma$ and $GP(\varepsilon_3)$ are not equivalent. Hence σ is outer by Lemma 4.7 (i). □

However, we can not prove the following claim in [**16**](Corollary B): *If α is a non trivial automorphism on $\mathcal{O}_N$ satisfying $\alpha(s_1) = \lambda s_1$ for some complex number λ with modulus one, then α is outer.* This statement is more general because the case $\lambda = 1$ is included.

By using results in § 4.1 and § 4.2, we show the followings:

Corollary 4.11. (i) *For $k \geq 1$, let $(l_2(\mathbf{N}), \pi_k)$ be in Example* 2.7 *(ii) and $g = (g_{ij}) \in U(N)$. Then $\pi_k \circ \alpha_g \sim \pi_k$ if and only if $(g_{11})^k = 1$.*

(ii) *Let $(l_2(\mathbf{Z}\times\mathbf{N}), \pi_f)$ be in Example* 2.7 *(iii) and $g = (g_{ij}) \in U(N)$. Then $\pi_f \circ \alpha_g \sim \pi_f$ if and only if $|g_{11}| = 1$.*

Proof. Note $\Gamma_{g^*}(\pi) = \pi \circ \alpha_g$ for a representation π of $\mathcal{O}_N$ when $g \in U(N) \subset U_{\mathbf{N}}(N)$. Hence (i) and (ii) follow from Lemma 4.1 and Lemma 4.4, respectively. □

Proposition 4.12. *Let $z \in TS(\mathbf{C}^N)$ and $g \in U(N)$. Then the followings hold:*

(i) *$GP(z) \circ \alpha_g = GP(T_{g^*}z)$.*

(ii) *$GP(z) \circ \alpha_g = GP(z)$ if and only if there is $p \in \mathbf{Z}$ such that $W_p z = T_g z$.*

Proof. (i) Let $(\mathcal{H}, \pi, \Omega) = GP(z)$. Then $(\pi \circ \alpha_g)(\alpha_{g^*})(s(z))\Omega = \Omega$ by definition. We see $\alpha_{g^*}(s(z)) = s(T_{g^*}z)$. Hence $(\mathcal{H}, \pi \circ \alpha_g, \Omega) = GP(T_{g^*}z)$. In consequence, we have the statement.

(ii) By (i) and Theorem 2.5 (iii), $GP(z)\circ\alpha_g = GP(z)$ if and only if $z \sim T_{g^*}z$. This is equivalent that there is $q \in \mathbf{Z}$ such that $W_q z = T_{g^*}z$. Since W and T commute, $T_{g^*} = (T_g)^{-1}$ and $W_{-q} = (W_q)^{-1}$, we have $W_p z = T_g z$ for $p \equiv -q$. □

In Proposition 4.12, when $z = \varepsilon_1 \otimes \varepsilon_2$ and $V_{\varepsilon_1\otimes\varepsilon_2} \equiv \{g \in U(N) : GP(\varepsilon_1 \otimes \varepsilon_2) \circ \alpha_g \sim GP(\varepsilon_1 \otimes \varepsilon_2)\}$, we have

$$V_{\varepsilon_1\otimes\varepsilon_2} = \left\{ \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}, \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} : c \in U(1) \right\} \times U(N-2).$$

In the same way, we have

$$V_{\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_2} = \left\{ \begin{pmatrix} c & 0 \\ 0 & \bar{c}^2 \end{pmatrix} : c \in U(1) \right\} \times U(N-2).$$

In this way, isotropy subgroups of $U(N)$ with respect to spectrums of $\mathcal{O}_N$ are not same in general.

Proposition 4.13. *Let $z \in S(\mathbf{C}^N)^\infty$ and $g \in U(N)$. Then the followings hold:*

(i) *$GP(z) \circ \alpha_g = GP(T_{g^*}z)$.*
(ii) *$GP(z) \circ \alpha_g = GP(z)$ if and only if there are $p, q \geq 0$ and $w \in \mathbf{T}^\infty$ such that $W_p T_{g^*} z = \tau_w W_q z$.*

Proof. (i) Let $(\mathcal{H}, \pi, \Omega) = GP(z)$ and $\Omega_n = \pi(s(z[n])^*)\Omega$ for $n \geq 1$. Put $y^{(n)} = g^* z^{(n)}$ for $n \geq 1$. Then $(\pi \circ \alpha_g)(s(y[n])^*)\Omega = \Omega_n$ for $n \geq 1$. Hence $(\mathcal{H}, \pi\circ\alpha_g, \Omega) = GP(T_{g^*}z)$. In consequence, $GP(z) \circ \alpha_g = GP(T_{g^*}z)$.
(ii) By (i), $GP(z) \circ \alpha_g = GP(z)$ if and only if $T_{g^*}z \sim z$. This is equivalent that there are $p, q \geq 0$ and $w \in \mathbf{T}^\infty$ such that $W_p T_{g^*} z = \tau_w W_q z$. □

5. Examples

5.1. Representation of $\mathcal{O}_2$ on $U(1)$. Let η be the Haar measure on $U(1)$. We consider the gauge transformation on a measure space $(U(1), \eta)$. For $n, m \in \mathbf{Z}$, put $g \in U_{U(1)}(2)$ defined by $g(w) \equiv \begin{pmatrix} w^n & 0 \\ 0 & w^m \end{pmatrix}$. Let $(L_2(U(1), \eta), \pi)$ be a representation of $\mathcal{O}_2$ defined by

$$(\pi(s_i)\phi)(w) \equiv \sqrt{2}\chi_{D_i}(w)\phi(w^2) \quad (i = 1, 2) \tag{5.1}$$

for $w \in U(1)$ and $\phi \in L_2(U(1), \eta)$. Note $(L_2(U(1), \eta), \pi, \mathbf{1})$ is $GP(2^{-1/2}(1, 1))$. where $D_1 \equiv \{e^{2\pi\sqrt{-1}\theta} \in U(1) : \theta \in [0, 1/2]\}$, $D_2 \equiv \{e^{2\pi\sqrt{-1}\theta} \in U(1) : \theta \in [1/2, 1]\}$ and $\mathbf{1}$ is the constant function on $U(1)$ with value 1. Then the gauge transformation of π in (5.1) by g is following:

$$(5.2)\quad \begin{aligned} ((\Gamma_g(\pi))(s_1)\phi)(w) = &\ \sqrt{2}\chi_{D_1}(w)w^{n/2}\phi(w^2), \\ ((\Gamma_g(\pi))(s_2)\phi)(w) = &\ \sqrt{2}\chi_{D_2}(w)(-w^{1/2})^m\phi(w^2). \end{aligned}$$

Lemma 5.1. *Let $(L_2(U(1),\eta),\Gamma_g(\pi))$ be a representation of $\mathcal{O}_2$ in (5.2). If $n = m = 2k$, $k \in \mathbf{Z}$, then $(L_2(U(1),\eta),\Gamma_g(\pi))$ contains $GP((2^{-1/2},2^{-1/2}))$ as a subrepresentation.*

Proof. Put $\pi_g \equiv \Gamma_g(\pi)$ and $z_0 \equiv 2^{-1/2}(1,1) \in S(\mathbf{C}^2)$. If $n = m = 2k$, then $(\pi_g(s(z_0))\phi)(w) = w^k\phi(w^2)$ for $\phi \in L_2(U(1),\eta)$. Hence $\pi_g(s(z_0))\zeta_c = \zeta_{2c+k}$ where $\zeta_c(w) \equiv w^c$ for $c \in \mathbf{R}$. From this,

$$\pi_g(s(z_0))\zeta_{-k} = \zeta_{-k} \quad (k \in \mathbf{Z}).$$

Therefore $(\mathcal{V},\pi_g,\zeta_{-k})$ is $GP(z_0)$ where $\mathcal{V} \equiv \pi_g(\mathcal{O}_2)\zeta_{-k}$. □

5.2. Heegaard splitting of S^3 and GP representations of $\mathcal{O}_2$. We show a relation between the Heegaard splitting of 3-dimensional sphere $S^3 \equiv \{(z_1,z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$([**7**]) and the orbit of non canonical $U(2)$-gauge action in the set of GP representations of $\mathcal{O}_2$. For $z = (z_1,z_2) \in S^3$, consider a cyclic representation $(\mathcal{H}_z,\pi_z)$ of $\mathcal{O}_2$ with a cyclic unit vector $v \in \mathcal{H}_z$ which satisfies the following condition:

$$(5.3)\qquad D_z v = v$$

where $D_z \equiv \pi_z(s_1(z_1s_1 + z_2s_2))$.

Question: *For $z \in S^3$, solve the equation (5.3) with respect to v by finding $(\mathcal{H}_z,\pi_z)$.*

This problem is the origin of the study of GP representations of $\mathcal{O}_N$ for us. By generalize this problem and solution, we obtain [**8**, **9**, **10**]. In consequence, GP representations are formulated without gauge transformation.

We give the answer of this question in the following:

Theorem 5.2. *For $z \in S^3$, assume that $(\mathcal{H}_z,\pi_z)$ is in (5.3).*

(i) *(Existence and realization) For $z \in S^3$, $(\mathcal{H}_z,\pi_z)$ is realized on $\mathcal{H}_z = l_2(\mathbf{N})$ as a pair t_1,t_2 of operators:*

$$\begin{cases} t_1e_2 = & \bar{z}_1e_1 + \bar{w}_1e_4, \\ t_2e_2 = & \bar{z}_2e_1 + \bar{w}_2e_4, \end{cases} \quad \begin{cases} t_1e_1 = & e_2, \\ t_2e_1 = & e_3, \end{cases} \quad \begin{cases} t_1e_n = & e_{2n-1}, \\ t_2e_n = & e_{2n}, \end{cases}$$

for $n \geq 3$ where $w = (w_1,w_2) \in S^3$ is chosen as $< w|z >= 0$.

(ii) *(Irreducibility) For* $z = (z_1, z_2) \in S^3$, $(\mathcal{H}_z, \pi_z)$ *is irreducible if and only if* $z_2 \neq 0$.

(iii) *(Classification) For* $z = (z_1, z_2) \in S^3$, *all of possibilities are only followings:*

a) *If* $z_2 = 0$, *then the following irreducible decomposition holds:* $\pi_z \sim GP((\sqrt{z_1}, 0)) \oplus GP((-\sqrt{z_1}, 0))$.

b) *If* $z_1 z_2 \neq 0$, *then* π_z *is a representation of* $\mathcal{O}_2$ *which is not equivalent to any permutative representation and its rotation by* $U(2)$*-automorphism.*

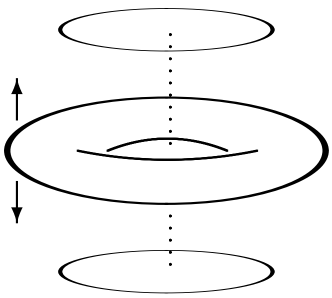
c) *If* $z_1 = 0$, *then* $\pi_z \sim GP(z_2 \varepsilon_1 \otimes \varepsilon_2)$.

(iv) *(Equivalence) For* $z, z' \in S^3$, $\pi_z \sim \pi_{z'}$ *if and only if* $z = z'$.

Proof. By Definition 2.4 (i) and (5.3), π_z is just $GP(\varepsilon_1 \otimes z)$. Let $(l_2(\mathbf{N}), \pi_2)$ be the case $k = 2$ of Example in 2.7 (ii). We see that $\pi_2(s_1)e_1 = e_2$, $\pi_2(s_1)e_2 = e_1$, $\pi_2(s_2)e_1 = e_3$, $\pi_2(s_2)e_2 = e_4$, $\pi_2(s_i)e_n = e_{2(n-1)+i}$ for $n \geq 3$ and $i = 1, 2$.
(i) Choose $g \in U(2)$ as $g_{j1} = z_j$ for $j = 1, 2$. Then $\Gamma_{\hat{g}}(\pi_2) \sim GP(\varepsilon_1 \otimes z)$ by Lemma 4.1 (i). Put $\pi' \equiv \Gamma_{\hat{g}}(\pi_2)$. Then $\pi'(s_i)e_2 = (g^*_{1i}\pi_2(s_1) + g^*_{2i}\pi_2(s_2))e_2$, $\pi'(s_i)e_n = \pi(s_i)e_n$ for $n \geq 1, n \neq 2$. We see $t_i = \pi'(s_i)$ for $i = 1, 2$ when $w_1 = g_{12}$ and $w_2 = g_{22}$. Hence the assertion holds.
(ii),(iii) and (iv) follow from properties of GP representations immediately. □

By Theorem 5.2, we illustrate solutions of (5.3). Consider an action of a torus $T^2 \equiv U(1) \times U(1)$ on S^3 by $\tau_{w_1,w_2}(z_1, z_2) \equiv (w_1 z_1, w_2 z_2)$ for $(w_1, w_2) \in T^2$ and $(z_1, z_2) \in S^3$. The orbit decomposition of S^3 by T^2 is just Heegaard splitting of S^3. On the other hand, we have a gauge transformation $\Gamma_{\hat{w}}(\pi_2)$ of π_z by $w \in T^2 \subset U(2)$. Then the orbit of T^2 in $\mathrm{Rep}(\mathcal{O}_2, l_2(\mathbf{N}))$ is corresponded to the Heegaard splitting of S^3 in the following:

Heegaard splitting of S^3	Representations of $\mathcal{O}_2$
S^1	$GP((e^{\sqrt{-1}\theta/2}, 0)) \oplus GP((-e^{\sqrt{-1}\theta/2}, 0))$
$T^2 \times (0,1)$	$GP((1,0) \otimes z)$ $z = (ae^{\sqrt{-1}\theta_1}, \sqrt{1-a^2}e^{\sqrt{-1}\theta_2})$ $0 < a < 1$
S^1	$GP((1,0) \otimes (0, e^{\sqrt{-1}\theta}))$



In this way, S^3 is regarded as the set of equivalence classes of GP representations of $\mathcal{O}_2$ in a sense of $U(2)$-homogeneous space with same Heegaard splitting by T^2. Any point in S^3 is corresponded with a representation of $\mathcal{O}_2$ and any two points in S^3 are corresponded with inequivalent representations. The north polar circle of S^3 in the above is corresponded with reducible representations of $\mathcal{O}_2$ and others are irreducible.

References

[1] M.Abe and K.Kawamura, *Recursive Fermion System in Cuntz Algebra. I — Embeddings of Fermion Algebra into Cuntz Algebra —*, Comm. Math. Phys. 228,85-101 (2002).

[2] O.Bratteli and P.E.T.Jorgensen, *Iterated function Systems and Permutation Representations of the Cuntz algebra*, Memories Amer. Math. Soc. No.663 (1999).

[3] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173-185.

[4] K.R.Davidson and D.R.Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras*, Math.Ann. 311, 275-303 (1998).

[5] K.R.Davidson and D.R.Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proc. London Math. Soc. (3) 78 (1999) 401-430.

[6] M.Enomoto, H.Takehaha and Y.Watatani, *Automorphisms on Cuntz algebras*, Math.Japonica **24** (1979), 231-234.

[7] J.Hempel, 3-*manifolds*, Ann.Math.Stud., Princeton Univ. Press, 1976.

[8] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. I — Generalization of cycle type—*, preprint RIMS-1380 (2002).

[9] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. II — Irreducible decomposition of periodic cycle—*, preprint RIMS-1388 (2002).

[10] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. III — Generalization of chain type—* , preprint RIMS-1423 (2003).

[11] K.Kawamura, *Representations of the Cuntz algebra $\mathcal{O}_2$ arising from real quadratic transformations*, preprint RIMS-1396 (2003).

[12] K.Kawamura and O.Suzuki, *Construction of orthonormal basis on self-similar sets by generalized permutative representations of the Cuntz algebras*, preprint RIMS-1408 (2003).

[13] K.Kawamura, *Representations of the Cuntz algebra $\mathcal{O}_3$ arising from real cubic transformations*, preprint RIMS-1412 (2003).

[14] K.Kawamura, *Three representations of the Cuntz algebra $\mathcal{O}_2$ by a pair of operators arising from a $\mathbf{Z}_2$-graded dynamical system*, preprint RIMS-1415 (2003).

[15] K.Kawamura, *Representations of the Cuntz algebra $\mathcal{O}_2$ arising from complex quadratic transformations —Annular basis of $L_2(\mathbf{C})$—*, preprint RIMS-1418 (2003).

[16] K.Matsumoto and J.Tomiyama, *Outer automorphisms on Cuntz algebras*, Bull.London Math.Soc. **25** (1993), 64-66.

[17] R.Mills and C.Yang, *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96** 1 (1954).

[18] N.Nakanishi and I.Ojima, *Covariant operator formalism of gauge theories and quantum gravity*, World Scientific (1990).