The Geometry of Anabelioids

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Introduction

The goal of the present manuscript is to consider the following question:

> *To what extent can the* **fundamental group** *of a Galois category be constructed in a* **canonical** *fashion which is* **independent of a choice of basepoint***?*

Put another way, we would like to consider the extent to which the elements (considered, say, up to conjugation) of the fundamental group may be assigned **canonical names**, or **labels**.

In §1, §2, we consider this issue from a *very general point of view*. That is to say, we develop the general theory of **"anabelioids"** — i.e., "multi-Galois categories" in the terminology of [SGA1]— with an eye to giving an answer to this question. We use the new terminology "anabelioid", partly because it is shorter than "(multi-)Galois category", and partly because we wish to emphasize that we would like to treat such objects from a *fundamentally different point of view* from the point of view taken in [SGA1]: Namely, we would like to regard anabelioids as the *primary geometric objects of interest*, which themselves form a category [i.e., not as a category containing as objects the primary geometric objects of interest].

Our *main result* in §1, §2, is *Theorem 2.4.3*, which states that:

> When an anabelioid possesses a *"faithful quasi-core"* (cf. Definition 2.3.1), then its fundamental group may be constructed in a *canonical fashion* as a *profinite group*.

The notion of a "quasi-core" is motivated by the notion of a *"hyperbolic core"* (cf. [Mzk3]). The condition for a quasi-core states, roughly speaking, that a certain *"forgetful functor"* from a category of geometric objects equipped with some special auxiliary structure to the category of the same geometric objects not equipped with this auxiliary structure is, in fact, an *equivalence*. Indeed, this general pattern of considering such forgetful functors which are, in fact, equivalences is an important *theme* in the present manuscript (cf. Definition 2.3.1, as well as Theorem 2.4.2). One elementary example of this sort of phenomenon — which was, in fact, one of the main motivations for the introduction of the notion of a "quasi-core" — is the following example from *elementary complex analysis*:

Motivating Example: Metrics on Hyperbolic Riemann Surfaces. A connected Riemann surface is called *hyperbolic* if its universal covering is biholomorphic to the *upper half plane*. An arbitrary Riemann surface will be called hyperbolic if every connected component of this Riemann surface is hyperbolic. Let us write

$$\mathrm{Loc}^{\mathrm{hyp}}$$

for the *category* whose *objects* are *hyperbolic Riemann surfaces* and whose *morphisms* are *étale morphisms* [i.e., holomorphic maps with everywhere nonvanishing derivative]. If $X \in \mathrm{Ob}(\mathrm{Loc}^{\mathrm{hyp}})$ is a object of $\mathrm{Loc}^{\mathrm{hyp}}$, then we shall refer to the metric on its tangent bundle determined by the standard *Poincaré metric* on the upper half plane [which is biholomorphic to the universal covering of every connected component of X] as the *canonical metric* on X. If $f : X \to Y$ is a morphism in $\mathrm{Loc}^{\mathrm{hyp}}$, then we shall say that this morphism f is *integral* if the norm of its derivative [when measured with respect to the canonical metrics on the tangent bundles of X, Y] is ≤ 1. Let us write

$$\mathrm{Loc}^{\mathrm{hyp}}_{\mathrm{int}} \subseteq \mathrm{Loc}^{\mathrm{hyp}}$$

for the *subcategory* whose *objects* are the objects of $\mathrm{Loc}^{\mathrm{hyp}}$ and whose *morphisms* are the *integral* morphisms of $\mathrm{Loc}^{\mathrm{hyp}}$. Then it follows from the *"theory of the Kobayashi hyperbolic metric"* that the natural inclusion functor

$$\mathrm{Loc}^{\mathrm{hyp}}_{\mathrm{int}} \hookrightarrow \mathrm{Loc}^{\mathrm{hyp}}$$

is, in fact, an *equivalence*. At a more concrete level, one verifies easily that the essential substantive fact that one needs to show this equivalence is the well-known *Schwarz lemma* of elementary complex analysis [to the effect that any holomorphic function $\phi : D \to \mathbb{C}$ on the open unit disc D in the complex plane satisfying $\phi(0) = 0$, $|\phi(z)| \leq 1$ (for all $z \in D$), necessarily satisfies $|\phi'(0)| \leq 1$]. This lemma of Schwarz in turn may be regarded as a formal consequence of the well-known *"maximum modulus principle"* of elementary complex analysis.

This example also suggests an interesting relationship between the notions of *uniformization* and of *canonical labels* for elements of the fundamental group: Namely, the Koebe uniformization theorem for hyperbolic Riemann surfaces gives rise to "canonical labels" (up to an ambiguity arising from some sort of conjugation action) as 2 by 2 matrices since it induces an embedding of the topological fundamental group of the Riemann surface into $PSL_2(\mathbb{R})$.

This leads us to the content of §3: In §3, we discuss the theory of §1, §2, in the case of *hyperbolic curves* over p-adic and number fields. In this case, our *main result* — *Theorem 3.1.6* — states that:

> If a non-proper hyperbolic curve over such a field is a "geometric core" (i.e., a core as in [Mzk3]), then its associated anabelioid admits a *faithful quasi-core.*

This allows us to assign **"canonical names"** to the elements of its arithmetic fundamental group in a fashion reminiscent of the way in which the Koebe uniformization theorem allows one to assign "canonical names" to the elements of the topological fundamental group of a hyperbolic Riemann surface. This main result is, in essence, a formal consequence of Theorem A of [Mzk7], and may be regarded as an interpretation of the main result of [Mzk9], §2, via the *geometry of anabelioids.*

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Section 0: Notations and Conventions

Numbers:

We will denote by $\mathbb{N}$ the set of *natural numbers*, by which we mean the set of integers $n \geq 0$. A *number field* is defined to be a finite extension of the field of rational numbers $\mathbb{Q}$

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall \ h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \bigcap H \text{ has finite index in } H, \ g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions*

$$H, \ Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$$

and (iii) H is *normal* in $N_G(H)$.

Note that $Z_G(H)$, $N_G(H)$ are *always closed in* G, while $C_G(H)$ is *not necessarily closed in* G.

Indeed, one may construct such an example as follows: Let

$$M \stackrel{\text{def}}{=} \prod_{\mathbb{N}} \mathbb{Z}_p$$

endowed with the *product topology* (of the various copies of $\mathbb{Z}_p$ equipped with their usual topology). Thus, M is a *Hausdorff topological group*. For $n \in \mathbb{N}$, write $F^n(M) \subseteq M$ for the sub-topological group given by the product of the copies of $\mathbb{Z}_p$ indexed by $m \geq n$. Write $\text{Aut}_F(M)$ for the *set of automorphisms of the topological group* M *that preserve the filtration* $F^*(M)$ on M. If $\alpha \in \text{Aut}_F(M)$, then for every $n \in \mathbb{N}$, α induces a continuous homomorphism $\alpha_n : M/F^n(M) \to M/F^n(M)$ which is clearly *surjective*, hence an *isomorphism* (since $M/F^n(M)$ is profinite and topologically finitely generated — cf. [FJ], Proposition 15.3). It thus follows that α induces an isomorphism $F^n(M) \xrightarrow{\sim} F^n(M)$, hence that the inverse of α also lies in $\text{Aut}_F(M)$. In particular, we conclude that $\text{Aut}_F(M)$ is a *group*. Equip $\mathrm{Aut}_F(M)$ with the *coarsest topology* for which all of the homomorphisms $\mathrm{Aut}_F(M) \to \mathrm{Aut}(M/F^n(M))$ (where $\mathrm{Aut}(M/F^n(M)) \cong GL_n(\mathbb{Z}_p)$ is equipped with its usual topology) are continuous. Note that relative to this topology, $\mathrm{Aut}_F(M)$ forms a *Hausdorff topological group*. Now define G to be the *semi-direct product* of M with $\mathrm{Aut}_F(M)$ (so G is a Hausdorff topological group), and H to be

$$\prod_{n \in \mathbb{N}} p^n \cdot \mathbb{Z}_p \subseteq \prod_{\mathbb{N}} \mathbb{Z}_p = M$$

(so $H \subseteq G$ is a closed subgroup). Then $C_G(H)$ is *not closed in* G. For instance, if one denotes by $e_n \in \prod_{\mathbb{N}} \mathbb{Z}_p$ the vector with a 1 in the n-th place and zeroes elsewhere, then the *limit* A_∞ (where

$$A_\infty(e_n) \stackrel{\text{def}}{=} e_n + e_{n+1}$$

for all $n \in \mathbb{N}$) *of the automorphisms* $A_m \in C_G(H)$ (where $A_m(e_n) \stackrel{\text{def}}{=} e_n + e_{n+1}$ if $n \leq m$, $A_m(e_n) \stackrel{\text{def}}{=} e_n$ if $n > m$) *is not contained in* $C_G(H)$.

Curves:

Suppose that $g \geq 0$ is an *integer*. Then a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. We shall denote the *moduli stack of* r*-pointed stable curves of genus* g (where we assume the points to be *unordered*) by $\overline{\mathcal{M}}_{g,r}$ (cf. [DM], [Knud] for an exposition of the theory of such curves; strictly speaking, [Knud] treats the finite étale covering of $\overline{\mathcal{M}}_{g,r}$ determined by *ordering* the marked points). The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth* r*-pointed stable curves of genus* g or, alternatively, as the *moduli stack of hyperbolic curves of type* (g, r).

A *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism*. (Indeed, when S is the spectrum of a field, this fact is well-known from the elementary theory of algebraic curves. Next, we consider an arbitrary *connected normal* S on which a prime l is *invertible* (which, by Zariski localization, we may assume without loss of generality). Denote by $S' \to S$ the finite étale covering parametrizing *orderings of the marked points* and *trivializations* *of the l-torsion points of the Jacobian of* Y. Note that $S' \to S$ is *independent* of the choice of (Y, D), since (by the normality of S), S' may be constructed as the *normalization* of S in the function field of S' (which is independent of the choice of (Y, D) since the restriction of (Y, D) to the generic point of S has already been shown to be unique). Thus, the uniqueness of (Y, D) follows by considering the classifying morphism (associated to (Y, D)) from S' to the finite étale covering of $(\mathcal{M}_{g,r})_{\mathbb{Z}[\frac{1}{l}]}$ parametrizing orderings of the marked points and trivializations of the l-torsion points of the Jacobian [since this covering is well-known to be a scheme, for l sufficiently large].) We shall refer to Y (respectively, D; D; D) as the *compactification* (respectively, *divisor at infinity*; *divisor of cusps*; *divisor of marked points*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above.

Next, we would like to consider *"orbicurves"*. We shall say that an algebraic stack is *generically scheme-like* if it admits an open dense algebraic substack which is isomorphic to a scheme. Let X be a *smooth, geometrically connected, generically scheme-like algebraic stack of finite type* over a field k of *characteristic zero*. Then we shall say that X is an *orbicurve* if it is of dimension 1. We shall say that X is a *hyperbolic orbicurve* if it is an orbicurve which admits a compactification $X \hookrightarrow \overline{X}$ (necessarily unique!) by a *proper orbicurve* $\overline{X}$ over k such that if we denote the reduced divisor $\overline{X} \backslash X$ by $D \subseteq \overline{X}$, then $\overline{X}$ is *scheme-like* near D, and, moreover, the line bundle $\omega_{\overline{X}/k}(D)$ on $\overline{X}$ has *positive degree.*

Now suppose that

$$X$$

is a *hyperbolic orbicurve* over a field k (of characteristic zero), with compactification $X \hookrightarrow \overline{X}$. Let $\overline{k}$ be an *algebraic closure* of k. Write

$$\overline{X} \to \overline{X}'$$

for the *"coarse moduli space"* (cf. [FC], Chapter I, Theorem 4.10) associated to $\overline{X}$. Thus, $\overline{X}'$ is a smooth, proper, geometrically connected curve over k. Denote the open subscheme of $\overline{X}'$ which is the image of X by X'. Write:

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} (\mathbb{N}\backslash\{0,1\}) \bigcup \{\infty\}$$

Then we shall say that the hyperbolic curve X is *of type*

$$(g, \vec{r})$$

if $\overline{X}'$ is of genus g and $\vec{r} : \mathbb{N}_\infty \to \mathbb{N}$ is the function with finite support [i.e., which is 0 away from some finite subset of $\mathbb{N}_\infty$] defined as follows: $\vec{r}(\infty)$ is the cardinality of $(\overline{X}'\backslash X')(\overline{k})$. For every positive integer $e \in \mathbb{N}_\infty$, $\vec{r}(e)$ is the cardinality of the set of $\overline{k}$-valued points of X' over which X is (necessarily tamely) ramified with *ramification index* e.

When $k = \overline{k}$, it is well-known (and easily verified) that the *isomorphism class* of the algebraic fundamental group $\pi_1(X)$ is *completely determined by the type* $(g, \vec{r})$.

Categories:

We shall say that two arrows $f_i : A_i \to B_i$ (where $i = 1, 2$) in a category $\mathcal{C}$ are *abstractly equivalent* — and write $f_1 \overset{\text{abs}}{\approx} f_2$ — if there exists a commutative diagram:

$$\begin{array}{ccc} A_1 & \xrightarrow{\sim} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\sim} & B_2 \end{array}$$

(where the horizontal arrows are isomorphisms in $\mathcal{C}$).

We shall refer to a *natural transformation* between functors all of whose component morphisms are *isomorphisms* as an *isomorphism between the functors* in question. A functor $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ between categories $\mathcal{C}_1$, $\mathcal{C}_2$ will be called *rigid* if ϕ has no nontrivial automorphisms.

A diagram of functors between categories will be called 1-*commutative* if the various composite functors in question are rigid and isomorphic. When such a diagram "commutes in the literal sense" we shall say that it 0-*commutes*. Note that when a diagram "1-*commutes*", it follows from the *rigidity* hypothesis that any isomorphism between the composite functors in question is necessarily *unique*. Thus, to state that the diagram 1-commutes does not result in any "loss of information" by comparison to the datum of a *specific isomorphism* between the various composites in question.

We shall say that two rigid functors $\phi_i : \mathcal{C}_i \to \mathcal{C}'_i$ (where $i = 1, 2$; the $\mathcal{C}_i$, $\mathcal{C}'_i$ are categories) are *abstractly equivalent* — and write $\phi_1 \overset{\text{abs}}{\approx} \phi_2$ — if there exists a 1-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\sim} & \mathcal{C}_2 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ \mathcal{C}'_1 & \xrightarrow{\sim} & \mathcal{C}'_2 \end{array}$$

(in which the horizontal arrows are equivalences of categories).

Section 1: Anabelioids

§1.1. The Notion of an Anabelioid

We begin by fixing a *(Grothendieck) universe* V, in the sense of set-theory (cf., e.g., [McLn1]; [McLr], §12.1), in which we shall work. Also, let us assume that we are given a *V-small category* $\mathfrak{Ens}^{\mathrm{f}}$ of *finite sets.*

Let G be a (V-small) *profinite group* — that is to say, the underlying profinite set of G is an inverse limit of V-sets indexed by a V-set. Then to G, we may associate the (V-small) *category*

$$\mathcal{B}(G)$$

of (V-small) finite sets $\in Ob(\mathfrak{Ens}^{\mathrm{f}})$ *with continuous G-action.* This category is a(n) *(elementary) topos* (in the sense of topos theory). In fact, it forms a rather special kind of topos called a *Galois category* (cf. [John1] for an exposition of the general theory of topoi and, in particular, of Galois categories; cf. also [SGA1], Exposé V).

Definition 1.1.1. We shall refer to as a *connected anabelioid* any category $\mathcal{X}$ which is equivalent to a category of the form $\mathcal{B}(G)$ for some profinite group G.

Remark 1.1.1.1. Thus, a "connected anabelioid" is the same as a *Galois category* (as defined, for instance, in [John1], p. 285) — i.e., a "Boolean topos" that admits an "exact, isomorphism reflecting functor" to the category of finite sets.

Let $\mathcal{X}$ be a *connected anabelioid.* Then recall (cf. [SGA1], Exposé V, §5) the notion of a *fundamental functor*

$$\beta^* : \mathcal{X} \to \mathfrak{Ens}^{\mathrm{f}}$$

— i.e., an *exact* functor. Here, we recall that an exact functor is a functor that preserves finite limits and finite colimits. Note that (since $\mathcal{X}$ is assumed to be a connected anabelioid) an exact functor $\beta^* : \mathcal{X} \to \mathfrak{Ens}^{\mathrm{f}}$ is necessarily *isomorphism reflecting* (i.e., a morphism α of $\mathcal{X}$ is an isomorphism if and only if $\beta^*(\alpha)$ is). Recall, moreover, that if $\mathcal{X} \stackrel{\mathrm{def}}{=} \mathcal{B}(G)$, and $\beta^* : \mathcal{B}(G) \to \mathfrak{Ens}^{\mathrm{f}}$ is the functor defined by forgetting the G-action, then G may be *recovered*, up to *inner automorphism*, from $\mathcal{X}$, β as the group:

$$\mathrm{Aut}(\beta^*)$$

Also, let us recall that any two fundamental functors are isomorphic. Note that $\mathfrak{Ens}^{\mathrm{f}}$ itself is a connected anabelioid (i.e., the result of applying $\mathcal{B}(-)$ to the trivial group), so we may think of fundamental functors as "basepoints" in the following way:

Definition 1.1.2.

(i) If $\mathcal{X}$ and $\mathcal{Y}$ are connected anabelioids, then we define a *morphism* $\phi : \mathcal{X} \to \mathcal{Y}$ to be an exact functor $\phi^* : \mathcal{Y} \to \mathcal{X}$ (cf. [SGA1], Exposé V, Proposition 6.1). An *isomorphism* between connected anabelioids is a morphism whose corresponding functor in the opposite direction is an equivalence of categories.

(ii) We define a *basepoint* of a connected anabelioid $\mathcal{X}$ to be a morphism $\beta : \mathfrak{Ens}^{\mathrm{f}} \to \mathcal{X}$. If β is a basepoint of $\mathcal{X}$, then we refer to the group $\mathrm{Aut}(\beta)$ as the *fundamental group* $\pi_1(\mathcal{X}, \beta)$ *of* $(\mathcal{X}, \beta)$.

Remark 1.1.2.1. Thus, the *"category of (V-small) connected anabelioids"* is a 2-*category* (cf., e.g., [John1], §0.1; [McLr], Chapter 12; [McLn2], XII), hence requires special care, for instance, when considering composites, etc. Also, we remark, relative to the standard terminology of category theory, that if $\phi : \mathcal{X} \to \mathcal{Y}$ is an *isomorphism* (of connected anabelioids), it will *not*, in general, be the case that ϕ^* is an *isomorphism of categories* (i.e., an equivalence for which the correspondence between classes of objects in the two categories is a *bijection* — cf. [McLn2], IV, §4).

Remark 1.1.2.2. Since the isomorphism class of the fundamental group $\pi_1(\mathcal{X}, \beta)$ is independent of the choice of basepoint β, we will also speak of the *"fundamental group* $\pi_1(\mathcal{X})$ *of* $\mathcal{X}$*"* when the choice of basepoint is irrelevant to the issue under discussion.

Remark 1.1.2.3. Note that a functor $\phi^* : \mathcal{Y} \to \mathcal{X}$ which is an *equivalence* is always *necessarily exact*. Thus, an isomorphism of anabelioids $\phi : \mathcal{X} \to \mathcal{Y}$ is simply an equivalence $\phi^* : \mathcal{Y} \to \mathcal{X}$ in the opposite direction.

Example 1.1.3. Anabelioids Associated to Schemes. Let X be a (V-small) *connected locally noetherian scheme*. Then we shall denote by

$$\acute{\mathrm{E}}\mathrm{t}(X)$$

the *category* whose *objects* are (V-small) finite étale coverings of X and whose *morphisms* are morphisms of schemes over X. Then it is well-known (cf. [SGA1], Exposé V, §7) that $\acute{\mathrm{E}}\mathrm{t}(X)$ is a connected *anabelioid*.

If G is a profinite group, then we shall use the notation

$$\mathrm{Aut}(G); \quad \mathrm{Inn}(G); \quad \mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$$

to denote the group of (continuous) *automorphisms* (respectively, *inner automorphisms*; (continuous) *outer automorphisms*) of G. If H is another profinite group, then we shall write $\mathrm{Hom}(G, H)$ for the *set of continuous homomorphisms* $G \to H$, and

$$\mathrm{Hom}^{\mathrm{Out}}(G, H)$$

for the *set of continuous outer homomorphisms* $G \to H$, i.e., the quotient of $\mathrm{Hom}(G, H)$ by the natural action of H from the right. Also, we shall write

$$\mathfrak{Hom}^{\mathfrak{Out}}(G, H)$$

for the *(V-small) category* whose objects are the elements of the set $\mathrm{Hom}(G, H)$ and for which the morphisms

$$\mathrm{Mor}_{\mathfrak{Hom}^{\mathfrak{Out}}}(\psi_1, \psi_2)$$

from an object $\psi_1 : G \to H$ to an object $\psi_2 : G \to H$ are the elements $h \in H$ such that $\psi_2(g) = h \cdot \psi_1(g) \cdot h^{-1}$, $\forall g \in G$. Thus, $\mathrm{Hom}^{\mathrm{Out}}(G, H)$ may be thought of as the set of isomorphism classes of the category $\mathfrak{Hom}^{\mathfrak{Out}}(G, H)$.

Proposition 1.1.4. (The "Grothendieck Conjecture" for Connected Anabelioids) *Let* $\mathcal{X} \stackrel{\mathrm{def}}{=} \mathcal{B}(G)$, $\mathcal{Y} \stackrel{\mathrm{def}}{=} \mathcal{B}(H)$ *(where* G, H *are profinite groups), and* $\beta : \mathfrak{Ens}^{\mathrm{f}} \to \mathcal{X}$, $\gamma : \mathfrak{Ens}^{\mathrm{f}} \to \mathcal{Y}$ *be the tautological basepoints of* $\mathcal{X}$, $\mathcal{Y}$*, respectively, determined by the definition of the notation "*$\mathcal{B}(-)$*". Then:*

(i) There is a natural equivalence of categories:

$$\mathfrak{Hom}^{\mathfrak{Out}}(G, H) \stackrel{\sim}{\to} \mathfrak{Mor}(\mathcal{X}, \mathcal{Y})$$

which induces a natural bijection:

$$\mathrm{Hom}^{\mathrm{Out}}(G, H) \stackrel{\sim}{\to} \mathrm{Mor}(\mathcal{X}, \mathcal{Y})$$

Here, $\mathfrak{Mor}(\mathcal{X}, \mathcal{Y})$ *(respectively,* $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$*) denotes the category (respectively, set of isomorphism classes) of morphisms* $\mathcal{X} \to \mathcal{Y}$*.*

(ii) There is a natural bijection:

$$\mathrm{Hom}(G, H) \stackrel{\sim}{\to} \mathrm{Mor}\{(\mathcal{X}, \beta); (\mathcal{Y}, \gamma)\}$$

Here, $\mathrm{Mor}\{(\mathcal{X}, \beta); (\mathcal{Y}, \gamma)\}$ *denotes the set of (isomorphism classes of) morphisms* $\phi : \mathcal{X} \to \mathcal{Y}$ *such that* $\phi \circ \beta = \gamma$*.*

Proof. Let us first consider the situation of (2). Given a homomorphism $\psi : G \to H$, composition with ψ induces a continuous action of G on any finite set with continuous H-action. Moreover, this operation does not affect the underlying finite set, so we get an element $\psi_{\mathrm{Mor}} \in \mathrm{Mor}\{(\mathcal{X}, \beta); (\mathcal{Y}, \gamma)\}$. This defines the morphism of (2). On the other hand, given an element $\phi \in \mathrm{Mor}\{(\mathcal{X}, \beta); (\mathcal{Y}, \gamma)\}$, it follows from the definitions that ϕ induces a morphism $\mathrm{Aut}(\beta) \to \mathrm{Aut}(\gamma)$. One checks easily that this correspondence defines a two-sided inverse (well-defined up to composition with an inner automorphism of $\mathrm{Aut}(\gamma) \cong H$) to the correspondence $\psi \mapsto \psi_{\mathrm{Mor}}$.

Next, we consider the situation of (1). By the above paragraph, we get a morphism

$$\mathrm{Hom}(G, H) \rightarrow \mathrm{Mor}(\mathcal{X}, \mathcal{Y})$$

Let us first verify that this morphism is a *surjection*. Denote by S the pro-object of $\mathcal{Y}$ whose underlying profinite set $S_{\mathrm{set}} = H$ and whose H-action is given by the usual action on $S_{\mathrm{set}} = H$ from the *left*. Note that the group of automorphisms $\mathrm{Aut}_{\mathcal{Y}}(S)$ of S (as a pro-object of $\mathcal{Y}$) may be identified with H via the action of H on $S_{\mathrm{set}} = H$ from the *right*. In fact, this action of H on S (from the right) endows S with a structure of *"H-torsor object"* of $\mathcal{Y}$. Thus, if $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is a *morphism*, then $T \stackrel{\mathrm{def}}{=} \phi^*(S)$ is an H-torsor object of $\mathcal{X}$. If we think of T as a profinite set T_{set} equipped with a G-action from the *left* and an H-action from the *right*, then let us observe that, by fixing some element $t \in T_{\mathrm{set}}$, we may identify the group of automorphisms $\mathrm{Aut}_H(T_{\mathrm{set}})$ of the profinite set T_{set} that commute with the H-action from the *right* with H via its action from the *left*. Here, we observe that such an identification

$$\mathrm{Aut}_H(T_{\mathrm{set}}) \cong H$$

is determined by the *choice* of a "basepoint" $t \in T_{\mathrm{set}}$, hence is well-defined, up to composition with an inner automorphism of H. It thus follows that the action of G on T_{set} from the *left* determines a *continuous outer homomorphism* $G \rightarrow \mathrm{Aut}_H(T_{\mathrm{set}}) = H$ which (cf. the preceding paragraph) gives rise to a morphism $\mathcal{X} \rightarrow \mathcal{Y}$ isomorphic to ϕ. This completes our verification of surjectivity.

Thus, to complete the proof of (1), it suffices to verify that there is a *natural bijection* between the set of isomorphisms between the morphisms $\phi_1, \phi_2 : \mathcal{X} \rightarrow \mathcal{Y}$ arising from two continuous homomorphisms

$$\psi_1, \psi_2 : G \rightarrow H$$

and the subset $\mathrm{Mor}_{\mathfrak{Hom}^{\mathfrak{Out}}}(\psi_1, \psi_2) \subseteq H$. To verify this, let us observe that if we pull-back the H-torsor object S of $\mathcal{Y}$ (cf. the preceding paragraph) via ϕ_1, ϕ_2 to obtain H-torsor objects $T_1 \stackrel{\mathrm{def}}{=} \phi_1^*(S)$, $T_2 \stackrel{\mathrm{def}}{=} \phi_2^*(S)$ of $\mathcal{X}$, then it is a tautology that isomorphisms $\phi_1 \xrightarrow{\sim} \phi_2$ are in natural bijective correspondence with isomorphisms $T_1 \xrightarrow{\sim} T_2$ of *H-torsor objects of* $\mathcal{X}$. Thus, the desired bijection is a consequence of Lemma 1.1.5 below. ◯

Lemma 1.1.5. (Two-Sided Group Actions) *Let*

$$\psi_1, \psi_2 : G \rightarrow H$$

be continuous homomorphisms. For $i = 1, 2$, *denote by* Y_i *a copy of* H *equipped with the usual action of* H *from the* **right** *and the action of* G *determined by composing the usual action of* H *from the* **left** *with* ψ_i; *write* t_i *for the copy of "1" in* Y_i. *Then*

$$\xi \mapsto h \in H$$

— *where* h *satisfies* $\xi(t_1) = t_2 \cdot h$ — *determines a* **bijection** *from the set of* (G, H)-*equivariant bijections* $\xi : Y_1 \xrightarrow{\sim} Y_2$ *to the subset* $\mathrm{Mor}_{\mathfrak{Hom}^{\mathfrak{Out}}}(\psi_1, \psi_2) \subseteq H$.

Proof. Indeed,

$$t_2 \cdot h \cdot \psi_1(g) = \xi(t_1) \cdot \psi_1(g) = \xi(t_1 \cdot \psi_1(g)) = \xi(g \cdot t_1) = g \cdot \xi(t_1) = t_2 \cdot \psi_2(g) \cdot h$$

i.e., $h \cdot \psi_1(g) \cdot h^{-1} = \psi_2(g)$, $\forall\, g \in G$. Thus, $h \in \mathrm{Mor}_{\mathfrak{Hom}^{\mathfrak{Out}}}(\psi_1, \psi_2) \subseteq H$. Similarly, if ψ_1 and ψ_2 differ by composition with an inner automorphism of H defined by an element $h \in \mathrm{Mor}_{\mathfrak{Hom}^{\mathfrak{Out}}}(\psi_1, \psi_2)$, then $t_1 \mapsto t_2 \cdot h$ defines a (G, H)-equivariant bijection ξ, as desired. ○

Remark 1.1.4.1. Many readers may feel that Proposition 1.1.4 is "trivial" and "well-known". The reason that we nevertheless chose to give a detailed exposition of this fact here is that it represents the *essential spirit* that we wish to convey in the term *"anabelioid"*. That is to say, we wish to think of anabelioids $\mathcal{X}$ as *generalized spaces* (which is natural since they are, after all, *topoi* — cf. [John2]) whose geometry just happens to be *"completely determined by their fundamental groups"* (albeit somewhat tautologically!). This is meant to recall the notion of an *anabelian variety* (cf. [Groth]), i.e., a variety whose geometry is determined by its fundamental group. The point here (which will become clear as the manuscript progresses) is that:

> *The introduction of anabelioids allows us to work with both "algebro-geometric anabelioids" (i.e., anabelioids arising from (anabelian) varieties — cf. Example 1.1.3) and "abstract anabelioids" (i.e., those which do not necessarily arise from an (anabelian) variety) as* **geometric objects** *on an equal footing.*

The reason that it is important to deal with "geometric objects" as opposed to groups, is that:

> *We wish to study what happens as one* **varies the basepoint** *of one of these geometric objects.*

That is to say, groups are determined only once one *fixes* a basepoint. Thus, it is difficult to describe what happens when one varies the basepoint solely in the language of groups.

Next, let

$$\phi : \mathcal{X} \to \mathcal{Y}$$

be a *morphism between connected anabelioids.* Write

$$\mathcal{I}_\phi \subseteq \mathcal{X}$$

for the smallest *subcategory* of $\mathcal{X}$ that contains all subquotients of objects in the essential image of the pull-back functor ϕ^*. One verifies immediately that $\mathcal{I}_\phi$ is a *connected anabelioid*. Note that the morphism $\phi : \mathcal{X} \to \mathcal{Y}$ factors naturally as a *composite*

$$\mathcal{X} \to \mathcal{I}_\phi \to \mathcal{Y}$$

with the property that if we choose a *basepoint* $\beta_\mathcal{X}$ of $\mathcal{X}$ and denote the resulting basepoints of $\mathcal{I}_\phi$, $\mathcal{Y}$, by $\beta_{\mathcal{I}_\phi}$, $\beta_\mathcal{Y}$, respectively, then the induced morphisms of fundamental groups

$$\pi_1(\mathcal{X}, \beta_\mathcal{X}) \twoheadrightarrow \pi_1(\mathcal{I}_\phi, \beta_{\mathcal{I}_\phi}) \hookrightarrow \pi_1(\mathcal{Y}, \beta_\mathcal{Y})$$

are a *surjection* followed by an *injection*. Moreover, we note the following consequence of Proposition 1.1.4, (i):

Corollary 1.1.6. (Automorphism of an Arrow Between Connected Anabelioids) *The set of automorphisms* $\mathrm{Aut}(\mathcal{X} \to \mathcal{Y})$ *of a* 1*-arrow* $\mathcal{X} \to \mathcal{Y}$ *between connected anabelioids is in natural bijective correspondence with the* **centralizer** *in the fundamental group of* $\mathcal{Y}$ *of the image of the fundamental group of* $\mathcal{X}$.

Definition 1.1.7.

(i) We shall refer to $\mathcal{I}_\phi$ as the *image of* $\mathcal{X}$ *in* $\mathcal{Y}$.

(ii) We shall refer to a morphism $\phi : \mathcal{X} \to \mathcal{Y}$ between connected anabelioids as a π_1*-epimorphism* (respectively, π_1*-monomorphism*) if the morphism $\mathcal{I}_\phi \to \mathcal{Y}$ (respectively, $\mathcal{X} \to \mathcal{I}_\phi$) is an equivalence.

Now let I be a *finite set*. Assume that for each $i \in I$, we are given a *connected anabelioid* $\mathcal{X}_i$. Write

$$\mathcal{X}_I \stackrel{\mathrm{def}}{=} \prod_{i \in I} \mathcal{X}_i$$

for the product of the categories $\mathcal{X}_i$. In the terminology of [SGA1], Exposé V, §9, this $\mathcal{X}_I$ is a *"multi-Galois category"*. In particular, $\mathcal{X}_I$ is a *topos*.

Definition 1.1.8. Let $\mathcal{X}$ be a topos, and $S \in \mathrm{Ob}(\mathcal{X})$ an object of $\mathcal{X}$. Write $\mathbf{0}_\mathcal{X}$ (respectively, $\mathbf{1}_\mathcal{X}$) for the initial (respectively, terminal) object of $\mathcal{X}$. Then any collection of data

$$S \cong \coprod_{a \in A} S_a$$

(where $\mathbf{0}_\mathcal{X} \not\cong S_a \in \mathrm{Ob}(\mathcal{X})$; the index set A is finite) will be called a *decomposition of* S. The object S will be called *connected* if the index set of any decomposition of S has cardinality one. The topos $\mathcal{X}$ will be called *connected* if $\mathbf{1}_\mathcal{X}$ is connected.

Next, let us observe that:

The set I and the categories $\mathcal{X}_i$ (for $i \in I$), as well as the equivalence of categories between $\mathcal{X}_I$ with the product of the $\mathcal{X}_i$ may be recovered entirely from the abstract category $\mathcal{X}_I$.

Indeed, let us denote (for $i \in I$) the object of $\mathcal{X}_I$ obtained by taking the product of $\mathbf{1}_{\mathcal{X}_i}$ with the $\mathbf{0}_{\mathcal{X}_j}$ (for $j \neq i$) by ϵ_i. Thus, we obtain a *decomposition*

$$\mathbf{1}_{\mathcal{X}_I} = \coprod_{i \in I} \epsilon_i$$

of the object $\mathbf{1}_{\mathcal{X}_I}$. Moreover, this decomposition is clearly *maximal* with respect to the partial ordering on decompositions of $\mathbf{1}_{\mathcal{X}_I}$ determined by the (obviously defined) notion of *refinements* of decompositions of $\mathbf{1}_{\mathcal{X}_I}$. Thus, we see that this decomposition may be recovered *solely from internal structure of the category $\mathcal{X}_I$.* In particular, the *finite set I* may be recovered category-theoretically from the category $\mathcal{X}_I$. Moreover, the *category $\mathcal{X}_i$* may be recovered category-theoretically from the category $\mathcal{X}_I$ as the subcategory of objects over ϵ_i. Finally, it is clear that these subcategories determine the equivalence of categories between $\mathcal{X}_I$ with the product of the $\mathcal{X}_i$.

Definition 1.1.9. We shall refer to the $\mathcal{X}_i$ as the *connected components* of $\mathcal{X}_I$ and to $\pi_0(\mathcal{X}_I) \stackrel{\text{def}}{=} I$ as the *(finite) index set of connected components.*

Definition 1.1.10.

(i) We shall refer to a category equivalent to a category of the form $\mathcal{X}_I$ as an *anabelioid.* We shall denote the 2-category of V-small anabelioids by

$$\mathfrak{Anab}^V$$

(or simply $\mathfrak{Anab}$, when there is no danger of confusion).

(ii) A *morphism* between anabelioids is defined to be an exact functor in the opposite direction. An *isomorphism* between anabelioids is a morphism whose corresponding functor in the opposite direction is an equivalence of categories.

Next, let us observe that if we are given a finite set J, together with connected anabelioids $\mathcal{Y}_j$ for each $j \in J$, and *morphisms*

$$\zeta : I \to J; \quad \phi_i : \mathcal{X}_i \to \mathcal{Y}_{\zeta(j)}$$

we get an exact functor $\phi_I^* : \mathcal{Y}_J \to \mathcal{X}_I$ (by forming the product of the ϕ_i), which we would like to regard as a *morphism* $\phi_I : \mathcal{X}_I \to \mathcal{Y}_J$.

Proposition 1.1.11. (Morphisms of Not Necessarily Connected Anabelioids) *The association*

$$\{\zeta, \phi_i\} \mapsto \phi_I$$

defines an equivalence between the category of data on the left and the category of arrows $\psi : \mathcal{X}_I \to \mathcal{Y}_J$.

Proof. Indeed, this follows immediately by considering the pull-back of $\mathbf{1}_{\mathcal{Y}_J}$, as well as of its maximal decomposition, via the *exact functor* ψ^*, in light of the fact (observed above) that

$$\mathbf{1}_{\mathcal{X}_I} = \coprod_{i \in I} \epsilon_i$$

is the *maximal* (relative to refinement) decomposition of $\mathbf{1}_{\mathcal{X}_I}$. ○

Definition 1.1.12. We shall refer to a morphism between anabelioids as a π_1-*epimorphism* (respectively, π_1-*monomorphism*) if each of the component morphisms (cf. Proposition 1.1.11) between connected anabelioids is a π_1-epimorphism (respectively, π_1-monomorphism).

§1.2. Finite Étale Morphisms

In this §, we consider the notion of a *"finite étale morphism"* in the context of anabelioids.

Let $\mathcal{X}$ be an *anabelioid*. Let $S \in \mathrm{Ob}(\mathcal{X})$. We will denote the category of *objects over* S by

$$\mathcal{X}_S$$

(i.e., the objects of $\mathcal{X}_S$ are arrows $T \to S$ in $\mathcal{X}$; the arrows of $\mathcal{X}_S$ between $T \to S$ and $T' \to S$ are S-morphisms $T \to T'$). Let us write

$$j_S : \mathcal{X}_S \to \mathcal{X}$$

for the *forgetful functor* (i.e., the functor that maps $T \to S$ to T) and

$$i_S^* : \mathcal{X} \to \mathcal{X}_S$$

for the functor given by taking the *product with* S.

Proposition 1.2.1. (The Extension Functor)

(i) The category $\mathcal{X}_S$ is an **anabelioid** *whose connected components are in natural bijective correspondence with the connected components of S.*

(ii) The functor j_S is **left adjoint** *to the functor i_S^*.*

(iii) The functor i_S^* is **exact***, hence defines a morphism of anabelioids $i_S : \mathcal{X}_S \to \mathcal{X}$.*

(iv) Suppose that S is the coproduct of a finite number of copies of $\mathbf{1}_{\mathcal{X}}$ (indexed, say, by a (V-)set A). Then each connected component of $\mathcal{X}_S$ may be identified with

$\mathcal{X}$; the set of connected components of $\mathcal{X}_S$ may be identified with A. Moreover, j_S maps a collection of objects $\{S_a\}_{a\in A}$ of $\mathcal{X}$ indexed by A to the coproduct object

$$\coprod_{a\in A} S_a$$

in $\mathcal{X}$.

(v) Suppose that $\mathcal{X} = \mathcal{B}(G)$ (where G is a (V-small) profinite group) and that S is given by the G-set G/H, where $H \subseteq G$ is an **open subgroup***. Then $i_S : \mathcal{X}_S \to \mathcal{X}$ may be identified with [i.e., is "abstractly equivalent" (cf. §0) — in a natural fashion — to] the morphism*

$$\mathcal{B}(H) \to \mathcal{B}(G)$$

induced by the inclusion $H \hookrightarrow G$. Moreover, if $T \in Ob(\mathcal{X}_S)$ is represented by an H-set T_{set}, then $j_S(T)$ is isomorphic to the G-set given by

$$(G \times T_{\text{set}})/H$$

where $H \ni h$ acts on $G \times T_{\text{set}} \ni (g,t)$ via $(g,t) \mapsto (hg, ht)$, and the G-action is the action induced on $(G \times T_{\text{set}})/H$ by letting $G \ni g$ act on G by multiplication by g^{-1} from the right.

Proof. These assertions all follow immediately from the definitions. ○

Thus, Proposition 1.2.1, (ii), shows that if $\phi : \mathcal{Y} \to \mathcal{X}$ *factors as the composite* of an isomorphism $\alpha : \mathcal{Y} \xrightarrow{\sim} \mathcal{X}_S$ with the morphism $i_S : \mathcal{X}_S \to \mathcal{X}$ for some $S \in \text{Ob}(\mathcal{X})$, then there is a *natural choice for the isomorphism* α, namely, the isomorphism induced by the *left adjoint* $\phi_! : \mathcal{Y} \to \mathcal{X}$ to the functor ϕ^*. Indeed, it follows from Proposition 1.2.1, (ii), that such a left adjoint $\phi_!$ always *exists* and that $\phi_!$ induces an isomorphism $\mathcal{Y} \xrightarrow{\sim} \mathcal{X}_{S_\phi}$, where $S_\phi \stackrel{\text{def}}{=} \phi_!(\mathbf{1}_\mathcal{Y})$.

Definition 1.2.2.

(i) A morphism of anabelioids $\phi : \mathcal{Y} \to \mathcal{X}$ will be called *finite étale* if it factors as the composite of an isomorphism $\alpha : \mathcal{Y} \xrightarrow{\sim} \mathcal{X}_S$ with the morphism $i_S : \mathcal{X}_S \to \mathcal{X}$ for some $S \in \text{Ob}(\mathcal{X})$.

(ii) Suppose that $\phi : \mathcal{Y} \to \mathcal{X}$ is a finite étale morphism. Then we shall refer to the left adjoint functor $\phi_!$ to the pull-back functor ϕ^* as the *extension functor* associated to ϕ.

Remark 1.2.2.1. Thus, the morphism $\mathcal{B}(H) \to \mathcal{B}(G)$ induced by a continuous homomorphism $\phi : H \to G$ is *finite étale* if and only if ϕ is an injection onto an open subgroup of G. Moreover, any finite étale morphism of connected anabelioids may be written in this form (by choosing appropriate basepoints for the domain and range). The characterization of Definition 1.2.2, (i), however, has the virtue of being independent of choices of basepoints.

Definition 1.2.3. Let $\phi : \mathcal{Y} \to \mathcal{X}$ be a finite étale morphism of anabelioids. Then we shall say that ϕ is a *covering* (respectively, *relatively connected*) if the induced morphism $\pi_0(\mathcal{Y}) \to \pi_0(\mathcal{X})$ on connected components (cf. Definition 1.1.9) is surjective (respectively, an bijective).

Definition 1.2.4.

(i) Let G be a profinite group. Then we shall say that G is *slim* if the centralizer $Z_G(H)$ of any open subgroup $H \subseteq G$ in G is trivial.

(ii) Let $\mathcal{X}$ be an anabelioid. Then we shall say that $\mathcal{X}$ is *slim* if the fundamental group $\pi_1(\mathcal{X}_i)$ of every connected component $i \in \pi_0(\mathcal{X})$ of $\mathcal{X}$ is slim.

(iii) A morphism of anabelioids whose corresponding pull-back functor is rigid will be called *rigid*. A 2-category of anabelioids will be called *slim* if every 1-morphism in the 2-category is rigid.

(iv) If $\mathcal{C}$ is a 2-*category*, we shall write

$$|\mathcal{C}|$$

for the *associated* 1-*category* whose *objects* are objects of $\mathcal{C}$ and whose *morphisms* are *isomorphism classes* of morphisms of $\mathcal{C}$. We shall also refer to $|\mathcal{C}|$ as the *coarsification* of $\mathcal{C}$.

Remark 1.2.4.1. The name *"coarsification"* is motivated by the theory of "coarse moduli spaces" associated to (say) "fine moduli stacks".

Remark 1.2.4.2. Thus, a diagram of rigid morphisms of anabelioids *"1-commutes"* (cf. §0) if and only if it *commutes in the coarsification.*

In a word, *the theory of coverings of anabelioids is easiest to understand when the anabelioid in question is slim.* For instance:

Proposition 1.2.5. (Slim Anabelioids) *Let $\mathcal{X}$ be a* **slim** *anabelioid. Then:*

(i) The pull-back and extension functors associated to a finite étale morphism between slim anabelioids are **rigid** *(cf. §0). In particular, if we write*

$$\mathfrak{Et}(\mathcal{X}) \subseteq \mathfrak{Anab}$$

for the 2*-category whose* **(0-)objects** *are finite étale morphisms* $\mathcal{Y} \to \mathcal{X}$ *and whose* **(1-)morphisms** *are finite étale arrows* $\mathcal{Y}_1 \to \mathcal{Y}_2$ *"over"* $\mathcal{X}$ *[i.e., in the sense of "*1*-commutativity" — cf. §0], then* $\mathfrak{Et}(\mathcal{X})$ *is* **slim**. *Write:* $\text{Ét}(\mathcal{X}) \stackrel{\text{def}}{=} |\mathfrak{Et}(\mathcal{X})|$.

(ii) The functor

$$\mathfrak{F}_{\mathcal{X}} : \mathcal{X} \to \text{Ét}(\mathcal{X})$$
$$S \mapsto (\mathcal{X}_S \to \mathcal{X})$$

(where $S \in Ob(\mathcal{X})$*) is an* **equivalence** *(i.e., fully faithful and essentially surjective).*

Proof. Indeed, (i) follows formally from Corollary 1.1.6 and Definition 1.2.4, (i), (ii), (iii). As for (ii), *essential surjectivity* follows formally from Definition 1.2.2, (i). To prove *fully faithfulness*, it suffices to compute, when $\mathcal{X} = \mathcal{B}(G)$, $\mathcal{Y}_1 = \mathcal{B}(H_1)$, $\mathcal{Y}_2 = \mathcal{B}(H_2)$, and H_1, H_2 are open subgroups of G, the subset

$$\text{Mor}_{\mathcal{X}}(\mathcal{Y}_1, \mathcal{Y}_2) \subseteq \text{Mor}(\mathcal{Y}_1, \mathcal{Y}_2)$$

[i.e., of isomorphism classes of morphisms "over" $\mathcal{X}$] via Proposition 1.1.4, (i). This computation yields that the set in question is equal to the *quotient*, via the conjugation action by H_2, of the set of morphisms $H_1 \to H_2$ induced by conjugation by an element $g \in G$ such that $H_1 \subseteq g \cdot H_2 \cdot g^{-1}$. But this quotient may be identified with the subset of elements $g \cdot H_2 \in G/H_2$ such that $H_1 \subseteq g \cdot H_2 \cdot g^{-1}$. Note that here we must apply the assumption of *slimness*, to conclude that it is not necessary to quotient G/H_2 any further by various centralizers in G of conjugates of H_1. On the other hand, this quotient is simply another description of the set

$$\text{Hom}_G(G/H_1, G/H_2)$$

as desired. ○

Remark 1.2.5.1. By Proposition 1.2.5, (i), it follows that, at least when we restrict our attention to *finite étale morphisms of slim anabelioids*, we do not "lose any essential information" by **working in the coarsification** (of $\mathfrak{Anab}$). Thus, in the following discussion, we shall often do this, since this simplifies things substantially. For instance, if $\phi : \mathcal{Y} \to \mathcal{X}$ and $\psi : \mathcal{Z} \to \mathcal{X}$ are *arbitrary finite étale morphism of slim anabelioids*, then [if we work in the coarsification] it makes sense to speak of the *pull-back* (of ϕ via ψ), or *fiber product* (of $\mathcal{Y}$, $\mathcal{Z}$ over $\mathcal{X}$):

$$\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$$

Indeed, such an object may be defined by the formula:

$$\mathcal{Z}_{\psi^*(\phi_!(\mathbf{1}_{\mathcal{Y}}))} \to \mathcal{Z}$$

By thinking of $\phi : \mathcal{Y} \to \mathcal{X}$ as some "$\mathcal{X}_S \to \mathcal{X}$" as in the above discussion, one verifies easily that this definition satisfies all the expected properties. One verifies easily that all conceivable compatibilities are satisfied [e.g., when one interchanges the roles of ϕ and ψ].

Remark 1.2.5.2. In fact, *essentially all of the anabelioids that we shall actually deal with in this paper will be* **slim**. Thus, in some sense, it might have been more natural to take the notion of a "slim anabelioid" as our definition of the term "anabelioid". There are *two reasons* why we chose not to do this: First, this would *require* us to prove slimness every time that we wish to use term "anabelioid", which would, in some sense, be rather unnatural, just as having to prove *separatedness* every time one uses the term "scheme" (if, as in the earlier terminology, one defines a scheme to be a "separated scheme" (in the current terminology)). Second, just as with the separatedness of schemes, which is *not* a Zariski local notion, the notion of slimness of an anabelioid is *not (finite) étale local.* (That is to say, a non-slim anabelioid may admit a finite étale covering which is slim.) Thus, requiring anabelioids to be slim would mean that the notion of an anabelioid is not "finite étale local", which would again be unnatural.

Remark 1.2.5.3. Note that although $\mathfrak{F}_{\mathcal{X}}$ is fully faithful and essentially surjective, *substantial care* should be exercised when speaking of $\mathfrak{F}_{\mathcal{X}}$ as an *"equivalence".* The reason for this is that:

> *The collection of objects of* $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ *or* $\mathfrak{Et}(\mathcal{X})$ *necessarily belongs to a* **larger Grothendieck universe** *— that is to say, unlike* $\mathcal{X}$*, the category* $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ *is no longer* V*-small — than the collection of objects of* $\mathcal{X}$*.*

Put another way, $\mathfrak{F}_{\mathcal{X}}$, i.e., the passage from $\mathcal{X}$ to $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$, may be thought of as a sort of *"change of Grothendieck universe, while keeping the internal category structure intact".*

Just as in the theory of schemes, one often wishes to work not just with finite étale coverings, but also with *"profinite étale coverings"* (i.e., projective systems of étale coverings). In the case of anabelioids, we make the following

Definition 1.2.6. We shall refer to as a *pro-anabelioid* any "pro-object" (indexed by a set)

$$\mathcal{X} = \varprojlim_{\alpha} \mathcal{X}_\alpha$$

relative to the coarsified category

$$\mathrm{Anab} \stackrel{\mathrm{def}}{=} |\mathfrak{Anab}|$$

in which all of the transition morphisms $\mathcal{X}_\alpha \to \mathcal{X}_\beta$ are finite étale coverings of slim anabelioids. Here, by *"pro-object",* we mean an equivalence class of projective systems (relative to the evident notion of equivalence).

Remark 1.2.6.1. Thus, (for us) pro-anabelioids only exist at the *"coarsified level"* (unlike anabelioids, which may be treated *either* in $\mathfrak{Anab}$ *or* in Anab).

Remark 1.2.6.2. Given a *pro-anabelioid*

$$\mathcal{X} = \varprojlim_{\alpha} \mathcal{X}_\alpha$$

it is natural to define the *set of connected components* of $\mathcal{X}$ by:

$$\pi_0(\mathcal{X}) \stackrel{\text{def}}{=} \varprojlim_{\alpha} \pi_0(\mathcal{X}_\alpha)$$

In general, $\pi_0(\mathcal{X})$ will be a *profinite set.* Moreover, for each $i \in I$, one obtains a *connected pro-anabelioid*

$$\mathcal{X}_i$$

by forming

$$\varprojlim_{\alpha}$$

of the compatible system of connected components of the $\mathcal{X}_\alpha$ *indexed by* i.

Remark 1.2.6.3. Given two pro-anabelioids $\mathcal{X} = \varprojlim_\alpha \mathcal{X}_\alpha$; $\mathcal{Y} = \varprojlim_\beta \mathcal{Y}_\beta$, by the definition of a *"pro-object"*, it follows that:

$$\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Mor}(\mathcal{X}_\alpha, \mathcal{Y}_\beta)$$

Note that this formula also applies in the case when one or both of $\mathcal{X}$, $\mathcal{Y}$ is an *anabelioid*, by thinking of anabelioids as pro-anabelioids indexed by the set with one element.

Suppose that we are given a *connected anabelioid* $\mathcal{X} \stackrel{\text{def}}{=} \mathcal{B}(G)$ (where G is a profinite group). Let us write $\beta : \mathfrak{Ens}^{\mathrm{f}} \to \mathcal{X}$ for the *tautological basepoint* of $\mathcal{B}(G)$. Then one important example of a pro-anabelioid which forms a *profinite étale covering* of $\mathcal{X}$ is the *"universal covering"* $\widetilde{\mathcal{X}}_\beta$, defined as follows: For each open subgroup $H \subseteq G$, let us write $\mathcal{X}_H \stackrel{\text{def}}{=} \mathcal{B}(H)$. (In other words, $\mathcal{X}_H$ is the category $\mathcal{X}_S$ associated to the object $S \in \mathrm{Ob}(\mathcal{X})$ determined by the G-set G/H.) Thus, if $H' \subseteq H$, then we have a natural morphism $\mathcal{X}_{H'} \to \mathcal{X}_H$. Moreover, these morphisms form a projective system whose transition morphisms are clearly finite étale coverings. Hence, we obtain a pro-anabelioid

$$\widetilde{\mathcal{X}}_\beta \stackrel{\text{def}}{=} \varprojlim_{H} \mathcal{X}_H$$

(where H ranges over the open subgroups of G), together with a *"profinite étale covering"*

$$\widetilde{\mathcal{X}}_\beta \to \mathcal{X}$$

which (just as in the case of schemes) has the property that the pull-back via this covering of any finite étale covering $\mathcal{Y} \to \mathcal{X}$ *splits* (i.e., is isomorphic to the coproduct of a finite number of copies of the base).

Definition 1.2.7. Let $\mathcal{X}$ be an anabelioid, and $\mathcal{Y}$ a pro-anabelioid. Then a profinite étale covering $\mathcal{Y} \to \mathcal{X}$ will be referred to as a *universal covering* of $\mathcal{X}$ if it is relatively connected [i.e., given by a projective system of relatively connected finite étale morphisms] and satisfies the property that the pull-back to $\mathcal{Y}$ of any finite étale covering of $\mathcal{X}$ splits.

Note that by Proposition 1.2.5, (ii), it follows that when $\mathcal{X} = \mathcal{B}(G)$ is *slim*, the set

$$\mathrm{Mor}_{\mathcal{X}}(\widetilde{\mathcal{X}}_\beta, \mathcal{X}_H)$$

may be identified with G/H. In particular, we obtain the result that the basepoint β is naturally equivalent to the restriction to the image of the functor $\mathfrak{F}_{\mathcal{X}}$ of Proposition 1.2.5, (ii), of the basepoint of $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ defined by the formula:

$$\mathrm{Mor}_{\mathcal{X}}(\widetilde{\mathcal{X}}_\beta, \mathcal{Y})$$

(where $\mathcal{Y} \to \mathcal{X}$ is an object of $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$).

Proposition 1.2.8. **(Basic Properties of Universal Coverings)** *Let $\mathcal{X}$ be a* **slim anabelioid***. Then:*

(i) There exists a universal covering $\mathcal{Y} \to \mathcal{X}$.

(ii) Any two universal coverings $\mathcal{Y} \to \mathcal{X}$, $\mathcal{Y}' \to \mathcal{X}$ are isomorphic over $\mathcal{X}$.

(iii) Suppose that $\mathcal{X}$ is **connected***. Then the formula*

$$\beta_{\widetilde{\mathcal{X}}}(S) \stackrel{\mathrm{def}}{=} \mathrm{Mor}_{\mathcal{X}}(\widetilde{\mathcal{X}}, \mathcal{X}_S)$$

(where $S \in Ob(\mathcal{X})$) defines an **equivalence of categories** *between the* **category of universal coverings** *$\widetilde{\mathcal{X}} \to \mathcal{X}$ (whose morphisms are isomorphisms $\widetilde{\mathcal{X}} \xrightarrow{\sim} \widetilde{\mathcal{X}}'$ over $\mathcal{X}$) and the* **category of basepoints** *$\beta : \mathfrak{Ens}^{\mathrm{f}} \to \mathcal{X}$ (whose morphisms $\beta \xrightarrow{\sim} \beta'$ are isomorphisms of functors $(\beta')^* \xrightarrow{\sim} \beta^*$). In particular, if $\widetilde{\mathcal{X}} \to \mathcal{X}$ determines the basepoint $\beta_{\widetilde{\mathcal{X}}}$, then*

$$\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}}) = \mathrm{Aut}(\beta_{\widetilde{\mathcal{X}}}) = \pi_1(\mathcal{X}, \beta_{\widetilde{\mathcal{X}}})$$

(where $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$ is the set of automorphisms relative to the category of universal coverings just defined).

(iv) Suppose that $\mathcal{X} \stackrel{\mathrm{def}}{=} \mathcal{B}(G)$, $\mathcal{X}' \stackrel{\mathrm{def}}{=} \mathcal{B}(G')$ are slim connected anabelioids. Let $\widetilde{\mathcal{X}} \to \mathcal{X}$, $\widetilde{\mathcal{X}}' \to \mathcal{X}'$ be the universal coverings determined by the tautological basepoints β, β'. Then, if we denote by $\mathrm{Isom}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{X}}')$ the set of isomorphisms $\widetilde{\mathcal{X}} \xrightarrow{\sim} \widetilde{\mathcal{X}}'$

which do **not** *necessarily lie over some isomorphism $\mathcal{X} \xrightarrow{\sim} \mathcal{X}'$, we have a* **natural isomorphism**

$$\mathrm{Isog}((\mathcal{X}, \beta); (\mathcal{X}', \beta')) \stackrel{\mathrm{def}}{=} \mathrm{Isom}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{X}}') \xrightarrow{\sim} \varinjlim_{H} \{\text{open injections } H \hookrightarrow G'\}$$

(where H ranges over the open subgroups of G).

Proof. Assertions (i), (ii) follow formally from the definitions and the above discussion. Now let us consider assertion (iv). Suppose that we are given an isomorphism $\phi : \widetilde{\mathcal{X}} \xrightarrow{\sim} \widetilde{\mathcal{X}}'$. By Proposition 1.1.4, (i), such a morphism arises from some homomorphism $H \to H'$, determined up to conjugation with an inner automorphism arising from H'. Here, we take $H \subseteq G$, $H' \subseteq G'$ to be normal open subgroups. If $K' \subseteq G$ is another normal open subgroup contained in H', then there exists a normal open subgroup $K \subseteq G$ contained in H, together with a homomorphism $K \to K'$ (determined by ϕ, up to conjugation with an inner automorphism arising from K') such that the outer homomorphism $H \to H'$ is compatible with the outer homomorphism $K \to K'$. Note, moreover, that since $\mathcal{X}'$ is *slim*, a *unique* homomorphism $H \to H'$ up to conjugation with an inner automorphism arising from K' is determined by the homomorphism $K \to K'$. (Indeed, this follows by considering the *faithful* actions (by conjugation) of H, H' on K, K', respectively.) Thus, by taking K' to be arbitrarily small, we see that ϕ *determines a unique homomorphism* $H \to H' \subseteq G'$. Consideration of the inverse to ϕ shows that this homomorphism $H \to G'$ is necessarily an open injection. On the other hand, any open injection $H \hookrightarrow G'$ clearly determines an isomorphism ϕ. This completes the proof of (iv).

Finally, we consider property (iii). Since it is clear that any isomorphism between universal coverings induces an isomorphism of the corresponding basepoints, it suffices to prove property (iii) in the "automorphism" case. For simplicity, we shall write $\mathcal{X} = \mathcal{B}(G)$, and assume that the basepoint β in question is the tautological basepoint. By property (iv), any isomorphism $\phi : \widetilde{\mathcal{X}} \xrightarrow{\sim} \widetilde{\mathcal{X}}$ arises from an open injection $H \hookrightarrow G$. The fact that the composite of ϕ with $\widetilde{\mathcal{X}} \to \mathcal{X}$ is isomorphic to $\widetilde{\mathcal{X}} \to \mathcal{X}$ implies (cf. Proposition 1.1.4, (i)) that this open injection $H \hookrightarrow G$ is induced by conjugation by a *unique* (by slimness) element of G. On the other hand, conjugation by an element of G clearly determines an element of $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$. Thus, $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}}) = G$, as desired. ○

Remark 1.2.8.1. When (cf. Proposition 1.2.8, (iv)) β, β' are *fixed* throughout the discussion, we shall write

$$\mathrm{Isog}(\mathcal{X}, \mathcal{X}')$$

for $\mathrm{Isog}((\mathcal{X}, \beta); (\mathcal{X}', \beta'))$. When $(\mathcal{X}, \beta) = (\mathcal{X}', \beta')$, we shall write $\mathrm{Isog}(\mathcal{X})$ for $\mathrm{Isog}(\mathcal{X}, \mathcal{X}')$.

Finally, before proceeding, we present the following:

Definition 1.2.9.

(i) We shall say that a continuous homomorphism of Hausdorff topological groups $G \to H$ is *relatively slim* if the centralizer in H of the image of every open subgroup of G is trivial.

(ii) We shall say that a morphism of anabelioids $\mathcal{U} \to \mathcal{V}$ is *relatively slim* if the induced morphism between fundamental groups of corresponding connected components of $\mathcal{U}$, $\mathcal{V}$ is relatively slim.

Remark 1.2.9.1. Thus, $\mathcal{X}$ is *slim* if and only if the *identity morphism* $\mathcal{X} \to \mathcal{X}$ is *relatively slim*. Also, if $\mathcal{U} \to \mathcal{V}$ is *relatively slim*, then the arrow $\mathcal{U} \to \mathcal{V}$ is *rigid* [i.e., has no nontrivial automorphisms — cf. Corollary 1.1.6]. If $\mathcal{U} \to \mathcal{V}$ is a *relatively slim* morphism between *connected anabelioids*, then it follows that $\mathcal{V}$ is *slim*; if, moreover, $\mathcal{U} \to \mathcal{V}$ is a π_1-*monomorphism*, then it follows that $\mathcal{U}$ is also *slim*.

Remark 1.2.9.2. The construction of a *pull-back, or fiber product,* discussed in Remark 1.2.5.1 generalizes immediately to the case where $\phi : \mathcal{Y} \to \mathcal{X}$ is a *finite étale morphism of slim anabelioids*, and $\psi : \mathcal{Z} \to \mathcal{X}$ is an *arbitrary relatively slim morphism of slim anabelioids*, via the formula of *loc. cit.*:

$$\mathcal{Z}_{\psi^*(\phi_!(\mathbf{1}_{\mathcal{Y}}))} \to \mathcal{Z}$$

One verifies immediately that all conceivable compatibilities are satisfied.

Section 2: Cores and Quasi-Cores

§2.1. Localizations and Cores

In this §, we discuss the notion of a *core* in the context of slim anabelioids. This notion will play a *central* role in the theory of the present paper.

Let $\mathcal{X}$ be a *slim anabelioid.* Let us write

$$\mathfrak{Loc}(\mathcal{X}) \ (\subseteq \mathfrak{Anab})$$

for the 2-*category* whose *(*0-*)objects* are (necessarily slim) anabelioids $\mathcal{Y}$ that admit a finite étale morphism to $\mathcal{X}$, and whose *(*1-*)morphisms* are finite étale morphisms $\mathcal{Y}_1 \to \mathcal{Y}_2$ (that do not necessarily lie over $\mathcal{X}$!). Note that given an object of $\mathfrak{Loc}(\mathcal{X})$, the *set of connected components of this object* may be recovered *entirely category-theoretically* from the *coarsification*

$$\mathrm{Loc}(\mathcal{X}) \stackrel{\mathrm{def}}{=} |\mathfrak{Loc}(\mathcal{X})|$$

of the 2-category $\mathfrak{Loc}(\mathcal{X})$ (cf. Proposition 1.1.11).

Proposition 2.1.1. (Categories of Localizations) *Let $\mathcal{X}$ be a* **slim anabelioid***. Then:*

(i) $\mathfrak{Loc}(\mathcal{X})$ is slim.

(ii) Denote by

$$\overline{\mathfrak{Loc}}(X)$$

the **2-category** *whose* **(0-)objects** *$\mathcal{Z}$ are* **slim** *anabelioids which arise as finite étale quotients of objects in $\mathfrak{Loc}(\mathcal{X})$ [i.e., there exists a finite étale morphism $\mathcal{Y} \to \mathcal{Z}$, where $\mathcal{Y} \in Ob(\mathfrak{Loc}(\mathcal{X}))$] and whose* **(1-)morphisms** *are finite étale morphisms. Then the 2-category $\overline{\mathfrak{Loc}}(\mathcal{X})$ is* **slim**. *Write: $\overline{\mathrm{Loc}}(\mathcal{X}) \stackrel{\mathrm{def}}{=} |\overline{\mathfrak{Loc}}(\mathcal{X})|$.*

(iii) The 2-category $\overline{\mathfrak{Loc}}(\mathcal{X})$ (respectively, category $\overline{\mathrm{Loc}}(\mathcal{X})$) may be reconstructed **entirely category-theoretically** *from $\mathfrak{Loc}(\mathcal{X})$ (respectively, $\mathrm{Loc}(\mathcal{X})$) by considering the "2-category (respectively, category) of objects of $\mathfrak{Loc}(\mathcal{X})$ (respectively, $\mathrm{Loc}(\mathcal{X})$) equipped with a finite étale equivalence relation".*

(iv) Suppose that we **arbitrarily choose** *finite étale structure morphisms to $\mathcal{X}$ for all of the objects of $\mathfrak{Loc}(\mathcal{X})$. Then every morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ of $\mathfrak{Loc}(\mathcal{X})$ may be written as the* **composite** *of an* **isomorphism** *$\mathcal{Y}_1 \stackrel{\sim}{\to} \mathcal{Y}_3$ with a finite étale morphism $\mathcal{Y}_3 \to \mathcal{Y}_2$* **over** *$\mathcal{X}$.*

Proof. Assertions (i) and (ii) are formal consequences of Corollary 1.1.6. Assertions (iii) and (iv) follow formally from the definitions. ○

Let $\mathcal{X}$ be a *slim* anabelioid. Then:

Definition 2.1.2.

(i) We shall say that $\mathcal{X}$ is a(n) *(absolute) core* if $\mathcal{X}$ is a terminal object in $\text{Loc}(\mathcal{X})$.

(ii) We shall say that $\mathcal{X}$ *admits a(n) (absolute) core* if there exists a terminal object $\mathcal{Z}$ in $\overline{\text{Loc}}(\mathcal{X})$. In this case, $\overline{\text{Loc}}(\mathcal{X}) = \overline{\text{Loc}}(\mathcal{Z}) = \text{Loc}(\mathcal{Z})$, so we shall say that $\mathcal{Z}$ is a *core*.

Remark 2.1.2.1. Note that in Proposition 2.1.1, (ii), it is important to assume that the quotients $\mathcal{Z}$ that one considers are *slim*. Indeed, if one did not impose this condition, then by "forming quotients of slim anabelioids by the trivial actions of finite groups", one verifies easily that the 1-category associated to the resulting 2-category *never* admits a terminal object — i.e., "no slim anabelioid would admit a core". From the point of view of anabelian varieties — e.g., *hyperbolic orbicurves* — this condition of slimness amounts to the condition that the algebraic stacks that one works with are *generically schemes* (cf. [Mzk9], §2).

Remark 2.1.2.2. Note that the *definability of* $\text{Loc}(\mathcal{X})$, $\overline{\text{Loc}}(\mathcal{X})$ is one of the most *fundamental differences* between the theory of finite étale coverings of *anabelioids* as discussed in §1.2 and the theory of finite étale coverings from the point of view of *"Galois categories"*, as given in [SGA1]. Indeed, from the point of view of the theory of [SGA1], it is only possible to consider "$\text{Ét}(\mathcal{X})$" — i.e., finite étale coverings and morphisms that *always lie over* $\mathcal{X}$. That is to say, in the context of the theory of [SGA1], it is not possible to consider diagrams such as:

$$\begin{array}{ccc} & \mathcal{Z} & \\ \swarrow & & \searrow \\ \mathcal{X} & & \mathcal{Y} \end{array}$$

(where the arrows are finite étale) that do *not necessarily lie over any specific geometric object.* We shall refer to such a diagram as a *correspondence* or *isogeny* between $\mathcal{X}$ and $\mathcal{Y}$. When there exists an isogeny between $\mathcal{X}$ and $\mathcal{Y}$, we shall say that $\mathcal{X}$ and $\mathcal{Y}$ are *isogenous.*

Next, we would like to consider *universal coverings.* Let β, γ be *basepoints* of a *connected slim anabelioid* $\mathcal{X}$. Write

$$\pi_\beta : \widetilde{\mathcal{X}}_\beta \to \mathcal{X}; \quad \pi_\gamma : \widetilde{\mathcal{X}}_\gamma \to \mathcal{X}$$

for the associated *universal coverings* (cf. the discussion of §1.2). In the following discussion, we would also like to consider an *isomorphism*

$$\xi : \widetilde{\mathcal{X}}_\beta \xrightarrow{\sim} \widetilde{\mathcal{X}}_\gamma$$

(cf. Proposition 1.2.8, (iv)).

Definition 2.1.3. We shall refer to an isomorphism $\xi : \widetilde{\mathcal{X}}_\beta \xrightarrow{\sim} \widetilde{\mathcal{X}}_\gamma$ as above as an *outer path* from β to γ. If ξ arises from a commutative [i.e., at the coarsified level] diagram of anabelioids

$$\begin{array}{ccc} \widetilde{\mathcal{X}}_\beta & \xrightarrow{\xi} & \widetilde{\mathcal{X}}_\gamma \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\mathrm{id}_{\mathcal{X}}} & \mathcal{X} \end{array}$$

then we shall refer to ξ as an *inner path* from β to γ. An outer (respectively, inner) path from β to itself will be referred to as an *($\widetilde{\mathcal{X}}_\beta$-valued) open (respectively, closed) path.*

Remark 2.1.3.1. Thus, *inner paths* are precisely the paths of [SGA1], Exposé V, §7. Note that the difference between an *"inner"* path and an *"outer" path* depends essentially on the *"identity" of β, γ* — i.e., what appears to be an outer path if one thinks of β and γ as in fact being *"equal"* may appear to be an inner path if one thinks of β and γ as *"distinct"*. Put another way:

> The distinction between inner and outer paths *depends essentially on the* **"model of set theory"** *under consideration* — i.e., on the **labels** that one uses to describe the various sets involved in the discussion.

It is the hope of the author to pursue this point of view in more detail in a future paper.

Remark 2.1.3.2. Note that an inner path is a special case of an outer path. The difference between an inner path and an arbitrary outer path is easiest to analyze when $\beta = \gamma$ (but cf. Remark 2.1.3.1!). In this case, an ($\widetilde{\mathcal{X}}_\beta$-valued) *closed path* is simply an element of the fundamental group $\pi_1(\mathcal{X}, \beta)$.

On the other hand, the motivation for the terminology *"open path"* is the following. Let K be a *perfect field*; L a *finite Galois extension of* K; and $\overline{K}$ an *algebraic closure* of K. Then to give a $\overline{K}$-valued basepoint β of L is to give an embedding $\iota_\beta : L \hookrightarrow \overline{K}$. If we are then given a *K-linear isomorphism* $\sigma : \overline{K} \xrightarrow{\sim} \overline{K}$ (i.e., an element $\sigma \in \mathrm{Gal}(\overline{K}/K)$), then the composite of σ with ι_β determines another embedding $\iota_\gamma : L \hookrightarrow \overline{K}$. Of course, the two basepoints β, γ of $\mathrm{Spec}(L)$ defined by $\iota_\beta, \iota_\gamma$ map to the *same basepoint* of $\mathrm{Spec}(K)$ — i.e., "if one applies the projection $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$, then σ becomes a *closed path* in $\mathrm{Spec}(K)$". This is intended to be reminiscent of the *analogy between Galois groups in field theory and fundamental groups in algebraic topology* (where we recall that the theory of the latter may be formulated not just in terms of *covering groups*, but also in terms of *literal closed paths*, i.e., topological images of the circle $\mathbb{S}^1$, in the space in question). Thus, it is natural to regard σ — when working with σ as an object associated to $\mathrm{Spec}(L)$ — as an *open path* (valued in $\overline{K}$), i.e., the analogue of a topological image of the interval $[0, 2\pi]$ as opposed to the circle $\mathbb{S}^1$, on $\mathrm{Spec}(L)$.

Incidentally, this example also shows the reason for the choice of terminology *"inner/outer path"*. That is to say, inner/closed paths induce (via "parallel transport") *inner automorphisms* of the fundamental group, while outer/open paths arise from *arbitrary (outer) automorphisms, or even isogenies,* of the fundamental group.

Proposition 2.1.4. **(The Totality of Basepoints)** *Let* $\mathcal{X}$ *be a* **connected slim anabelioid**. *Let* $\widetilde{\mathcal{X}} \to \mathcal{X}$ *be a* **universal covering** *of* $\mathcal{X}$, *that determines some basepoint* β *of* $\mathcal{X}$. *Then:*

(i) The subgroup

$$\Pi_{\mathcal{X}} \stackrel{\text{def}}{=} \pi_1(\mathcal{X}, \beta) = \text{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}}) \subseteq \text{Aut}(\widetilde{\mathcal{X}}) = \text{Isog}(\mathcal{X})$$

is **commensurable** *with all of its conjugates in* $\text{Isog}(\mathcal{X})$. *Moreover, the open subgroups of* $\Pi_{\mathcal{X}}$ *define a basis for a topology on* $\text{Isog}(\mathcal{X})$ *with respect to which* $\text{Isog}(\mathcal{X})$ *forms a* **Hausdorff topological group**. *Finally, the subgroup* $\Pi_{\mathcal{X}} \subseteq \text{Isog}(\mathcal{X})$ *is both open and closed with respect to this topology.*

(ii) $\text{Isog}(\mathcal{X})$ *acts* **transitively** *on the set of* $\widetilde{\mathcal{X}}$**-valued basepoints** *— i.e., (isomorphism classes of) profinite étale morphisms* $\widetilde{\mathcal{X}} \to \mathcal{X}$ *— of* $\mathcal{X}$. *Moreover, this action determines a bijection between the set of* $\widetilde{\mathcal{X}}$*-valued basepoints and the coset space:*

$$\text{Isog}(\mathcal{X})/\Pi_{\mathcal{X}}$$

(iii) Suppose that $\mathcal{X}$ *is a* **core**. *Then* $\Pi_{\mathcal{X}} = \text{Isog}(\mathcal{X})$. *That is to say,* $\mathcal{X}$ *admits* **precisely one** $\widetilde{\mathcal{X}}$*-valued basepoint. In particular, all open paths on* $\mathcal{X}$ *are, in fact, closed. Moreover, the natural functors*

$$\mathfrak{Et}(\mathcal{X}) \to \mathfrak{Loc}(\mathcal{X}) \to \overline{\mathfrak{Loc}}(\mathcal{X}); \quad \text{Ét}(\mathcal{X}) \to \text{Loc}(\mathcal{X}) \to \overline{\text{Loc}}(\mathcal{X})$$

are **equivalences**.

Proof. These assertions are all formal consequences of the definitions (cf. also Proposition 1.2.8, (iv)). ○

Remark 2.1.4.1. Note, however, that the subgroup of $\text{Isog}(\mathcal{X})$ generated by $\Pi_{\mathcal{X}}$ and some conjugate of $\Pi_{\mathcal{X}}$ *does not necessarily contain either of these two groups as a finite index subgroup.* Perhaps the most famous example of this phenomenon is the *theorem of Ihara* (cf., e.g., [Serre1], II, §1.4, Corollary 1) expressing $SL_2(\mathbb{Q}_p)$ as an *amalgam of two copies of* $SL_2(\mathbb{Z}_p)$, amalgamated along a subgroup which is open in both copies of $SL_2(\mathbb{Z}_p)$. In the notation of the present discussion, this example corresponds to the case

$$\mathcal{X} \stackrel{\text{def}}{=} \mathcal{B}(SL_2^{\pm}(\mathbb{Z}_p))$$

(where, instead of $SL_2(\mathbb{Z}_p)$, we use its quotient $SL_2^{\pm}(\mathbb{Z}_p)$ by ± 1 to ensure that $\mathcal{X}$ is *slim*). Note that this example shows that $\mathrm{Isog}(\mathcal{X})$ does not necessarily admit a natural structure of *profinite group*. Indeed, in the case of $SL_2^{\pm}(\mathbb{Z}_p)$, one checks easily (by applying the theory of p-adic Lie groups — cf., e.g., [Serre2], Chapter V, §7) that $\mathrm{Isog}(\mathcal{X}) = PGL_2(\mathbb{Q}_p)$ (which is not profinite).

Remark 2.1.4.2. The above example of $SL_2(\mathbb{Z}_p)$ highlights one of the *major themes* of the present paper, i.e., that:

$$\textit{open paths} \iff \mathrm{Isog}(\mathcal{X}) \iff \textit{correspondences}$$

— that is to say, just as (in the "classical theory" of the étale fundamental group given in [SGA1]) *closed paths* (i.e., elements of π_1) correspond to elements of $\Pi_{\mathcal{X}}$, *open paths* corresponds to elements of $\mathrm{Isog}(\mathcal{X})$, i.e., *"correspondences"*.

Remark 2.1.4.3. It is interesting to note relative to Proposition 2.1.4, (ii) (cf. also Proposition 1.2.8, (iii); Remark 2.1.3.1) that the *cardinality* of the collection of *basepoints* $\mathfrak{Ens}^{\mathrm{f}} \to \mathcal{X}$ is the same as that of the collection of *profinite étale morphisms* $\widetilde{\mathcal{X}} \to \mathcal{X}$. Indeed, both collections have the same cardinality as the collection of morphisms $\mathfrak{Ens}^{\mathrm{f}} \to \mathfrak{Ens}^{\mathrm{f}}$.

§2.2. Holomorphic Structures and Commensurable Terminality

In this §, we wish to discuss a *relative version* of the theory of §2.1. Let $\mathcal{X}$, $\mathcal{Q}$ be *slim anabelioids*.

Definition 2.2.1.

(i) A *$\mathcal{Q}$-holomorphic structure* on $\mathcal{X}$ is the datum of a relatively slim morphism (cf. Definition 1.2.9, (ii)) $\mathcal{X} \to \mathcal{Q}$, which we shall refer to as the *structure morphism*.

(ii) A slim anabelioid equipped with a $\mathcal{Q}$-holomorphic structure will be referred to as a *$\mathcal{Q}$-anabelioid*.

(iii) A *$\mathcal{Q}$-holomorphic morphism* (or *"$\mathcal{Q}$-morphism"* for short) between $\mathcal{Q}$-anabelioids is a morphism of anabelioids compatible with the $\mathcal{Q}$-holomorphic structures.

(iv) A $\mathcal{Q}$-holomorphic structure/$\mathcal{Q}$-anabelioid will be called *faithful* if its structure morphism is a π_1-monomorphism.

Remark 2.2.1.1. Here, we note that the term *"compatible"* in Definition 2.2.1, (iii), makes sense, precisely because of the assumption of *relative slimness* in Definition 2.2.1, (i) (cf. Corollary 1.1.6).

Let us write

$$\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})$$

for the 2-*category* whose *(*0*-)objects* $\mathcal{Y} \to \mathcal{Q}$ are $\mathcal{Q}$-anabelioids that admit a $\mathcal{Q}$-holomorphic finite étale morphism $\mathcal{Y} \to \mathcal{X}$ to $\mathcal{X}$, and whose *(*1*-)morphisms* are arbitrary finite étale $\mathcal{Q}$-morphisms (that do not necessarily lie over $\mathcal{X}$!). Now we have the *"$\mathcal{Q}$-holomorphic analogue"* of Proposition 2.1.1:

Proposition 2.2.2. **(Categories of Holomorphic Localizations)** *Let $\mathcal{Q}$ be a* **slim, connected anabelioid***; $\mathcal{X}$ a* **$\mathcal{Q}$-anabelioid***. Then:*

(i) $\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})$ is slim. Write: $\text{Loc}_{\mathcal{Q}}(\mathcal{X}) \stackrel{\text{def}}{=} |\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})|$.

(ii) Denote by

$$\overline{\mathfrak{Loc}}_{\mathcal{Q}}(X)$$

the **2-category** *whose* **(**0**-)objects** *$\mathcal{Z} \to \mathcal{Q}$ are $\mathcal{Q}$-anabelioids which arise as finite étale quotients of objects in $\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})$ [i.e., there exists a finite étale $\mathcal{Q}$-morphism $\mathcal{Y} \to \mathcal{Z}$, where $\mathcal{Y} \in Ob(\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X}))$] and whose* **(**1**-)morphisms** *are finite étale $\mathcal{Q}$-morphisms. Then the* 2*-category $\overline{\mathfrak{Loc}}_{\mathcal{Q}}(\mathcal{X})$ is* **slim**. *Write:* $\overline{\text{Loc}}_{\mathcal{Q}}(\mathcal{X}) \stackrel{\text{def}}{=} |\overline{\mathfrak{Loc}}_{\mathcal{Q}}(\mathcal{X})|$.

(iii) The 2*-category $\overline{\mathfrak{Loc}}_{\mathcal{Q}}(\mathcal{X})$ (respectively, category $\overline{\text{Loc}}_{\mathcal{Q}}(\mathcal{X})$) may be reconstructed* **entirely category-theoretically** *from $\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})$ (respectively, $\text{Loc}_{\mathcal{Q}}(\mathcal{X})$) by considering the "*2*-category (respectively, category) of objects of $\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})$ (respectively, $\text{Loc}_{\mathcal{Q}}(\mathcal{X})$) equipped with a finite étale equivalence relation".*

(iv) Suppose that we **arbitrarily choose** *finite étale structure morphisms to $\mathcal{X}$ for all of the objects of $\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})$. Then every morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ of $\mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X})$ may be written as the* **composite** *of an* **isomorphism** *$\mathcal{Y}_1 \xrightarrow{\sim} \mathcal{Y}_3$ (over $\mathcal{Q}$) with a finite étale morphism $\mathcal{Y}_3 \to \mathcal{Y}_2$* **over** *$\mathcal{X}$.*

Let $\mathcal{X}$ be a *$\mathcal{Q}$-anabelioid*. Then:

Definition 2.2.3.

(i) We shall say that $\mathcal{X}$ is a *$\mathcal{Q}$-core* if $\mathcal{X}$ [i.e., $\mathcal{X} \to \mathcal{Q}$] is a terminal object in $\text{Loc}_{\mathcal{Q}}(\mathcal{X})$.

(ii) We shall say that $\mathcal{X}$ *admits a $\mathcal{Q}$-core* if there exists a terminal object $\mathcal{Z}$ in $\overline{\text{Loc}}_{\mathcal{Q}}(\mathcal{X})$. In this case, $\overline{\text{Loc}}_{\mathcal{Q}}(\mathcal{X}) = \overline{\text{Loc}}_{\mathcal{Q}}(\mathcal{Z}) = \text{Loc}_{\mathcal{Q}}(\mathcal{Z})$, so we shall say that $\mathcal{Z}$ is a *$\mathcal{Q}$-core*.

Definition 2.2.4.

(i) We shall say that a closed subgroup $H \subseteq G$ of a profinite group G is *commensurably* (respectively, *normally*) *terminal* if the commensurator $C_G(H)$ (respectively, normalizer $N_G(H)$) is equal to H.

(ii) We shall say that a π_1-monomorphism of anabelioids $\mathcal{U} \to \mathcal{V}$ is *commensurably* (respectively, *normally*) *terminal* if the image of the induced morphism between fundamental groups of corresponding connected components of $\mathcal{U}$, $\mathcal{V}$ is *commensurably* (respectively, *normally*) *terminal.*

Remark 2.2.4.1. Thus, it is a formal consequence of the definitions that:

$$\text{commensurably terminal} \implies \text{normally terminal}$$

and that

$$\text{commensurably terminal with slim domain} \implies \text{relatively slim}$$

(where the "domain" is the group H (respectively, anabelioid $\mathcal{U}$) in Definition 2.2.4, (i) (respectively, (ii))).

Proposition 2.2.5. (Commensurable Terminality and Holomorphic Cores) *Let $\mathcal{X}$ be a* **connected faithful $\mathcal{Q}$-anabelioid***; assume that $\mathcal{Q}$ is also connected. Then $\mathcal{X}$ is a $\mathcal{Q}$-***core** *if and only if its structure morphism is* **commensurably terminal**.

Proof. Without loss of generality, we may write $\mathcal{X} = \mathcal{B}(H)$, $\mathcal{Q} = \mathcal{B}(G)$, where $H \subseteq G$ is a *closed subgroup*. First, we verify *sufficiency*. By Proposition 1.1.4, it suffices to prove that, if $H' \subseteq H$ is an open subgroup, then any continuous homomorphism $\phi : H' \to G$ whose image lies in H and which factors as the composite of the natural inclusion $H' \hookrightarrow G$ with conjugation by an element $g \in G$ is, in fact, equal to the to composite of the natural inclusion $H' \hookrightarrow G$ with conjugation by an element $h \in H$. But this follows immediately from Definition 2.2.4, (i), which implies that $g \in H$. Finally, *necessity* follows by reversing the preceding argument in the evident fashion. ○

Let $\mathcal{X}$ be a *connected $\mathcal{Q}$-anabelioid.* For simplicity, we also assume that $\mathcal{Q}$ is *connected.* Suppose that we are given a *universal covering* $\widetilde{\mathcal{Q}} \to \mathcal{Q}$ of $\mathcal{Q}$ and consider the resulting *cartesian diagram*:

$$\begin{array}{ccc} \widetilde{\mathcal{Q}}|_{\mathcal{X}} & \longrightarrow & \widetilde{\mathcal{Q}} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Q} \end{array}$$

Note that $\Pi_{\mathcal{Q}} \stackrel{\text{def}}{=} \text{Aut}_{\mathcal{Q}}(\widetilde{\mathcal{Q}})$ acts (compatibly) on $\widetilde{\mathcal{Q}}$ over $\mathcal{Q}$, as well as on $\widetilde{\mathcal{Q}}|_{\mathcal{X}}$ over $\mathcal{X}$. On the other hand, if we consider a connected component $\widetilde{\mathcal{X}}$ of $\widetilde{\mathcal{Q}}|_{\mathcal{X}}$ as an *independent geometric object*, even if the *$\mathcal{Q}$-holomorphic structure on $\widetilde{\mathcal{X}}$ remains fixed*, in general $\widetilde{\mathcal{X}}$ will admit *distinct* (profinite) étale morphisms to $\mathcal{X}$. Put another way, *in general,* $\mathcal{X}$ *admits distinct* $\widetilde{\mathcal{X}}$*-valued* $\mathcal{Q}$*-holomorphic basepoints.* That is to say, we have the $\mathcal{Q}$-holomorphic analogue of Proposition 2.1.4:

Proposition 2.2.6. (The Totality of $\mathcal{Q}$-Holomorphic Basepoints) *Let* $\mathcal{X}$ *be a* **connected faithful $\mathcal{Q}$-anabelioid**, *where* $\mathcal{Q}$ *is also connected. Let* $\widetilde{\mathcal{Q}} \to \mathcal{Q}$ *be a* **universal covering** *of* $\mathcal{Q}$*;* $\widetilde{\mathcal{X}} \to \mathcal{X}$ *a connected component of* $\widetilde{\mathcal{Q}}|_{\mathcal{X}} \to \mathcal{X}$*. Write* $\Pi_{\mathcal{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{Q}}(\widetilde{\mathcal{Q}})$, $\Pi_{\mathcal{X}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$. *Thus, we have a natural inclusion* $\Pi_{\mathcal{X}} \subseteq \Pi_{\mathcal{Q}}$*. Then:*

(i) The subgroup

$$\Pi_{\mathcal{X}} = \mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}}) \subseteq \mathrm{Isog}_{\mathcal{Q}}(\mathcal{X}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{Q}}(\widetilde{\mathcal{X}}) = C_{\Pi_{\mathcal{Q}}}(\Pi_{\mathcal{X}})$$

is **commensurable** *with all of its conjugates in* $\mathrm{Isog}_{\mathcal{Q}}(\mathcal{X})$*. Moreover, the open subgroups of* $\Pi_{\mathcal{X}}$ *define a basis for a topology on* $\mathrm{Isog}_{\mathcal{Q}}(\mathcal{X})$ *with respect to which* $\mathrm{Isog}_{\mathcal{Q}}(\mathcal{X})$ *forms a* **Hausdorff topological group**. *Finally, the subgroup* $\Pi_{\mathcal{X}} \subseteq \mathrm{Isog}_{\mathcal{Q}}(\mathcal{X})$ *(respectively,* $\mathrm{Isog}_{\mathcal{Q}}(\mathcal{X}) \subseteq \mathrm{Isog}(\mathcal{X})$*) is both open and closed (respectively, open) with respect to this topology.*

(ii) $\mathrm{Isog}_{\mathcal{Q}}(\mathcal{X})$ *acts* **transitively** *on the set of* **$\widetilde{\mathcal{X}}$-valued $\mathcal{Q}$-holomorphic basepoints** *— i.e., (isomorphism classes of) profinite étale* $\mathcal{Q}$*-morphisms* $\widetilde{\mathcal{X}} \to \mathcal{X}$ *— of* $\mathcal{X}$*. Moreover, this action determines a bijection between the set of* $\widetilde{\mathcal{X}}$*-valued* $\mathcal{Q}$*-holomorphic basepoints and the coset space:*

$$\mathrm{Isog}_{\mathcal{Q}}(\mathcal{X})/\Pi_{\mathcal{X}}$$

(iii) Suppose that $\mathcal{X}$ *is a* **$\mathcal{Q}$-core**. *Then* $\Pi_{\mathcal{X}} = \mathrm{Isog}_{\mathcal{Q}}(\mathcal{X})$*. That is to say,* $\mathcal{X}$ *admits* **precisely one** $\widetilde{\mathcal{X}}$*-valued* $\mathcal{Q}$*-holomorphic basepoint. In particular, all "*$\mathcal{Q}$*-holomorphic" open paths on* $\mathcal{X}$ *are, in fact, closed. Moreover, the natural functors*

$$\mathfrak{Et}(\mathcal{X}) \to \mathfrak{Loc}_{\mathcal{Q}}(\mathcal{X}) \to \overline{\mathfrak{Loc}}_{\mathcal{Q}}(\mathcal{X}); \quad \acute{\mathrm{E}}\mathrm{t}(\mathcal{X}) \to \mathrm{Loc}_{\mathcal{Q}}(\mathcal{X}) \to \overline{\mathrm{Loc}}_{\mathcal{Q}}(\mathcal{X})$$

are **equivalences**.

Remark 2.2.6.1. Thus, at a more intuitive level, just as "(absolute) cores have essentially only one basepoint", *if* $\mathcal{X}$ *is a* $\mathcal{Q}$*-core, then every basepoint of* $\mathcal{Q}$ *determines an essentially* **unique** *(up to renaming)* $\mathcal{Q}$*-holomorphic basepoint of* $\mathcal{X}$*.*

Remark 2.2.6.2. The topology of Proposition 2.2.6, (i), is *not* to be confused with the topology on $C_{\Pi_{\mathcal{Q}}}(\Pi_{\mathcal{X}})$ induced by the topology of $\Pi_{\mathcal{Q}}$. For instance, if $\Pi_{\mathcal{X}}$ is the *profinite free group on* 2 *generators* (which is easily seen to be *slim* — cf., e.g., [Mzk8], Lemma 1.3.1) and $\Pi_{\mathcal{Q}} = \mathrm{Aut}(\Pi_{\mathcal{X}})$ (which also has a natural structure of profinite group), then $\Pi_{\mathcal{Q}} = C_{\Pi_{\mathcal{Q}}}(\Pi_{\mathcal{X}})$, but $\Pi_{\mathcal{X}}$ is *not open* [i.e., relative to the profinite topology of $\Pi_{\mathcal{Q}}$] in $\Pi_{\mathcal{Q}}$. Here, we note that $\mathrm{Out}(\Pi_{\mathcal{X}}) = \mathrm{Aut}(\Pi_{\mathcal{X}})/\Pi_{\mathcal{X}}$, hence also $\Pi_{\mathcal{Q}}$, is *infinite* and *slim.* [Indeed, the *slimness* of $\Pi_{\mathcal{Q}}$ may be shown, for instance, as follows: By [Tama], Theorem 0.4, applied to the projective line minus three points over the field of rational numbers, it follows that the centralizer of any open subgroup of $\mathrm{Out}(\Pi_{\mathcal{X}})$ is contained in the subgroup of $\mathrm{Out}(\Pi_{\mathcal{X}})$ obtained by considering the permutation group of the three points. On the other hand, by projecting to $\mathrm{Out}(\Pi_{\mathcal{X}}^{\mathrm{ab}}) \cong GL_2(\widehat{\mathbb{Z}})$, one sees that any element of this permutation group that centralizes an open subgroup of $GL_2(\widehat{\mathbb{Z}})$ must be trivial.]

§2.3. Quasi-Cores and Intrinsic Exhaustivity

In order to define the *fundamental group* of a (connected slim) anabelioid $\mathcal{X}$, it is necessary to choose a *basepoint* for $\mathcal{X}$. As we saw in Proposition 1.2.8, this is equivalent to choosing a *universal cover* $\widetilde{\mathcal{X}} \to \mathcal{X}$ of $\mathcal{X}$. On the other hand, in general, there is **nothing special** that distinguishes a given profinite étale $\widetilde{\mathcal{X}} \to \mathcal{X}$ from another $\widetilde{\mathcal{X}} \to \mathcal{X}$ obtained from the first by composition with some element of $\mathrm{Aut}(\widetilde{\mathcal{X}}) = \mathrm{Isog}(\mathcal{X})$. That is to say, the difference between these two $\widetilde{\mathcal{X}} \to \mathcal{X}$ is a **"matter of arbitrary choices of labels"**. Thus, the question naturally arises:

> *To what extent is it possible to construct the fundamental group of a (connected slim) anabelioid in a* **canonical fashion** *that does not depend on such arbitrary choices?*

In this § and the next, we would like to analyze this issue in more detail. Our main result (cf. Theorem 2.4.3 below) states that when the anabelioid in question *admits a "faithful quasi-core"* (cf. Definition 2.3.1), then its *fundamental group* can indeed be constructed in a rather *canonical fashion*. In addition to quasi-cores, we also consider the notion of *intrinsic exhaustivity*, which provides a *convenient, intrinsic necessary condition* for an anabelioid to admit a faithful quasi-core.

In the following, we shall always consider *morphisms* between anabelioids in the *coarsification* Anab of $\mathfrak{Anab}$.

Definition 2.3.1. Let $\mathcal{X}$ be a $\mathcal{Q}$*-anabelioid* (so $\mathcal{X}$, $\mathcal{Q}$ are *slim*). For simplicity, we also assume that the fundamental group of every irreducible component of $\mathcal{Q}$ is *countably (topologically) generated.*

(i) We shall say that $\mathcal{X}$ *admits ($\mathcal{Q}$ as) a quasi-core* if the natural functor

$$\mathrm{Loc}_{\mathcal{Q}}(\mathcal{X}) \to \mathrm{Loc}(\mathcal{X})$$

(given by forgetting the $\mathcal{Q}$-holomorphic structure) is an equivalence.

(ii) We shall say that $\mathcal{X}$ *admits ($\mathcal{Q}$ as) a faithful quasi-core* if $\mathcal{X}$ admits $\mathcal{Q}$ as a quasi-core, and, moreover, the $\mathcal{Q}$-structure on $\mathcal{X}$ is faithful.

Next, let us recall that if G is a *slim profinite group*, then it admits a natural injection

$$G \hookrightarrow \mathrm{Isog}(G) \stackrel{\mathrm{def}}{=} \mathrm{Isog}(\mathcal{B}(G))$$

(cf. Proposition 2.1.4, (i)). Thus, in the following discussion, we shall regard G as a *subgroup* of $\mathrm{Isog}(G)$.

Definition 2.3.2. We shall refer to as a *profinite subgroup* $K \subseteq \mathrm{Isog}(G)$ a subgroup K of the *abstract group* $\mathrm{Isog}(G)$ which is equipped with a structure of profinite group such that the intersection $K \bigcap G$ is a *closed* subgroup of both G and K whose induced topologies from G and K coincide.

Remark 2.3.2.1. If $K \subseteq \mathrm{Isog}(G)$ is a *profinite subgroup* which is, moreover, *commensurable to a closed subgroup* $F \subseteq G$ (i.e., $K \bigcap F$ is open in F, K), then one verifies easily that the topology on K is the *unique topology* with respect to which $K \subseteq \mathrm{Isog}(G)$ is a *profinite subgroup*.

Remark 2.3.2.2. One verifies immediately that if $G' \subseteq \mathrm{Isog}(G)$ is a *profinite subgroup commensurable to* G — so that one has a *natural identification* $\mathrm{Isog}(G) = \mathrm{Isog}(G')$ — then the profinite subgroups of $\mathrm{Isog}(G)$ are the *same* (relative to this identification) as the profinite subgroups of $\mathrm{Isog}(G')$.

We will also make use of the following definitions:

Definition 2.3.3.

(i) A profinite group G will be called *weakly intrinsically exhaustive* if for every open subgroup $H \subseteq G$ and every open embedding $\iota : H \hookrightarrow G$, we have:

$$[G : H] = [G : \iota(H)]$$

(ii) A slim profinite group G will be called *intrinsically exhaustive* if there exists a filtration

$$\ldots \subseteq G_{n+1} \subseteq G_n \subseteq \ldots \subseteq G$$

(where n ranges over the positive integers) of open normal subgroups G_n of G such that

$$\bigcap_n G_n = \{1\}$$

and, moreover, for any *profinite subgroup* $K \subseteq \mathrm{Isog}(G)$ *commensurable to* G, there exists an integer n_K — depending only on the profinite subgroup K — such that $G_n \subseteq K$ for $n \geq n_K$, and, for any open subgroup $H \subseteq G_n$ (where $n \geq n_K$) and any open embedding $\iota : H \hookrightarrow K$, we have $\iota(H) \subseteq G_n$ $(\subseteq K)$.

(ii) An anabelioid will be called *intrinsically exhaustive* (respectively, *weakly intrinsically exhaustive*) if the fundamental group of every connected component of the anabelioid is intrinsically exhaustive (respectively, weakly intrinsically exhaustive).

Definition 2.3.4. Let $\mathcal{X}$ be a $\mathcal{Q}$-anabelioid. Then we shall refer to a finite étale (necessarily Galois) covering $\mathcal{Y} \to \mathcal{X}$ obtained as the direct summand of the pull-back via the structure morphism $\mathcal{X} \to \mathcal{Q}$ of a finite étale Galois covering $\mathcal{R} \to \mathcal{Q}$ as $\mathcal{Q}$-*Galois*.

Proposition 2.3.5. (Basic Properties of Quasi-Cores and Intrinsic Exhaustivity)

(i) Suppose that a slim anabelioid $\mathcal{X}$ admits a quasi-core $\mathcal{X} \to \mathcal{Q}$. Then the natural functor

$$\overline{\text{Loc}}_{\mathcal{Q}}(\mathcal{X}) \to \overline{\text{Loc}}(\mathcal{X})$$

(given by forgetting the $\mathcal{Q}$-holomorphic structure) is an equivalence. Moreover, any relatively slim composite $\mathcal{X} \to \mathcal{Q}'$ of $\mathcal{X} \to \mathcal{Q}$ with a morphism $\mathcal{Q} \to \mathcal{Q}'$ of slim anabelioids is also a quasi-core for $\mathcal{X}$.

(ii) If a slim anabelioid $\mathcal{X}$ admits a core $\mathcal{X} \to \mathcal{Q}$, then $\mathcal{X} \to \mathcal{Q}$ is a faithful quasi-core for $\mathcal{X}$.

(iii) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are slim, connected anabelioids which are **isogenous**. *Then $\mathcal{X}$ admits a quasi-core (respectively, admits a faithful quasi-core) if and only if $\mathcal{Y}$ does.*

(iv) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are slim, connected anabelioids which are **isogenous**. *Then $\mathcal{X}$ is intrinsically exhaustive (respectively, weakly intrinsically exhaustive) if and only if $\mathcal{Y}$ is.*

(v) If $\mathcal{X}$ is intrinsically exhaustive, then it is weakly intrinsically exhaustive.

(vi) Suppose that $\mathcal{X}$ is **weakly intrinsically exhaustive**. *Then there is a unique map*

$$\deg_{\mathcal{X}} : Ob(\mathfrak{Loc}(\mathcal{X})) \to \mathbb{Q}_{>0}$$

such that

$$\deg_{\mathcal{X}}(\mathcal{X}) = 1; \quad \deg(\mathcal{Y}_1/\mathcal{Y}_2) = \deg_{\mathcal{X}}(\mathcal{Y}_1)/\deg_{\mathcal{X}}(\mathcal{Y}_2)$$

for all morphisms $\mathcal{Y}_1 \to \mathcal{Y}_2$ of $\mathfrak{Loc}(\mathcal{X})$. In particular, if $\mathcal{Y} \to \mathcal{X}$ is a finite étale morphism of connected anabelioids of degree > 1, then $\mathcal{Y}$ is not isomorphic to $\mathcal{X}$.

(vii) Let $\mathcal{X}$ be a slim, connected, **weakly intrinsically exhaustive** *anabelioid that admits a* **quasi-core** *$\mathcal{X} \to \mathcal{Q}$. Let*

$$\phi : \mathcal{Y} \to \mathcal{X}$$

be a connected **$\mathcal{Q}$-Galois** *covering. Then any finite étale (not necessarily Galois!) morphism $\psi : \mathcal{Y} \to \mathcal{X}$ is* **abstractly equivalent** *(cf. §0) to ϕ.*

(viii) If $\mathcal{X}$ admits a **faithful quasi-core**, *then $\mathcal{X}$ is* **intrinsically exhaustive**. *In particular, if $\mathcal{X}$ admits a core, then $\mathcal{X}$ is intrinsically exhaustive.*

Proof. Assertions (i), (ii), (iv), and (vi) are immediate from the definitions. Assertion (iii) follows from the definitions and assertion (i). Next, we verify assertion (v). Let $H \subseteq G$ be an open subgroup, and $\iota : H \hookrightarrow G$ be an open embedding.

Suppose that (for some large n) G_n (as in Definition 2.3.3) is contained in H, so $\iota(G_n) \subseteq G_n$. Then:

$$\begin{aligned}\infty > [G : \iota(H)] \cdot [H : G_n] &= [G : \iota(H)] \cdot [\iota(H) : \iota(G_n)] \\ &= [G : \iota(G_n)] = [G : G_n] \cdot [G_n : \iota(G_n)] \\ &\geq [G : G_n] = [G : H] \cdot [H : G_n]\end{aligned}$$

Thus, $[G : \iota(H)] \geq [G : H]$. On the other hand, if we apply this inequality to $\iota^{-1} : \iota(H) \hookrightarrow G$, then we obtain the *reverse inequality*. This implies equality, as desired.

Next, we turn to assertion (vii). Suppose that $\mathcal{X} \to \mathcal{Q}$ is a *quasi-core* for $\mathcal{X}$. Without loss of generality, we may assume that $\mathcal{X} = \mathcal{B}(G)$, $\mathcal{Q} = \mathcal{B}(A)$, and that $\mathcal{X} \to \mathcal{Q}$ is induced by a continuous homomorphism $G \to A$ which factors:

$$G \twoheadrightarrow G_A \subseteq A$$

If $B \subseteq A$ is an *open normal subgroup* of A, and $H_A \stackrel{\mathrm{def}}{=} G_A \bigcap B$, $H \stackrel{\mathrm{def}}{=} G \times_A B$, then for any open embedding $\iota : H \hookrightarrow G$, it follows from Definition 2.3.1, (i), that the *image of the composite* of ι with the homomorphism $G \to A$ is equal to $a \cdot H_A \cdot a^{-1}$ (for some element $a \in A$). Thus, since B is *normal* in A, we conclude that $a \cdot H_A \cdot a^{-1} \subseteq G_A \bigcap B = H_A$ (for some $a \in A$). On the other hand, this implies that ι factors through H, hence — by assertion (vi) — that $\iota(H) = H$, as desired.

Finally, we turn to assertion (viii). Suppose that $\mathcal{X} \to \mathcal{Q}$ is a *faithful quasi-core* for $\mathcal{X}$. Without loss of generality, we may assume that $\mathcal{X} = \mathcal{B}(G)$, $\mathcal{Q} = \mathcal{B}(A)$, where $G \subseteq A$ is a closed subgroup of a profinite group A. Let

$$\ldots \subseteq A_{n+1} \subseteq A_n \subseteq \ldots \subseteq A$$

(where n ranges over the positive integers) be a descending sequence of *open normal subgroups* of A (which exists since A is assumed to be *countably (topologically) generated* — cf. Definition 2.3.1) such that:

$$\bigcap_n A_n = \{1\}$$

Let $G_n \stackrel{\mathrm{def}}{=} G \bigcap A_n$. Then for any *profinite subgroup* $K \subseteq \mathrm{Isog}(G)$ *commensurable to* G, it follows from assertion (i) that $K \bigcap G \subseteq G \subseteq A$ *extends uniquely* to an inclusion $K \subseteq A$. Now take n_K to be sufficiently large that $G_n = K_n \stackrel{\mathrm{def}}{=} K \bigcap A_n$ $(\subseteq K)$, for all $n \geq n_K$. Then for any open subgroup $H \subseteq G_n$ (where $n \geq n_K$) and any open embedding $\iota : H \hookrightarrow K$, it follows from Definition 2.3.1, (i), that the composite of ι with the inclusion $K \subseteq A$ is induced by *conjugation by an element* $a \in A$. Thus, (since A_n is *normal* in A) we obtain the desired inclusion:

$$\iota(H) = a \cdot H \cdot a^{-1} \subseteq K \bigcap A_n = K_n = G_n$$

○

Remark 2.3.5.1. Thus, in words (cf. Definition 2.3.3; Proposition 2.3.5, (vi)), *weak intrinsic exhaustivity* means, with respect to finite étale localization on $\mathcal{B}(G)$, that:

> The property of "being sufficiently local as to be finite étale over $\mathcal{B}(G)$ of *degree N*" is *intrinsic*.

On the other hand, *intrinsic exhaustivity* means that:

> The property of "being sufficiently local as to be *finite étale over* $\mathcal{B}(G_n)$" is *intrinsic*.

Moreover, we have *implications* (cf. Proposition 2.3.5, (v), (viii)):

$$\begin{aligned}\text{existence of a faithful quasi-core} &\implies \text{intrinsic exhaustivity}\\ &\implies \text{weak intrinsic exhaustivity}\end{aligned}$$

Here, the *second implication* is *strict* (cf. Example 2.3.7, (ii), (iii), below), but it is not clear to the author at the time of writing to what extent the first implication is strict (but cf. Theorem 3.1.3, (iii); Corollary 3.1.7).

Proposition 2.3.6. **(Quasi-Cores and the Group of Isogenies)** *Let G be a* **slim profinite group**.

(i) Suppose that $\text{Isog}(G)$ *is* **profinite** *(i.e., "*$\text{Isog}(G) \subseteq \text{Isog}(G)$ *is a profinite subgroup" — cf. Definition 2.3.2). Then* $\mathcal{B}(G) \to \mathcal{B}(\text{Isog}(G))$ *is a* **quasi-core**.

(ii) Suppose that G is **intrinsically exhaustive***; let $\{G_n\}$ be as in Definition 2.3.3, (ii). Then the natural inclusions* $\ldots \subseteq \text{Aut}(G_n) \subseteq \text{Aut}(G_{n+1}) \subseteq \ldots \subseteq \text{Isog}(G)$ *(where $n \geq n_G$) induce an isomorphism of abstract groups:*

$$\varinjlim_n \ \text{Aut}(G_n) \xrightarrow{\sim} \text{Isog}(G)$$

(iii) Suppose that G is a **closed subgroup** *of a slim profinite group A such that the inclusion $G \hookrightarrow A$ is relatively slim. Then the following are equivalent:*

(a) $\mathcal{B}(G) \to \mathcal{B}(A)$ *is a* **faithful quasi-core**.

(b) The natural inclusion $C_A(G) \hookrightarrow \text{Isog}(G)$ *is* **surjective**.

(c) The homomorphism of abstract groups $G \hookrightarrow A$ **factors** *through* $G \hookrightarrow \text{Isog}(G)$.

Proof. These assertions are all formal consequences of the definitions. ○

Remark 2.3.6.1. Relative to Proposition 2.3.6, (ii), we note that $\mathrm{Aut}(G_n)$ is also equal to the *normalizer* of G_n in $\mathrm{Isog}(G)$. When G (hence also the G_n) is *topologically finitely generated*, then it follows that G admits an exhaustive descending sequence of *characteristic open subgroups* $\ldots \subseteq H_m \subseteq \ldots \subseteq G$, hence that $\mathrm{Aut}(G)$ (hence also the G_n) admits a natural structure of *profinite group* (by considering the inverse limit of the images of $\mathrm{Aut}(G)$ in the various $\mathrm{Aut}(G/H_m)$). On the other hand, this profinite topology on $\mathrm{Aut}(G_n)$ does not, in general, coincide with the topology induced by the topology of $\mathrm{Isog}(G)$ discussed in Proposition 2.1.4, (i) — cf. Remark 2.2.6.2. Moreover, (relative to Proposition 2.3.6, (ii)) the work of [TSH] — involving inductive limits of topological groups whose inductive limit topology (in the category of topology spaces) is not necessarily compatible with the group structure of the inductive limit — shows that the topology of inductive limits of topological groups can, in general, be a rather *subtle* issue.

Remark 2.3.6.2. The observations given in Proposition 2.3.6, (i), (iii); Remark 2.3.6.1 were related to the author by *A. Tamagawa.*

Example 2.3.7. Non-Intrinsically Exhaustive Profinite Groups. Let p be a prime number.

(i) Take $A \stackrel{\mathrm{def}}{=} \mathbb{Z}_p^{\times}$, $B \stackrel{\mathrm{def}}{=} \mathbb{Z}_p$. Let A act on B in the usual fashion. Take $G \stackrel{\mathrm{def}}{=} B \rtimes A$. Note that G is *slim.* Then the open subgroup

$$H \stackrel{\mathrm{def}}{=} (p \cdot B) \rtimes A \subseteq G$$

is clearly *isomorphic* to G, hence violates Proposition 2.3.5, (vi). Thus, G *fails to be weakly intrinsically exhaustive.*

(ii) Let $G \stackrel{\mathrm{def}}{=} PGL_2(\mathbb{Z}_p)$. Note that G is *slim.* For m a positive integer, write $C_m \subseteq G$ for the subgroup determined by the matrices congruent to the identity matrix modulo p^m. Then G *fails to be intrinsically exhaustive.* Indeed, if $\{G_n\}$ is as in Definition 2.3.3, then there exist positive integers $m \geq n \geq n_G$ such that:

$$C_m \subseteq G_n \subseteq C_1$$

Thus, for all open embeddings $\iota : C_m \hookrightarrow G$, we should have: $\iota(C_m) \subseteq C_1$. But this inclusion fails to hold if we take ι to be the embedding given by conjugation by the matrix $\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$. On the other hand, (it is an easy exercise to show that) in this case, the *unimodularity* of the action by conjugation of $GL_2(\mathbb{Q}_p)$ on $M_2(\mathbb{Q}_p)$ implies that G *is weakly intrinsically exhaustive.*

(iii) For $n \geq 2$, let $G \stackrel{\mathrm{def}}{=} \widehat{F}_n$, the *free profinite group on* n *generators.* Then G is *slim* (cf., e.g., [Mzk8], Lemma 1.3.1). Moreover, since, for any $n, m \geq 2$, $\widehat{F}_n$,

$\widehat{F}_m$ admit *isomorphic open subgroups*, in order to prove that G *is not intrinsically exhaustive* for *all* n, it suffices to prove that G fails to be intrinsically exhaustive for *some* n (cf. Proposition 2.3.5, (iv)). On the other hand, there exists an n such that G is isomorphic to an open subgroup of the profinite completion $SL_2(\mathbb{Z})^\wedge$ of $SL_2(\mathbb{Z})$. Thus, one may show that to assume the intrinsic exhaustivity of any open subgroup of such a G leads to a contradiction by conjugating by *"Hecke operator-type matrices"* — an operation which preserves the quotient $SL_2(\mathbb{Z})^\wedge \twoheadrightarrow SL_2(\mathbb{Z}_p)$ — as in (ii), above. Note, however, that in this case, the *Nielsen-Schreier formula* (cf., e.g., [FJ], Proposition 15.25) implies that G *is weakly intrinsically exhaustive.*

(iv) The anabelioid $\text{Ét}(\mathbb{A}^1_{\mathbb{F}_p})$ (notation as in Example 1.1.3) associated to the affine line over $\mathbb{F}_p$ *fails to be weakly intrinsically exhaustive.* Indeed, the existence of the *finite étale morphism* $\mathbb{A}^1_{\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}$ defined by

$$T \mapsto T^p + T$$

(where T is the standard coordinate on $\mathbb{A}^1_{\mathbb{F}_p}$) contradicts Proposition 2.3.5, (vi).

(v) If K is a *finite extension* of $\mathbb{Q}_p$, then the associated anabelioid $\text{Ét}(K)$ is *weakly intrinsically exhaustive* (cf., e.g., [Mzk6], Proposition 1.2), but *fails to be intrinsically exhaustive*, at least when $p > 2$. Indeed, to see that G_K (the absolute Galois group of K) fails to be intrinsically exhaustive, let us first recall the following *theorem of [JR]*:

> Let K_1, K_2 be *finite extensions of* $\mathbb{Q}_p$ (where $p > 2$) which contain the roots of unity of order p. Then $G_{K_1} \xrightarrow{\sim} G_{K_2}$ if and only if $[K_1 : \mathbb{Q}_p] = [K_2 : \mathbb{Q}_p]$ and $K_1 \bigcap (\mathbb{Q}_p{}^{\text{ab}}) = K_2 \bigcap (\mathbb{Q}_p{}^{\text{ab}})$ (where $\mathbb{Q}_p{}^{\text{ab}}$ is the maximal abelian extension of $\mathbb{Q}_p$).

Now suppose that $\{G_n\}$ is a sequence of open normal subgroups of G_K as in Definition 2.3.3, (ii). Without loss of generality (cf. Proposition 2.3.5, (iv)), we may assume that K contains the roots of unity of order p, and that $[K : \mathbb{Q}_p] \geq 3$. Let L be the finite Galois extension of K corresponding to some G_n. Write $M \subseteq L$ for the *maximal tamely ramified subextension* of L over K. By taking n to be sufficiently large, we may assume that the extension L of M is *not* cyclotomic, i.e., that $L \neq L \bigcap (M \cdot \mathbb{Q}_p{}^{\text{ab}})$. Since $[M : \mathbb{Q}_p] \geq [K : \mathbb{Q}_p] \geq 3$, it thus follows from *local class field theory* (cf., e.g., [Serre3]) that the *wild inertia subgroup* of G_M^{ab} has rank ≥ 3 over $\mathbb{Z}_p$, hence that there exists a *wildly ramified abelian extension* L' of M such that:

$$[L' : M] = [L : M]; \quad L' \neq L; \quad L' \bigcap (M \cdot \mathbb{Q}_p{}^{\text{ab}}) = L \bigcap (M \cdot \mathbb{Q}_p{}^{\text{ab}})$$

Thus, (by the theorem of [JR] quoted above) we conclude that $G_{L'} \xrightarrow{\sim} G_L = G_n$ despite the fact that $G_{L'} \neq G_L$. But this *contradicts* Definition 2.3.3, (ii).

Remark 2.3.7.1. Examples (iv) and (v) were related to the author by *A. Tamagawa.*

§2.4. Canonical Construction of the Fundamental Group

Let $\mathcal{X}$ be a *slim, connected anabelioid.* In this §, we would like to examine the extent to which the fundamental group of $\mathcal{X}$ may be constructed in a **canonical fashion**, *independent of a choice of basepoint.*

We begin by introducing some notation. Let us write

$$\mathrm{Loc}_{\mathrm{bp}}(\mathcal{X})$$

for the *category* each of whose *objects* is an arrow $\mathcal{U} \to \mathcal{Y}$, where $\mathcal{Y}$ is a connected object of $\mathrm{Loc}(\mathcal{X})$, and $\mathcal{U} \to \mathcal{Y}$ is a *universal covering* of $\mathcal{Y}$ (cf. Definition 1.2.7), and whose *morphisms* from an arrow $\mathcal{U}_1 \to \mathcal{Y}_1$ to an arrow $\mathcal{U}_2 \to \mathcal{Y}_2$ are pairs of arrows $\alpha_{\mathcal{U}} : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$, $\alpha_{\mathcal{Y}} : \mathcal{Y}_1 \to \mathcal{Y}_2$ such that the diagram

$$\begin{array}{ccc} \mathcal{U}_1 & \xrightarrow{\alpha_{\mathcal{U}}} & \mathcal{U}_2 \\ \downarrow & & \downarrow \\ \mathcal{Y}_1 & \xrightarrow{\alpha_{\mathcal{Y}}} & \mathcal{Y}_2 \end{array}$$

commutes; $\alpha_{\mathcal{U}}$ is an *isomorphism*; and $\alpha_{\mathcal{Y}}$ is *finite étale.* Thus, in particular, by mapping $\mathcal{U} \to \mathcal{Y}$ to $\mathcal{Y}$, we obtain a *functor*

$$\Phi_{\mathcal{X}} : \mathrm{Loc}_{\mathrm{bp}}(\mathcal{X}) \to \mathrm{Loc}(\mathcal{X})^0$$

— where the superscript "0" is to denote the full subcategory consisting of connected objects — which (by definition) is *surjective on objects.*

On the other hand, if we define

$$\mathfrak{SGp}$$

to be the *category* whose *objects* are pairs (G, H), where G is a group, and H is a subgroup of G, and whose *morphisms* from (G_1, H_1) to (G_2, H_2) are homomorphisms $\phi : G_1 \to G_2$ such that $\phi(H_1) \subseteq H_2$, then we obtain a natural *functor*

$$\Psi_{\mathcal{X}} : \mathrm{Loc}_{\mathrm{bp}}(\mathcal{X}) \to \mathfrak{SGp}$$

by mapping an arrow $\mathcal{U} \to \mathcal{Y}$ to the pair

$$(\mathrm{Aut}(\mathcal{U}), \mathrm{Aut}_{\mathcal{Y}}(\mathcal{U}) \subseteq \mathrm{Aut}(\mathcal{U}))$$

and a morphism from $\mathcal{U}_1 \to \mathcal{Y}_1$ to $\mathcal{U}_2 \to \mathcal{Y}_2$ to the isomorphism $\mathrm{Aut}(\mathcal{U}_1) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{U}_2)$. Thus:

> $\mathrm{Loc}_{\mathrm{bp}}(\mathcal{X})$ may be thought of as the *"category of objects of* $\mathrm{Loc}(\mathcal{X})$ *equipped with a basepoint"* and $\Psi_{\mathcal{X}}$ may be thought of as the *standard construction of the fundamental group* (in the presence of a basepoint).

When it is necessary to *specify the universe* V relative to which we are working — i.e., relative to which we take all of our (pro-)anabelioids (respectively, groups) to be V*-small* (respectively, V-sets) — we shall write $\mathrm{Loc}^V_{\mathrm{bp}}(\mathcal{X})$, $\mathrm{Loc}^V(\mathcal{X})$ (respectively, $\mathfrak{S}\mathfrak{G}\mathfrak{p}^V$). [Similarly, we shall write $\Phi^V_{\mathcal{X}}$, $\Psi^V_{\mathcal{X}}$.] Thus, we observe, in particular, that the categories $\mathrm{Loc}^V_{\mathrm{bp}}(\mathcal{X})$, $\mathrm{Loc}^V(\mathcal{X})$ are *not* V*-small.*

Proposition 2.4.1. (Dependence of the Fundamental Group on the Choice of Universal Covering) *Let* V *be a* **universe** *[which is, therefore, in particular, a "set" in some ambient model of set theory]. Let* $\mathcal{X}$ *be a* V*-small slim, connected anabelioid such that the subgroup* $\Pi_{\mathcal{X}} \subseteq \mathrm{Isog}(\mathcal{X})$ *(cf. Proposition 2.1.4, (i)) is* **not normal**. *Then there exist distinct objects of* $\mathrm{Loc}^V_{\mathrm{bp}}(\mathcal{X})$ *that map via* $\Phi^V_{\mathcal{X}}$ *to the same object of* $\mathrm{Loc}^V(\mathcal{X})^0$*, but via* $\Psi^V_{\mathcal{X}}$ *to distinct objects of* $\mathfrak{S}\mathfrak{G}\mathfrak{p}^V$. *In particular, the functor* $\Psi^V_{\mathcal{X}}$ **does not factor** *through* $\Phi^V_{\mathcal{X}}$.

Proof. Indeed, let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a universal covering; let $\alpha \in \mathrm{Aut}(\widetilde{\mathcal{X}})$ be an element that does not normalize $\Pi_{\mathcal{X}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$. Then $\pi' \stackrel{\mathrm{def}}{=} \pi \circ \alpha^{-1}$ is also a universal covering of $\mathcal{X}$. Moreover, we have

$$\Psi^V_{\mathcal{X}}(\pi) = (\mathrm{Aut}(\widetilde{\mathcal{X}}), \Pi_{\mathcal{X}}) \neq \Psi^V_{\mathcal{X}}(\pi') = (\mathrm{Aut}(\widetilde{\mathcal{X}}), \alpha \cdot \Pi_{\mathcal{X}} \cdot \alpha^{-1})$$

but $\Phi^V_{\mathcal{X}}(\pi) = \Phi^V_{\mathcal{X}}(\pi') = \mathcal{X}$. ○

Remark 2.4.1.1. Thus, the proof of Proposition 2.4.1 suggests, in particular, that, in order to obtain a *factorization* of $\Psi^V_{\mathcal{X}}$ through $\Phi^V_{\mathcal{X}}$ — i.e., to obtain a "canonical construction" of the fundamental group that does not depend on the choice of basepoint — it is necessary to modify $\Psi^V_{\mathcal{X}}$ so that it takes values in some sort of **"quotient"** in which subgroups of $\mathrm{Aut}(\widetilde{\mathcal{X}})$ are *identified* with their conjugates. This motivates the following discussion.

Let V, $\mathcal{X}$ be as in Proposition 2.4.1. Then let us denote by

$$\mathfrak{S}\mathfrak{G}\mathfrak{p}^V_{\mathcal{X}}$$

the *category* each of whose *objects* is an *assignment* A

$$\mathcal{U} \mapsto A_{\mathcal{U}}$$

— where $\mathcal{U}$ ranges over *all* V*-small universal coverings* of $\mathcal{X}$ [i.e., all domains of arrows in $\mathrm{Loc}_{\mathrm{bp}}(\mathcal{X})$], and $A_{\mathcal{U}}$ is a *collection of subgroups of* $\mathrm{Aut}(\mathcal{U})$ — such that for every isomorphism $\mathcal{U}_1 \stackrel{\sim}{\to} \mathcal{U}_2$, the induced isomorphism $\mathrm{Aut}(\mathcal{U}_1) \stackrel{\sim}{\to} \mathrm{Aut}(\mathcal{U}_2)$ maps $A_{\mathcal{U}_1}$ onto $A_{\mathcal{U}_2}$; and whose *morphisms* $\mathrm{Hom}(A, A')$ are defined as follows: The *cardinality* of $\mathrm{Hom}(A, A')$ is always ≤ 1; we take the cardinality of $\mathrm{Hom}(A, A')$ to be 1 if and only if the following condition is satisfied: for every $\mathcal{U}$, every $H \in A_{\mathcal{U}}$, there exists an $H' \in A'_{\mathcal{U}}$ such that $H \subseteq H'$. Note that this category $\mathfrak{S}\mathfrak{G}\mathfrak{p}^V_{\mathcal{X}}$ is *not* V*-small.*

Thus, we obtain a natural *functor*

$$\widetilde{\Xi}^V_{\mathcal{X}} : \mathrm{Loc}^V_{\mathrm{bp}}(\mathcal{X}) \to \mathfrak{SGp}^V_{\mathcal{X}}$$

by mapping an arrow $\mathcal{U} \to \mathcal{Y}$ to the assignment that maps a universal covering $\mathcal{V}$ to the *conjugacy class of subgroups* of $\mathrm{Aut}(\mathcal{V})$ determined by the subgroup $\mathrm{Aut}_{\mathcal{Y}}(\mathcal{U}) \subseteq \mathrm{Aut}(\mathcal{U})$ and an isomorphism $\mathrm{Aut}(\mathcal{U}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{V})$ which is induced by an isomorphism $\mathcal{U} \xrightarrow{\sim} \mathcal{V}$. [Note that this conjugacy class is independent of the choice of isomorphism $\mathcal{U} \xrightarrow{\sim} \mathcal{V}$.] Moreover, it is evident from the definition of $\widetilde{\Xi}^V_{\mathcal{X}}$ that:

Theorem 2.4.2. (Canonical Fundamental Groups up to Isogeny) *Let V, $\mathcal{X}$ be as in Proposition 2.4.1. Then there exists a functor*

$$\Xi^V_{\mathcal{X}} : \mathrm{Loc}^V(\mathcal{X})^0 \to \mathfrak{SGp}^V_{\mathcal{X}}$$

such that $\Xi^V_{\mathcal{X}} = \widetilde{\Xi}^V_{\mathcal{X}} \circ \Phi^V_{\mathcal{X}}$.

Remark 2.4.2.1. Thus, the functor of Theorem 2.4.2 yields a *functorial* [i.e., with respect to finite étale coverings] *construction of the fundamental group* as a *group of transformations* of some geometric object [i.e., the universal covering], albeit up to a certain **indeterminacy**, given by the action of $\mathrm{Isog}(\mathcal{X})$. On the other hand, this functor has the drawback that it only constructs the fundamental group as an *"abstract group"*, i.e., *not as a* **profinite group**, as one might ideally wish.

Now let us assume that $\mathcal{X}$ is a *connected $\mathcal{Q}$-anabelioid*. For simplicity, we assume that $\mathcal{Q}$ is also *connected*. In the following discussion, we would like to show that (certain quotients) of the fundamental group of $\mathcal{X}$ may be constructed in a very *canonical* fashion complete with their *profinite structure*, under the *assumption* that $\mathcal{X} \to \mathcal{Q}$ is a **quasi-core** for $\mathcal{X}$.

First, let us choose an *explicit system of finite étale Galois coverings*

$$\ldots \to \mathcal{Q}_{n+1} \to \mathcal{Q}_n \to \ldots \to \mathcal{Q}$$

of $\mathcal{Q}$ which, when regarded as a *pro-anabelioid* $\mathcal{Q}_\infty$, forms a *universal covering* of $\mathcal{Q}$. For each n, choose a coherent system of *connected components*

$$\mathcal{X}_n \hookrightarrow \mathcal{Q}_n|_{\mathcal{X}}$$

of $\mathcal{Q}_n|_{\mathcal{X}}$ (cf. the proof of Proposition 2.3.5, (viii)). This system thus defines a *pro-anabelioid* $\mathcal{X}_\infty$, together with a morphism $\mathcal{X}_\infty \to \mathcal{Q}_\infty$.

Now observe that, since $\mathcal{X} \to \mathcal{Q}$ is a *quasi-core*, it follows that *any automorphism* $\alpha : \mathcal{X}_n \xrightarrow{\sim} \mathcal{X}_n$ necessarily lies over $\mathcal{Q}$, hence that the natural morphism

$$\mathcal{X}_n \to \mathcal{Q}_n$$

[obtained by composing the inclusion $\mathcal{X}_n \hookrightarrow \mathcal{Q}_n|_{\mathcal{X}}$ with the projection $\mathcal{Q}_n|_{\mathcal{X}} \to \mathcal{Q}_n$] is *preserved* by composition on the the left with arbitrary automorphisms of $\mathcal{X}_n$, up to the action of a (unique) element of $\mathrm{Gal}(\mathcal{Q}_n/\mathcal{Q}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{Q}}(\mathcal{Q}_n)$. That is to say, there is a *unique element* $\alpha_{\mathcal{Q}_n} \in \mathrm{Gal}(\mathcal{Q}_n/\mathcal{Q})$ for which the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X}_n & \to & \mathcal{Q}_n \\ \big\downarrow \alpha & & \big\downarrow \alpha_{\mathcal{Q}_n} \\ \mathcal{X}_n & \to & \mathcal{Q}_n \end{array}$$

Moreover, the uniqueness of this element implies that the assignment $\alpha \mapsto \alpha_{\mathcal{Q}_n}$ is a *homomorphism*. Thus, in summary, we see that we obtain an *outer homomorphism*

$$\rho_n^{\mathrm{Aut}} : \mathrm{Aut}(\mathcal{X}_n) \to \mathrm{Gal}(\mathcal{Q}_n/\mathcal{Q})$$

which is *entirely determined* (as an outer homomorphism) by the *isomorphism class of* $\mathcal{X}_n$. In particular, restricting to $\mathrm{Gal}(\mathcal{X}_n/\mathcal{X}) \subseteq \mathrm{Aut}(\mathcal{X}_n)$, we obtain an *outer homomorphism*

$$\rho_n^{\mathrm{Gal}} : \mathrm{Gal}(\mathcal{X}_n/\mathcal{X}) \to \mathrm{Gal}(\mathcal{Q}_n/\mathcal{Q})$$

which is *entirely determined* (as an outer homomorphism) by the abstract equivalence class of the morphism $\mathcal{X}_n \to \mathcal{X}$, hence, in particular, by the isomorphism class of $\mathcal{X}$ plus the covering $\mathcal{Q}_n \to \mathcal{Q}$ (since $\mathcal{X} \to \mathcal{Q}$ is a *quasi-core*).

Since the above construction is clearly "functorial in n", by passing to the limit over n, we thus obtain an *outer homomorphism*

$$\rho_\infty^{\mathrm{Gal}} : \mathrm{Gal}(\mathcal{X}_\infty/\mathcal{X}) \to \Pi_{\mathcal{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\mathcal{Q}_\infty/\mathcal{Q})$$

whose image is *entirely determined* (up to conjugacy) by the *isomorphism class of* $\mathcal{X}$. Let us denote this *image* (well-defined up to conjugacy) by:

$$\Pi_{\mathcal{X}/\mathcal{Q}} \subseteq \Pi_{\mathcal{Q}}$$

Moreover, since the above construction is determined entirely by the *isomorphism class of* $\mathcal{X}$, it follows (cf. Proposition 2.1.1, (iv)) that the assignment $\mathcal{X} \mapsto \Pi_{\mathcal{X}/\mathcal{Q}}$ is *functorial with respect to finite étale coverings* $\mathcal{X}_1 \to \mathcal{X}_2$ of connected objects of $\mathrm{Loc}(\mathcal{X})$ in the sense that such a covering induces an *inclusion*

$$\Pi_{\mathcal{X}_1/\mathcal{Q}} \subseteq \Pi_{\mathcal{X}_2/\mathcal{Q}} \ (\subseteq \Pi_{\mathcal{Q}})$$

which is *well-defined up to conjugation by elements of* $\Pi_{\mathcal{Q}}$. (That is to say, one allows an *indeterminacy* with respect to distinguishing between, say, a given inclusion $\Pi_{\mathcal{X}_1/\mathcal{Q}} \subseteq \Pi_{\mathcal{X}_2/\mathcal{Q}}$ and some other inclusion $\Pi_{\mathcal{X}_1/\mathcal{Q}} \subseteq \pi \cdot \Pi_{\mathcal{X}_2/\mathcal{Q}} \cdot \pi^{-1}$, where $\pi \in \Pi_{\mathcal{Q}}$.)

If G is a *Hausdorff topological group*, then let us write

$$\mathfrak{Sub}(G)$$

for the category whose *objects* are *conjugacy classes of closed subgroups* $H \subseteq G$, and whose *morphisms* $H \to H'$ are inclusions of H into a (conjugate of) H'. That is to say, the *cardinality* of the set of morphisms between two objects of $\mathfrak{Sub}(G)$ is *either* 0 *or* 1.

Then the above discussion may be summarized as follows:

Theorem 2.4.3. (Canonically Constructed Fundamental Groups via Quasi-Cores) *Let $\mathcal{Q}$ be a slim, connected anabelioid. Suppose that $\mathcal{X}$ is a connected $\mathcal{Q}$-anabelioid for which $\mathcal{X} \to \mathcal{Q}$ is a* **quasi-core***. Then there is a* **functor**

$$\begin{array}{ccc} \mathrm{Loc}(\mathcal{X})^0 & \to & \mathfrak{Sub}(\Pi_{\mathcal{Q}}) \\ \mathcal{Y} & \mapsto & \{\Pi_{\mathcal{Y}/\mathcal{Q}} \subseteq \Pi_{\mathcal{Q}}\} \end{array}$$

such that $\mathcal{B}(\Pi_{\mathcal{Y}/\mathcal{Q}})$ is isomorphic to the image of $\mathcal{Y}$ in $\mathcal{Q}$ (cf. Definition 1.1.7). In particular, if $\mathcal{X} \to \mathcal{Q}$ is a **faithful quasi-core***, then $\mathcal{Y} \cong \mathcal{B}(\Pi_{\mathcal{Y}/\mathcal{Q}})$.*

Remark 2.4.3.1. Thus, Theorem 2.4.3 yields a **canonical construction** of the fundamental group of a slim, connected $\mathcal{X}$ which admits a *faithful quasi-core* $\mathcal{Q}$. Moreover, this construction has the virtue that it is compatible [cf. the above discussion!] with the **profinite structure** of the fundamental group of $\mathcal{X}$. That is to say, more concretely:

> *The functor of Theorem 2.4.3 may be written as an* **inverse limit** *of a compatible system of functors to the categories*
>
> $$\mathfrak{Sub}(\Pi_{\mathcal{Q}}/H_n)$$
>
> *where $\ldots \subseteq H_n \subseteq \ldots \subseteq \Pi_{\mathcal{Q}}$ is an exhaustive descending sequence of open normal subgroups of $\Pi_{\mathcal{Q}}$.*

This compatibility with the profinite structure is closely related to the *the* **intrinsicity** *of "knowing how local one is"* (cf. Remark 2.3.5.1).

On the other hand, one drawback of the construction of Theorem 2.4.3 is that it depends on the **arbitrary choice** *of a universal covering $\mathcal{Q}_\infty \to \mathcal{Q}$ for $\mathcal{Q}$* as an *"input datum"*. This motivates the following definition:

Definition 2.4.4. Let $\mathcal{X}$ be a slim, connected anabelioid. Then we shall refer to a closed subgroup $\Delta \subseteq \Pi_{\mathcal{X}}$, considered as a subgroup of $\mathrm{Isog}(\mathcal{X})$, as an *intrinsic profinite subgroup* if it is topologically finitely generated, normal in $\mathrm{Isog}(\mathcal{X})$, and, moreover, the continuous inclusion of Hausdorff topological groups $\Delta \hookrightarrow \mathrm{Isog}(\mathcal{X})$ is relatively slim.

Remark 2.4.4.1. Note that since Δ is *topologically finitely generated*, it follows (cf. Remark 2.3.6.1) that $\mathrm{Aut}(\Delta)$ has a natural structure of *profinite group*.

Proposition 2.4.5. (The Faithful Quasi-Core Associated to an Intrinsic Profinite Subgroup) *Let* $\Delta \subseteq \text{Isog}(\mathcal{X})$ *be an* **intrinsic profinite subgroup**. *Then the action by conjugation of* $\Pi_{\mathcal{X}}$ *on* Δ *yields a morphism*

$$\mathcal{X} \cong \mathcal{B}(\Pi_{\mathcal{X}}) \to \mathcal{B}(\text{Aut}(\Delta))$$

which is a **faithful quasi-core** *for* $\mathcal{X}$.

Proof. Indeed, this is a formal consequence of Proposition 2.3.6, (iii), (a) $\iff$ (c). ○

Remark 2.4.5.1. Thus, when the quasi-core of Theorem 2.4.3 is obtained as in Proposition 2.4.5, one can replace the functor of Theorem 2.4.3 by a *functor in the style of Proposition 2.4.2*: That is to say, instead of considering a conjugacy class of subgroups of a *particular profinite group* $\Pi_{\mathcal{Q}}$, we observe that for any universal covering $\mathcal{U}$, we obtain a *natural profinite subgroup*

$$\Delta_{\mathcal{U}} \subseteq \text{Aut}(\mathcal{U})$$

(determined by conjugating Δ by some isomorphism $\mathcal{U} \xrightarrow{\sim} \widetilde{\mathcal{X}}$ of $\mathcal{U}$ to the universal covering $\widetilde{\mathcal{X}}$ used to define $\text{Isog}(\mathcal{X})$) such that any isomorphism $\mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$ maps $\Delta_{\mathcal{U}_1}$ to $\Delta_{\mathcal{U}_2}$. In particular, we obtain an assignment

$$\mathcal{U} \mapsto \mathcal{A}_{\mathcal{U}} \stackrel{\text{def}}{=} \text{Aut}(\Delta_{\mathcal{U}})$$

which is *functorial* in isomorphisms $\mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$. Then instead of obtaining a conjugacy class of subgroups [as in Theorem 2.4.3] in a particular $\Pi_{\mathcal{Q}}$, we obtain a conjugacy class of subgroups of $\mathcal{A}_{\mathcal{U}}$, for each $\mathcal{U}$, which is compatible with all isomorphisms $\mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$. [We leave the routine details to the reader.] At any rate, this yields a *construction of the canonical fundamental groups of Theorem 2.4.3 which is independent of the choice of any universal covering of* $\mathcal{Q}$.

Section 3: Anabelioids Arising from Hyperbolic Curves

§3.1. Anabelioid-Theoretic Interpretation of Scheme-Theoretic Cores

In the following discussion, we wish to *translate* the *scheme-theoretic* theory of *cores* in the context of *hyperbolic curves* (cf. [Mzk9], §2) into the *language of anabelioids* (cf. the profinite group-theoretic approach to such a translation given in [Mzk9], §2). The main technical tool that will enable us to do this is the *"Grothendieck Conjecture"* — i.e., Theorem A of [Mzk7].

For $i = 1, 2$, let F_i be either $\mathbb{Q}$ or $\mathbb{Q}_{p_i}$ (for some *prime number* p_i). Let K_i be a *finite extension of* F_i. Let $(X_i)_{K_i}$ be a *hyperbolic orbicurve* over K_i. Assume that we have chosen basepoints of the $(X_i)_{K_i}$, which thus induce basepoints/algebraic closures $\overline{K}_i$ of the K_i and determine *fundamental groups* $\Pi_{(X_i)_{K_i}} \stackrel{\mathrm{def}}{=} \pi_1((X_i)_{K_i})$ and *Galois groups* $G_{K_i} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}_i/K_i)$. Thus, for $i = 1, 2$, we have an *exact sequence*:

$$1 \to \Delta_{X_i} \to \Pi_{(X_i)_{K_i}} \to G_{K_i} \to 1$$

(where $\Delta_{X_i} \subseteq \Pi_{(X_i)_{K_i}}$ is defined so as to make the sequence exact). Here, we shall think of G_{K_i} as a *quotient* of $\Pi_{(X_i)_{K_i}}$ (i.e., not as an independent group to which $\Pi_{(X_i)_{K_i}}$ happens to surject). One knows (cf. [Mzk8], Lemma 1.3.8) that this quotient $\Pi_{(X_i)_{K_i}} \to G_{K_i}$ is an *intrinsic invariant* of the profinite group $\Pi_{(X_i)_{K_i}}$.

Next, we would like to introduce *anabelioids* into our discussion. Write:

$$\mathcal{X}_i \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}((X_i)_{K_i}); \quad \mathcal{S}_i \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}(K_i)$$

Note that $\mathcal{X}_i$, $\mathcal{S}_i$ are *slim* (cf. [Mzk8], Theorem 1.1.1, (ii); [Mzk8], Lemma 1.3.1), and that the structure morphisms $\mathcal{X}_i \to \mathcal{S}_i$ are *relatively slim* (cf. [Mzk8], Theorem 1.1.1, (ii)). Thus, we may think of $\mathcal{X}_i$ as an $\mathcal{S}_i$*-anabelioid* (cf. §2.2). In particular, we may consider the categories

$$\mathrm{Loc}_{\mathcal{S}_i}(\mathcal{X}_i); \quad \overline{\mathrm{Loc}}_{\mathcal{S}_i}(\mathcal{X}_i)$$

of §2.2. In the following discussion, we shall work with anabelioids *"at the coarsified level"* [i.e., in Anab].

Corollary 3.1.1. (Anabelioid-Theoretic Preservation of Arithmetic Quotients) *Any finite étale morphism*

$$\alpha : \mathcal{X}_1 \to \mathcal{X}_2$$

induces a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_1 & \stackrel{\alpha}{\longrightarrow} & \mathcal{X}_2 \\ \downarrow & & \downarrow \\ \mathcal{S}_1 & \stackrel{\alpha_{\mathcal{S}}}{\longrightarrow} & \mathcal{S}_2 \end{array}$$

(where the horizontal morphisms are finite étale), hence pull-back

$$\mathrm{Loc}_{\mathcal{S}_2}(\mathcal{X}_2) \to \mathrm{Loc}_{\mathcal{S}_1}(\mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_1); \quad \overline{\mathrm{Loc}}_{\mathcal{S}_2}(\mathcal{X}_2) \to \overline{\mathrm{Loc}}_{\mathcal{S}_1}(\mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_1)$$

and extension functors

$$\mathrm{Loc}_{\mathcal{S}_1}(\mathcal{X}_1) \hookrightarrow \mathrm{Loc}_{\mathcal{S}_1}(\mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_1); \quad \overline{\mathrm{Loc}}_{\mathcal{S}_1}(\mathcal{X}_1) \xrightarrow{\sim} \overline{\mathrm{Loc}}_{\mathcal{S}_1}(\mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_1)$$

which are equivalences whenever α is an isomorphism. Here, the extension functor on "$\mathrm{Loc}(-)$'s" (respectively, "$\overline{\mathrm{Loc}}(-)$'s") is a full embedding (respectively, equivalence).

Proof. Indeed, this is a formal consequence of [Mzk8], Lemma 1.3.8 (and Proposition 2.2.2, (iv)). ○

Theorem 3.1.2. **(Anabelioid-Theoreticity of Correspondences)** *Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{Q}$; X_K a* **hyperbolic orbicurve** *over K; write $\mathcal{X} \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}(X_K)$, $\mathcal{S} \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}(K)$. Then the natural functor*

$$\begin{array}{ccc} \mathrm{Loc}_K(X_K) & \longrightarrow & \mathrm{Loc}_{\mathcal{S}}(\mathcal{X}) \\ Z & \mapsto & \acute{\mathrm{E}}\mathrm{t}(Z) \end{array}$$

(defined by applying "$\acute{\mathrm{E}}\mathrm{t}(-)$") is an <u>equivalence of categories</u>. A similar assertion holds for "$\mathrm{Loc}(-)$" replaced by "$\overline{\mathrm{Loc}}(-)$". In particular, X_K is (respectively, admits) a K-core if and only if $\mathcal{X}$ is (respectively, admits) an $\mathcal{S}$-core.

Proof. Since "$\overline{\mathrm{Loc}}(-)$" may be categorically reconstructed from "$\mathrm{Loc}(-)$" via the *same recipe* for both schemes and anabelioids, it suffices to prove the asserted equivalence in the case of "$\mathrm{Loc}(-)$".

In this case, it is immediate from the definitions that the functor in question is *essentially surjective.* It follows from the injectivity of [Mzk7], Theorem A (cf. also Proposition 1.1.4) that this functor is *faithful.* Thus, it suffices to prove that this functor is *full.* Since "fullness" follows from Proposition 1.2.5, (ii), for morphisms over $\mathcal{X}$, it suffices (by Proposition 2.2.2, (iv)) to prove that every $\mathcal{S}$-isomorphism

$$\mathcal{Y} \xrightarrow{\sim} \mathcal{Z}$$

(where $\mathcal{Y}$, $\mathcal{Z}$ are anabelioids representing objects of $\mathrm{Loc}_{\mathcal{S}}(\mathcal{X})$) arises from a morphism of schemes in $\mathrm{Loc}_K(X_K)$. But this is a *formal consequence* of [Mzk7], Theorem A (cf. also Proposition 1.1.4). ○

Theorem 3.1.3. **(Absolute Cores over Number Fields)** *Let K be a* **number field**; *X_K a* **hyperbolic orbicurve** *over K; write $\mathcal{X} \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}(X_K)$. Then:*

(i) X_K is an [absolute] **core** *if and only if X_K is a K-core, and, moreover, K is a* **minimal field of definition** *for X_K.*

(ii) Applying "Ét(−)" induces an equivalence of categories:

$$\mathrm{Loc}(X_K) \xrightarrow{\sim} \mathrm{Loc}(\mathcal{X})$$

In particular, $\mathcal{X}$ is (respectively, admits) an [absolute] **core** *if and only if X_K is (respectively, admits) an [absolute]* **core**.

(iii) Suppose that X_K is **non-proper**. *Then $\mathcal{X}$ admits a* **core** *if and only if it is* **intrinsically exhaustive**.

Proof. Assertion (i) follows formally from [Mzk9], Definition 2.1 and [Mzk9], Remark 2.1.1. Assertion (ii) follows, in light of [Mzk8], Theorem 1.1.3, by the same argument as that used to prove Theorem 3.1.2. To prove assertion (iii), let us recall from the theory of [Mzk3] (cf. [Mzk9], Remark 2.1.2) that X_K [or, equivalently, by assertion (ii), $\mathcal{X}$] fails to admit a core if and only if X_K is *isogenous to a Shimura curve*. Since X_K is assumed to be *non-proper*, this Shimura curve may be taken to be the *moduli stack of hemi-elliptic curves* (cf. [Take], p. 396, second paragraph). Thus, if X_K fails to admit a core, one may show that $\mathcal{X}$ fails to be intrinsically exhaustive by using *Hecke correspondences* on the moduli stack of hemi-elliptic curves, as in Example 2.3.7, (ii), (iii) (cf. Proposition 2.3.5, (iv)). On the other hand, if $\mathcal{X}$ admits a core, then it follows from Proposition 2.3.5, (ii), (viii), that $\mathcal{X}$ is intrinsically exhaustive. ◯

Remark 3.1.3.1. One expects that the assumption that X_K be *non-proper* in Theorem 3.1.3, (iii), is inessential. We made this assumption only to technically simplify the proof that $\mathcal{X}$ fails to be intrinsically exhaustive (when it is assumed to fail to admit a core). The point of Theorem 3.1.3, (iii), was to give an *example* where the existence of a core is equivalent to intrinsic exhaustivity (cf. Remark 2.3.5.1), since this *contrasts* with the situation that occurs in the *p-adic case* (cf. Remark 3.1.6.1, Corollary 3.1.7 below).

Corollary 3.1.4. (Anabelioid-Theoreticity of Cores) *Let*

$$\alpha : \mathcal{X}_1 \to \mathcal{X}_2$$

be a finite étale morphism. Then:

(i) α induces — in a fashion **functorial** *with respect to α — a pull-back functor*

$$\overline{\mathrm{Loc}}_{K_2}((X_2)_{K_2}) \to \overline{\mathrm{Loc}}_{K_1}((X_1)_{K_1})$$

which is an **equivalence** *whenever $\mathcal{S}_1 \to \mathcal{S}_2$ is an isomorphism, and is equal to the usual scheme-theoretic pull-back functor whenever α arises from a finite étale morphism of schemes $(X_1)_{K_1} \to (X_2)_{K_2}$.*

(ii) $(X_1)_{K_1}$ *is* K_1-**arithmetic** *if and only if* $(X_2)_{K_2}$ *is* K_2-**arithmetic**. *Similarly, if* $\mathcal{X}_1 \to \mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_1$ *is an isomorphism, then* $(X_1)_{K_1}$ *is a* K_1-**core** *if and only if* $(X_2)_{K_2}$ *is a* K_2-**core**.

(iii) If a finite étale morphism $(X_2)_{K_2} \to (Z_2)_{K_2}$ *to a* K_2*-core* $(Z_2)_{K_2}$ *maps (via the functor of (i)) to a finite étale morphism* $(X_1)_{K_1} \to (Z_1)_{K_1}$*, then* $(Z_1)_{K_1}$ *is a* K_1-**core**, *and, moreover, the morphism* $\mathcal{X}_1 \to \mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_1$ **extends uniquely** *to a commutative diagram:*

$$\begin{array}{ccccc} \mathcal{X}_1 & \to & \mathcal{X}_2 \times_{\mathcal{S}_2} \mathcal{S}_1 & \to & \mathcal{X}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}_1 & \stackrel{\sim}{\to} & \mathcal{Z}_2 \times_{\mathcal{S}_2} \mathcal{S}_1 & \to & \mathcal{Z}_2 \end{array}$$

(where $\mathcal{Z}_i \stackrel{\mathrm{def}}{=} \mathrm{\acute{E}t}((Z_i)_{K_i})$*, and the lower horizontal arrow on the left is an isomorphism).*

Proof. The functor of (i) is obtained by composing the pull-back functor on "$\overline{\mathrm{Loc}}(-)$'s" of Corollary 3.1.1 with an inverse to the extension functor on "$\overline{\mathrm{Loc}}(-)$'s" of Corollary 3.1.1 (which is an equivalence), and then applying the equivalences of Theorem 3.1.2 to the domain and codomain of this composite. Assertion (ii) is a formal consequence of assertion (i); [Mzk9], Definition 2.1; [Mzk9], Remark 2.1.1; and [Mzk9], Proposition 2.3, (i). To prove assertion (iii), we may assume, for simplicity, (cf. Proposition 2.1.1, (iv)) that $\mathcal{S}_1 \to \mathcal{S}_2$ is an *isomorphism.* Then it follows that the pull-back functor on "$\overline{\mathrm{Loc}}(-)$'s" of (i) is an *equivalence*:

$$\overline{\mathrm{Loc}}_{K_2}((X_2)_{K_2}) \stackrel{\sim}{\to} \overline{\mathrm{Loc}}_{K_1}((X_1)_{K_1})$$

Thus, the *existence of an extension* as in assertion (iii) follows formally by thinking of $\mathcal{X}_i$, $\mathcal{Z}_i$ as *subcategories* of $\overline{\mathrm{Loc}}_{K_i}((X_i)_{K_i})$ (cf. Proposition 1.2.5, (ii)). The *uniqueness of such an extension* is a formal consequence of the *slimness* of $\mathcal{Z}_i$. ○

Proposition 3.1.5. **(Absolute Degrees)** *For* $i = 1, 2$*, set:*

$$\deg_{\mathrm{arith}}(\mathcal{X}_i) \stackrel{\mathrm{def}}{=} [K_i : F_i]$$

and $\deg_{\mathrm{geo}}(\mathcal{X}_i)$ *equal to the* **Euler characteristic** *of* $(X_i)_{K_i}$*. [That is to say, if* $(X_i)_{K_i}$ *is a hyperbolic curve of type* (g_i, r_i)*, then we set* $\deg_{\mathrm{geo}}(\mathcal{X}_i)$ *equal to* $2g_i - 2 + r_i$*; more generally, if* $(X_i)_{K_i}$ *is only an orbicurve, then we take its* $\deg_{\mathrm{geo}}(-)$ *to be the* $\deg_{\mathrm{geo}}(-)$ *of some degree* d *finite étale covering of* $(X_i)_{K_i}$ *which is a curve, divided by* d*.] Then for any finite étale morphism* $\alpha : \mathcal{X}_1 \to \mathcal{X}_2$ *(which thus induces a commutative diagram as in Corollary 3.1.1), we have:*

$$\deg_{\mathrm{geo}}(\mathcal{X}_1) = \deg_{\mathrm{geo}}(\mathcal{X}_2) \cdot (\deg(\alpha)/\deg(\alpha_{\mathcal{S}})); \quad \deg_{\mathrm{arith}}(\mathcal{X}_1) = \deg_{\mathrm{arith}}(\mathcal{X}_2) \cdot \deg(\alpha_{\mathcal{S}})$$

In particular, $\mathcal{X}_i$ *is* **weakly intrinsically exhaustive**. *We shall refer to* $\deg_{\mathrm{geo}}(\mathcal{X}_i)$ *(respectively,* $\deg_{\mathrm{arith}}(\mathcal{X}_i)$*) as the* **absolute geometric** *(respectively,* **absolute arithmetic***)* **degree** *of* $\mathcal{X}_i$.

Proof. Indeed, this follows from [Mzk8], Lemma 1.3.9, (for the *absolute geometric degree*) and [Mzk8], Proposition 1.2.1, (i), (v) (for the *absolute arithmetic degree*). ○

Remark 3.1.5.1. Proposition 3.1.5 already suggests the possibility that, under the further assumption that $(X_i)_{K_i}$ admits a K_i-*core*, $\mathcal{X}_i$ should *admit a faithful quasi-core*. In the remainder of the present §, we shall show that this is, in fact, the case (at least when $(X_i)_{K_i}$ is *non-proper*) — cf. Theorem 3.1.6 below. In light of Proposition 2.3.5, (ii); Theorem 3.1.3, (ii), this fact is primarily of interest in the case where K_i is a *p-adic local field* (although we shall not assume this to be the case in the following discussion).

In the following discussion, we would like to *assume* that:

(a) The hyperbolic orbicurve $(X_i)_{K_i}$ *admits a* K_i-*core* $(Z_i)_{K_i}$.

(b) The anabelioids $\mathcal{X}_1$, $\mathcal{X}_2$ are *isogenous*.

Choose basepoints for $(Z_i)_{K_i}$, so that we obtain, for $i = 1, 2$, *exact sequences*:

$$1 \to \Delta_{Z_i} \to \Pi_{(Z_i)_{K_i}} \to G_{K_i} \to 1$$

Write $\mathcal{Z}_i \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}((Z_i)_{K_i})$. Then assumptions (a), (b); Corollary 3.1.4, (iii); and [Mzk8], Lemma 1.3.9, imply that $(Z_1)_{K_1}$, $(Z_2)_{K_2}$ are hyperbolic orbicurves of the *same type* $(g, \vec{r})$. Let us choose once and for all a *model*

$$\widehat{\Pi}_{g,\vec{r}}$$

of the *geometric fundamental group of a hyperbolic orbicurve of type* $(g, \vec{r})$ (in characteristic 0). To simplify notation, in the following discussion, we shall simply write $\widehat{\Pi}$ for $\widehat{\Pi}_{g,\vec{r}}$.

Thus, we have (noncanonical) *isomorphisms* $\widehat{\Pi} \cong \Delta_{Z_i}$. Such isomorphisms induce an *outer homomorphism* $\Pi_{(Z_i)_{K_i}} \to \mathrm{Aut}(\widehat{\Pi})$ which is *independent* (as an outer homomorphism) of the choice of such isomorphism and, moreover, fits into a *commutative diagram*:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_i} & \longrightarrow & \Pi_{(X_i)_{K_i}} & \longrightarrow & G_{K_i} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{Z_i} & \longrightarrow & \Pi_{(Z_i)_{K_i}} & \longrightarrow & G_{K_i} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \widehat{\Pi} & \longrightarrow & \mathrm{Aut}(\widehat{\Pi}) & \longrightarrow & \mathrm{Out}(\widehat{\Pi}) & \longrightarrow & 1
\end{array}$$

Here, we observe that the vertical arrows *between the first and second lines* are always *injective*. If, moreover, $(X_i)_{K_i}$ is *non-proper*, then the vertical arrows *between* *the second and third lines* are also *injective* (by the theory of [Mtmo] — cf. [Mzk8], Theorem 1.3.6). If we then set

$$\mathcal{Z}_{\text{com}} \stackrel{\text{def}}{=} \mathcal{B}(\text{Aut}(\widehat{\Pi})) \to \mathcal{M}_{\text{com}} \stackrel{\text{def}}{=} \mathcal{B}(\text{Out}(\widehat{\Pi}))$$

— i.e., we wish to think of $\mathcal{Z}_{\text{com}} \to \mathcal{M}_{\text{com}}$ as a sort of *"universal combinatorial model"* of $\mathcal{Z}_i \to \mathcal{S}_i$ — then we obtain a *commutative diagram of connected slim anabelioids*

$$\begin{array}{ccccc} \mathcal{X}_i & \longrightarrow & \mathcal{Z}_i & \longrightarrow & \mathcal{Z}_{\text{com}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_i & \stackrel{\text{id}}{\longrightarrow} & \mathcal{S}_i & \longrightarrow & \mathcal{M}_{\text{com}} \end{array}$$

in which the horizontal arrows are all *relatively slim* (cf. [Mzk8], Theorem 1.1.1, (ii); [Mzk8], Lemma 1.3.1; [Mzk7], Theorem A). Next, let us observe that the *intrinsic nature of the anabelioid associated to a geometric core* (cf. Corollary 3.1.4, (iii)) implies that the morphism $\mathcal{X}_i \to \mathcal{Z}_{\text{com}}$ is *functorial* with respect to *arbitrary finite étale morphisms* $\mathcal{X}_1 \to \mathcal{X}_2$.

Finally, let us observe that $\text{Aut}(\widehat{\Pi})$ (hence also $\text{Out}(\widehat{\Pi})$) is *countably (topologically) generated*. Indeed, to show this, it suffices to show the existence of a *descending sequence of open subgroups*

$$\ldots \subseteq A_{n+1} \subseteq A_n \subseteq \ldots \subseteq \text{Aut}(\widehat{\Pi})$$

such that $\bigcap_n A_n = \{1\}$. To this end, let us note that $\widehat{\Pi}$ admits a *descending sequence of open* **characteristic** *subgroups*

$$\ldots \subseteq \widehat{\Pi}[n+1] \subseteq \widehat{\Pi}[n] \subseteq \ldots \subseteq \text{Aut}(\widehat{\Pi})$$

such that $\bigcap_n \widehat{\Pi}[n] = \{1\}$. Thus, if we set

$$A_n \stackrel{\text{def}}{=} \text{Ker}(\text{Aut}(\widehat{\Pi}) \to \text{Aut}(\widehat{\Pi}/\widehat{\Pi}[n]))$$

we obtain a sequence $\{A_n\}$ with the desired properties.

Thus, in summary, we see that we have proven (most of) the following:

Theorem 3.1.6. (The Quasi-Core Associated to a Geometric Core) *Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{Q}$; X_K a* **hyperbolic orbicurve** *over K which admits a K-***core** *Z_K of type $(g, \vec{r})$. Write:*

$$\mathcal{X} \stackrel{\text{def}}{=} \text{Ét}(X_K); \quad \mathcal{Z} \stackrel{\text{def}}{=} \text{Ét}(Z_K); \quad \mathcal{Z}_{\text{com}} \stackrel{\text{def}}{=} \mathcal{B}(\text{Aut}(\widehat{\Pi}_{g,\vec{r}})); \quad \mathcal{M}_{\text{com}} \stackrel{\text{def}}{=} \mathcal{B}(\text{Out}(\widehat{\Pi}_{g,\vec{r}}))$$

Then:

(i) Z_K determines a $\mathcal{Z}_{\text{com}}$-holomorphic structure $\mathcal{X} \to \mathcal{Z}_{\text{com}}$ on $\mathcal{X}$ which is a **quasi-core** *for $\mathcal{X}$. In particular, the theory of §2.3, 2.4 may be applied to $\mathcal{X}$.*

(ii) If X_K is **non-proper**, *then this quasi-core is* **faithful** *and, moreover, obtained as the quasi-core associated to an* **intrinsic profinite subgroup** *(cf. Proposition 2.4.5). Finally, if K is a* **number field** *which is a* **minimal field of definition** *for Z_K, then the morphism $\mathcal{Z} \to \mathcal{Z}_{\text{com}}$ is* **commensurably terminal**.

Proof. It remains only to observe that the final part of (ii) is a formal consequence of Theorem 3.1.3, (i), (ii); Proposition 2.2.5. ○

Remark 3.1.6.1. In the case of p-adic local fields, one does not expect $\mathcal{Z} \stackrel{\text{def}}{=} \acute{\mathrm{E}}\mathrm{t}(Z_K)$ to be a *core* (even if K is a minimal extension of $\mathbb{Q}_p$ over which Z_K is defined). Nevertheless, Theorem 3.1.6 shows that $\mathcal{Z}$ has the interesting property of being *"closer to being a core"* than, for instance, $PGL_2(\mathbb{Q}_p)$ (cf. Example 2.3.7, (ii), (iii); Theorem 3.1.3, (iii); Corollary 3.1.7 below).

Remark 3.1.6.2. Our use of [Mzk8], Theorem 1.3.6 [i.e., the main result of [Mtmo]] in the above construction of a faithful quasi-core — which (by the theory of §2.4) allows us to construct *"canonical fundamental groups"*, i.e., to assign **canonical names, or labels** (up to conjugacy) to the elements of the fundamental group — is reminiscent of the essential idea lying behind the theory of the *Grothendieck-Teichmüller group*, which applies this same *injectivity* to assign canonical names (up to conjugacy) to the elements of $G_{\mathbb{Q}}$. It is interesting to note, however, that although this theory of the Grothendieck-Teichmüller group is typically applied to analyzing $G_{\mathbb{Q}}$, in fact, (by the *"Neukirch-Uchida Theorem"* — cf., e.g., [Mzk8], Theorem 1.1.3) *the elements of $G_{\mathbb{Q}}$ already possess* **intrinsic, canonically determined names** (up to conjugacy). Thus, the ability to assign canonically determined names has *much greater significance* in the case of p-adic local fields.

Remark 3.1.6.3. Relative to Remark 3.1.6.2, it is also interesting to note that, just as the theory of §2.4 only applies in the case where the curve in question admits a *geometric core*, the theory of the Grothendieck-Teichmüller group centers around considering *not just* the projective line minus three points — a curve which *fails* to admit a geometric core — but instead a certain *system of moduli stacks of hyperbolic curves*, which includes, for instance, the moduli stack of hyperbolic curves of type $(0, 5)$ which (by [Mzk3], Theorem C) *does admit a geometric core.*

Finally, we have the following analogue of Theorem 3.1.3, (iii), which is valid in the local p-adic case as well:

Corollary 3.1.7. (Criteria for the Existence of a Geometric Core) *Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{Q}$; X_K a* **non-proper hyperbolic orbicurve** *over K; write $\mathcal{X} \stackrel{\text{def}}{=} \acute{\mathrm{E}}\mathrm{t}(X_K)$. Then the following assertions are equivalent:*

(i) X_K admits a K-core.

(ii) $\mathcal{X}$ *admits a faithful quasi-core.*

(iii) $\mathcal{X}$ *is intrinsically exhaustive.*

Proof. This is a formal consequence of Theorem 3.1.6, (i), (ii); Proposition 2.3.5, (viii); and the existence of *Hecke correspondences* (cf. the proof of Theorem 3.1.3, (iii)) when X_K does not admit a K-core. ○

Remark 3.1.7.1. The implication (i) $\Longrightarrow$ (iii) of Corollary 3.1.7 (in the p-adic case) is somewhat *surprising* in light of Example 2.3.7, (v). That is to say, Corollary 3.1.7 implies that (when X_K admits a K-core) the rigidity of $\acute{\mathrm{E}}\mathrm{t}(X_K)$ is *sufficiently strong* to eliminate the non-intrinsic exhaustivity of $\acute{\mathrm{E}}\mathrm{t}(K)$. In particular, we conclude in this case that the natural inclusion

$$C_{\mathrm{Out}(\Delta_Z)}(G_K) \hookrightarrow \mathrm{Isog}(G_K)$$

fails to be surjective (cf. Propositions 2.3.5, (viii); 2.3.6, (iii)).

§3.2. The Logarithmic Special Fiber via Quasi-Cores

In this §, we interpret the results of [Mzk8], §2, from the point of view of the *theory of quasi-cores* — cf. §2.3, 2.4.

Let X_K be a *hyperbolic curve* over a *finite extension* K of $\mathbb{Q}_p$. Denote the *ring of integers* (respectively, *residue field*) of K by $\mathcal{O}_K$ (respectively, k); also we shall use notation such as "$k^{\log}$", "$(k^{\log})^{\sim}$", as in [Mzk8], §2.

Assume that X_K admits a *stable model* over $\mathcal{O}_K$ (cf. [Mzk8], §2), as well as a *K-core* Z_K, and that X_K is *Galois* over Z_K. Then we *define* the *stable model* of Z_K to be the quotient — in the *sense of [log] stacks* — of the stable model of X_K by $\mathrm{Gal}(X_K/Z_K)$. Let us denote the *logarithmic special fibers* of the stable models of X_K, Z_K by $X_k^{\log}$, $Z_k^{\log}$, respectively. Write:

$$\mathcal{X} \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}(X_K); \quad \mathcal{Z} \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}(Z_K)$$

Also, let us write $\mathcal{Z}_{\mathrm{com}}$ for the *quasi-core* (for $\mathcal{X}$, $\mathcal{Z}$) of Theorem 3.1.6.

Now recall from [Mzk8], the discussion following Remark 2.5.3, the *"universal admissible covering"*

$$\widetilde{X}_k^{\log} \to X_k^{\log}$$

of $X_k^{\log}$ determined by the *admissible quotient* $\Pi_{X_K} \twoheadrightarrow \Pi_{X_K}^{\mathrm{adm}}$. Put another way, this covering is the composite of all *admissible coverings* (cf. [Mzk4], §3) of $X_k^{\log} \times_{k^{\log}} (k^{\log})^{\sim}$. In the following discussion, let us denote the *category of* (disjoint unions of coverings isomorphic to) *subcoverings of this universal admissible covering* (respectively, *subcoverings of the "geometric universal admissible covering"* $\widetilde{X}_k^{\log} \to X_k^{\log} \times_{k^{\log}} (k^{\log})^{\sim}$) by:

$$\acute{\mathrm{E}}\mathrm{t}^{\mathrm{adm}}(X_k^{\log}) \text{ (respectively, } \acute{\mathrm{E}}\mathrm{t}^{\mathrm{adm}}(X_k^{\log} \times_{k^{\log}} (k^{\log})^{\sim}))$$

To keep the notation simple, we set:

$$\mathcal{X}_0 \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}^{\mathrm{adm}}(X_k^{\log}); \quad \overline{\mathcal{X}}_0 \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}^{\mathrm{adm}}(X_k^{\log} \times_{k^{\log}} (k^{\log})^{\sim})$$

[so the fundamental group of $\mathcal{X}_0$ (respectively, $\overline{\mathcal{X}}_0$) may be identified with $\Pi_{X_K}^{\mathrm{adm}}$ (respectively, the geometric portion $\Delta_{X_K}^{\mathrm{adm}} \subseteq \Pi_{X_K}^{\mathrm{adm}}$ of $\Pi_{X_K}^{\mathrm{adm}}$ — cf. [Mzk8], §2)].

Similarly, we may construct

$$\mathcal{Z}_0 \stackrel{\mathrm{def}}{=} \acute{\mathrm{E}}\mathrm{t}^{\mathrm{adm}}(Z_k^{\log})$$

[for instance, as the quotient of $\mathcal{X}_0$ by the faithful action on $\mathcal{X}_0$ of the finite group $\mathrm{Gal}(X_K/Z_K)$].

Next, let us write

$$\mathcal{Q}_0$$

for the *"anabelioid quotient"* of $\overline{\mathcal{X}}_0$ by the natural action on $\overline{\mathcal{X}}_0$ by the *profinite group*

$$\mathrm{Aut}(X_k^{\log} \times_{k^{\log}} (k^{\log})^{\sim})$$

[i.e., the group of automorphisms of the *abstract log scheme* which do not necessarily lie over $k^{\log}$ or $(k^{\log})^{\sim}$!]. That is to say, at the level of *profinite groups*, the fundamental group of the anabelioid $\mathcal{Q}_0$ is the extension of the profinite group $\mathrm{Aut}(X_k^{\log} \times_{k^{\log}} (k^{\log})^{\sim})$ by the fundamental group of $\overline{\mathcal{X}}_0$ determined by the *natural outer action* of the former profinite group on the latter. Note that by the definition of "Aut", the slimness of $\mathcal{X}_0$ [cf. [Mzk8], Lemma 2.2, (i)], and the slimness of $\mathrm{Gal}((k^{\log})^{\sim}/k^{\log})$ [cf. [Mzk8], Proposition 1.2.3, (iii)], it follows that $\mathcal{Q}_0$ is also *slim*.

Thus, we have a commutative diagram of natural *relatively slim* morphisms of *slim, connected anabelioids*

$$\begin{array}{ccccc} \mathcal{X} & \to & \mathcal{Z} & & \\ \downarrow & & \downarrow & & \\ \mathcal{X}_0 & \to & \mathcal{Z}_0 & \to & \mathcal{Q}_0 \end{array}$$

in which the horizontal morphisms are all *finite étale*.

Theorem 3.2.1. (The Admissible Quotient as Quasi-Core) *Assume that* $\mathcal{X} \to \mathcal{Z}$ *is* $\mathcal{Z}_{\text{com}}$-**Galois**. *Then the morphism*

$$\mathcal{X} \to \mathcal{Q}_0$$

is a **quasi-core**. *In particular, the theory of §2.3, 2.4 may be applied to this morphism.*

Proof. By the *functoriality of the anabelioid associated to a geometric core* [(cf. Corollary 3.1.4, (iii)] and our hypothesis that $\mathcal{X} \to \mathcal{Z}$ is $\mathcal{Z}_{\text{com}}$-*Galois* [cf. Proposition 2.3.5, (vii)], it follows that it suffices to consider, for K' a finite extension of K, the behavior of automorphisms of the quotient

$$(\Pi_{\mathcal{X}} \supseteq) \ \Pi_{X_{K'}} \twoheadrightarrow \Pi^{\text{adm}}_{X_{K'}} \ (\subseteq \Pi_{\mathcal{X}_0})$$

induced by *arbitrary automorphisms* of $\Pi_{X_{K'}}$. By [Mzk8], Theorem 2.7, it follows that such automorphisms of $\Pi^{\text{adm}}_{X_{K'}}$ necessarily arise from automorphisms of the *logarithmic special fiber* of $X_{K'}$. Thus, we conclude by the definition of $\mathcal{Q}_0$ and the easily verified fact that base-change to totally wildly ramified extensions K'' of K' *does not affect the automorphism group of the logarithmic special fiber*. ○

Remark 3.2.1.1. Note that the *anabelian nature of the logarithmic special fiber* (i.e., [Mzk8], Theorem 2.7) is applied in Theorem 3.2.1 in a fashion similar to the way in which the *anabelian nature of hyperbolic curves over number fields* is applied in Theorem 3.1.3, (ii).

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