Polynomial endomorphisms of the Cuntz algebras arising from permutations. I
—General theory—

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We introduce a class of endomorphisms of the Cuntz algebras which are defined by polynomials of generators. We show their classification under unitary equivalence by help of permutative representations.

1. Introduction

1.1. Main theorem. In usual, the irreducible decomposition of a representation of an operator algebra does not make sense because there is no uniqueness of such decomposition in general. This fact disturbs an intention to study an ordinary representation theory of operator algebras like that of semisimple Lie algebras and quantum groups. In spite of this, *permutative representations* of the Cuntz algebra $\mathcal{O}_N$([**3**, **5**, **6**]) are completely reducible and their irreducible decompositions are unique up to unitary equivalences. Roughly speaking, there are two kinds of (cyclic)permutative representations, *"cycle"* and *"chain"*. This remarkable property assists to characterize endomorphisms of $\mathcal{O}_N$, too, in the following way: For $N \geq 2$, let $s_1, \ldots, s_N$ be generators of $\mathcal{O}_N$ and $\{1, \ldots, N\}^k \equiv \{(j_l)_{l=1}^k : j_l = 1, \ldots, N,\ l = 1, \ldots, k\}$ for $k \geq 1$.

Theorem 1.1. *For a permutation σ on $\{1, \ldots, N\}^k$, $k \geq 1$, let ψ_σ be an endomorphism of $\mathcal{O}_N$ defined by*

$$\psi_\sigma(s_i) \equiv u_\sigma s_i \quad (i = 1, \ldots, N) \tag{1.1}$$

where $u_\sigma \equiv \sum_{J\in\{1,\ldots,N\}^k} s_{\sigma(J)}(s_J)^*$ and $s_J \equiv s_{j_1} \cdots s_{j_k}$ when $J = (j_1, \ldots, j_k)$. If $(\mathcal{H}, \pi)$ is a permutative representation, then $(\mathcal{H}, \pi \circ \psi_\sigma)$ is, too. Specially, if $(\mathcal{H}, \pi)$ has only cycles, then $(\mathcal{H}, \pi \circ \psi_\sigma)$ does, too.

Theorem 1.1 assures the completely reducibility of $(\mathcal{H}, \pi \circ \psi_\sigma)$ for any permutative representation $(\mathcal{H}, \pi)$ and any permutation σ.

The first aim of this article is a preparation of tools of analysis of endomorphisms of $\mathcal{O}_N$ by representations.

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1.2. Origin of study of endomorphisms. Endomorphisms of operator algebras are studied in fields of operator algebras and quantum field theory([**2**, **7**, **8**, **9**, **10**]) for aims of computation and construction of the indices of subalgebras, and formulation of super selection sectors. In these theories, we are interest in concrete endomorphisms of $\mathcal{O}_N$ which are defined by polynomials of generators $s_1, \ldots, s_N$ and their conjugates. For example, we showed the branching law of representations of the CAR algebra which are associated with endomorphisms of $\mathcal{O}_N$ in [**1**]. In fact, ψ_σ in (1.1) is not only an endomorphism of $\mathcal{O}_N$ but also that of $UHF_N = \mathcal{O}_N^{U(1)}$ because ψ_σ is covariant with respect to the gauge action of $\mathcal{O}_N$. This tame class of endomorphisms of $\mathcal{O}_N$, $N \geq 2$, was obtained by generalizing the following endomorphism ρ_ν of $\mathcal{O}_3$:

$$\left\{\begin{array}{l}\rho_\nu(s_1) \equiv s_1 s_2 s_3^* + s_2 s_3 s_1^* + s_3 s_1 s_2^*, \\ \\ \rho_\nu(s_2) \equiv s_2 s_1 s_3^* + s_3 s_2 s_1^* + s_1 s_3 s_2^*, \\ \\ \rho_\nu(s_3) \equiv s_1 s_1 s_1^* + s_2 s_2 s_2^* + s_3 s_3 s_3^*.\end{array}\right. \tag{1.2}$$

N.Nakanishi found ρ_ν in (1.2) by trial and error([**18**]). (Reader can check that three elements $\rho_\nu(s_1), \rho_\nu(s_2), \rho_\nu(s_3)$ satisfy the relations of generators of $\mathcal{O}_3$. Therefore they define an endomorphism ρ_ν of $\mathcal{O}_3$.) Such wild type of endomorphism of C*-algebra is beyond someone's reach by well known method because there is no general assumption from index theory and quantum field theory. In other words, we need new approach for ρ_ν which depends on just the definition of ρ_ν. Fortunately, we develop tools of analysis of ρ_ν and show the following:

Theorem 1.2. *ρ_ν in (1.2) is a unital $*$-endomorphism of $\mathcal{O}_3$ which is irreducible, that is, $\rho_\nu(\mathcal{O}_3)' \cap \mathcal{O}_3 = \mathbf{C}I$ and not an automorphism. Specially, ρ_ν is not unitarily equivalent to the canonical endomorphism of $\mathcal{O}_3$.*

In this way, we can construct many naive nontrivial examples of endomorphisms of $\mathcal{O}_N$ systematically as ψ_σ in (1.1). The second aim of this article is an introduction of our studies of endomorphisms and a notion of *sector* of C*-algebras in the next work([**16**]). For this purpose, we show elementary examples and naive methods of classification of endomorphisms of $\mathcal{O}_N$. After discovery of ρ_ν in (1.2), we studied the following endomorphisms of $\mathcal{O}_2$.

Theorem 1.3. *Put $E_{2,2}$ the set of ψ_σ in (1.1) by a permutation σ on a set $\{1,2\}^2 = \{(1,1),(1,2),(2,1),(2,2)\}$. Then the followings hold:*

(i) *The number of unitary equivalence classes of elements in $E_{2,2}$ is 16.*
(ii) *$G_2 \equiv \mathrm{Aut}\mathcal{O}_2 \cap E_{2,2}$ is a subgroup of the automorphism group $\mathrm{Aut}\mathcal{O}_2$ of $\mathcal{O}_2$ which is isomorphic to the Klein's four-group. G_2 consists of two outer, and two inner automorphisms.*

(iii) $E_{2,2}\setminus G_2$ *consists of* 10 *irreducible and* 10 *reducible endomorphisms. Numbers of equivalence classes in them are* 5 *and* 9*, respectively. Specially, the (class of)canonical endomorphism of* $\mathcal{O}_2$ *belongs to the set of reducible classes in* $E_{2,2}$.

For example, the following irreducible endomorphisms ρ, $\bar{\rho}$, η of $\mathcal{O}_2$ are in $E_{2,2}\setminus G_2$:

$$\begin{array}{llll}\rho(s_1)\equiv & s_{12,1}+s_{11,2}, & \rho(s_2)\equiv & s_2,\\[1ex] \bar{\rho}(s_1)\equiv & s_{21,1}+s_{12,2}, & \bar{\rho}(s_2)\equiv & s_{11,1}+s_{22,2},\\[1ex] \eta(s_1)\equiv & s_{22,1}+s_{11,2}, & \eta(s_2)\equiv & s_{21,1}+s_{12,2}\end{array}$$

where $s_{ij,k}\equiv s_is_js_k^*$ for $i,j,k=1,2$.(ρ and $\bar{\rho}$ are *"conjugate"* each other in a sense of super selection sector. We show relations among $\rho,\bar{\rho},\eta$ in [**17**].) Endomorphisms in Theorem 1.3 are called the *second order permutative endomorphisms* of $\mathcal{O}_2$(§ 6-1 [**1**]). They play an important role in representation theory of CAR algebra.

In § 2, we prepare branching function systems and transformation of them by permutations. In § 3, we review the permutation representation of $\mathcal{O}_N$. In § 4, we introduce a generalization of ρ_ν and $E_{2,2}$ for $\mathcal{O}_N$ for $N\geq 2$, that is, ψ_σ in (1.1), and show Theorem 1.1. In § 5, we prove Theorem 1.2 and Theorem 1.3 as examples of Theorem 1.1.

2. Action of permutations on branching function systems

We introduce several sets of multi indices which consist of numbers $1,\ldots,N$ for $N\geq 2$. Put

$$\{1,\ldots,N\}^*\equiv\coprod_{k\geq 0}\{1,\ldots,N\}^k,\quad \{1,\ldots,N\}^*_1\equiv\coprod_{k\geq 1}\{1,\ldots,N\}^k,$$

$\{1,\ldots,N\}^0\equiv\{0\}$, $\{1,\ldots,N\}^k\equiv\{(j_l)_{l=1}^k : j_l=1,\ldots,N,\ l=1,\ldots,k\}$ for $k\geq 1$. For $J\in\{1,\ldots,N\}^*$, the length $|J|$ of J is defined by $|J|\equiv k$ when $J\in\{1,\ldots,N\}^k$, $k\geq 0$. For $J_1,J_2\in\{1,\ldots,N\}^*$, $J_1\cup J_2\equiv(j_1,\ldots,j_k,j_1^{'},\ldots,j_l^{'})$ when $J_1=(j_1,\ldots,j_k)$ and $J_2=(j_1^{'},\ldots,j_l^{'})$. Specially, we define $J\cup\{0\}=\{0\}\cup J=J$ for $J\in\{1,\ldots,N\}^*$ and $(i,J)\equiv(i)\cup J$ for convention. For $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k$ and $\tau\in\mathbf{Z}_k$, denote $\tau(J)=(j_{\tau(1)},\ldots,j_{\tau(k)})$.

Definition 2.1. (i) $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k$ *is periodic if there is* $\tau\in\mathbf{Z}_k\setminus\{0\}$ *such that* $\tau(J)=J$.

(ii) *For $J_1, J_2 \in \{1,\ldots,N\}_1^*$, $J_1 \sim J_2$ if there are $k \geq 1$ and $\tau \in \mathbf{Z}_k$ such that $J_1, J_2 \in \{1,\ldots,N\}^k$ and $\tau(J_1) = J_2$.*

2.1. Branching function systems. Let Λ be an infinite set and $N \geq 2$.

Definition 2.2. *$f = \{f_i\}_{i=1}^N$ is a branching function system on Λ if f_i is an injective transformation on Λ for $i = 1,\ldots,N$ such that a family of their images coincides a partition of Λ.*

A branching function system was introduced by [**3**] in order to study representation of $\mathcal{O}_N$. It is convenient to construct concrete examples of representations easily. It is possible to consider branching function systems on any set with infinite cardinality. About the measure theoretical generalization of branching function system, see [**15**]. We often treat cases $\Lambda = \mathbf{N} \equiv \{1,2,3,\ldots\}, \mathbf{Z}$. Put $\mathrm{BFS}_N(\Lambda)$ the set of all branching function systems on Λ. For $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$, we denote $f_J \equiv f_{j_1} \circ \cdots \circ f_{j_k}$ when $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, $k \geq 1$, and define $f_0 \equiv id$.

Definition 2.3. *Let $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$.*

(i) *For $x, y \in \Lambda$, $x \sim y$(with respect to f) if there are $J_1, J_2 \in \{1,\ldots,N\}^*$ and $z \in \Lambda$ such that $f_{J_1}(z) = x$ and $f_{J_2}(z) = y$.*
(ii) *For $x \in \Lambda$, denote $A_f(x) \equiv \{y \in \Lambda : x \sim y\}$.*
(iii) *f is cyclic if there is an element $x \in \Lambda$ such that $\Lambda = A_f(x)$.*
(iv) *For $k \geq 1$, $R = \{n_1,\ldots,n_k\} \subset \Lambda$ is a k-cycle of f if $n_l \neq n_{l'}$ when $l \neq l'$ and there is $J \in \{1,\ldots,N\}^k$ such that $f_{j_l}(n_l) = n_{\tau(l)}$ for $l = 1,\ldots,k$ where τ is a shift on $\mathbf{Z}_k$.*
(v) *$R = \{n_l\}_{l\in\mathbf{N}} \subset \Lambda$ is a chain of f if $n_l \neq n_{l'}$ when $l \neq l'$ and there is $\{j_l \in \{1,\ldots,N\} : l \in \mathbf{N}\}$ such that $f_{j_l}^{-1}(n_l) = n_{l+1}$ for $l \in \mathbf{N}$.*
(vi) *f has a k-cycle(chain) if there is a k-cycle(resp. chain) of f in Λ. Specially, we call simply that f has a cycle if f has a k-cycle some $k \geq 1$.*

The definition of cyclicity of branching function system is corresponded with that of representation of $\mathcal{O}_N$. In order to treat the decomposition of representations, we prepare the followings:

Definition 2.4. *Let Ξ be a set.*

(i) *For a branching function system $f^{[\omega]} = \{f_i^{[\omega]}\}_{i=1}^N$ on an infinite set Λ_ω for $\omega \in \Xi$, f is the direct sum of $\{f^{[\omega]}\}_{\omega\in\Xi}$ if $f = \{f_i\}_{i=1}^N$ is a branching function system on a set $\Lambda \equiv \coprod_{\omega\in\Xi} \Lambda_\omega$ which is defined by $f_i(n) \equiv f_i^{[\omega]}(n)$ when $n \in \Lambda_\omega$ for $i = 1,\ldots,N$ and $\omega \in \Xi$.*
(ii) *For a branching function system $f \in \mathrm{BFS}_N(\Lambda)$, $f = \oplus_{\omega\in\Xi} f^{[\omega]}$ is a decomposition of f into a family $\{f^{[\omega]}\}_{\omega\in\Xi}$ if there is a family $\{\Lambda_\omega\}_{\omega\in\Xi}$ of subsets of Λ such that f is the direct sum of $\{f^{[\omega]}\}_{\omega\in\Xi}$.*

Proposition 2.5. *Let* $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$.

(i) *There is a decomposition* $\Lambda = \coprod_{\omega\in\Xi} \Lambda_\omega$ *such that* $\#\Lambda_\omega = \infty$, $f|_{\Lambda_\omega} \equiv \{f_i|_{\Lambda_\omega}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_\omega)$ *and* $f|_{\Lambda_\omega}$ *is cyclic for each* $\omega \in \Xi$.

(ii) *Assume that* f *is cyclic. Then there is only one case in the followings: a)* f *has just one cycle. b)* f *has just one chain where we identify two chains* $R = \{n_l \in \Lambda : l \in \mathbf{N}\}$ *and* $R' = \{m_l \in \Lambda : l \in \mathbf{N}\}$ *when there are* $M, L \geq 0$ *such that* $n_{l+L} = m_l$ *for each* $l > M$.

Proof. See Appendix A. □

Definition 2.6. (i) *For* $J \in \{1, \ldots, N\}^k$, $k \geq 1$, $f \in \mathrm{BFS}_N(\Lambda)$ *is* $P(J)$ *if* f *is cyclic and has a cycle* $R = \{n_1, \ldots, n_k\}$ *such that* $f_J(n_k) = n_k$.

(ii) *For* $N \geq 2$, $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_1)$ *and* $g = \{g_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_2)$ *are equivalent if there is a bijection* φ *from* Λ_1 *to* Λ_2 *such that* $\varphi \circ f_i \circ \varphi^{-1} = g_i$ *for* $i = 1, \ldots, N$.

By Proposition 2.5, Definition 2.6 (i) makes sense. In this article, we treat only cycle case.

Lemma 2.7. *Let* Λ_1 *and* Λ_2 *be infinite sets.*

(i) *If* $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_1)$ *and* φ *is a bijection from* Λ_1 *to* Λ_2, *then* $\varphi \circ f \circ \varphi^{-1} \equiv \{\varphi \circ f_i \circ \varphi^{-1}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_2)$.

(ii) *If* $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_1)$ *with a cycle* R *and* φ *is a bijection from* Λ_1 *to* Λ_2, *then* $\varphi(R)$ *is a cycle of* $\varphi \circ f \circ \varphi^{-1}$.

(iii) *Let* $f \in \mathrm{BFS}_N(\Lambda_1)$ *and* $g \in \mathrm{BFS}_N(\Lambda_2)$. *Assume that there are* $J_1, J_2 \in \{1, \ldots, N\}_1^*$ *such that* f *is* $P(J_1)$ *and* g *is* $P(J_2)$. *Then* f *and* g *are equivalent if and only if* $J_1 \sim J_2$.

Proof. (i) By checking the condition of branching function system for $\varphi \circ f \circ \varphi^{-1}$, we have the assertion.
(ii) By direct computation, we have the statement.
(iii) Let R and R' be cycles of f and g', respectively.

If f and g are equivalent, then there is a bijection φ from Λ_1 to Λ_2 such that $\varphi \circ f_i \circ \varphi^{-1} = g_i$ for $i = 1, \ldots, N$. By (ii) and $\varphi \circ f_{J_1} \circ \varphi^{-1} = g_{J_1}$, $\varphi(R)$ is a cycle of g such that $\varphi(R)$ satisfies cycle condition of g by J_1. Furthermore $\varphi(R) = R'$ by Proposition 2.5 (ii). From this, we see $J_1 \sim J_2$.

Assume $J_1 \sim J_2$. Then we have a map φ_0 from R to R' such that $\varphi_0 \circ f \circ \varphi_0^{-1} = g$ on R'. Because of cyclicity of f on Λ_1 and that of g on Λ_2, we can extend φ_0 to a map φ such that $\varphi \circ f \circ \varphi^{-1} = g$. Therefore $f \sim g$. □

In order to show the completely reducibility about the action of permutative endomorphisms on permutative representations(Theorem 1.1, Theorem 4.11), we use the order structure of a set $\Lambda = \mathbf{N}$.

Lemma 2.8. *Let $f = \{f_i\}_{i=1}^N$ be a branching function system on $\mathbf{N}$. If there is a subset C of $\mathbf{N}$ such that $f_i(C) \subset C$ and f_i is strictly monotone increasing on C for each $i = 1, \ldots, N$, then f has neither cycle nor chain in C.*

Proof. Assume that f has a cycle R in C. Put n_0 the minimum number in R. Then there are $i \in \{1, \ldots, N\}$ and $m \in R$ such that $f_i(m) = n_0 < m$. This contradicts the assumption of f. Hence f has no cycle in C. In the same way, we see that f has no chain in C, too. $\square$

Lemma 2.9. *Let $f = \{f_i\}_{i=1}^N$ be a branching function system on $\mathbf{N}$. Assume that there is a subset $C \subset \mathbf{N}$ such that*

$$f_i(C) \subset C, \quad f_i(n) = N(n-1) + i \quad (n \in C) \tag{2.1}$$

for $i = 1, \ldots, N$. Then the followings hold:

(i) *For $n \in C$, $N^{l+k} + 1 \le f_J(n) \le N^{l+k+1}$ when $J \in \{1, \ldots, N\}^l$, $l \ge 1$ and $N^k + 1 \le n \le N^{k+1}$.*
(ii) *For $J, J^{'} \in \{1, \ldots, N\}_1^*$ and $n \in C$, $f_J(n) < f_{J^{'}}(n)$ when $|J| < |J^{'}|$.*
(iii) *f has neither cycle nor chain in C.*

Proof. (i) Note that $f_1(N^l + 1) = N^{l+1} + 1$ and $f_N(N^l) = N^{l+1}$ when $N^l, N^l + 1 \in C$. Because $f_1(n) \le f_i(n) \le f_N(n)$ for $i = 1, \ldots, N$ and $n \in C$, $f_J(n) \ge f_1^l(N^k + 1) = N^{k+l} + 1$, $f_J(n) \le f_N^l(N^{k+1}) = N^{k+l+1}$.
(ii) Assume $N^k + 1 \le n \le N^{k+1}$. By (i), $f_J(n) \le N^{|J|+k+1} \le N^{|J^{'}|+k} < N^{|J|^{'}+k} + 1 \le f_{J^{'}}(n)$.
(iii) By Lemma 2.8, it holds. $\square$

2.2. Transformation of branching function systems. Let $\mathfrak{S}_{N,k}$ be the set of all bijective transformations on $\{1, \ldots, N\}^k$ for $k \ge 1$. Put a bijective map κ from $\{1, \ldots, N\}^k$ to a set $\Sigma_{N^k} \equiv \{1, 2, 3, \ldots, N^k - 1, N^k\}$ by $\kappa(i_1, \ldots, i_k) \equiv \sum_{l=1}^k N^{k-l}(i_l - 1) + 1$. We often identify $\mathfrak{S}_{N,k}$ and the (symmetric)group $\mathfrak{S}_{N^k}$ of all permutations on Σ_{N^k} by corresponding $\sigma \in \mathfrak{S}_{N,k}$ and $\kappa \circ \sigma \circ \kappa^{-1} \in \mathfrak{S}_{N^k}$. Specially, $\kappa = id$ on $\{1, \ldots, N\} = \Sigma_N$. By a natural identification $\mathfrak{S}_{N,k}$ and a subset $\mathfrak{S}_{N,k} \times \{id\}$ of $\mathfrak{S}_{N,k+1}$, $k \ge 1$, we can consider $\mathfrak{S}_{N,*} \equiv \lim_{\to k} \mathfrak{S}_{N,k}$.

For $\sigma \in \mathfrak{S}_{N,k}$ and $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$, put $f^{(\sigma)} = \{f_i^{(\sigma)}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$ by

$$f_i^{(\sigma)} \equiv f_{\sigma(i)} \quad (k=1), \quad f_i^{(\sigma)}(f_J(n)) \equiv f_{\sigma(i,J)}(n) \quad (k \ge 2) \tag{2.2}$$

for $n \in \Lambda$, $i = 1, \ldots, N$ and $J \in \{1, \ldots, N\}^{k-1}$. For $\sigma \in \mathfrak{S}_{N,*}$, define a transformation Φ_σ on $\mathrm{BFS}_N(\Lambda)$ by

$$\Phi_\sigma(f) \equiv f^{(\sigma)} \quad (f \in \mathrm{BFS}_N(\Lambda)). \tag{2.3}$$

Remark $\Phi_\sigma \circ \Phi_{\sigma'} \neq \Phi_{\sigma\circ\sigma'}$ in general.

Lemma 2.10. *Let $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $k \geq 1$ and $\sigma \in \mathfrak{S}_N = \mathfrak{S}_{N,1}$. If $f \in \mathrm{BFS}_N(\Lambda)$ is $P(J)$ in Definition 2.6, then $f^{(\sigma)}$ is $P(J_{\sigma^{-1}})$. where $J_{\sigma^{-1}} \equiv (\sigma^{-1}(j_1), \ldots, \sigma^{-1}(j_k))$.*

Proof. Let $\{n_j\}_{j=1}^k \subset \Lambda$ be unique cycle of f such that $f_{j_l}(n_l) = n_{\tau(l)}$ for $l = 1, \ldots, k$. Then $f^{(\sigma)}_{\sigma^{-1}(j_l)}(n_l) = f_{\sigma(\sigma^{-1}(j_l))}(n_l) = f_{j_l}(n_l) = n_{\tau(l)}$ for $l = 1, \ldots, k$. Hence $\{n_j\}_{j=1}^k$ is a cycle of $f^{(\sigma)}$, too. From this, the assertion holds. $\square$

Lemma 2.11. *Let $f = \{f_i\}_{i=1}^N$ be a branching function system on $\mathbf{N}$. Assume that there is a subset C of $\mathbf{N}$ which satisfies (2.1).*

(i) *For $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 2$, $f_i^{(\sigma)}$ in (2.2) is strictly monotone increasing on $\hat{C} \equiv \bigcup_{J\in\{1,\ldots,N\}^{l-1}} f_J(C)$ for $i = 1, \ldots, N$.*

(ii) *In (i), $f^{(\sigma)}$ has neither cycle nor chain in $\hat{C}$.*

Proof. (i) By definition of $f_i^{(\sigma)}$ and Lemma 2.9 (ii), $f_i^{(\sigma)}(f_J(n)) = f_{\sigma(i,J)}(n) > f_J(n)$ for $n \in C$, $J \in \{1, \ldots, N\}^{l-1}$. Therefore $f_i^{(\sigma)}(m) > m$ for each $m \in \hat{C}$.
(ii) By definition and the choice of C, $f_j^{(\sigma)}(\hat{C}) \subset \hat{C}$ for each $j = 1, \ldots, N$. By (i) and Lemma 2.8, the statement holds. $\square$

Next we show concrete examples of branching function system on $\mathbf{N}$ and its transformation by permutations.

Lemma 2.12. *For $J \in \{1, \ldots, N\}^*_1$, define a branching function system $f = \{f_i\}_{i=1}^N$ on $\mathbf{N}$ defined as follows: When $J = (j) \in \{1, \ldots, N\}$, put*

$$f_i(1) \equiv \begin{cases} i+1 & (1 \leq i < j), \\ 1 & (i = j), \\ i & (j \leq i \leq N), \end{cases} \qquad f_i(n) \equiv N(n-1) + i \quad (n \geq 2)$$

for $i = 1, \ldots, N$. *When* $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $k \geq 2$, *put*

$$f_i(1) \equiv \begin{cases} k+i & (1 \leq i < j_1), \\ k & (i = j_1), \\ k+i-1 & (j_1 \leq i \leq N), \end{cases}$$

$$f_i(l) \equiv \begin{cases} k+(N-1)(l-1)+m & (1 \leq i < j_l), \\ l-1 & (i = j_l), \\ k+(N-1)(l-1)+i-1 & (j_l \leq i \leq N), \end{cases}$$

$$f_i(n) \equiv \quad N(n-1)+i$$

for $l = 2, \ldots, k$, $n \geq k+1$ *and* $i = 1, \ldots, N$. *Then the followings hold:*

(i) f *is* $P(J)$.
(ii) *For* $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, $f^{(\sigma)}$ *has no chain.*
(iii) *For* $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, $f^{(\sigma)}$ *is decomposed into a finite direct sum of cycles.*

Proof. (i) Note $\{f_i(n) : i = 1, \ldots, N, n = 1, \ldots, k\} = \{1, \ldots, Nk\}$ and $\{f_{J'}(1) : J' \in \{1, \ldots, N\}^*\} = \mathbf{N}$. Furthermore $f_J(k) = k$. Hence the assertion holds by definition of $P(J)$.
(ii) Let $C \equiv \{n \in \mathbf{N} : n \geq k+1\}$. Then f satisfies (2.1). Hence $f^{(\sigma)}$ has neither cycle nor chain in $\hat{C} = \{n \in \mathbf{N} : n \geq N^{l-1}k+1\}$ by Lemma 2.11 (ii). Because $\mathbf{N} \setminus \hat{C} = \{1, \ldots, N^{l-1}k\}$ is a finite set, $f^{(\sigma)}$ has no chain in $\mathbf{N} \setminus \hat{C}$. Therefore $f^{(\sigma)}$ has no chain in $\mathbf{N}$.
(iii) By (ii) and Proposition 2.5 (i), $f^{(\sigma)}$ is decomposed into a direct sum of cyclic branching function systems with cycle. By proof of (ii), if $f^{(\sigma)}$ has a cycle, then it is in $\{1, \ldots, N^{l-1}k\}$. Hence $f^{(\sigma)}$ has finite number of cycles in $\mathbf{N}$ at most. Therefore the statement holds. □

Theorem 2.13. *For* $J \in \{1, \ldots, N\}^*$, $|J| \geq 1$, *let* f *be* $P(J)$ *on an infinite set* Λ. *Then for each* $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, $f^{(\sigma)}$ *is a direct finite sum of cycles.*

Proof. Let f be $P(J)$. Then the assumption of cyclicities of f on Λ, Λ is countable. Hence we can take a bijection φ from Λ to $\mathbf{N}$ such that f is equivalent to a branching function system $f' \equiv \varphi \circ f \circ \varphi^{-1}$. By Lemma 2.12 and Lemma 2.7 (iii), the statement holds. □

3. Permutative representations

For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**4**]), that is, it is a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad (i, j = 1, \ldots, N), \quad s_1 s_1^* + \cdots + s_N s_N^* = I. \tag{3.1}$$

In this paper, any representation and endomorphism are assumed unital and $*$-preserving. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1, \ldots, S_N$ on an infinite dimensional Hilbert space which satisfy (3.1) in order to construct a representation of $\mathcal{O}_N$. In the same reason, it is sufficient to define elements $T_1, \ldots, T_N$ in $\mathcal{O}_N$ which satisfy (3.1) in order to construct an endomorphism of $\mathcal{O}_N$.

Put α an action of a unitary group $U(N)$ on $\mathcal{O}_N$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^N g_{ji} s_j$ for $i = 1, \ldots, N$. Specially we denote $\gamma_w \equiv \alpha_{g(w)}$ when $g(w) = w \cdot I \subset U(N)$ for $w \in U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$. We denote $UHF_N = \{x \in \mathcal{O}_N : \gamma_w(x) = x,\, w \in U(1)\}$. For multiindices $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, we denote $s_J = s_{j_1} \cdots s_{j_k}$ and $s_J^* = s_{j_k}^* \cdots s_{j_1}^*$.

In order to consider properties of endomorphisms of $\mathcal{O}_N$, we review the permutative representations of $\mathcal{O}_N$. The permutative representation was introduced by [**3**, **5**, **6**]. We generalize and give another characterization of them in [**11**, **12**, **13**, **14**]. In this article, we treat only cyclic permutative representation with cycle.

Definition 3.1. (i) *$(\mathcal{H}, \pi)$ is a permutative representation of $\mathcal{O}_N$ if there are a complete orthonormal basis $\{e_n\}_{n \in \Lambda}$ of $\mathcal{H}$ and a branching function system $f = \{f_i\}_{i=1}^N$ on Λ such that $\pi(s_i) e_n = e_{f_i(n)}$ for $n \in \Lambda$ and $i = 1, \ldots, N$.*

(ii) *$(\mathcal{H}, \pi)$ is a generalized permutative(=GP) representation of $\mathcal{O}_N$ with cycle by $J \in \{1, \ldots, N\}^k$, $k \geq 1$ if there is a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(s_J)\Omega = \Omega$ and $\{\pi(s_{j_1} \cdots s_{j_l})\Omega : l = 1, \ldots, k\}$ is an orthonormal family in $\mathcal{H}$. We denote $P(J) = (\mathcal{H}, \pi, \Omega)$ simply.*

(iii) *$(l_2(\Lambda), \pi_f)$ is the permutative representation of $\mathcal{O}_N$ by $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$ if $\pi_f(s_i) e_n \equiv e_{f_i(n)}$ for $n \in \Lambda$ and $i = 1, \ldots, N$.*

(i) in Definition 3.1 contains non-cyclic cases. (ii) is another characterization of cyclic case in (i). (iii) is a realization of (i) by branching function system. In Definition 3.1 (ii), we use a symbol $P(J)$ for representation again because such definition is justified in later.

Recall Definition 2.1 about multiindices.

Theorem 3.2. (i) *Any permutative representation is completely reducible.*

(ii) *For each $J \in \{1, \ldots, N\}_1^*$, $P(J)$ exists and unique up to unitary equivalences. Furthermore, $P(J)$ is a permutative representation.*

(iii) *For $J \in \{1, \ldots, N\}_1^*$, $P(J)$ is irreducible if and only if J is non periodic.*

(iv) *For* $J_1, J_2 \in \{1, \ldots, N\}_1^*$, $P(J_1) \sim P(J_2)$ *if and only if* $J_1 \sim J_2$.

Proof. Note $P(J) = GP(\varepsilon_J)$ in [**11**] where $\{\varepsilon_i\}_{i=1}^N$ is the canonical coordinate of $\mathbf{C}^N$ and $\varepsilon_J = \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}$ when $J = (j_1, \ldots, j_k)$.
(i) This follows from [**3**, **5**, **6**].
(ii) The existence is shown in Proposition 3.4 in [**11**]. If J is non periodic, then the uniqueness is shown in Proposition 5.4 in [**11**]. If J is periodic, the uniqueness is shown in Corollary 5.6 (v) in [**12**].

In § 4 in [**11**], we construct a canonical basis of $P(J) = GP(\varepsilon_J)$. By this basis, $P(J)$ satisfies the condition of permutative representation.
(iii) The irreducibility is proved in Proposition 5.5 in [**11**].
(iv) This is shown in Proposition 5.11 in [**11**]. $\square$

By Theorem 3.2 (i), it is sufficient for a statement about $P(J)$ to shown by a suitable concrete representation which is $P(J)$.

The characterization of permutative representations are given by terminology of branching function systems.

Proposition 3.3. *Let* f *be a branching function system on an infinite set* Λ.

(i) *If* g *is a branching function system on an infinite set* $\Lambda^{'}$ *such that* $f \sim g$, *then* $(l_2(\Lambda), \pi_f) \sim (l_2(\Lambda^{'}), \pi_g)$.
(ii) *If* f *is cyclic, then* $(l_2(\Lambda), \pi_f)$ *is cyclic.*
(iii) *If* f *is* $P(J)$, *then* $(l_2(\Lambda), \pi_f)$ *is* $P(J)$, *too.*
(iv) *If* $f = f^{(1)} \oplus f^{(2)}$ *and* $\Lambda = \Lambda_1 \sqcup \Lambda_2$ *is the associated decomposition, then* $(l_2(\Lambda), \pi_f) \sim (l_2(\Lambda_1), \pi_{f^{(1)}}) \oplus (l_2(\Lambda_2), \pi_{f^{(2)}})$.

Here $\sim$ *means the unitary equivalence of representations.*

Proof. (i) Put φ a bijection from Λ to $\Lambda^{'}$ such that $\varphi \circ f \circ \varphi^{-1} = g$. Let U_φ be a unitary from $l_2(\Lambda)$ to $l_2(\Lambda^{'})$ naturally defined from φ. Then we see $\mathrm{Ad}U_\varphi \circ \pi_f = \pi_g$.
(ii) For $J, J^{'} \in \{1, \ldots, N\}^*$, we see $\pi_f(s_J s_{J'}^*) e_n = e_{(f_J \circ f_{J'}^{-1})(n)}$ for $n \in \Lambda$. Hence the statement holds.
(iii) If R is a cycle in Λ, then $\{e_n \in l_2(\Lambda) : n \in R\}$ is an orthonormal family which satisfies the condition of $P(J)$. By (ii) and this, the statement holds.
(iv) By definition of the direct sum decomposition of branching function system and π_f, it holds. $\square$

However it is not sufficient to show properties of a representation $(l_2(\Lambda), \pi_f)$ by using only those of a branching function system f, a branching function system is convenient to study of representation of $\mathcal{O}_N$.

4. Permutative endomorphisms

4.1. General properties of endomorphisms of C*-algebras. In order to classify endomorphisms of C*-algebras, we prepare several notions about properties of endomorphisms. Assume that End$\mathcal{A}$ is the set of all unital $*$-endomorphisms of a unital $*$-algebra $\mathcal{A}$ and $\rho, \rho_1, \rho_2 \in \mathrm{End}\mathcal{A}$ in this subsection.

Definition 4.1. (i) *ρ is proper if $\rho(\mathcal{A}) \neq \mathcal{A}$.*
(ii) *ρ is irreducible if $\rho(\mathcal{A})' \cap \mathcal{A} = \mathbf{C}I$.*
(iii) *ρ is reducible if ρ is not irreducible.*
(iv) *ρ_1 and ρ_2 are equivalent if there is a unitary $u \in \mathcal{A}$ such that $\rho_2 = \mathrm{Ad}u \circ \rho_1$. In this case, we denote $\rho_1 \sim \rho_2$.*

Of course an automorphism is a special endomorphism, but we are mainly interest in an endomorphism which is not an automorphism. Hence the notion of *"proper"* is important and we treat an automorphism as a trivial endomorphism. The notion of *"reducible"* is more reasonably explained in [**17**]. One of the aim of study of endomorphisms is an analogy of representation theory. The minimal object of endomorphism is an *"irreducible"* endomorphism. In [**17**], we show the analogy of tensor product, irreducible decomposition and representation ring of endomorphisms.

Immediately, we have the followings without topological argument:

Lemma 4.2. (i) *If ρ_1 is proper and ρ_2 is not proper, then $\rho_1 \not\sim \rho_2$.*
(ii) *If ρ_1 is irreducible and ρ_2 is not irreducible, then $\rho_1 \not\sim \rho_2$.*
(iii) *If ρ is an automorphism, then ρ is irreducible and not proper. If $\mathcal{A}$ is simple, ρ is an automorphism if and only if ρ is not proper.*

Our method of study of endomorphism is a practical use of representation. By using representation, we can look endomorphisms more closely and easily in some situation. Let Rep$\mathcal{A}$ be the set of all unital $*$-representations of $\mathcal{A}$. We simply denote π for $(\mathcal{H}, \pi) \in \mathrm{Rep}\mathcal{A}$,

Lemma 4.3. (i) *Assume that $\mathcal{A}$ is simple. If there is $\pi \in \mathrm{Rep}\mathcal{A}$ such that π is irreducible and $\pi \circ \rho$ is irreducible, too, then ρ is irreducible.*
(ii) *Assume that $\mathcal{A}$ is simple. If there is $\pi \in \mathrm{Rep}\mathcal{A}$ such that π is irreducible and equivalent to $\pi \circ \rho$, then $\rho^n \equiv \underbrace{\rho \circ \cdots \circ \rho}_{n}$, $n \geq 1$, is irreducible.*
(iii) *If there is $\pi \in \mathrm{Rep}\mathcal{A}$ such that $\pi \circ \rho_1$ and $\pi \circ \rho_2$ are not unitarily equivalent, then $\rho_1 \not\sim \rho_2$.*
(iv) *If there is $\pi \in \mathrm{Rep}\mathcal{A}$ such that π is irreducible and $\pi \circ \rho$ is not irreducible, then ρ is proper.*

(i) By assumption, $\pi_1 \equiv \pi \circ \rho$ is irreducible. $\mathbf{C}I = \pi_1(\mathcal{A})' = \pi(\rho(\mathcal{A}))' \supset \pi(\rho(\mathcal{A}))' \cap \pi(\mathcal{A})$. Therefore $\pi(\rho(\mathcal{A})' \cap \mathcal{A}) = \pi(\rho(\mathcal{A}))' \cap \pi(\mathcal{A}) = \mathbf{C}I$. On the other hand, ρ is injective since $\mathcal{A}$ is simple. Therefore $\rho(\mathcal{A})' \cap \mathcal{A} = \mathbf{C}I$.

(ii) By assumption, there is a unitary u on $\mathcal{H}$ such that $\pi \circ \rho = \mathrm{Ad}u \circ \pi$. Therefore $\pi \circ \rho^n = (\mathrm{Ad}u)^n \circ \pi$. Because $(\mathrm{Ad}u)^n \circ \pi$ is irreducible and (i), the assertion holds for each $n \geq 1$.
(iii) For $\rho_1, \rho_2 \in \mathrm{End}\mathcal{A}$, if $\rho_1 \sim \rho_2$, then $\pi \circ \rho_1 \sim \pi \circ \rho_2$. Hence the statement holds.
(iv) For an irreducible representation, if $\rho(\mathcal{A}) = \mathcal{A}$, then $\pi \circ \rho$ is irreducible. Hence the assertion holds. $\square$

4.2. Definition of permutative endomorphisms. Consider the following canonical inclusions:

$$\mathfrak{S}_{N^k} \subset U(N^k) \subset M_{N^k}(\mathbf{C}) \subset UHF_N = \mathcal{O}_N^{U(1)} \subset \mathcal{O}_N \tag{4.1}$$

where $\mathfrak{S}_{N^k}$ is the symmetric group with order N^k. We identify $\mathfrak{S}_{N^k}$ and $\mathfrak{S}_{N,k}$ the set of all bijective transformations on $\{1, \ldots, N\}^k$ for $k \geq 1$ by the method in § 2.2. The inclusion of $\mathfrak{S}_{N,k} \cong \mathfrak{S}_{N^k}$ into $\mathcal{O}_N$ in (4.1) is defined by

$$\sigma \mapsto u_\sigma = \sum_{J \in \{1,\ldots,N\}^k} s_{\sigma(J)} s_J^*. \tag{4.2}$$

Definition 4.4. *For $\sigma \in \mathfrak{S}_{N,k}$, $\psi_\sigma \in \mathrm{End}\mathcal{O}_N$ is defined by*

$$\psi_\sigma(s_i) \equiv u_\sigma s_i \quad (i = 1, \ldots, N).$$

ψ_σ is called the permutative endomorphism of $\mathcal{O}_N$ by σ where u_σ is in (4.2).

Reader can check that $\{\psi_\sigma(s_i)\}_{i=1}^N$ satisfies (3.1). By the first paragraph in § 3, ψ_σ is an endomorphism of $\mathcal{O}_N$. There are many other methods of construction of endomorphism of $\mathcal{O}_N$([**1**]). We treat only this type in this article.

Put the following sets:

$$E_{N,k} \equiv \{\psi_\sigma \in \mathrm{End}\mathcal{O}_N : \sigma \in \mathfrak{S}_{N,k}\} \quad (k \geq 1). \tag{4.3}$$

Note $\#E_{N,k} = N^k!$. Before a complete characterization of ψ_σ by σ, we aspire a goal to classify elements in $E_{N,k}$ for concrete N and k.

Immediately, we see the followings:

Proposition 4.5. (i) *If $\sigma = id$, then $\psi_{id} = id$ on $\mathcal{O}_N$.*
(ii) *If $\sigma \in \mathfrak{S}_N$, then ψ_σ is an automorphism of $\mathcal{O}_N$ which satisfies $\psi_\sigma(s_i) = s_{\sigma(i)}$ for $i = 1, \ldots, N$.*
(iii) *If $\sigma \in \mathfrak{S}_{N,2}$ is defined by $\sigma(i,j) \equiv (j,i)$ for $i,j = 1, \ldots, N$, then ψ_σ is the canonical endomorphism of $\mathcal{O}_N$.*
(iv) *$\gamma_z \circ \psi_\sigma = \psi_\sigma \circ \gamma_z$ for each $z \in U(1)$ and $\sigma \in \mathfrak{S}_{N,*}$.*

When $\sigma, \eta \in \mathfrak{S}_N = \mathfrak{S}_{N,1}$, then $\psi_\sigma \circ \psi_\eta = \psi_{\sigma\circ\eta}$. However

$$\psi_\sigma \circ \psi_\eta \neq \psi_{\sigma\circ\eta}$$

for $\sigma \in \mathfrak{S}_{N,k}$, $\eta \in \mathfrak{S}_{N,l}$, $k, l \geq 1$, $(k,l) \neq (1,1)$ in general. In order to consider the composition $\psi_\sigma \circ \psi_\eta$, we introduce a new product on a set of permutations. When $M \geq m$, for $F \in \mathfrak{S}_{N,m}$, define $F_j \in \mathfrak{S}_{N,M}$ by

$$F_1 \equiv F \times id^{M-m}, \quad F_j \equiv id^{j-1} \times F \times id^{M-m-j+1} \tag{4.4}$$

for $2 \leq j \leq M - m + 1$.

Proposition 4.6. *(product rule) For $\sigma \in \mathfrak{S}_{N,k}$ and $\eta \in \mathfrak{S}_{N,l}$, $k, l \geq 1$, define $\sigma * \eta \in \mathfrak{S}_{N,k+l-1}$ by*

$$\sigma * \eta \equiv \begin{cases} \sigma \circ \eta_1 & (l = 1), \\ \\ \sigma_1 \circ (\mathrm{Ad}(\sigma_2 \circ \cdots \circ \sigma_l))(\eta_1) & (l \geq 2) \end{cases} \tag{4.5}$$

where $\sigma_1, \ldots, \sigma_l$ and η_1 are in $\mathfrak{S}_{N,k+l-1}$ which are defined by (4.4) with respect to σ and η. Then $\sigma * (\eta * \zeta) = (\sigma * \eta) * \zeta$ and the followings hold:

$$\Phi_\sigma \circ \Phi_\eta = \Phi_{\sigma * \eta}, \tag{4.6}$$

$$\psi_{\sigma_j} = \psi_\sigma \quad (j = 1, \ldots, l), \tag{4.7}$$

$$\psi_\sigma \circ \psi_\eta = \psi_{\sigma * \eta} \tag{4.8}$$

where Φ is in (2.3). Specially $(\mathfrak{S}_{N,*}, *)$ is a semigroup and $\psi_\sigma \circ \psi_\eta = \psi_{\sigma\circ\eta}$ when $k = l = 1$.

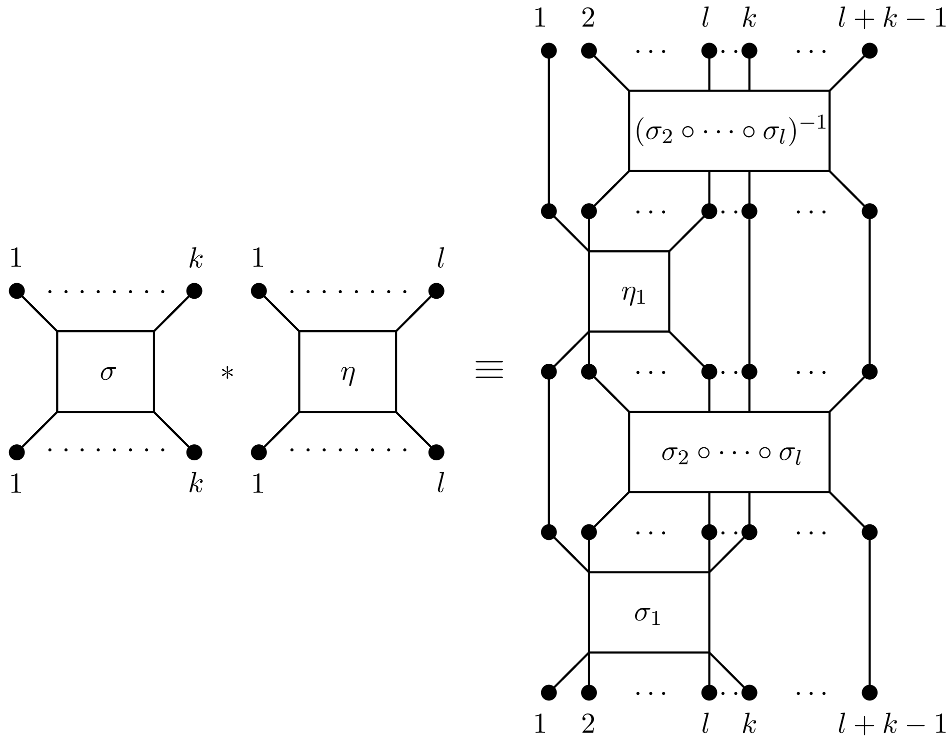
Proof. The associativity of $*$-product and (4.6) can be checked by (4.5). (4.7) holds by (3.1). By checking both sides of (4.8), it follows by definition of $*$-product and ψ_σ and ψ_η. The compatibility between inclusion $\mathfrak{S}_{N,k} \hookrightarrow \mathfrak{S}_{N,k+1}$, $k \geq 1$, and $*$-product is verified directly. □

We see $E_{N,k} \subset E_{N,k+1}$ for each $k \geq 1$ by (4.7) when $l = 2$. From this, we can define $E_{N,*} \equiv \varinjlim_k E_{N,k}$. Recall $\mathfrak{S}_{N,*}$ in § 2.2. Although $\mathfrak{S}_{N,*}$ is a semigroup by ordinary composition of transformations, it is important to consider another product " $*$ " in (4.5) on $\mathfrak{S}_{N,*}$ in the following sense:

Corollary 4.7. $(\mathfrak{S}_{N,*}, *)$ *and* $(E_{N,*}, \circ)$ *are isomorphic as a semigroup with unit.*

Proof. By Proposition 4.6, $\sigma \mapsto \psi_\sigma$ is a homomorphism between $(\mathfrak{S}_{N,*}, *)$ and $(E_{N,*}, \circ)$. Because $\psi_\sigma = \psi_\eta$ if and only if $u_\sigma = u_\eta$, the injectivity of a map ψ is checked. □

The meaning of (4.5) is illustrated as follows:



where we use an electronic circuit-like figure by explaining a permutation on the set $\{1,\ldots,N\}^k$, $k \geq 1$. There are k wirings and $\sigma \in \mathfrak{S}_{N,k}$ as a part of the electronic circuit changes the input signals into the output. $\sigma * \eta$ means a king of composition of two circuits σ and η. From top down, the electronic signal is changed by this integrated circuit.

In § 4.1, we introduce equivalence and irreducibility of endomorphisms of C*-algebras. In this sense, we consider the following problem:

Problem 4.8. *For $N, k \geq 2$, classify elements in $E_{N,k}$. More concretely, solve the following questions:*

(i) *When is $\rho \in E_{N,k}$ proper ?*
(ii) *When is $\rho \in E_{N,k}$ irreducible ?*
(iii) *When are $\rho, \rho^{'} \in E_{N,k}$ equivalent ?*

Since we are interest in proper endomorphisms of $\mathcal{O}_N$, we neglect the case $k = 1$ by Proposition 4.5 (ii). We solve Problem 4.8 by using permutative representation in § 3. By fixing $N \geq 2$ and $k \geq 2$, we check elements in $E_{N,k}$ individually.

4.3. Action of permutative endomorphisms on permutative representations. For a representation $(\mathcal{H}, \pi)$ and an endomorphism ρ of $\mathcal{O}_N$,

$(\mathcal{H}, \pi\circ\rho)$ is a representations, too. That is, an endomorphism brings a transformation of representation. In order to show the properties of permutative endomorphisms, we consider this transformation.

Recall $\mathrm{BFS}_N(\Lambda)$ in § 2.1.

Lemma 4.9. *Let Λ be an infinite set. For $\sigma \in \mathfrak{S}_{N,k}$, $k \geq 1$, and $f \in \mathrm{BFS}_N(\Lambda)$, let $(l_2(\Lambda), \pi_f)$ be in Definition* 3.1 *(iii) and $f^{(\sigma)}$ in (*2.2*). Then we have $\pi_f \circ \psi_\sigma = \pi_{f^{(\sigma)}}$.*

Proof. By definition, we can directly check $(\pi_f\circ\psi_\sigma)(s_i)e_n = \pi_{f^{(\sigma)}}(s_i)e_n$ for $n \in \Lambda$ and $i = 1, \ldots, N$. □

Theorem 4.10. (i) *If ρ is a permutative endomorphism and $(\mathcal{H}, \pi)$ is a permutative representation of $\mathcal{O}_N$, then $\pi \circ \rho$ is a permutative representation, too.*

(ii) *If ρ is a permutative endomorphism of $\mathcal{O}_N$, then the restriction of any permutative representation on a subalgebra $\rho(\mathcal{O}_N) \subset \mathcal{O}_N$ is completely reducible.*

Proof. (i) By Lemma 4.9, it holds immediately.
(ii) Because any permutative representation is completely reducible, it holds from (i). □

Theorem 4.11. *If $(\mathcal{H}, \pi)$ is $P(J)$ for $J \in \{1, \ldots, N\}_1^*$ and $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, then there are a finite family $\{J_{\sigma,i}\}_{i=1}^M \subset \{1, \ldots, N\}_1^*$ and a family $\{(\mathcal{H}_i, \pi_i)\}_{i=1}^M$ of subrepresentations of $(\mathcal{H}, \pi\circ\psi_\sigma)$ such that*

$$(\mathcal{H}, \pi\circ\psi_\sigma) = \bigoplus_{i=1}^M (\mathcal{H}_i, \pi_i) \tag{4.9}$$

and $(\mathcal{H}_i, \pi_i)$ is $P(J_{\sigma,i})$ for $i = 1, \ldots, M$.

Proof. By definition, $(\mathcal{H}, \pi)$ is equivalent to $(l_2(\mathbf{N}), \pi_f)$ for a suitable branching function system on $\mathbf{N}$. Hence we identify $(\mathcal{H}, \pi)$ and $(l_2(\mathbf{N}), \pi_f)$. By Lemma 4.9, $\pi_f\circ\psi_\sigma = \pi_{f^{(\sigma)}}$. By Theorem 2.13, the statement holds. □

In consequence, Theorem 1.1 is proved. From this and uniqueness of irreducible decomposition of permutative representations, it is worth to consider the branching law of an endomorphism on a representation and characterization of an endomorphism by its branching law. We simply denote (4.9) as

$$P(J)\circ\psi_\sigma = \bigoplus_{i=1}^M P(J_{\sigma,i}). \tag{4.10}$$

Specially, if $\sigma \in \mathfrak{S}_N = \mathfrak{S}_{N,1}$, then $P(J) \circ \psi_\sigma = P(J_{\sigma^{-1}})$ by Lemma 2.10. Roughly speaking, we can say that a permutative endomorphism transforms cycles to cycles.

Problem 4.12. *For $\sigma \in \mathfrak{S}_{N,*}$ and $J \in \{1,\ldots,N\}_1^*$, find $\{J_{\sigma,i}\}_{i=1}^M \subset \{1,\ldots,N\}_1^*$ in (4.10).*

The solution of Problem 4.12 is the *branching law* of ψ_σ. We show concrete examples of Theorem 4.11 in § 5 and treat the subject about branching law in [**16**] for real.

5. Examples

5.1. Properties of ρ_ν. Recall ρ_ν in (1.2) and § 4.2.

Proposition 5.1. (i) *ρ_ν is in $E_{3,2}$.*
(ii) *If α is an action of $\mathbf{Z}_3$ on $\mathcal{O}_3$ defined by $\alpha_\tau(s_i) \equiv s_{\tau(i)}$ for $i = 1,2,3$ and $\tau \in \mathbf{Z}_3$, then $\alpha_\tau \circ \rho_\nu = \rho_\nu$ for each $\tau \in \mathbf{Z}_3$.*
(iii) *ρ_ν is proper.*

Proof. (i) Put σ_0 a transformation on $\{1,2,3\}^2$ defined by the following:

$$\sigma_0 : \begin{pmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{pmatrix} \mapsto \begin{pmatrix} (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \\ (1,1) & (2,2) & (3,3) \end{pmatrix}. \tag{5.1}$$

Then we see $\rho_\nu = \psi_{\sigma_0}$.
(ii) By definition of ρ_ν in (1.2), we can check the assertion directly.
(iii) By (ii), $\alpha_\tau(\rho_\nu(x)) = \rho(x)$. for each $x \in \mathcal{O}_3$ and $\tau \in \mathbf{Z}_3$. Hence $\rho_\nu(\mathcal{O}_3) \subset \mathcal{O}_3^\alpha \equiv \{x \in \mathcal{O}_3 : \alpha_\tau(x) = x,\ \text{for each } \tau \in \mathbf{Z}_3\} \neq \mathcal{O}_3$. Therefore ρ_ν is proper. $\square$

In order to show Theorem 1.2, we prepare a permutative representation of $\mathcal{O}_3$. Recall Definition 3.1.

Lemma 5.2. *Let σ_0 be in (5.1) and $f = \{f_1, f_2, f_3\}$ a branching function system on $\mathbf{N}$ defined by*

$$f_1(1) \equiv 2, \quad f_1(2) \equiv 5, \quad f_2(1) \equiv 4, \quad f_2(2) \equiv 1, \quad f_3(1) \equiv 3, \quad f_3(2) \equiv 6,$$
$$f_i(n) \equiv 3(n-1) + i \quad (i = 1,2,3,\ n \geq 3).$$

Then the followings hold:
(i) *$(l_2(\mathbf{N}), \pi_f)$ in Definition 3.1 (iii) is equivalent to $P(12)$ of $\mathcal{O}_3$.*
(ii) *$f^{(\sigma_0)}$ has neither cycle nor chain in $\{n \in \mathbf{N} : n \geq 7\}$.*

Proof. (i) Note $(f_1 \circ f_2)(2) = 2$ and $\{f_J(2) : J \in \{1,2,3\}^*\} = \mathbf{N}$. From these, $\pi_f(s_1 s_2)e_2 = e_2$ and e_2 is a cyclic vector of $(l_2(\mathbf{N}), \pi_f)$. Therefore $(l_2(\mathbf{N}), \pi_f, e_2)$ is $P(12)$.

(ii) Let $C \equiv \{n \in \mathbf{N} : n \geq 3\}$. By Lemma 2.11, $f^{(\sigma_0)}$ has neither cycle nor chain in $\hat{C} = \cup_{j=1}^{3} f_j(C) = \{n \in \mathbf{N} : n \geq 7\}$. □

Proposition 5.3. $P(12) \circ \rho_\nu = P(113223)$.

Proof. Put $(l_2(\mathbf{N}), \pi_f)$ in Lemma 5.2. By Lemma 5.2 (i), it is sufficient to show that $(l_2(\mathbf{N}), \pi_f \circ \rho_\nu)$ is $P(113223)$. Because $\pi_f \circ \rho_\nu = \pi_f \circ \psi_{\sigma_0} = \pi_{f^{(\sigma_0)}}$, compute the value of $h_i \equiv f_i^{(\sigma_0)}$ on a subset $\{1, \ldots, 6\} \subset \mathbf{N}$ for $i = 1, 2, 3$. Then we have the following:

n	$h_1(n)$	$h_2(n)$	$h_3(n)$
1	12	16	5
2	8	15	4
3	13	1	9
4	17	3	10
5	6	7	14
6	2	11	18

From this, we can find the following cycle by $h = \{h_1, h_2, h_3\}$:

$$1 \overset{h_3}{\mapsto} 5 \overset{h_1}{\mapsto} 6 \overset{h_1}{\mapsto} 2 \overset{h_3}{\mapsto} 4 \overset{h_2}{\mapsto} 3 \overset{h_2}{\mapsto} 1. \tag{5.2}$$

Therefore $f^{(\sigma_0)}_{(113223)}(2) = h_{(113223)}(2) = 2$. From this, $(\pi_f \circ \psi_{\sigma_0})(s_J)e_2 = e_2$. By Lemma 5.2 (ii), any $n \geq 7$ belongs to $K \equiv \{f_J^{(\sigma_0)}(m) : m = 1, \ldots, 6,\ J \in \{1, 2, 3\}^*\}$. Hence $\mathbf{N} = K$ and there is no cycle except (5.2). Because $\pi_f \circ \rho_\nu$ is cyclic and has a cycle (5.2), $(l_2(\mathbf{N}), \pi_f \circ \rho_\nu)$ is $P(113223)$. □

Corollary 5.4. *ρ_ν is irreducible.*

Proof. Note that both $P(12)$ and $P(113223)$ are irreducible because (12) and (113223) are non periodic. By Lemma 4.3 (i) and Proposition 5.3, the statement holds. □

From these, Theorem 1.2 is proved.

5.2. Classification of $E_{2,2}$. We show the complete classification of elements in $E_{2,2}$ in (4.3). A classification of $E_{2,2}$ in the point of view from quantum field theory is already shown by [**1**]. Note $\#E_{2,2} = 2^2! = 24$. Despite of small number of elements of $E_{2,2}$, $E_{2,2}$ contains sufficiently various examples of non trivial inequivalent endomorphisms of $\mathcal{O}_2$. By the map κ in § 2.2, we identify between $\{1, 2, 3, 4\}$ and $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$ by $\kappa^{-1}(1) = (1, 1)$, $\kappa^{-1}(2) = (1, 2)$, $\kappa^{-1}(3) = (2, 1)$, $\kappa^{-1}(4) = (2, 2)$. For $\sigma \in \mathfrak{S}_4$, we identify σ and $\kappa^{-1} \circ \sigma \circ \kappa$. Denote $s_{ij,k} \equiv s_i s_j s_k^*$ for $i, j, k = 1, 2$.

We show the classification of 24 endomorphisms of $\mathcal{O}_2$ in $E_{2,2}$. First, we introduce a rough classification of them by the following:

Table 5.5. *(Elements in $E_{2,2}$)*

ψ_σ	$\psi_\sigma(s_1)$	$\psi_\sigma(s_2)$	property	$\mathrm{Ad}u\circ\psi_\sigma$
ψ_{id}	s_1	s_2	*inn.aut*	$\psi_{(14)(23)}$
ψ_{12}	$s_{12,1}+s_{11,2}$	s_2	*irr.end*	ψ_{1324}
ψ_{13}	$s_{21,1}+s_{12,2}$	$s_{11,1}+s_{22,2}$	*irr.end*	ψ_{1432}
ψ_{14}	$s_{22,1}+s_{12,2}$	$s_{21,1}+s_{11,2}$	*red.end*	ψ_{14}
ψ_{23}	$s_{11,1}+s_{21,2}$	$s_{12,1}+s_{22,2}$	*red.end*	ψ_{23}
ψ_{24}	$s_{11,1}+s_{22,2}$	$s_{21,1}+s_{12,2}$	*irr.end*	ψ_{1234}
ψ_{34}	s_1	$s_{22,1}+s_{21,2}$	*irr.end*	ψ_{1423}
ψ_{123}	$s_{12,1}+s_{21,2}$	$s_{11,1}+s_{22,2}$	*red.end*	ψ_{243}
ψ_{132}	$s_{21,1}+s_{11,2}$	$s_{12,1}+s_{22,2}$	*red.end*	ψ_{132}
ψ_{124}	$s_{12,1}+s_{22,2}$	$s_{21,1}+s_{11,2}$	*red.end*	ψ_{124}
ψ_{142}	$s_{22,1}+s_{11,2}$	$s_{21,1}+s_{12,2}$	*irr.end*	ψ_{134}
ψ_{134}	$s_{21,1}+s_{12,2}$	$s_{22,1}+s_{11,2}$	*irr.end*	ψ_{142}
ψ_{143}	$s_{22,1}+s_{12,2}$	$s_{11,1}+s_{21,2}$	*red.end*	ψ_{143}
ψ_{234}	$s_{11,1}+s_{21,2}$	$s_{22,1}+s_{12,2}$	*red.end*	ψ_{234}
ψ_{243}	$s_{11,1}+s_{22,2}$	$s_{12,1}+s_{21,2}$	*red.end*	ψ_{123}
ψ_{1234}	$s_{12,1}+s_{21,2}$	$s_{22,1}+s_{11,2}$	*irr.end*	ψ_{24}
ψ_{1243}	$s_{12,1}+s_{22,2}$	$s_{11,1}+s_{21,2}$	*red.end*	ψ_{1243}
ψ_{1324}	s_2	$s_{12,1}+s_{11,2}$	*irr.end*	ψ_{12}
ψ_{1342}	$s_{21,1}+s_{11,2}$	$s_{22,1}+s_{12,2}$	*red.end*	ψ_{1342}
ψ_{1423}	$s_{22,1}+s_{21,2}$	s_1	*irr.end*	ψ_{34}
ψ_{1432}	$s_{22,1}+s_{11,2}$	$s_{12,1}+s_{21,2}$	*irr.end*	ψ_{13}
$\psi_{(12)(34)}$	$s_{12,1}+s_{11,2}$	$s_{22,1}+s_{21,2}$	*out.aut*	$\psi_{(13)(24)}$
$\psi_{(13)(24)}$	s_2	s_1	*out.aut*	$\psi_{(12)(34)}$
$\psi_{(14)(23)}$	$s_{22,1}+s_{21,2}$	$s_{12,1}+s_{11,2}$	*inn.aut*	ψ_{id}

where "inn.aut", "out.aut", "irr.end" and "red.end" mean an inner automorphism, an outer automorphism, a proper irreducible endomorphism and a proper reducible endomorphism, respectively, and $u \equiv s_1s_2^* + s_2s_1^*$.

We prove Table 5.5 step by step. The column of $\mathrm{Ad}u\circ\psi_\sigma$ follows from direct computation. From this, there are 16 unitary equivalence classes at most in $E_{2,2}$. Because notions "inn.aut", "out.aut", "irr.end" and "red.end" are preserving unitary equivalence, it is sufficient for the column of property to representative elements of 16 endomorphisms ψ_σ for $\sigma =$

$$id,(12),(13),(14),(23),(24),(34),(123),(132),(124),(142),(143),(234),$$
$$(1243),(1342),(12)(34).$$

We see the following immediately:

Lemma 5.6. *Let $V_4 \equiv \{id,(12)(34),(13)(24),(14)(23)\} \subset \mathfrak{S}_4$ be the Klein's four-group. Then $G_2 \equiv \{\psi_\sigma : \sigma \in V_4\}$ is a family of automorphisms of $\mathcal{O}_2$. Specially $\psi_\sigma \circ \psi_\eta = \psi_{\sigma\circ\eta}$ for each $\sigma,\eta \in V_4$.*

Hence it is sufficient to consider only 14 endomorphisms ψ_σ except $\sigma = (id), (12)(34)$. The results of branching law of ψ_σ in (4.10) are followings:

Table 5.7.

ψ_σ	$P(1)\circ\psi_\sigma$	$P(2)\circ\psi_\sigma$	$P(12)\circ\psi_\sigma$
ψ_{id}	$P(1)$	$P(2)$	$P(12)$
$\psi_{(12)(34)}$	$P(2)$	$P(1)$	$P(12)$
ψ_{12}	$P(12)$	$P(1)\oplus P(2)$	$P(1122)$
ψ_{13}	$P(2)$	$P(2)$	$P(11)$
ψ_{24}	$P(1)$	$P(1)$	$P(22)$
ψ_{34}	$P(1)\oplus P(2)$	$P(12)$	$P(1122)$
ψ_{142}	$P(12)$	$P(12)$	$P(11)\oplus P(22)$
ψ_{14}	$P(22)$	$P(11)$	$P(12)\oplus P(12)$
ψ_{23}	$P(1)\oplus P(1)$	$P(2)\oplus P(2)$	$P(12)\oplus P(12)$
ψ_{123}	$P(1)\oplus P(2)$	$P(1)\oplus P(2)$	$P(12)\oplus P(12)$
ψ_{124}	$P(22)$	$P(1)\oplus P(1)$	$P(1212)$
ψ_{132}	$P(11)$	$P(2)\oplus P(2)$	$P(1212)$
ψ_{143}	$P(2)\oplus P(2)$	$P(11)$	$P(1212)$
ψ_{234}	$P(1)\oplus P(1)$	$P(22)$	$P(1212)$
ψ_{1243}	$P(2)\oplus P(2)$	$P(1)\oplus P(1)$	$P(12)\oplus P(12)$
ψ_{1342}	$P(11)$	$P(22)$	$P(12)\oplus P(12)$

These branching laws are computed as the case may be in the same way with ρ_ν in § 5.1. For example, we show the sketch of proof about $P(2)\circ\psi_{12} = P(1)\oplus P(2)$. For a branching function system $f = \{f_1, f_2\}$ on $\mathbf{N}$ which is $P(2)$ defined by lhs of Table 5.8, its transformation $f^{(12)} = \{f_1^{(12)}, f_2^{(12)}\}$ by $\sigma = (12)$ is rhs of Table 5.8 where the symbol $*$ means a suitable number in $\mathbf{N}$. We see that there are two cycles $1 \overset{f_2^{(12)}}{\mapsto} 1$ and $2 \overset{f_1^{(12)}}{\mapsto} 2$ in the table of $f^{(12)}$. From this, we see that $\pi_f\circ\psi_{12}$ is $P(1)\oplus P(2)$. By uniqueness of $P(1), P(2)$, this shows $P(2)\circ\psi_{12} = P(1)\oplus P(2)$.

Table 5.8.

n	$f_1(n)$	$f_2(n)$
1	2	1
2	3	4
$*$	$*$	$*$

n	$f_1^{(12)}(n)$	$f_2^{(12)}(n)$
1	3	1
2	2	4
$*$	$*$	$*$

Lemma 5.9. (i) *All equivalence classes in $E_{2,2}$ is*

$$\left\{[\psi_\sigma] : \sigma = \begin{array}{l} id, (12), (13), (14), (23), (24), (34), \\ (123), (132), (124), (142), (143), (234), \\ (1243), (1342), (12)(34) \end{array}\right\} \tag{5.3}$$

where $[\psi_\sigma] = \{\rho\in E_{2,2} : \rho\sim\psi_\sigma\}$.

(ii) *ψ_σ is proper except $\sigma = id, (12)(34)$.*

(iii) *If* $\sigma \in \{(12),(13),(24),(34),(142)\}$*, then* ψ_σ *is irreducible.*

Proof. (i) Because any two branching laws in Table 5.7 are different, they are inequivalent each other by Lemma 4.3 (iii).
(ii) Note that all of $P(11), P(22), P(1212)$ is reducible by Theorem 3.2 (iii). Any ψ_σ in (5.3) except $\sigma = id, (12)(34)$ transforms one of irreducible representations $P(1), P(2), P(12)$ to reducible one. By Lemma 4.3 (iv), the assertion holds.
(iii) For each $\sigma = (12),(13),(24),(34),(142)$, we see that there is a representation π of $\mathcal{O}_2$ in $\{P(1), P(2), P(12)\}$ such that $\pi \circ \psi_\sigma$ is irreducible. By Lemma 4.3 (i), the statement holds. □

By the proof of Lemma 5.9, we see that branching law is an excellent tool to understand the difference among many endomorphisms at a glance.

Lemma 5.10. *Any element in*

$$\{\psi_\sigma \in E_{2,2} : \sigma = (14),(23),(123),(124),(132),(143),(234),(1243),(1342)\}$$

is reducible.

Proof. Put $u_1 \equiv s_1 s_1^* + s_2 s_2^*$, $\quad u_2 \equiv \delta_1\delta_1^* + \delta_2\delta_2^*$, $\delta_1 \equiv 2^{-1/2}(s_1 - s_2)$, $\delta_2 \equiv 2^{-1/2}(s_1 + s_2)$. Then we see that $\psi_\sigma(s_1)$ and $\psi_\sigma(s_2)$ commute u_1 for $\sigma = (23),(123),(1243)$ and $\psi_\sigma(s_1)$ and $\psi_\sigma(s_2)$ commute u_2 for $\sigma = (14),(124),(132),(143),(234)$. Furthermore $\psi_{1342} = \alpha \circ \psi_{14}$ for $\alpha \in \mathrm{Aut}\mathcal{O}_2$, $\alpha(s_1) \equiv s_2$, $\alpha(s_2) \equiv s_1$. Since u_1, u_2 are self adjoint, they belong to $\psi_\sigma(\mathcal{O}_2)' \cap \mathcal{O}_2$, respectively. Hence the statement holds. □

In consequence, the column of property in Table 5.5 is proved. Toward Problem 4.8, we put the following *"homework"* for readers:

Problem 5.11. (i) Classify elements in $E_{3,2}$ in (4.3). Note $\#E_{3,2} = 3^2! = 362880$. The cardinality of $E_{3,2}$ is too many to classify by the same method for $E_{2,2}$. How many is the number of irreducible proper endomorphisms in $E_{3,2}$? How many is the number of equivalence classes in $E_{3,2}$?

(ii) For ρ_ν in (1.2), characterize an inclusion

$$\rho_\nu(\mathcal{O}_3) \subset \mathcal{O}_3$$

by index theory of C*-algebras. We show this answer and compute fusion rules about elements in $E_{2,2}$ in [**17**].

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Appendix A. Proof of Proposition 2.5

(i) For $x \in \Lambda$, f is a cyclic branching function system on $\Lambda_x \equiv A_f(x)$. For $x, y \in \Lambda$, if $A_f(x) \cap A_f(y) \neq \emptyset$, then $A_f(x) = A_f(y)$. We see that $x \sim y$ is equivalent to $A_f(x) = A_f(y)$. In this way, $\sim$ is an equivalence relation on Λ and we can choose $\{x_\omega\}_{\omega \in \Xi} \subset \Lambda$ such that $\Lambda = \coprod_{\omega \in \Xi} A_f(x_\omega)$ and $A_f(x_\omega) \neq A_f(x_{\omega'})$ when $\omega \neq \omega'$ where $\Xi \equiv \Lambda/\sim$.

(ii) Assume that f has a cycle $R \subset \Lambda$. For $x, y \in \Lambda$, we call that there is a path from x to y when there is $J \in \{1, \ldots, N\}^*$ such that $f_J(x) = y$. Note there is no incoming path into R because of injectivity of f_i for each $i = 1, \ldots, N$. If f has another cycle $R' \subset \Lambda$, then there is a finite path in Λ from R to R' by f because f is cyclic. However this is forbidden because there is no incoming path into both R and R' from outside of them. Therefore there is no cycle except R.

Assume that f has no cycle. Fix $x \in \Lambda$. By definition of branching function system, there are unique $j \in \{1, \ldots, N\}$ and $y \in \Lambda$, such that $f_j(y) = x$. Denote $j_1 \equiv j$ and $y_1 \equiv y$. For y_1, we can take a pair (j_2, y_2) such that $f_{j_2}(y_2) = y_1$ in the same way. In this way, we have $\{j_l\}_{l \in \mathbf{N}}$ and $S_x \equiv \{y_l\}_{l \in \mathbf{N}}$. If $y_l = y_{l'}$ for $l \neq l'$, then, there is a cycle in R. Therefore $y_l \neq y_{l'}$ when $l \neq l'$. Hence f has a chain R. If f has another chain R', then there is only one path from R and R' in Λ because f is cyclic and f has no cycle. Therefore R and R' are identified in the statement of proposition. In consequence, the assertion is verified.

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