tt^* GEOMETRY OF RANK 2

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Abstract. A notion of a Frobenius manifold with a nice real structure was introduced by Hertling. It is called CDV structure (Cecotti-Dubrovin-Vafa structure). In this paper, we introduce a "positivity condition" on CDV structures and show that any Frobenius manifold of rank two with real spectrum can be equipped with a positive CDV structure. We extend naturally the symmetries of Frobenius structures given by Dubrovin to symmetries of CDV structures, which will play an important role.

1. Introduction

A family of 2-dimensional topological field theories endow its base space with a nice geometric structure. This structure is mathematically axiomatized by Dubrovin and called the Frobenius structure (see [D1]):

Definition 1.1. A Frobenius manifold of rank μ is a tuple $(M, \eta, \circ, E, e)$ where

- M is a complex manifold of dimension μ,
- $\eta : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M$ is a non-degenerate $\mathcal{O}_M$-bilinear form,
- $\circ$ is an associative commutative multiplication on T_M which is η-invariant:

$$\eta(\delta \circ \delta', \delta'') = \eta(\delta, \delta' \circ \delta''), \quad \delta, \delta', \delta'' \in \mathcal{T}_M, \tag{1.1}$$

- $\nabla\!\!\!/ : \mathcal{T}_M \to \mathcal{T}_M \otimes \Omega^1_M$ is a Levi–Civita connection of η which is flat:

$$[\nabla\!\!\!/_\delta, \nabla\!\!\!/_{\delta'}] = \nabla\!\!\!/_{[\delta,\delta']}, \quad \delta, \delta' \in \mathcal{T}_M, \tag{1.2a}$$

$$\nabla\!\!\!/_\delta \delta' - \nabla\!\!\!/_{\delta'} \delta = [\delta, \delta'], \quad \delta, \delta' \mathcal{T}_M, \tag{1.2b}$$

$$\delta\eta(\delta', \delta'') = \eta(\nabla\!\!\!/_\delta \delta', \delta'') + \eta(\delta', \nabla\!\!\!/_\delta \delta''), \quad \delta, \delta', \delta'' \in \mathcal{T}_M, \tag{1.2c}$$

  and satisfies the potentiality condition $\nabla\!\!\!/ C = 0$, where C is the $\mathcal{O}_M$-linear map $C : \mathcal{T}_M \to \mathcal{T}_M \otimes \Omega^1_M$ defined by $C_\delta \delta' := \delta \circ \delta'$, $\delta, \delta' \in \mathcal{T}_M$,
- E is a holomorphic vector field called Euler vector field satisfying

$$Lie_E(\circ) = \circ, \quad Lie_E(\eta) = (2-d)\eta, \quad \text{for some } d \in \mathbb{C}, \tag{1.3}$$

- e is a $\nabla\!\!\!/$-flat holomorphic vector field which is an identity with respect to the multiplication $\circ$, i.e., $e \circ \delta = \delta$, for all $\delta \in \mathcal{T}_M$.

Since $\nabla\!\!\!/$ is flat and torsion free, there exist *flat coordinates* on M, i.e., there exists a local coordinate system $(t^1,\dots,t^\mu)$ such that

$$\mathcal{T}_M \supset \mathcal{T}_M^f := \mathrm{Ker}\nabla\!\!\!/ = \bigoplus_{i=1}^{\mu} \mathbb{C}\frac{\partial}{\partial t^i} \quad \text{and} \quad \mathcal{T}_M \simeq \mathcal{T}_M^f \otimes \mathcal{O}_M. \tag{1.4}$$

Since e is flat, we can find a flat coordinate t^0 such that $e = \partial/\partial t^0$. We shall keep "0" to denote the special direction corresponding to e. It is easy to see that $\eta(\delta,\delta')$ is constant for $\delta,\delta' \in \mathcal{T}_M^f$.

The condition $\nabla\!\!\!/ C = 0$ implies the existence of the "potential":

Definition 1.2. (**Frobenius potential**)
Let $(M,\eta,\circ,E,e)$ be a Frobenius manifold and $(t^0,\dots,t^{\mu-1})$ be its flat coordinates. A local holomorphic function F is called the *Frobenius potential* of the Frobenius manifold M if

$$\eta(\partial_i\circ\partial_j,\partial_k)=\eta(\partial_i,\partial_j\circ\partial_k)=\partial_i\partial_j\partial_k F, \quad {}^\forall i,j,k\in\{0,\dots,\mu-1\}, \quad \text{where } \partial_i := \partial/\partial t^i. \tag{1.5}$$

In particular, $\eta_{ij} := \eta(\partial_i,\partial_j) = \partial_0\partial_i\partial_j F$.

The gradient of the Euler vector field ∇E of a Frobenius manifold defines a $\mathbb{C}$-linear map on the $\mathbb{C}$-vector space of flat vector fields $\mathcal{T}_M^f$. We shall only consider Frobenius manifolds with semi-simple ∇E. Then we see that there exist flat coordinates $(t^0,\dots,t^{\mu-1})$ and complex numbers q_i and r_i such that the Euler vector field is given by

$$E = \sum_{i=0}^{\mu-1}\{(1-q_i)t^i + r_i\}\partial_i, \tag{1.6}$$

where $q_0 = 0$ and $r_i \neq 0$ only if $q_i = 1$. Note that $Lie_E(\eta) = (2-d)\eta$ implies that $\eta_{ij} = 0$ if $q_i + q_j \neq d$.

It is easy to classify Frobenius manifolds of rank one and two with a flat identity e and a semisimple endomorphism ∇E. See [D1] and [M] for details.

Theorem 1.1. (i) *Any Frobenius manifold of rank* 1 *with a flat identity* $e = \partial_0$ *is locally isomorphic to the Frobenius manifold defined by*

$$F(t^0) = \frac{1}{6}a(t^0)^3, \quad a\in\mathbb{C}, \quad E = t^0\partial_0. \tag{1.7}$$

(ii) *Any Frobenius manifold of rank* 2 *with a flat identity and a semisimple endomorphism* ∇E *is locally isomorphic to a direct sum of two* 1*-dimensional Frobenius manifolds or to one* *of the Frobenius manifolds defined by the following potentials* $(e = \partial_0)$:

$$F(t^0, t^1) = \frac{1}{2}(t^0)^2 t^1, \qquad E = t^0\partial_0, \quad d = 0, \tag{1.8a}$$

$$F(t^0, t^1) = \frac{1}{2}(t^0)^2 t^1 + ce^{\frac{2}{r}t^1}, \qquad E = t^0\partial_0 + r\partial_1, \quad r \neq 0 \quad d = 0, \tag{1.8b}$$

$$F(t^0, t^1) = \frac{1}{2}(t^0)^2 t^1 + c(t^1)^{\frac{3-d}{1-d}}, \qquad E = t^0\partial_0 + (1-d)t^1\partial_1, \quad d \in \mathbb{C}\backslash\{-1, 1, 3\}, \tag{1.8c}$$

$$F(t^0, t^1) = \frac{1}{2}(t^0)^2 t^1 + c(t^1)^2 \log t^1, \qquad E = t^0\partial_0 + 2t^1\partial_1, \quad d = -1, \tag{1.8d}$$

$$F(t^0, t^1) = \frac{1}{2}(t^0)^2 t^1 + c \log t^1, \qquad E = t^0\partial_0 - 2t^1\partial_1, \quad d = 3, \tag{1.8e}$$

$$F(t^0, t^1) = \frac{1}{6}\eta_{00}(t^0)^3 + \frac{1}{2}(t^0)^2 t^1, \qquad E = t^0\partial_0 + t^1\partial_1, \quad d = 1, \tag{1.8f}$$

where c, r, η_{00} *are complex numbers.*

□

Remark. (i) In (1.8b), t^1 is defined up to addition of a constant.

(ii) Frobenius manifolds given by (1.8b), (1.8c), (1.8d), (1.8e) with $c \neq 0$ are semi-simple, i.e., there exists a local basis (e_1, e_2) of $\mathcal{T}_M$ in which the multiplication $\circ$ takes the form $e_i \circ e_j = \delta_{ij} e_j$ where δ_{ij} is the Kronecker's delta.

It is a very interesting problem to equip with Frobenius manifolds with nice real structures. The following structure was first discovered in the study of the geometry of the moduli space of N=2 supersymmetric quantum field theories in 2-dimension by Cecotti–Vafa [CV1]. It is mathematically axiomatized by Dubrovin [D2] and Hertling [H]:

Definition 1.3. (tt^***-geometry, CDV structure**)

Let us denote by $\mathcal{A}_M$ the sheaf of real analytic functions on a complex manifold M and put $\mathcal{T}_M^{1,0} := \mathcal{T}_M \otimes \mathcal{A}_M$. A *CDV structure* $(M, \eta, \circ, E, e, \kappa)$ is a Frobenius manifold $(M, \eta, \circ, E, e)$ with an $\mathcal{A}_M$-antilinear involution $\kappa : \mathcal{T}_M^{1,0} \to \mathcal{T}_M^{1,0}$ satisfying the following conditions:

- $h(\cdot, \cdot) := \eta(\cdot, \kappa\cdot)$ is a Hermitian form on $\mathcal{T}_M^{1,0}$ satisfying

$$h(C_\delta \delta', \delta'') = h(\delta', \kappa C_\delta \kappa \delta''), \quad \delta, \delta', \delta'' \in \mathcal{T}_M^{1,0}, \tag{1.9a}$$

$$Lie_e(h) = 0, \quad Lie_{E-\bar{E}}(h) = 0, \tag{1.9b}$$

where $\bar{E}$ is the usual complex conjugate of E.

- The metric connection D for h respects κ, i.e.,

$$D(\kappa) = 0, \quad D(h) = 0 \quad \text{and} \quad D(\eta) = 0. \tag{1.10}$$

- Let $\mathbb{P}_z^1$ be the completion of Spec $\mathbb{C}[z, z^{-1}]$ and $\pi : \mathbb{P}_z^1 \times M \to M$ be the natural projection. Consider the canonical lifts of D and $Q := Lie_E - D_E + (1 - d/2) \cdot id.$ to $\pi^* T_M$. Then the connection ∇^{CV} on $\pi^* T_M|_{\mathbb{C}\backslash\{0\} \times M}$ defined by

$$\nabla^{CV} := D + zC + z^{-1}\kappa C \kappa + (zC_E - Q - z^{-1}\kappa C_E \kappa)\frac{dz}{z}, \tag{1.11}$$

is flat.

It is proved in [H] that

(i) M can be extended uniquely to a manifold M^{ext} ($\supset M$) such that all E-orbits in M^{ext} are isomorphic to $\mathbb{C}$, $\mathbb{C}^*$ or $\{pt\}$,

(ii) outside of a real analytic subvariety $R \subset M^{ext}$, there exists a CDV structure on M^{ext} which extends canonically [1] the CDV structure on M.

Note that if κ is the real structure of a CDV structure on a Frobenius manifold M, then $-\kappa$ defines another CDV structure on M. Therefore we shall introduce the following "positivity condition" on CDV structures based on the flows of the real vector field $E+\overline{E}$ on M^{ext} which is motivated by Hertling's paper [H]:

Definition 1.4. We shall call a CDV structure $(M,\eta,\circ,E,e,\kappa)$ *positive* when the following condition is satisfied:

If one starts at any point $t \in M^{ext}$ and goes sufficiently far along the flow of the real vector field $E+\overline{E}$, then the Hermitian form h is positive definite there.

Remark. In general, the Hermitian form h is not positive definite on M^{ext}. We shall see such examples in section 3.

It is easy to see that any Frobenius manifold of rank one (1.7) can be equipped with the positive CDV structure defined by $\kappa(\partial_0) = a^{-1}|a|\partial_0$, i.e., $h(\partial_0,\partial_0) = |a|$. Then the natural problem is whether there exists a positive CDV structure on any Frobenius manifold. We have the following result for Frobenius manifolds of rank two:

Theorem 1.2. (**Main Theorem**)
Any Frobenius manifold of rank two can be equipped with a positive CDV structure if and only if $d \in \mathbb{R}$.

We shall prove our main theorem based on the classification of Frobenius manifolds since CDV structures change very much according to their multiplication structure. Therefore we shall discuss

(i) in section 3, Frobenius manifolds with trivial potentials: (1.8a), (1.8b), (1.8c), (1.8d), (1.8e) with $c=0$ and (1.8f),

(ii) in section 5, semisimple Frobenius manifolds: (1.8a), (1.8b), (1.8c), (1.8d), (1.8e) with $c \neq 0$.

We shall also show in section 4 that there exist discrete symmetries of CDV structures which is a natural extension of those of Frobenius manifolds discovered by Dubrovin [D1].

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[1] in the sense that the underlying $(TERP)$-structure on M^{ext} is the canonical extension of the one on M. See [H] for details.

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2. CDV structures of dimension 2

In this section, we shall give basic properties of CDV structures of rank two. Let (t^0, t^1) be flat coordinates of the Frobenius manifold of a CDV structure $(M, \eta, \circ, E, e, \kappa)$. The Euler field is given by $E = t^0\partial_0 + [(1-d)t^1 + r]\partial_1$ for some $r, d \in \mathbb{C}$ and $r \neq 0$ only if $d = 1$.

Lemma 2.1. *If $d \neq 0$ and h is positive definite at a point in $M^{ext} \backslash R$ [2], then*

$$(h_{i\bar{j}}) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \tag{2.1}$$

for a real analytic function g on $M^{ext} \backslash R$.

Proof. Note that we have

$$\kappa^2 = id. \quad \Longleftrightarrow \quad \eta^{-1}h \cdot \overline{(\eta^{-1}h)} = id. \tag{2.2}$$

by definition $h(\cdot,\cdot) = \eta(\cdot, \kappa\cdot)$. On the other hand, from the potentials F in the classification,

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Since h is positive definite at a point in $M^{ext} \backslash R$, from the above facts we see that $(h_{i\bar{j}})$ must be a diagonal matrix and the determinant of $(h_{i\bar{j}})$ must be 1. □

Note that g does not depend on t^0 since $Lie_e(h) = 0$. $[E, e] = -e$ and $Lie_{E-\bar{E}}(h) = 0$ give $(\partial_E - \bar{\partial}_{\bar{E}})g = 0$. Therefore, $Q = Lie_E - D_E + (2-d)/2 \cdot id.$ is diagonal if $d \neq 0$ and given by

$$(Q)_i^j = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \quad \text{where} \quad q = -\frac{d}{2} - \partial_E \log(g). \tag{2.4}$$

We can also show that Q is Hermitian with respect to the Hermitian form h, in other words, $h(Q\delta, \delta') = h(\delta, Q\delta')$ for $\delta, \delta' \in \mathcal{T}_M^{1,0}$. These facts give the following necessary condition for a Frobenius manifold to be equipped with a CDV structure:

Lemma 2.2. *Let $(M, \eta, \circ, E, e, \kappa)$ be a CDV structure of rank 2 and $E = t^0\partial_0 + [(1-d)t^1 + r]\partial_1$ be its Euler vector field. Then $d \in \mathbb{R}$.*

Proof. Q is Hermitian if and only if $q = \bar{q}$. Since $(\partial_E - \bar{\partial}_{\bar{E}})g = 0$, this is equivalent to $d = \bar{d}$. □

$(\partial_E - \bar{\partial}_{\bar{E}})g = 0$ also implies that $(t^1\partial_1 - \bar{t}^{\bar{1}}\bar{\partial}_{\bar{1}})g = 0$ if $d \neq 1$ and $(r\partial_1 - \bar{r}\bar{\partial}_{\bar{1}})g = 0$ if $d = 1$ and $r \neq 0$. Now the following statement is obvious:

[2] One may also include the case (1.8f) with $\eta_{00} = 0$.

Lemma 2.3. *g depends only on $|t^1|^2$ ($|e^{t^1/r}|^2$) if $d \neq 1$ and $r = 0$ ($d = 1$ and $r \neq 0$, respectively).* □

It is very complicated in general to write down explicitly the flatness condition for ∇^{CV}. But for CDV structures of rank two, $(\nabla^{CV})^2 = 0$ gives only one non-trivial differential equation:

Lemma 2.4. *Let $(M, \eta, \circ, E, e, \kappa)$ be a CDV structure of rank 2 and assume $d \neq 0$. Then $(\nabla^{CV})^2 = 0$ implies the following*

$$\bar{\partial}_{\bar{1}}\partial_1 \log(g) = -g^{-2} + |\partial_1^3 F|^2 g^2, \tag{2.5}$$

where F is the Frobenius potential of the CDV structure.

Conversely, if a real analytic solution of (2.5) is given for a Frobenius manifold of rank two with $d(\neq 0) \in \mathbb{R}$, then g defines a real structure κ of a CDV structure over the given Frobenius manifold.

Proof. In our case, all equations given by $(\nabla^{CV})^2 = 0$ except

$$[D_{\partial_1}, \bar{\partial}_{\bar{1}}] = -\,[C_{\partial_1}, \kappa C_{\partial_1}\kappa], \tag{2.6}$$

$$[\bar{\partial}_{\bar{1}}, Q] = -\,[C_E, \kappa C_{\partial_1}\kappa], \tag{2.7}$$

$$[D_{\partial_1}, C_E] + [Q, C_{\partial_1}] - C_{\partial_1} = 0, \tag{2.8}$$

are trivially satisfied. The last two equations follow from the definition of Q, $Q = Lie_E - D_E + (2-d)/2 \cdot id.$. We see that $\kappa C_{\partial_1}\kappa$ has the following matrix elements (with respect to the basis (∂_0, ∂_1) of $\mathcal{T}_M^{1,0}$)

$$\kappa C_{\partial_1}\kappa = h^{-1} C_{\partial_1}^{\dagger} h = \begin{pmatrix} 0 & g^{-2} \\ \overline{\partial_1^3 F} \cdot g^2 & 0 \end{pmatrix}. \tag{2.9}$$

Then we have the statement by some easy calculations. □

Therefore, for a Frobenius manifold with $d \neq 0$, all we have to do is to show the existence of a real analytic solution g of (2.5) such that g is positive in the sense of the Definition 1.4.

3. Frobenius manifolds with trivial potentials

In this section, we consider the case when the equation (2.5) is given by

$$\bar{\partial}_{\bar{1}}\partial_1 \log(g) = -g^{-2}, \tag{3.1}$$

in other words, we consider Frobenius manifolds with trivial potentials $F = (t^0)^2 t^1/2$. This class has three subclasses according to the classification: 1) case (1.8a), 2) case (1.8b) with $c = 0$ and 3) cases (1.8c), (1.8d), (1.8e) with $c = 0$, and the case (1.8f) with $\eta_{00} = 0$.

3.1. **Extended moduli space of elliptic curves.** First we shall study the case 1) (1.8a) which should corresponds to the geometry of the extended moduli space of elliptic curves.

Theorem 3.1. *Let us set* $M := \{(t^0,\tau) \in \mathbb{C}^2 \mid \mathrm{Im}\tau > 0\}$. *Consider the following data on a holomorphic vector bundle of rank two* $\mathcal{H} := \mathcal{O}_M e_1 \oplus \mathcal{O}_M e_2$ [3]:

(i) *a Hermitian form* h *on* $\mathcal{H} \otimes \mathcal{A}_M$

$$(h_{i\bar{j}}) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad \text{where} \quad g := 2\mathrm{Im}\tau, \tag{3.2}$$

(ii) *an* $\mathcal{O}_M$*-bilinear form* η *on* $\mathcal{H}$

$$(\eta_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.3}$$

(iii) $\mathcal{O}_M$*-linear maps* $C_\delta : \mathcal{H} \to \mathcal{H}$, $\delta \in \mathcal{T}_M$

$$C_{\partial_0} = id., \quad C_{\frac{1}{\sqrt{-1}}\partial_\tau} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.4}$$

(iv) *An* $\mathcal{A}_M$*-endomorphism* Q *of* $\mathcal{H} \otimes \mathcal{A}_M$

$$(Q_i^j) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{3.5}$$

Then any choice of a point $(0,\tau_0) \in M \cup \{(0,\sqrt{-1}\infty)\}$ *gives a global section* ζ *of* $\mathcal{H}$, *which together with the Euler vector field* $E = t^0\partial_0$ *induces via an* $\mathcal{O}_M$*-isomorphism*

$$\zeta : \mathcal{T}_M \simeq \mathcal{H}, \quad \delta \mapsto C_\delta \zeta, \tag{3.6}$$

a positive CDV structure on $\mathcal{T}_M$ *isomorphic to the one defined by* (1.8a), *whose flat coordinates are given by* t^0 *and* $t^1 = (\tau - \tau_0)/(\tau - \bar{\tau}_0)$ *if* $\tau_0 \neq \sqrt{-1}\infty$ *and* $t^1 = \sqrt{-1}\tau$ *if* $\tau_0 = \sqrt{-1}\infty$.

Proof. Note that $C_\delta : \mathcal{H} \to \mathcal{H}$, $\delta \in \mathcal{T}_M$ defines a multiplication structure on $\mathcal{T}_M$ via the isomorphism (3.6) once we fix ζ. If we set $\zeta := e_1$, then one easily sees that $t^0, t^1 := \sqrt{-1}\tau$ define flat coordinates of Frobenius manifold of type (1.8a) and that $g := 2\mathrm{Im}\tau$ satisfies the equation (3.1). Thus we have a positive CDV structure since the real vector field $E + \overline{E} = t^0\partial_0 + \bar{t}^{\bar{0}}\bar{\partial}_{\bar{0}}$ fixes the τ-direction.

Note that $\partial_\tau^k \log(h(\zeta,\zeta)) = \partial_\tau^k \log(g) = 0$ for all $k \geq 1$ at $(0,\sqrt{-1}\infty)$. In this sense, we mean that the section $\zeta = e_1$ is given by the choice of the point $(0,\infty)$. For other points $(0,\tau_0) \in M$, since $g = 2\mathrm{Im}\tau$ is real analytic, there exists a unique holomorphic function f on M up to a constant term such that $\zeta := \exp(f)e_1$ satisfies $\partial_\tau^k \log(h(\zeta,\zeta)) = 0$ for all $k \geq 1$ at $(0,\tau_0) \in M$. Indeed, we can choose such a function f by $\exp(f) = (2\mathrm{Im}\tau_0)^{1/2}/(\tau - \overline{\tau_0})$.

Recall that for each point $\tau_0 \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$ there exists a bi-holomorphic map

$$f_{\tau_0} : \mathcal{H} \simeq D := \{z \in \mathbb{C} \mid |z| < 1\}, \quad \tau \mapsto t^1 = \frac{\tau - \tau_0}{\tau - \bar{\tau}_0}. \tag{3.7}$$

[3] Here we use the matrix representation with respect to the basis (e_1, e_2) of $\mathcal{H}$.

and that f_{τ_0} maps the Weil–Peterson metric $g^{-2} = (2\mathrm{Im}\tau)^{-2}$ on $\mathcal{H}$ to the Poincaré metric on D:

$$(f_{\tau_0}^{-1})^*(\frac{|d\tau|^2}{(2\mathrm{Im}\tau)^2}) = \frac{1}{(\tau-\bar{\tau})^2}\frac{(\tau-\bar{\tau}_0)^2(\bar{\tau}-\tau_0)^2}{(\tau_0-\bar{\tau}_0)^2}|dt^1|^2 = \frac{|dt^1|^2}{(1-|t^1|^2)^2}. \tag{3.8}$$

Note that $\partial_1^k(1-|t^1|^2)^{-2} = 0$ for all $k \geq 1$ at $t^1 = 0$. Considering the following isomorphism as holomorphic bundles with Hermitian metrics

$$(\mathcal{H}, h) \simeq (\mathcal{T}_M, G) \otimes (L, g), \quad L := \mathcal{O}_M e_1, \quad (G_{i\bar{j}}) = \mathrm{diag}(1, g^{-2}), \tag{3.9}$$

we see that $\zeta = \exp(f)e_1$ gives a CDV structure [4] $(\mathbb{C} \times D \simeq M, \eta, \circ, E = t^0\partial_0, e, \kappa)$ via the isomorphism

$$\zeta : (f_{\tau_0}^{-1})_*\mathcal{T}_{\mathbb{C}\times D} \simeq \mathcal{H}, \quad \delta \mapsto C_{(f_{\tau_0}^{-1})_*\delta}\zeta, \tag{3.10}$$

whose Frobenius manifold structure is given by (1.8a) with flat coordinates $t^0, t^1 = (\tau-\tau_0)/(\tau-\bar{\tau}_0)$. In particular, the Hermitian metric with respect to the basis (∂_0, ∂_1) is given by

$$\begin{pmatrix} 1-|t^1|^2 & 0 \\ 0 & (1-|t^1|^2)^{-1} \end{pmatrix}. \tag{3.11}$$

This CDV structure is positive since $E = t^0\partial_0$ fixes the t^1-direction and $|t^1| < 1$ for all $(t^0, t^1) \in \mathbb{C} \times D$. □

Remark. (i) This theorem is implicit in [BCOV].

(ii) ζ is a primitive form in the sense of Kyoji Saito [S].

(iii) If one considers the restriction of a CDV structure to the subspace of M where $C_E = 0$, then one gets a variation of Hodge structures (see [H]). In the above case, we have a variation of Hodge structure of rank 2 of weight 1, which is classified by the upper half plane $\mathcal{H}$. The above construction of CDV structures is based on the Hodge theory, and as a result it classifies the all CDV structures of type (1.8a).

3.2. **Case** (1.8b) **with** $c = 0$**.** In this case, since g depends only on $x := |e^{t^1/r}|^2$, the equation (2.5) becomes

$$x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\log g\right) = -|r|^2 g^{-2}. \tag{3.12}$$

It is not difficult to solve the above equation under some assumptions:

Lemma 3.2. $g = |r|\log x$ *is the unique solution* (*up to addition of a real constant*) *of the equation* (3.12) *satisfying* $\lim_{x\to 0} x\partial \log g/\partial x = 0$, $\lim_{x\to 0} g^{-1} = 0$ *and* $g > 0$ *for* $x >> 0$.

Proof. This follows from the fact that (3.12) can be rewritten as $\partial_x(\partial_x \log(g/|r|))^2 = \partial_x(g/|r|)^{-2}$. □

[4] One may check directly that $\eta(\zeta, C_{\partial_1}\zeta) = 1$, $h(\zeta,\zeta) = (1-|t^1|^2)$ and etc..

From this Lemma, we have the following CDV structure for the Frobenius manifold of type (1.8b) with $c = 0$:

Theorem 3.3. *Let us consider the Frobenius manifold $(M, \eta, \circ, E, e)$ defined by $F = (t^0)^2 t^1/2$ and $E = t^0\partial_0 + r\partial_1$, $r \neq 0$. Then it can be equipped with a positive CDV structure whose real structure κ is given by the following Hermitian form h :*

$$(h_{i\bar{j}}) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad \text{where} \quad g = \frac{|r|}{r}t^1 + \frac{|r|}{\bar{r}}\bar{t}^{\bar{1}} - g_0, \quad g_0 \in \mathbb{R}. \tag{3.13}$$

In this case, the real analytic variety $R \subset M^{ext}$ where we have no CDV structures is given by $g = 0$, i.e., $2\mathrm{Re}(\bar{r}t^1) = |rg_0|$. □

Remark. (i) The Hermitian endomorphism of the above CDV structure is given by

$$(Q_i^j) = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \quad \text{where } q = -\frac{1}{2} + \frac{|r|^2}{\bar{r}t^1 + r\bar{t}^{\bar{1}} - g_0|r|}. \tag{3.14}$$

(ii) The flat coordinate t^1 for the potential (1.8b) are defined up to addition of a constant. Hence one can make $g_0 = 0$ by a constant shift of t^1.

3.3. **Case** (1.8c)**,**(1.8d)**,**(1.8e) **with** $c = 0$**.** In this case, since g depends only on $x := |t^1|^2$, the equation (2.5) becomes

$$\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\log g\right) = -g^{-2}. \tag{3.15}$$

We can find the following solution to the above equation:

Lemma 3.4. *Assume that g has the power series expansion $g = \sum_{i\geq 0} g_i x^i$ at $x = 0$. Then $g_0 \cdot g_1 = 1$ and $g_i = 0$ for $i \geq 2$.*

Proof. One can show by induction. □

As a result, we have the following:

Theorem 3.5. *Let us consider the Frobenius manifold $(M, \eta, \circ, E, e)$ defined by $F = (t^0)^2 t^1/2$ and $E = t^0\partial_0 + (1-d)t^1\partial_1$, $d \in \mathbb{R}$. Then it can be equipped with a positive CDV structure whose real structure κ is given by the following Hermitian form h :*

$$(h_{i\bar{j}}) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad \text{where} \quad g = g_0 - g_0^{-1}|t^1|^2, \quad \begin{cases} g_0 > 0, & \text{if } d > 1, \\ g_0 < 0, & \text{if } d < 1. \end{cases} \tag{3.16}$$

In this case, the real analytic variety $R \subset M^{ext}$ where we have no CDV structures is given by $g = 0$, i.e., $|t^1| = |g_0|$. □

Remark. The Hermitian endomorphism of the above CDV structure is given by

$$(Q_i^j) = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \quad \text{where } q = -\frac{d}{2} + \frac{(1-d)|t^1|^2}{g_0^2 - |t^1|^2}. \tag{3.17}$$

Note that

$$\lim_{t^1\to 0} q = -\frac{d}{2}, \quad \lim_{t^1\to\infty} q = -\frac{2-d}{2}. \tag{3.18}$$

3.4. **Case** (1.8f)**.** Finally, let us consider the case (1.8f). By setting $\tilde{\partial}_0 := \partial_0 - \eta_{00}/2\cdot\partial_1$ and $\tilde{\partial}_1 := \partial_1$, we can reduce this case to the one given in the previous subsection. As a result we have the following positive CDV structure for this case:

Theorem 3.6. *Let us consider the Frobenius manifold $(M,\eta,\circ,E,e)$ defined by $F=\frac{1}{6}\eta_{00}(t^0)^3+\frac{1}{2}(t^0)^2t^1$ and $E=t^0\partial_0+t^1\partial_1$. Then it can be equipped with a positive CDV structure whose real structure κ is given by the following Hermitian form h:*

$$(h_{i\bar{j}}) = \begin{pmatrix} g+\frac{|\eta_{00}|^2}{4}g^{-1} & \frac{\eta_{00}}{2}g^{-1} \\ \frac{\overline{\eta_{00}}}{2}g^{-1} & g^{-1}\end{pmatrix}, \quad \text{where} \quad g=g_0-g_0^{-1}|t^1|^2, \quad g_0\in\mathbb{R}_{<0}. \tag{3.19}$$

In this case, the real analytic variety $R\subset M^{ext}$ where we have no CDV structures is given by $g=0$, i.e., $|t^1|=|g_0|$. □

Remark. The Hermitian endomorphism of the above CDV structure is given by

$$(Q_i^j) = \begin{pmatrix} q & 0 \\ -\eta_{00}\cdot q & -q\end{pmatrix}, \quad \text{where } q=\frac{|t^1|^2}{g_0^2-|t^1|^2}. \tag{3.20}$$

4. Symmetries of CDV structures

Symmetries of Frobenius manifolds are transformations which sends a Frobenius manifold to another one. There are two basic symmetries I and S_1 [D1].

4.1. **Inversion** I**.** First we consider a symmetry called the inversion I which is given by [5]

$$\widehat{t}^0 := t^0, \tag{4.1a}$$

$$\widehat{t}^1 := -(t^1)^{-1}, \tag{4.1b}$$

$$\widehat{F}(\widehat{t}) := (t^1)^{-2}\left[F(t)-(t^0)^2t^1\right], \tag{4.1c}$$

$$\widehat{\eta}_{01} := \eta_{01}. \tag{4.1d}$$

In particular, one sees that $\widehat{d}=2-d$.

We can extend this to the symmetry of CDV structures by setting

$$\widehat{g}(\widehat{t}) := g(t)|t|^{-2}. \tag{4.2}$$

Note that the CDV structures given in Theorem 3.5 with $g_0<0$, $d<1$ are the I-transforms of those of $g_0>0$, $d>1$. One can glue them and construct CDV structures on $\mathbb{C}\times\mathbb{P}^1$, whose underlying Frobenius structures are called twisted Frobenius manifold in [D1].

[5]We assume here $d\neq 0$ for simplicity.

4.2. **Symmetry S_1.** For cases (1.8c), (1.8d), (1.8e) with $c \neq 0$, we have also the following symmetry called S_1:

$$\widehat{t}^0 := t^0, \tag{4.3a}$$

$$\widehat{t}^1 := \partial_1^2 F, \tag{4.3b}$$

$$\widehat{\partial}_1^2 \widehat{F}(\widehat{t}) := t^1, \tag{4.3c}$$

$$\widehat{\eta}_{01} := \eta_{01}. \tag{4.3d}$$

In particular, one sees that $\widehat{d} = -d$.

We can also extend this to the symmetry of CDV structures by setting

$$\widehat{g}(\widehat{t}) := (g(t))^{-1}. \tag{4.4}$$

Thus by using I and S_1 we can reduce any semi-simple CDV structure [6] of rank two with $d \in \mathbb{R}$ to one with $0 \leq \widehat{d} \leq 1$.

Remark. (i) I and S_1 respects the positivity of CDV structure.
(ii) Note that I and S_1 define an action of $W(\widehat{A}_1)$, the affine Weyl group of type A_1, on the space of semisimple CDV structures of rank two.

5. Semisimple case

In this section, we consider the case when the equation (2.5) is given by

$$\bar{\partial}_{\bar{1}} \partial_1 \log(g) = -g^{-2} + |\partial_1^3 F|^2 g^2, \tag{5.1}$$

where $\partial_1^3 F$ is invertible at almost all points on M. This class consists of a direct sum of two CDV structures of rank one for which the existence of positive CDV structure is clear and the cases (1.8b), (1.8c), (1.8d), (1.8e) with $c \neq 0$.

Note that in the semisimple case, $g = |\partial_1^3 F|^{-1/2}$ is always a solution of (5.1) and defines a positive CDV structure. Since $\partial_E \partial_1^3 F = 2d\partial_1^3 F$, we see that $Q = 0$ for this solution. By a suitable re-definition of η and t^1, one can reduce this CDV structure to a direct sum of two CDV structures of rank one. So we have to find another nontrivial solution.

The key fact for semisimple CDV structures of rank two is that the differential equation (5.1) can be written in the form of the Painlevé III equation [CV1]:

$$\frac{d^2 u}{dz^2} + \frac{1}{z}\frac{\partial u}{\partial z} = 4\sinh(u), \tag{5.2}$$

where

$$u := \log(|\partial_1^3 F| g^2) \quad \text{and} \quad z = \begin{cases} C_1 |t^1|^{\frac{1}{1-d}}, & \text{if } d \neq 1, \\ C_2 |e^{\frac{t^1}{r}}|, & \text{if } d = 1, r \neq 0, \end{cases} \tag{5.3}$$

[6] We mean by semi-simple CDV structure that the multiplication $\circ$ is semi-simple.

for some non-zero constants C_1 and C_2. Note that under the symmetries I and S_1 of CDV structures (see the previous section), z is invariant and u transforms as

$$I : u \mapsto \widehat{u} = u, \quad S_1 : u \mapsto \widehat{u} = -u. \tag{5.4}$$

Therefore any semi-simple CDV structure of rank two with $d \in \mathbb{R}$ can be reduced to ones with $[d] \in [0,1] \simeq \mathbb{R}/W(\widehat{A}_1)$, where $W(\widehat{A}_1)$-action on $\mathbb{R}$ is given by $I : d \mapsto 2-d$ and $S_1 : d \mapsto -d$. On the other hand, the existence of positive CDV structures for Frobenius manifolds with $0 \le d \le 1$ is already shown by Cecotti–Vafa in [CV1][CV2]. Thus we get the following result:

Theorem 5.1. *Any semisimple Frobenius manifold of rank two $(M,\eta,\circ,E,e)$ with $d \in \mathbb{R}$ can be equipped with a positive CDV structure whose real structure is given by the following Hermitian metric h :*

$$(h_{i\bar{j}}) = \begin{pmatrix} e^{\frac{u(z)}{2}}|\partial_1^3 F|^{-\frac{1}{2}} & 0 \\ 0 & e^{-\frac{u(z)}{2}}|\partial_1^3 F|^{\frac{1}{2}} \end{pmatrix}, \tag{5.5}$$

where $u(z)$ is the unique real analytic solution $u(z)$ of (5.2) such that

(i) *$u(z)$ has the following asymptotic behavior as $z \to 0$:*

$$u(z) \sim s\log(z) + t + O(z^{2-|s|}), \quad 0 \le s < 2, \tag{5.6}$$

$$\sim 2\log(z) + 2\log[-\log(\frac{z}{2}+\gamma)] + O(z^4\log^2 z), \quad s = 2, \tag{5.7}$$

where γ is the Euler's constant and $s/2$ is the representative of d in $[0,1] \simeq \mathbb{R}/W(\widehat{A}_1)$,

(ii) *$u(z)$ has no poles on the positive real axis. This is equivalent to*

$$e^{\frac{t}{2}} = \frac{1}{2^s}\frac{\Gamma(\frac{1}{2}-\frac{s}{4})}{\Gamma(\frac{1}{2}+\frac{s}{4})}.$$

In particular, the CDV structures given by the above $u(z)$ are defined on whole M^{ext}, i.e., R is empty. □

Remark. The Hermitian endomorphism of the above CDV structure is given by

$$(Q_i^j) = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \quad \text{where} \quad q := -\frac{1}{4}z\frac{\partial u}{\partial z}. \tag{5.8}$$

Note that $E = z\partial_z/2$.

It is known that the Stokes matrix S defined over $\mathbb{R}$, a real upper triangular matrix with identity on the diagonals, describes a positive semisimple CDV structure [CV2][D2][H]. We see that the following conditions are equivalent for semisimple CDV structures of rank two:

(i) $S + {}^tS$ is positive semi-definite.

(ii) $R \subset M^{ext}$ where one can not have CDV structures is empty.

We expect[7] that this equivalence holds for semisimple CDV structures of rank greater than 2.

[7] Motivated by [CV2] and a private discussion with Claus Hertling at Sapporo, Sep. 2003.

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