COMPACT COMPLEX SURFACES ADMITTING NON-TRIVIAL SURJECTIVE ENDOMORPHISMS

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Abstract. Smooth compact complex surfaces admitting non-trivial surjective endomorphisms are classified up to isomorphisms. The algebraic case has been classified in [3], [19]. The following surfaces are listed in the non-algebraic case: a complex torus, a Kodaira surface, a Hopf surface with at least two curves, an Inoue surface with curves, and an Inoue surface without curves satisfying a rationality condition.

1. Introduction

A non-trivial surjective endomorphism of a compact complex variety X is a non-isomorphic surjective morphism $X \to X$ by definition. Projective surfaces X admitting non-trivial surjective endomorphisms are classified in [3], [19] as follows:

(1) X is a toric surface;
(2) X is a $\mathbb{P}^1$-bundle over an elliptic curve;
(3) X is a $\mathbb{P}^1$-bundle over a non-singular curve C of genus $g \geq 2$ such that $X \times_C C' \simeq \mathbb{P}^1 \times C'$ for a finite étale covering $C' \to C$;
(4) X is an abelian surface or a hyperelliptic surface;
(5) X is an elliptic surface with the Kodaira dimension $\kappa(X) = 1$ and the topological Euler number $e(X) = 0$.

The cases above correspond to the following numerical invariants: (1) $\kappa(X) = -\infty$ and the irregularity $q(X) = 0$; (2) $\kappa(X) = -\infty$ and $q(X) = 1$; (3) $\kappa(X) = -\infty$ and $q(X) \geq 2$; (4) $\kappa(X) = 0$; (5) $\kappa(X) = 1$. Note that a surface of general type does not admit non-trivial surjective endomorphisms. In this article, we study the case where X is non-algebraic. The following is our main result:

Theorem 1.1. *The non-algebraic non-singular compact complex surfaces X admitting non-trivial surjective endomorphisms are classified as follows*:

(1) *X is a complex torus*;
(2) *X is a primary Kodaira surface, a secondary Kodaira surface, or an elliptic Hopf surface*;

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(3) *X is a Hopf surface with two elliptic curves or one of the following Inoue surfaces without curves*: S_M, $S^{(+)}_{N,p,q,r;t}$ *satisfying a rationality condition* (cf. Theorem 8.6) *on the parameter* t, *and* $S^{(-)}_{N,p,q,r}$;

(4) *X is a successive blowups of one of the following surfaces whose centers are nodes of curves*: *a parabolic Inoue surface, a hyperbolic Inoue surface, and a half Inoue surface.*

The cases above correspond to the following numerical invariants: (1) the first Betti number $b_1(X)$ is even; (2) $b_1(X)$ is odd and the algebraic dimension $a(X) = 1$; (3) $a(X) = 0$, $b_1(X) = 1$, and $b_2(X) = 0$; (4) $a(X) = 0$, $b_1(X) = 1$, and $b_2(X) > 0$. In particular, if X is Kähler, then X is a complex torus. The definitions of Kodaira surfaces, Hopf surfaces, Inoue surfaces are given in [10], [5], [7] (cf. [1]). But we discuss the structures and the properties of these non-Kähler surfaces in Sections 2, 6–9 below. The Kodaira surfaces X are characterized by the conditions: $b_1(X)$ is odd and $c_1(X) = 0$ in $H^2(X, \mathbb{Q})$. A Hopf surface is a compact complex surface whose universal covering space is biholomorphic to $\mathbb{C}^2 \setminus \{(0, 0)\}$ by definition. A compact complex surface is called a surface of class VII if the first Betti number is 1. If it is minimal, furthermore, it is called a surface of class VII_0. Hopf surfaces and Inoue surfaces are typical examples of surfaces of class VII_0 with the algebraic dimension zero.

The idea of the proof of Theorem 1.1 is as follows: In the first step, we list the possible surfaces X admitting a non-trivial surjective endomorphism. We can show that, for such an X, the set $\mathcal{S}(X)$ of curves with negative self-intersection number is finite by the same argument as in [19]. This yields a strong condition on X. For example, it implies that if X is a non-algebraic elliptic surface, equivalently if $a(X) = 1$, then the singular fibers are multiple of elliptic curves (cf. Proposition 4.1). Furthermore by investigating the variation of Hodge structure, we infer that X is one of the surfaces listed in (2) of Theorem 1.1 (cf. Theorem 4.5). The finiteness of $\mathcal{S}(X)$ and some known results on surfaces of class VII_0 imply that if X is a surface of class VII, then its minimal model is one of the known examples (cf. Theorem 5.2). Thus we can make a list of the possible surfaces.

Conversely in the second step, we examine whether a non-trivial surjective endomorphism exists or not individually for the cases of surfaces listed as candidates. It seems to be difficult to determine the existence on Kodaira surfaces, on non-elliptic Hopf surfaces, and on Inoue surfaces without curves, because of their complicated construction from the universal covering space. We consider a lift of an expected endomorphism to the universal covering space and examine whether it really induces a non-trivial endomorphism by elementary and long calculations. In the case of Kodaira surfaces and Inoue surfaces without curves, we can describe the induced endomorphism of the fundamental group explicitly by using triangular matrices in $\mathrm{GL}(3, \mathbb{C})$ (cf. Proposition 6.4, Proposition 8.5). Our method is delicate but powerful enough for the investigation. For example, we find a remarkable condition on the parameter t for the existence of endomorphism on the Inoue surface $S^{(+)}_{N,p,q,r;t}$. Contrary to the above, in the case of elliptic Hopf surfaces, we look at the behavior of multiple fibers of the elliptic fibration. If it has three multiple fibers, then it is obtained as the quotient of an elliptic fiber bundle over $\mathbb{P}^1$ by a free action of a regular polyhedral group $G \subset \mathrm{PGL}(2, \mathbb{C})$. A G-equivariant endomorphism on the elliptic bundle is constructed by a similar method as in Lemma 6 in [19].

This paper is organized as follows: After explaining the classification theory of non-algebraic surfaces in Section 2, we recall and generalize the argument in [19] on the set $\mathcal{S}(X)$ of curves with negative self-intersection numbers in Section 3. The possible surfaces X are listed in Section 4 and Section 5, respectively for the cases $a(X) = 1$ and $a(X) = 0$. The existence of endomorphisms is studied individually for the cases of surfaces in Sections 6, 7, 8, and 9 according to Kodaira surfaces, Hopf surfaces, Inoue surfaces without curves, and Inoue surfaces with curves.

Notation

Throughout this paper, we call a compact complex analytic surface by a surface and a compact complex analytic curve by a curve, for short, if it causes no confusion.

Let X be a non-singular compact complex surface. For $u \in \mathrm{H}^i(X, \mathbb{Z})$, $v \in \mathrm{H}^{4-i}(X, \mathbb{Z})$, we denote by $u \cdot v$ the intersection number $\int u \cup v$, where $\cup$ is the cup-product and $\int$ is the trace map $\mathrm{H}^4(X, \mathbb{Z}) \to \mathbb{Z}$. A divisor D of X defines a homology class in $\mathrm{H}_2(X, \mathbb{Z})$ which corresponds to the first Chern class $c_1(D) = c_1(\mathcal{O}_X(D))$ associated with the line bundle $\mathcal{O}_X(D)$ by the Poincaré isomorphism $\mathrm{H}^2(X, \mathbb{Z}) \simeq \mathrm{H}_2(X, \mathbb{Z})$. The intersection number $c_1(D_1) \cdot c_1(D_2)$ of two divisors D_1 and D_2 is denoted by $D_1 \cdot D_2$. Note that $c_1(\mathcal{L}) \cdot C = \deg \mathcal{L}|_C$ for a line bundle $\mathcal{L}$ and for an irreducible curve C.

Let $f\colon Y \to X$ be a surjective morphism from another non-singular compact complex surface. It induces the pull-back $f^*\colon \mathrm{H}^i(X, \mathbb{Z}) \to \mathrm{H}^i(Y, \mathbb{Z})$ and the push-forward $f_*\colon \mathrm{H}_i(Y, \mathbb{Z}) \to \mathrm{H}_i(X, \mathbb{Z})$. By the Poincaré duality, the push-forward induces a homomorphism $\mathrm{H}^i(Y, \mathbb{Z}) \to \mathrm{H}^i(X, \mathbb{Z})$, which we also denote by f_*. Then the composite $f_* \circ f^*\colon \mathrm{H}^i(X, \mathbb{Z}) \to \mathrm{H}^i(X, \mathbb{Z})$ is the multiplication map by $\deg f$: the mapping degree of f. The projection formula $f_*(f^*x \cdot y) = x \cdot f_*y$ holds for $x \in \mathrm{H}^i(X, \mathbb{Z})$ and $y \in \mathrm{H}^{4-i}(Y, \mathbb{Z})$. For a divisor D on X and a divisor E on Y, we have $c_1(f^*D) = f^*c_1(D)$ and $c_1(f_*E) = f_*c_1(E)$, where f^*D and f_*E are the pull-back and the push-forward as divisors, respectively.

Contrary to the case of algebraic surfaces, the canonical line bundle $\omega_X = \Omega^2_X$ may not have a non-zero global meromorphic section. The divisor of the meromorphic section is called canonical and is denoted by K_X. Even if the canonical divisor does not exist, we use the same symbol K_X as the canonical divisor class virtually in order to simplify some formulas such as the canonical bundle formula of elliptic fibration, the adjunction formula, and the ramification formula. For example, we explain that the arithmetic genus

$p_a(D) = \dim \mathrm{H}^1(D, \mathcal{O}_D)$ for a connected reduced divisor D is calculated by $2p_a(D) - 2 = (K_X + D) \cdot D$, which is derived from the adjunction formula $K_D \sim (K_X + D)|_D$.

2. Non-algebraic surfaces

Let X be a non-singular compact complex surface. The algebraic dimension $a(X)$ is the transcendence degree of the meromorphic function field of X over $\mathbb{C}$. Here, $a(X) \leq 1$ if and only if X is non-algebraic. If $a(X) = 1$, then the algebraic reduction $\pi \colon X \to T$ is holomorphic and is an elliptic fibration. Moreover any curves on X are contained in fibers of π. If $a(X) = 0$, then there exist at most finitely many irreducible curves on X by Theorem 5.1 of [9, I]. We recall the following useful results:

Lemma 2.1. *Suppose that $a(X) \leq 1$. Then a line bundle $\mathcal{L}$ of X satisfies the following properties*:

(1) $c_1(\mathcal{L})^2 \leq 0$.
(2) *If $c_1(\mathcal{L})^2 = 0$, then $c_1(\mathcal{L}) \cdot c_1(\mathcal{L}') = 0$ for any line bundle $\mathcal{L}'$.*
(3) *If $p_g(X) = 0$ and $c_1(\mathcal{L})^2 = 0$, then $c_1(\mathcal{L})$ is torsion in $\mathrm{H}^2(X, \mathbb{Z})$.*

Proof. (1) Suppose that $c_1(\mathcal{L})^2 > 0$. The Riemann–Roch formula for $\chi(X, \mathcal{L}^{\otimes m})$ implies that $\mathrm{h}^0(X, \mathcal{L}^{\otimes m})$ or $\mathrm{h}^0(X, \mathcal{L}^{\otimes(-m)} \otimes \omega_X)$ increases of order m^2 as $m \to \infty$. But the former case does not occur since $\kappa(\mathcal{L}, X) \leq a(X) \leq 1$. Thus there exists a non-zero effective divisor D such that $\mathcal{O}_X(D) \simeq \omega_X \otimes \mathcal{L}^{\otimes(-n)}$ for some $n > 0$. The exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{L}^{\otimes(-m+n)}) \to \mathrm{H}^0(X, \omega_X \otimes \mathcal{L}^{\otimes(-m)}) \to \mathrm{H}^0(D, \omega_X \otimes \mathcal{L}^{\otimes(-m)}|_D)$$

implies $\kappa(\mathcal{L}^{-1}, X) = 2$ contradicting $\kappa(\mathcal{L}^{-1}, X) \leq a(X) \leq 1$.

(2) This is shown by (1) and by the inequalities

$$0 \geq (tc_1(\mathcal{L}) + c_1(\mathcal{L}'))^2 = 2tc_1(\mathcal{L}) \cdot c_1(\mathcal{L}') + c_1(\mathcal{L}')^2$$

for any rational number t.

(3) follows from (2), from the surjectivity of $c_1 \colon \mathrm{Pic}(X) \to \mathrm{H}^2(X, \mathbb{Z})$, and from the non-degeneracy of the intersection pairing on $\mathrm{H}^2(X, \mathbb{Q})$. □

Notation. Let C be an irreducible curve on a non-singular compact complex surface.

(1) If $C^2 < 0$, then C is called a negative curve.
(2) If $C^2 = 0$, then C is called a 0-curve.
(3) If $C \simeq \mathbb{P}^1$ and $C^2 = -d < 0$, then C is called a $(-d)$-curve.

An exceptional curve of the first kind is just a (-1)-curve. If $a(X) \leq 1$, then a non-negative irreducible curve is a 0-curve with $p_a = 1$ and does not intersect other curves.

Remark. A relative minimal model Y of X is, by definition, a non-singular compact complex surface bimeromorphic to X having no (-1)-curves. If X is non-algebraic, then Y is unique up to isomorphisms. This is shown as follows: Suppose that there exist a bimeromorphic morphism $\mu\colon X \to Y$ and a (-1)-curve $C \subset X$ such that $\mu(C)$ is not a point. Then $\mu(C)$ is a 0-curve with the arithmetic genus $p_a(\mu(C)) = 1$ by Lemma 2.1. Thus $\mu(C)$ has a node or a cusp. Let $Y' \to Y$ be the blowup at the singular point of $\mu(C)$. Then the self-intersection number of the proper transform of $\mu(C)$ is less than -1. Since μ factors through $Y' \to Y$, this is a contradiction. Thus, we call Y the minimal model of X in the non-algebraic case. Similarly, a non-algebraic surface without (-1)-curves is called a minimal surface.

If X is a non-Kähler elliptic surface with $\kappa(X) = 0$, then $b_1(X) = 3$ or 1. In the case $b_1(X) = 3$, the minimal model is the quotient space of $\mathbb{C}^2$ by the action of an affine transformation group and is called a primary Kodaira surface. In the case $b_1(X) = 1$, the minimal model has a primary Kodaira surface as a finite étale covering space and is called a secondary Kodaira surface.

Let X be a compact complex surface with $a(X) = 0$. If $b_1(X)$ is even, then the minimal model of X is either a complex torus or a K3 surface. If $b_1(X)$ is odd, then $b_1(X) = 1$.

In the classification theory of compact complex surfaces by Kodaira [10], the class VII is not completely classified. A compact complex surface belongs to the class VII if $b_1(X) = 1$. The class VII_0 consists of all the minimal surfaces of class VII. A surface X of class VII has the following invariants:

$$q(X) - 1 = p_g(X) = \chi(X, \mathcal{O}_X) = h^{1,0}(X) = 0, \quad b_2(X) = -K_X^2 \geq 0.$$

Moreover the intersection pairing on $\mathrm{H}^2(X, \mathbb{Q})$ is negative definite.

A Hopf surface is a surface whose universal covering space is isomorphic to $W := \mathbb{C}^2 \setminus \{(0,0)\}$, by definition. This is a surface of class VII_0 with $b_2 = 0$ containing an elliptic curve.

The classification of surfaces of class VII_0 after Kodaira [10] was started by the discovery of Inoue surfaces [5], [6], [7]. The Inoue surfaces S_M, $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$ contain no curves and have the vanishing second Betti number. The surfaces S_M are also found by Bombieri and are called Bombieri–Inoue surfaces. Inoue [5] showed that if a surface S of class VII_0 contains no curves, $b_2(S) = 0$, and has a line bundle $\mathcal{L}$ with $\mathrm{H}^0(S, \Omega^1_S \otimes \mathcal{L}) \neq 0$, then S is isomorphic to one of the Inoue surfaces above. The last condition on the existence of $\mathcal{L}$ is not required for the characterization. This was shown by [11], [21] in 1990's. The other Inoue surfaces: Parabolic Inoue surface $X_{\lambda,n}$, Hyperbolic Inoue surface $X_{\mathfrak{K},\mathsf{N}}$, Half Inoue surface $\widehat{X}_{\mathfrak{K},\mathsf{N}}$, are constructed in [7]. These surfaces contain curves and have positive second Betti numbers. A parabolic Inoue surface is related to Hirzebruch's cusp singularities and is called also a Hirzebruch–Inoue surface. Another construction of these Inoue surfaces with curves is given in [20] by the method of torus embedding theory (cf. Section 9).

There are many contributions to the classification of surfaces of class VII_0 by Kato [8], Enoki [2], Nakamura [13], [14], and others. The following surfaces are listed in the Table (10.3) of [13]:

FACT. The surfaces X of class VII_0 with $a(X) = 0$ are classified as follows:

(1) Hopf surface with $a(X) = 0$;
(2) A parabolic Inoue surface: It is characterized as a surface containing an elliptic curve and a cycle of rational curves;
(3) A hyperbolic Inoue surface: It is characterized as a surface containing two cycles of rational curves;
(4) An exceptional compactification with no elliptic curves (cf. [2]): It is characterized as a surface containing a cycle D of rational curves with $D^2 = 0$ and no elliptic curves;
(5) A half Inoue surface: It is characterized as a surface containing a cycle D of rational curves with $D^2 < 0$ and $b_2(X) = b_2(D)$;
(6) A surface with a cycle D of rational curves with $D^2 < 0$ and $b_2(X) > b_2(D)$;
(7) A surface with no elliptic curves and with no cycles of rational curves.

Here, by a cycle of rational curves, we mean a reduced connected divisor $D = \sum C_i$ satisfying one of the following conditions:

(1) D is an irreducible rational curve with exactly one node;
(2) Any irreducible component C_i is isomorphic to $\mathbb{P}^1$ and intersects with $D - C_i$ transversely at two points.

3. Curves of negative self-intersection number.

The argument of this section is almost parallel to that of Section 2 of [19], where the algebraic case was discussed.

LEMMA 3.1. *A surjective endomorphism $f\colon X \to X$ is a finite morphism. If $\kappa(X) \geq 0$, then f is étale.*

Proof. If an irreducible curve C is contracted to a point by f, then $C^2 < 0$. Since $f_*\colon \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(X, \mathbb{Q})$ is isomorphic, no irreducible curve is contracted by f. Hence f is finite. Suppose that $\kappa(X) \geq 0$. Then $K_X \sim f^*K_X + R$ for the ramification divisor $R \geq 0$. Thus $K_X \sim f^*f^*K_X + f^*R + R$. Since $f^*\colon \mathrm{H}^0(X, mK_X) \to \mathrm{H}^0(X, mK_X)$ is isomorphic, $R + f^*R + \cdots$ is contained in the fixed part of $|mK_X|$. Thus $R = 0$. □

LEMMA 3.2. *Let $f\colon X \to X$ be a surjective endomorphism. If C is a negative curve, then $f(C)$ is also negative and $f^{-1}(f(C)) = C$.*

Proof. Assume that $f(C) = f(C')$ for another irreducible curve C'. Then $af_*C = a'f_*C'$ for some a, $a' > 0$. Hence $c_1(aC - a'C') = 0$ in $\mathrm{H}^2(X, \mathbb{Q})$. In particular, $C \cdot C' < 0$ and thus $C = C'$. □

Let $f\colon X \to X$ be a non-trivial surjective endomorphism of degree $d > 1$. We consider the set $\mathcal{S}(X)$ of all the negative curves on X. Then $\mathcal{S}(X)$ is preserved by f and the mapping $\mathcal{S}(X) \ni C \mapsto f(C) \in \mathcal{S}(X)$ is injective. Let R be the ramification divisor of X and let $\mathcal{S}_0(X)$ be the set of all the negative curves contained in $\operatorname{Supp} R$.

LEMMA 3.3. *If $C \in \mathcal{S}(X) \setminus \mathcal{S}_0(X)$, then $|C^2| > |f(C)^2|$.*

Proof. There exist natural numbers a, b such that $f_*C = af(C)$ and $f^*f(C) = bC$. Here $b = 1$ since $C \not\subset \operatorname{Supp} R$. Thus $a = d$ and $af(C)^2 = C^2$. □

The proof of the following elementary Lemma is left to the reader:

LEMMA 3.4. *Let $\mathcal{S}$ be a set, $\mathcal{S}_0$ a finite subset, and let $h\colon \mathcal{S} \to \mathcal{S}$ be an injection. If*

$$\mathcal{S} = \bigcup_{m=1}^{\infty} (h^m)^{-1}(\mathcal{S}_0),$$

then $\mathcal{S}$ is finite and h^k is identical for some $k > 0$.

By Lemma 3.3 and Lemma 3.4, we have:

PROPOSITION 3.5. *$\mathcal{S}(X)$ is a finite set and there is a natural number k with $f^k(C) = C$ for any $C \in \mathcal{S}(X)$.*

Hence we assume in what follows that $f(C) = C$ for any $C \in \mathcal{S}(X)$. Then $f^*C = aC$ and $f_*C = aC$ for a natural number $a > 1$ with $a^2 = d$. Let N_X denote the reduced divisor $\sum_{C \in \mathcal{S}(X)} C$. Then $R = (a-1)N_X + \Delta$ for an effective divisor Δ whose irreducible component are not negative curves. In particular

$$K_X + N_X = f^*(K_X + N_X) + \Delta. \tag{3.1}$$

For any connected reduced curve $D \le N_X$, we have

$$K_D + (N_X - D)|_D = (f|_D)^* \left(K_D + (N_X - D)|_D\right) + \Delta|_D.$$

In particular, $p_a(D) = \mathrm{h}^1(D, \mathcal{O}_D) \le 1$. If $p_a(D) = 1$, then $\Delta \cap D = (N_X - D) \cap D = \emptyset$. If $p_a(D) = 0$, then $(N_X - D) \cdot D \le 2$, and if further $(N_X - D) \cdot D = 2$, then $\Delta \cap D = \emptyset$.

The induced morphism $f|_D\colon D \to D$ is an endomorphism of degree a. Moreover it is étale outside $\operatorname{Sing} D \cup \Delta|_D$ by the well-known Lemma 3.6 below. In particular, $f(\operatorname{Sing} D) \subset \operatorname{Sing} D \cup \Delta|_D$, and $\Delta|_D$ gives the ramification divisor of $f|_D$ over $D \setminus \operatorname{Sing} D$.

LEMMA 3.6. *Let $\tau\colon U \to V$ be a finite morphism between non-singular complex manifolds and let $C \subset V$ be a non-singular divisor such that τ is étale outside $\tau^{-1}C$. Then $\tau^{-1}C \to C$ is étale.*

Proof. We may assume that V is a d-dimensional polydisc and C is a hyperplane by considering the local situation. Then $V \setminus C$ is isomorphic to the product of the punctured disc and a $(d-1)$-dimensional polydisc. In particular, the finite étale covering $U \setminus \tau^{-1}C \to V \setminus C$ is cyclic and $U \to V$ is the cyclic covering branched along C. Hence $\tau^{-1}C \simeq C$. □

A reduced connected divisor D is called a straight chain of rational curves if $D = \sum_{i=1}^{l} C_i$ for irreducible curves $C_i \simeq \mathbb{P}^1$ such that

(1) $C_i \cdot C_j = 0$ for $|i-j| \geq 2$,
(2) $C_i \cap C_j = 1$ for $|i-j| = 1$.

LEMMA 3.7. *A negative curve C is either an elliptic curve, a rational curve with exactly one node, or a smooth rational curve. A reducible connected component of N_X is an elliptic curve, a straight chain of rational curves, or a cycle of rational curves.*

Proof. If $p_a(D) = 1$ for a connected reduced curve $D \leq N_X$, then $K_D = (f|_D)^* K_D$ and $f|_D \colon D \to D$ is étale outside $\operatorname{Sing} D$. Thus no rational curves with cusps are negative. If a negative curve C_1 intersects another negative C_2 at one point not transversely, then $p_a(C_1 + C_2) = 1$. This contradicts the property: no étale covering exists over $C_1 \setminus C_2 \simeq \mathbb{C}$. If three negative curves C_1, C_2, C_3 intersect transversely as $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \{P\}$ for a point P, then $p_a(C_1 + C_2 + C_3) = 1$. This contradicts the same property as above. These observations tell us that a reducible connected component D is a straight chain of rational curves or a cycle of rational curves. $\square$

Suppose that X contains a (-1)-curve C. Let $X \to X_1$ be the blowing down of C. Then an endomorphism of X_1 is induced since $f^{-1}C = C$. Therefore, an endomorphism is induced on a relative minimal model of X.

4. The case of elliptic surfaces

Let X be a non-singular compact complex surface admitting a non-trivial surjective endomorphism. Assume that $a(X) = 1$. Let $\pi \colon X \to T$ be the algebraic reduction which is an elliptic fibration onto a non-singular projective curve. A non-trivial surjective endomorphism f induces a surjective endomorphism h of T such that $h \circ \pi = \pi \circ f$.

PROPOSITION 4.1. *Under the situation, X is a minimal elliptic surface with $e(X) = 0$.*

Proof. The set of all the irreducible component of reducible fibers coincides with $\mathcal{S}(X)$. A 0-curve is the support of an irreducible fiber. We may assume that $f^{-1}C = C$ for negative curves C for the endomorphism f.

Step 1. We may assume that $f^{-1}C = C$ for any rational curves C.

We have to consider only rational 0-curves C. If C' is an irreducible component of $f^{-1}C$, then C' is not negative and $C' \to C$ is étale outside $\operatorname{Sing} C$ by Lemma 3.6. If C is a rational curve with a cusp, then $C' \simeq C$. If C is a rational curve with a node, then C' also has a node since f is branched along the normal crossing divisor around the node. The number of rational 0-curves are finite. Hence $f^{-1}C$ is irreducible and $(f^k)^{-1}C = C$ for some $k > 0$.

Step 2. X admits no curves with cusps.

Suppose that there exist an irreducible curve C with a cusp and set $P = \pi(C)$. Note that $C = \pi^*P$ is a singular fiber of type II. By the argument of *Step 1*, we infer that $f^*C = dC$ for $d = \deg f$. Hence $h^*P = dP$. In particular, $\deg h = d$. If D is a connected component of N_X, then $D = \pi^{-1}P''$ and $h^*P'' = dP'' = aP''$ for $a^2 = d$. Thus $N_X = 0$. In particular, π is a minimal elliptic fibration with only irreducible fibers. If $C' = \pi^{-1}(P')$ is another rational 0-curve, then $h^*P' = dP'$ since $f^{-1}C' = C'$. Considering the ramification formula for h, we infer that $T \simeq \mathbb{P}^1$ and there exist at most two rational curves on X. If C is the unique rational curve, then π is smooth outside P and the local constant system $R^1\pi_*\mathbb{Z}_X|_{C\setminus P}$ is trivial. The local monodromy corresponding to a singular fiber of type II is of order 6 in $\mathrm{SL}(2,\mathbb{Z})$. This is a contradiction. Hence there is another rational 0-curve $C' = \pi^{-1}(P')$. If C' has a node, then $J(P') = \infty$ for the J-function associated with π. However, π is smooth over $T \setminus \{P, P'\} \simeq \mathbb{C} \setminus \{0\}$. Thus the period function is constant, a contradiction. Hence there remains the case in which C' has a cusp. Let $\mathcal{U}$ and $\mathcal{U}'$ respectively be open discs with centers P and P'. A positive generator of $\pi_1(\mathcal{U} \setminus \{P\}) \simeq \mathbb{Z}$ corresponds to a negative generator of $\pi_1(\mathcal{U}' \setminus \{P'\})$ by the isomorphisms

$$\pi_1(\mathcal{U} \setminus \{P\}) \to \pi_1(T \setminus \{P, P'\}) \leftarrow \pi_1(\mathcal{U}' \setminus \{P'\}).$$

Thus the condition that C is of type II implies that C' is of type II*, a contradiction.

Step 3. X admits no rational curves

Assume the contrary. By *Step 1*, $f^*C = (\deg h)C$ for any rational curve C on X. If $\deg h = 1$, then $N_X = 0$ and f is étale along $f^{-1}C$ for a rational 0-curve C. Here, the mapping degree of $f^{-1}C \to C$ is $\deg f$. However, there exists only one point in $f^{-1}C$ over the node of C. This is a contradiction. Consequently, $\deg h \geq 2$. By the same argument as *Step 2*, we infer that $T \simeq \mathbb{P}^1$ and that the number of singular fibers supported on a union of rational curves is at most 2. Then the period map of π is constant. Hence no singular fibers of type ${}_m\mathrm{I}_b$ with $b > 0$ appear on the relative minimal model of $\pi\colon X \to T$. Therefore, X has no rational curves.

As a result, π is minimal and a singular fiber is a multiple of an elliptic curve. $\square$

The elliptic fibration $\pi\colon X \to T$ above defines a variation of Hodge structure H of weight 1 on T since the local monodromies around the image of singular fibers are trivial. Here, we have $R^1\pi_*\mathbb{Q}_X \simeq H \otimes \mathbb{Q}$ (cf. Lemma 5.4.4 of [18]). Here, $\mathrm{H}^0(T, H) \neq 0$ implies $H \simeq \mathbb{Z}_T^{\oplus 2}$ by Corollary 4.2.5 of [18] (cf. Theorem 11.7 of [9, III]). From Leray's exact sequence

$$0 \to \mathrm{H}^1(T,\mathbb{Q}) \to \mathrm{H}^1(X,\mathbb{Q}) \to \mathrm{H}^0(T, H \otimes \mathbb{Q}) \to \mathrm{H}^2(T,\mathbb{Q}) \to \mathrm{H}^2(X,\mathbb{Q}),$$

we infer that $b_1(X)$ is odd if and only if H is trivial and $\mathrm{H}^2(T,\mathbb{Q}) \to \mathrm{H}^2(X,\mathbb{Q})$ is zero. If $b_1(X)$ is even, then X is Kähler by Miyaoka [12]. Let $\mathcal{L}$ be the invertible sheaf $R^1\pi_*\mathcal{O}_X$. Then $\mathcal{L}$ is isomorphic to the graded piece Gr^0 for the Hodge filtration on $H \otimes \mathcal{O}_T$ and

$\pi_*\omega_X \simeq \omega_T \otimes \mathcal{L}^{-1}$. Moreover, $\mathcal{L}^{\otimes 12} \simeq \mathcal{O}_X$. Then $g(T) \leq q(X) = g(T) + \mathrm{h}^0(T, \mathcal{L}) \leq g(T)+1$ by

$$0 \to \mathrm{H}^1(T, \mathcal{O}_T) \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^0(T, \mathcal{L}) \to 0.$$

Hence $p_g(X) = g(T) - 1 + \mathrm{h}^0(T, \mathcal{L})$ by $\chi(X, \mathcal{O}_X) = 0$. If $\mathrm{h}^0(T, \mathcal{L}) = 0$, then $\mathrm{h}^0(T, H) = 0$, $b_1(X) = 2g(T)$, and X is Kähler. If $\mathrm{h}^0(T, \mathcal{L}) \neq 0$, or equivalently, $\mathcal{L} \simeq \mathcal{O}_T$, then the Weierstrass model [15] associated with H is isomorphic to the product of an elliptic curve and T, and hence H is trivial.

Lemma 4.2. *The induced endomorphism $h\colon T \to T$ is not identical.*

Proof. Assume the contrary. Then f is an endomorphism over T. Let Σ be a set of points $P \in T$ such that π^*P is a multiple fiber. Let m_P be the multiplicity of π^*P. Then we have a finite ramified covering $\tau\colon Z \to T$ such that $\tau^*P = m_P(\tau^*P)_{\mathrm{red}}$ for $P \in \Sigma$ and $g(Z) \geq 2$. Then the normalization of $X \times_T Z$ is smooth over Z and admits a non-trivial surjective endomorphism. Thus we may assume from the first that π is smooth and $g(T) \geq 2$. By considering the étale cyclic covering given by $\mathcal{L}^{\otimes k} \simeq \mathcal{O}_T$, we may also assume that $\mathcal{L} \simeq \mathcal{O}_T$ and hence the variation of Hodge structure H is trivial. Let E be the elliptic curve isomorphic to a fiber of π. We fix a point $0 \in E$ and give a group structure on E whose zero is 0. Let $\mathcal{O}_T(E)$ be the sheaf of germs of holomorphic mappings from T to E. Then we have an exact sequence

$$0 \to H \simeq \mathbb{Z}_T^{\oplus 2} \to \mathcal{O}_T \to \mathcal{O}_T(E) \to 0.$$

There is an element $\eta \in H^1(T, \mathcal{O}_T(E))$ such that π is obtained as the torsor of $E \times T$ over T defined by η. The endomorphism f induces an endomorphism $f_*\colon H \to H$ of variation of Hodge structures which corresponds to

$$H^1(\pi^{-1}(P), \mathbb{Z}) \simeq H_1(\pi^{-1}(P), \mathbb{Z}) \xrightarrow{f_*} H_1(\pi^{-1}(P), \mathbb{Z}) \simeq H^1(\pi^{-1}(P), \mathbb{Z}),$$

where the edge isomorphisms are the Poincaré duals. The endomorphism $f_*\colon E \to E$ keeps 0 and is the multiplication by a complex number λ. If we identify E as the quotient of $\mathbb{C}$ by the lattice $L_\theta = \mathbb{Z}\theta + \mathbb{Z}$ for some $\theta \in \mathbb{H}$, then $\lambda L_\theta \subset L_\theta$. Hence $1 \neq \lambda \in \mathbb{Z}$ or $\mathbb{Q}(\lambda)$ is an imaginary quadratic field. In the latter case, $t = \lambda + \overline{\lambda}$ and $d = |\lambda|^2$ are integers with $1 - t + d \neq 0$. The cohomology class η satisfies $\lambda_*\eta = \eta$. Hence $(\lambda - 1)\eta = 0$ or $(1 - t + d)\eta = 0$. Thus η is torsion, which implies that π is projective. This contradicts $a(X) = 1$. □

Corollary 4.3. $g(T) \leq 1$. *If $g(T) = 1$, then π is smooth.*

Proof. If $g(T) \geq 2$, then $h^k = \mathrm{id}_T$ for some $k > 0$. If $g(T) = 1$ and if there exists a multiple fiber $F = \pi^{-1}(P)$, then $\pi^{-1}(Q)$ is also multiple for any $Q \in h^{-1}(P)$, since $h\colon T \to T$ is étale. Thus h is isomorphic and h^k keeps P for some $k > 0$ since the set of multiple fibers is finite. Hence h^{kl} is identical for some $l > 0$, since the group of automorphisms of E keeping P is finite. □

Lemma 4.4. *Let $\pi\colon X \to T$ be an elliptic surface of class* VII_0. *Then $T \simeq \mathbb{P}^1$ and singular fibers are multiple of elliptic curves. In particular, $K_X \sim_{\mathbb{Q}} \pi^*(K_T + \Theta)$ for an effective $\mathbb{Q}$-divisor $\Theta = \sum(1 - m_i^{-1})P_i$ on T, where m_i is the multiplicity of the fiber $\pi^* P_i$. Furthermore the following assertions hold*:

1. *If $\deg \Theta > 2$, then any surjective endomorphism of X is isomorphic.*
2. *If $\deg \Theta = 2$, then X is a secondary Kodaira surface.*
3. *If $\deg \Theta < 2$, then X is an elliptic Hopf surface.*

Proof. T is rational by $b_1(T) \le b_1(X) = 1$. The variation of Hodge structure is trivial since $e(X) = 0$ and $\pi_1(T) = \{1\}$. Let $\Sigma = \{P_1, P_2, \ldots\}$ be the set of points P such that π^*P is a multiple fiber. Then $\pi^*P_i = m_iC_i$ for an elliptic curve C_i and $m_i \ge 2$. We assume that $m_1 \le m_2 \le \cdots$ Then

$$K_X \sim \pi^*K_T + \sum(m_i - 1)C_i \sim_{\mathbb{Q}} \pi^*(K_T + \Theta), \quad \text{for} \quad \Theta = \sum\left(1 - \frac{1}{m_i}\right)P_i.$$

In particular, $\kappa(X) = 1,\ 0,\ -\infty$ according as $\deg\Theta > 2,\ = 2,\ < 2$.

Suppose that $\kappa(X) = 1$. Let f be a surjective endomorphism of X and h the induced endomorphism of T with $\pi \circ f = h \circ \pi$. Then f is étale by Lemma 3.1. Thus $K_X \sim f^*K_X$ implies that $K_T + \Theta \sim_{\mathbb{Q}} h^*(K_T + \Theta)$. Thus h is an automorphism keeping the set Σ which consists at least three points. Hence some power h^k is identical and f is isomorphic by Lemma 4.2.

Suppose that $\kappa(X) = 0$. Then $(m_1, m_2, \ldots)$ is one of the followings:

$$(2,2,2,2), \quad (2,3,6), \quad (2,4,4), \quad (3,3,3).$$

In each case, there is a cyclic covering $\tau\colon A \to T$ from an elliptic curve such that $\tau^*P_i = m_i(\tau^*P_i)_{\mathrm{red}}$ for any i and that τ is étale outside Σ. Moreover, for a suitable choice of group structure of A, a generator of the Galois group of τ is given as the multiplication map $z \mapsto \alpha z$ by a primitive root α of unity of order 2, 6, 4, 3 according as $(2,2,2,2)$, $(2,3,6)$, $(2,4,4)$, $(3,3,3)$ above. The normalization Y of the fiber product $X \times_T A$ is smooth over A and étale over X. Hence Y is a primary Kodaira surface and X is secondary.

Finally suppose that $\deg\Theta < 2$. If $\Sigma \ne \emptyset$, then $(m_1, m_2, \ldots)$ is one of the followings:

$$(m_1), \quad (m_1, m_2), \quad (2,2,m_3), \quad (2,3,3), \quad (2,3,4), \quad (2,3,5).$$

If $\sharp\Sigma \le 2$, then X is Hopf by Lemma 8 of [10] (cf. Fact 7.2 below). If $\sharp\Sigma = 3$, then there is a finite Galois covering $\tau\colon \Gamma \to T$ from a non-singular rational curve Γ such that $\tau^*P_i = m_i(\tau^*P_i)_{\mathrm{red}}$ for any i and that τ is étale outside Σ. Moreover, τ is isomorphic to the quotient morphism by the standard action of the following finite group $G \subset \mathrm{Aut}(\Gamma)$ according to $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$: the dihedral group D_n of order $2n$, the tetrahedral group $\mathfrak{A}_4$, the octahedral group $\mathfrak{S}_4$, and the icosahedral group $\mathfrak{A}_5$. The normalization Y of the fiber product $X \times_T \Gamma$ is smooth over Γ and étale over X. Hence X is also a Hopf surface since Y is so. $\square$

THEOREM 4.5. *Let X be a non-singular compact complex surface admitting a non-trivial surjective endomorphisms. If $a(X) = 1$, then X is a complex torus, a primary Kodaira surface, a secondary Kodaira surface or an elliptic Hopf surface.*

Proof. Assume that $g(T) = 1$. If H is not trivial, then X is Kähler and $p_g(X) = 0$. This implies that X is projective, a contradiction. Hence H is trivial. Thus $\omega_X \simeq \mathcal{O}_X$ and $3 \leq b_1(X) \leq 4$. If $b_1(X) = 4$, then X is a complex torus. If $b_1(X) = 3$, then X is a primary Kodaira surface.

Next assume that $g(T) = 0$. Then $\mathcal{L} \simeq \mathcal{O}_T$ and H is trivial. In particular, $p_g(X) = 0$ and $q(X) = 1$. Thus X is a surface of class VII_0. It is a Hopf surface or a secondary Kodaira surface by Lemma 4.4. □

APPENDIX TO SECTION 4

The existence of non-trivial surjective endomorphisms on an algebraic surface X with $\kappa(X) = 1$, $e(X) = 0$ is proved in Proposition 3.3 of [3] by using the ∂-étale cohomology theory developed in [18]. Here, we shall give a more geometric proof.

Let $\pi\colon X \to T$ be the elliptic fibration obtained as the Iitaka fibration. Let Σ be the set of points $P \in T$ such that π^*P is a multiple fiber of multiplicity $m_P \geq 2$. Then $K_X \sim_{\mathbb{Q}} \pi^*(K_T + \Theta)$ for the $\mathbb{Q}$-divisor $\Theta = \sum_{P\in\Sigma}(1 - m_P^{-1})P$ as in Lemma 4.4. Note that $\deg(K_T + \Theta) > 0$ by $\kappa(X) = 1$. By applying Theorem 4.2 of [16], we have a finite Galois covering $Z \to T$ such that the normalization Y of $X \times_T Z$ is isomorphic to the product $C \times Z$ over Z for an elliptic curve C and is étale over X. We consider C as the torus $\mathbb{C}/L$ for the lattice $L = \mathbb{Z}\tau + \mathbb{Z}$ with $\operatorname{Im}\tau > 0$. We denote by $[x]$ the image of $x \in \mathbb{C}$ under $\mathbb{C} \to C$. Let G be the Galois group. Then the induced action of $g \in G$ on $Y \simeq C \times Z$ is written by

$$([x], z) \mapsto ([a_g x] + b_g(z),\ g \cdot z)$$

for some $a_g \in \mathbb{C}^\star$ and some holomorphic mapping $b_g\colon Z \to C$. Here, $\{a_g\}$ gives rise to a homomorphism $G \to \mathbb{C}^\star$ and L is a G-submodule of $\mathbb{C}$. In particular, the complex torus C is a G-module. The set $\operatorname{Hom}(Z, C)$ of holomorphic maps $\varphi\colon Z \to C$ also has a right G-module structure by $\varphi^g(z) = a_g^{-1}\varphi(g \cdot z)$. By the relation $a_g b_h(z) + b_g(h \cdot z) = b_{gh}(z)$ for g, $h \in G$, we infer that $\{a_g^{-1} b_g\}$ defines an element of $H^1(G, \operatorname{Hom}(Z, C))$. Since the cohomology group is torsion, there exist a positive integer n and a holomorphic mapping $c\colon Z \to C$ such that

$$n a_g^{-1} b_g(z) = c(z) - a_g^{-1} c(g \cdot z)$$

for any $g \in G$. The endomorphism $C \times Z \to C \times Z$ given by

$$([x], z) \mapsto ((n + 1)[x] + c(z), z)$$

commutes with the action of G on $C \times Z$. Thus it induces a non-trivial surjective endomorphism on X.

5. The case of algebraic dimension zero

Let X be a non-singular compact complex surface of $a(X) = 0$ admitting a non-trivial surjective endomorphism. Suppose that X is Kähler. Then $\kappa(X) = 0$. Thus the endomorphism is étale and hence X admits no negative curves. Hence X is minimal and is a complex torus. A complex torus admits a non-trivial surjective endomorphism as the multiplication map by an integer greater than 1.

Thus we assume that X is non-Kähler. Then X belongs to the class VII. We have $(K_X + N_X)^2 = 0$ by (3.1). Thus $p_a(D) = 1$ for any connected component D of N_X. Moreover, $K_X^2 = N_X^2 = \sum D_\lambda^2$ for the decomposition $N_X = \sum D_\lambda$ into the connected components.

Lemma 5.1. *If D is a reduced divisor with $(K_X + D) \cdot D = 0$, then D has at most two connected components.*

Proof. Since $a(X) = 0$, we have $\mathrm{h}^0(X, \mathcal{O}_X(K_X + D)) = \mathrm{h}^2(X, \mathcal{O}_X(-D)) \leq 1$. Hence $\mathrm{h}^1(X, \mathcal{O}_X(-D)) \leq 1$ by $(K_X + D) \cdot D = 0$. The exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(X, \mathcal{O}_D) \to \mathrm{H}^1(X, \mathcal{O}_X(-D))$$

implies $\mathrm{h}^0(D, \mathcal{O}_D) \leq 2$. □

Theorem 5.2. *Let X be a non-Kähler surface of $a(X) = 0$ admitting a non-trivial surjective endomorphism. Then the minimal model of X is one of the following surfaces: a parabolic Inoue surface; a hyperbolic Inoue surface; a half Inoue surface; a Hopf surface; an Inoue surface with no curves. Moreover, X is obtained from the minimal model by a succession of blowups whose centers are nodes of curves.*

Proof. One of the following cases occurs by Lemma 5.1:

Case 1. N_X has two connected components;
Case 2. N_X is connected;
Case 3. X contains a 0-curve but no negative curves;
Case 4. X contains no curves.

Let Y be the minimal model of X and let $\mu\colon X \to Y$ be the contraction. Then the endomorphism of X descends to Y and $N_Y \leq \mu_* N_X$.

Case 1. Any curve on X is contained in N_X by Lemma 5.1. Thus $f^*(K_X + N_X) \sim K_X + N_X$. We have $\mathrm{h}^2(X, \mathcal{O}_X(-N_X)) = 1$ by the exact sequence

$$\mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(N_X, \mathcal{O}_{N_X}) \to \mathrm{H}^2(X, \mathcal{O}_X(-N_X)) \to \mathrm{H}^2(X, \mathcal{O}_X) = 0.$$

Thus $K_X + N_X \sim E$ for an effective divisor E. Here $f^*E = E$. Therefore, $E = 0$, equivalently, $K_X + N_X \sim 0$. Let D_1 and D_2 be the two connected components of $\mu_* N_X \sim -K_Y$. Then $p_a(D_i) = 1$ for $i = 1, 2$. By Lemma (2.11) of [13], D_1 is an elliptic curve if and only if $D_2^2 = 0$. Hence, if $D_2^2 = 0$, then $D_1^2 < 0$. Otherwise, D_1 and D_2 are both elliptic curves and $\mu_* N_X$ has no nodes, which implies that μ is isomorphic and $N_X = 0$, a contradiction. Therefore if $D_1^2 = 0$ or $D_2^2 = 0$, then Y is a parabolic Inoue surface by Theorem (7.1) of [13] (cf. (7.12) of [13], [2]). If $D_1^2 < 0$ and $D_2^2 < 0$, then $N_Y = \mu_* N_X$ and Y is a parabolic Inoue surface by Theorem (8.1) of [13]. In both cases, $\mu \colon X \to Y$ is a successive blowups whose centers are nodes.

Case 2. Suppose that there is a curve C not contained in N_X. Then any curve on X is contained in $C \cup N_X$. The contraction $\mu \colon X \to Y$ is isomorphic along C. Since $C^2 = 0$, then $\mu_* N_X$ is an elliptic curve and C is a rational curve with a node by Lemma (2.11) of [13]. Then μ is isomorphic and X is a parabolic Inoue surface of $b_2 = 1$ by [2] or by Theorem (7.1) of [13].

Next suppose that any curve on X is contained in N_X. Then $f^*(K_X + N_X) \sim K_X + N_X$. Moreover, f induces a finite étale endomorphism on the complement $U = X \setminus N_X$. Therefore, $e(U) = 0$. Thus $e(X) = e(N_X)$. If N_X is an elliptic curve, then $-N_X^2 = -K_X^2 = e(X) = 0$, a contradiction. Thus N_X is a cycle of rational curves. Here, $e(X) = e(N_X)$ is equivalent to $b_2(X) = b_2(N_X)$. Thus $b_2(Y) = b_2(\mu_* N_X)$. If $N_Y \neq 0$, then $N_Y = \mu_* N_X$ and Y is a half Inoue surface by [13]. If $N_Y = 0$, then $\mu_* N_X$ is a rational curve with a node. This case does not occur by the argument in *Case 3* below. Therefore, X is obtained as a successive blowups of a half Inoue surface whose centers are nodes.

Case 3. We have $b_2(X) = e(X) = -K_X^2 = -(K_X + N_X)^2 = 0$. By Lemma (2.11) of [13], one of the following three possibilities remain:

(1) X contains two elliptic curves;
(2) X contains an elliptic curve as a unique curve.
(3) X contains a rational curve with a node as a unique curve.

For the complement U of the union of all the curves on X, we have $e(U) = 0$ since f induces a finite étale endomorphism on U. Hence the case (3) does not occur by $e(X \setminus U) = 0$. In the cases (1), (2), X is a Hopf surface by Lemma 8 of [10].

Case 4. Since $b_2(X) = -K_X^2 = 0$, X is one of Inoue surfaces without curves by [5], [11], [21]. □

6. Kodaira surfaces

A primary Kodaira surface X is defined as a surface with $K_X \sim 0$, $b_1(X) = 3$. The algebraic reduction $\pi \colon X \to T$ is an elliptic fibration over an elliptic curve T. This is smooth by $e(X) = 0$ and $K_X \sim 0$. Moreover the associated variation of Hodge structure H is trivial since $\pi_* \omega_{X/T} \simeq \mathcal{L}^{-1} \simeq \mathcal{O}_T$. For a fiber E, we fix a point 0 and give an abelian group structure on E with 0 being the identity. Then as in the proof of Lemma 4.2, $X \simeq (E \times T)^\eta$ as a torsor corresponding to some $\eta \in \mathrm{H}^1(T, \mathcal{O}_T(E))$, where $\mathcal{O}_T(E)$ is the sheaf of germs of holomorphic mappings from T to E. The image of η under $\mathrm{H}^1(T, \mathcal{O}_T(E)) \to \mathrm{H}^2(T, H) = \mathrm{H}^2(T, \mathbb{Z}^2)$ is not zero, since X is non-Kähler.

Let L_τ denote the lattice $\mathbb{Z}\tau + \mathbb{Z} \subset \mathbb{C}$ for $\tau \in \mathbb{H}$. We fix τ, $\theta \in \mathbb{H}$ and isomorphisms $T \simeq \mathbb{C}/L_\tau$, $E \simeq \mathbb{C}/L_\theta$. For $c \in L_\theta$ and $\delta \in \mathbb{C}$, let us consider the following automorphisms of $\mathbb{C} \times E$:

$$g_1 \colon (z, [\zeta]) \mapsto (z + \tau, [\zeta + cz + \delta]), \quad \text{and} \quad g_2 \colon (z, [\zeta]) \mapsto (z + 1, [\zeta]),$$

where $[\zeta]$ denotes $\zeta \mod L_\theta$. The quotient space of $\mathbb{C} \times E$ by g_1 and g_2 is denoted by $X_{c,\delta}$. Let $\pi \colon X_{c,\delta} \to T$ denote the induced smooth elliptic fibration from the first projection $\mathbb{C} \times E \to \mathbb{C}$.

Lemma 6.1. *A primary Kodaira surface is isomorphic to $X_{c,\delta}$ for some $c \neq 0$ and δ.*

Proof. We have an isomorphism $\mathrm{H}^1(T, \mathcal{O}_T(E)) \simeq \mathrm{H}^1(L_\tau, \mathrm{H}^0(\mathbb{C}, \mathcal{O}(E)))$ by Hochschild–Serre spectral sequence for the universal covering map $\mathbb{C} \to T$. Thus the cohomology class η is represented by a cocycle $\{x_u = x_u(z)\}$ of holomorphic functions on $\mathbb{C}$ for $u \in L_\tau$ such that $x_{u+v}(z) \equiv x_v(z) + x_u(z + v) \mod L_\theta$. Here, X is isomorphic to the quotient space of $\mathbb{C} \times E$ by the following action of $u \in L_\tau$:

$$(z, [\zeta]) \mapsto (z + u, [\zeta + x_u(z)]).$$

Thus we shall find a simple form of $x_u(z)$ up to coboundary. Note that $\{x_u\}$ is determined only by x_1 and x_τ which satisfy

$$x_\tau(z + 1) - x_\tau(z) \equiv x_1(z + \tau) - x_1(z) \mod L_\theta \tag{6.1}$$

We know that $\dim \mathrm{H}^1(T, \mathcal{O}_T) = 1$ and $\mathrm{H}^1(T, \mathbb{C}) \to \mathrm{H}^1(T, \mathcal{O}_T)$ is surjective. The homomorphism is isomorphic to $\mathrm{H}^1(L_\tau, \mathbb{C}) \to \mathrm{H}^1(L_\tau, \mathrm{H}^0(\mathbb{C}, \mathcal{O}))$. Hence, for a cocycle $\{y_u(z)\}$ of holomorphic functions on $\mathbb{C}$ satisfying $y_{u+v}(z) = y_v(z) + y_u(z + v)$, there exist constant c_1, c_2, and a holomorphic function $h(z)$ such that $y_1(z) = c_2 + h(z + 1) - h(z)$, $y_\tau(z) = c_1 + h(z + \tau) - h(z)$. Since $c_2(z + 1) - c_2 z = c_2$, we may assume $c_2 = 0$.

Applying the observation above to $(\mathrm{d}/\,\mathrm{d}z)x_u$, we have constant c, δ, and a holomorphic function $\phi(z)$ such that $x_\tau(z) = cz + \delta + \phi(z + \tau) - \phi(z)$ and $x_1(z) = \phi(z + 1) - \phi(z)$. The condition (6.1) is equivalent to $c \in L_\theta$. Hence $X \simeq X_{c,\delta}$. The homomorphism $\mathrm{H}^1(T, \mathcal{O}(E)) \to \mathrm{H}^2(T, H)$ is isomorphic to

$$\mathrm{H}^1(L_\tau, \mathrm{H}^0(\mathbb{C}, \mathcal{O}(E))) \to \mathrm{H}^2(L_\tau, L_\theta) \simeq L_\theta,$$

which sends η to c. Hence $c \neq 0$. □

Definition 6.2. (1) For three complex numbers x_1, x_2, x_3, let $T(x_1, x_2, x_3)$ denote the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_3 & x_2 & 1 \end{pmatrix}.$$

The matrices above form a subgroup of $\mathrm{GL}(3, \mathbb{C})$, which is denoted by $T_3(\mathbb{C})$.

(2) Let $D\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ be the skew symmetric form defined by

$$D((x_1, x_2), (x'_1, x'_2)) = x_1x'_2 - x'_1x_2.$$

(3) Let $\Delta_3(\mathbb{C})$ be the following group structure on $(\xi, y) \in \mathbb{C}^2 \times \mathbb{C}$:

$$(\xi, y) * (\xi', y') := (\xi + \xi', y + y' - (1/2)D(\xi, \xi')).$$

Note that the l-th power $(\xi, y)^l$ is equal to $(l\xi, ly)$ for $l \in \mathbb{Z}$, $(\xi, y) \in \Delta_3(\mathbb{C})$. There is an isomorphism $T_3(\mathbb{C}) \to \Delta_3(\mathbb{C})$ given by

$$T(x_1, x_2, x_3) \mapsto ((x_1, x_2), x_3 - (1/2)x_1x_2). \tag{6.2}$$

We have a homomorphism $\pi_1(X_{c,\delta}) \to T_3(\mathbb{C})$ by

$$g_1 \mapsto T(\tau, c, \delta), \quad g_2 \mapsto T(1, 0, 0), \quad g_3 \mapsto T(0, 0, \theta), \quad g_4 \mapsto T(0, 0, 1).$$

Therefore the composite $\pi_1(X_{c,\delta}) \to \Delta_3(\mathbb{C})$ is written by

$$g_1^{l_1}g_2^{l_2}g_3^{l_3}g_4^{l_4} \mapsto ((l_1\tau + l_2, l_1c), l_1\varepsilon + (1/2)l_1l_2c + l_3\theta + l_4), \quad \text{where} \quad \varepsilon := \delta - (1/2)c\tau.$$

Definition 6.3. (1) For a free abelian group L of finite rank and for $c \in L$, let $L[c/2]$ denotes the abelian group $L + \mathbb{Z}(c/2) \subset L \otimes \mathbb{Q}$.

(2) Let $D_\tau\colon L_\tau \times L_\tau \to \mathbb{Z}$ be the skew symmetric form defined by

$$D_\tau(m_1\tau + m_2, m'_1\tau + m'_2) = m_1m'_2 - m'_1m_2.$$

In other expressions,

$$D_\tau(x, y) = \frac{1}{\tau - \overline{\tau}}(x\overline{y} - \overline{x}y) = \frac{\operatorname{Im}(x\overline{y})}{\operatorname{Im}\tau}.$$

(3) For $c \in L_\theta$, let Π_c be the following group defined on $L_\tau \times L_\theta[c/2]$:

$$(x, y) * (x', y') := (x + x', y + y' + (c/2)D_\tau(x, x')).$$

Note that $D_\tau(x, 1) = \operatorname{Im} x / \operatorname{Im}\tau$ and $x = D_\tau(x, 1)\tau - D_\tau(x, \tau)$ for $x \in L_\tau$.

We have homomorphisms $\Pi_c \to \Delta_3(\mathbb{C})$ and $\pi_1(X_{c,\delta}) \to \Pi_c$, respectively, by

$$(x, y) \mapsto ((x,\ D_\tau(x, 1)c),\ y + D_\tau(x, 1)\varepsilon), \quad \text{and}$$
$$g_1^{l_1}g_2^{l_2}g_3^{l_3}g_4^{l_4} \mapsto (l_1\tau + l_2,\ l_3\theta + l_4 + (1/2)l_1l_2c).$$

Then we have the commutative diagram

$$\begin{array}{ccc} \pi_1(X_{c,\delta}) & \longrightarrow & \Pi_c \\ \Big\downarrow & & \Big\downarrow \\ T_3(\mathbb{C}) & \xrightarrow{\simeq} & \Delta_3(\mathbb{C}). \end{array}$$

The image of the injection $\pi_1(X_{c,\delta}) \hookrightarrow \Pi_c$ consists of all the elements (x, y) such that $y + (c/2)D_\tau(x, 1)D_\tau(x, \tau) \in L_\theta$. In particular, Π_c is generated by $\pi_1(X_{c,\delta})$ and $(0, c/2)$. The group Π_c acts on $\mathbb{C} \times \mathbb{C}$ by

$$(z, \zeta) \mapsto (z + x,\ \zeta + D_\tau(x, 1)cz + y + D_\tau(x, 1)(\varepsilon + (1/2)cx))$$

for $(x, y) \in \Pi_c$.

Proposition 6.4. *Let $f\colon X_{c,\delta} \to X_{c,\delta}$ be a surjective endomorphism and let $h\colon T \to T$ be the induced endomorphism with $\pi \circ f = h \circ \pi$. Suppose that*

$$h_*\colon \mathrm{H}^0(T, \Theta_T) \to \mathrm{H}^0(T, \Theta_T)$$

is the multiplication by $\alpha \in \mathbb{C}$ with $\alpha L_\tau \subset L_\tau$. Then f is induced from the automorphism

$$\Phi_{\alpha,v}\colon (z, \zeta) \mapsto (\alpha z + (1/c)(\alpha - 1)\varepsilon, |\alpha|^2\zeta + \varphi_{\alpha,v}(z))$$

of $\mathbb{C} \times \mathbb{C}$ for a holomorphic function

$$\varphi_{\alpha,v}(z) = \alpha D_\tau(\alpha, 1)\left(\frac{c}{2}z^2 + \varepsilon z\right) + v,$$

for $v \in \mathbb{C}$.

Proof. A lift Φ of f to $\mathbb{C} \times \mathbb{C}$ is written by $(z, \zeta) \mapsto (\alpha z + \beta, F(z, \zeta))$ for a holomorphic function $F(z, \zeta)$ and for a constant β. Here, $F(z, \zeta) = \rho\zeta + \varphi(z)$ for a holomorphic function $\varphi(z)$ and a constant ρ since $F(z, \zeta) \mod L_\theta$ depends only on $\zeta \mod L_\theta$. The endomorphism $f_*\colon \pi_1(X_{c,\delta}) \to \pi_1(X_{c,\delta})$ is induced from $g \mapsto \Phi \circ g \circ \Phi^{-1}$ and lifts to an endomorphism of Π_c. The image of $(x, y) \in \Pi_c$ is $(\alpha x, y_1)$ for some $y_1 \in L_\theta[c/2]$ in which the following equation holds:

$$\begin{aligned}(6.3)\quad & \rho\Big(D_\tau(x, 1)cz + y + D_\tau(x, 1)(\varepsilon + (1/2)cx)\Big) + \varphi(z + x) \\ &\qquad = \varphi(z) + D_\tau(\alpha x, 1)c(\alpha z + \beta) + y_1 + D_\tau(\alpha x, 1)(\varepsilon + (1/2)c\alpha x).\end{aligned}$$

By using $\mathrm{Im}(\alpha x) = x \,\mathrm{Im}\,\alpha + \overline{\alpha}\,\mathrm{Im}\,x$, we have $D_\tau(\alpha x, 1) = xD_\tau(\alpha, 1) + \overline{\alpha}D_\tau(x, 1)$, and

$$\begin{aligned}\varphi(z + x) - \varphi(z) = &\Big(D_\tau(\alpha, 1)\alpha x + D_\tau(x, 1)(|\alpha|^2 - \rho)\Big)\, cz + y_1 - \rho y \\ &+ (c/2)D_\tau(\alpha, 1)\alpha x^2 + D_\tau(\alpha, 1)(c\beta + \varepsilon)x \\ &+ D_\tau(x, 1)(\overline{\alpha}c\beta + (\overline{\alpha} - \rho)\varepsilon) + (cx/2)D_\tau(x, 1)(|\alpha|^2 - \rho).\end{aligned}$$

Hence $\varphi''(z)$ is a constant equal to $D_\tau(\alpha, 1)c\alpha$ and $\rho = |\alpha|^2$. If we write $\varphi(z) = (c\alpha/2)D_\tau(\alpha, 1)z^2 + uz + v$ for constants u, v, then

$$ux = D_\tau(\alpha, 1)(c\beta + \varepsilon)x + D_\tau(x, 1)\overline{\alpha}(c\beta + (1 - \alpha)\varepsilon) + y_1 - |\alpha|^2 y.$$

Therefore, $c\beta = (\alpha - 1)\varepsilon$ and $u = D_\tau(\alpha, 1)\alpha\varepsilon$. □

Theorem 6.5. *A primary Kodaira surface and a secondary Kodaira surface admit a non-trivial surjective endomorphism.*

Proof. For the primary Kodaira surface $X_{c,\delta}$, the morphism $\Phi_{l,0}$ for $l > 1$ induces a non-trivial surjective endomorphism of degree l^6. Let Y be a secondary Kodaira surface. Then by Lemma 4.4, there is a cyclic étale covering $X_{c,\delta} \to Y$ for some c, δ. Then a generator of the cyclic group acts on $X_{c,\delta}$ as $\Phi_{\alpha,v}$ for a root α of unity and for some $v \in \mathbb{C}$. We may assume $\alpha = \exp(2\pi\sqrt{-1}/k)$, where $k = 2$, 3, 4, or 6. The order of $\Phi_{\alpha,v}$ is just k. If $k > 2$, then $\tau \in \mathrm{SL}(2,\mathbb{Z})\alpha$ for the fractionally linear action on $\mathbb{H}$. Hence, we may assume $\alpha = \tau$ if $k > 2$. For $l \in \mathbb{Z}$, we define $v[l]$ by $\Phi^l_{\alpha,v} = \Phi_{\alpha^l, v[l]}$. Then

$$v[l] = lv + \alpha D_\tau(\alpha, 1)\frac{\varepsilon^2}{2c}\left(-l + \sum_{i=0}^{l-1}\alpha^{2i}\right)$$

for $l \geq 0$. In fact, it follows from that the term v_3 in the formula $\Phi_{\alpha_1,v_1} \circ \Phi_{\alpha_2,v_2} = \Phi_{\alpha_1\alpha_2,v_3}$ is calculated by

$$v_3 = v_1 + |\alpha_1|^2 v_2 + \alpha_1 D_\tau(\alpha_1, 1)\frac{\alpha_2^2 - 1}{2c}\varepsilon^2.$$

If $k = 2$, then $v[2] = 2v \in L_\theta$. If $k > 2$, then $\tau = \alpha$ implies $D_\tau(\alpha, 1) = 1$ and

$$v[k] = k\left(v - \frac{\alpha\varepsilon^2}{2c}\right) \in L_\theta.$$

Let l be an integer with $l > 1$ and $l^2 \equiv 1 \mod k$. Then, for w_1 and w_2 defined by $\Phi_{l,0} \circ \Phi_{\alpha,v} = \Phi_{l\alpha,w_1}$ and $\Phi_{\alpha,v} \circ \Phi_{l,0} = \Phi_{l\alpha,w_2}$, we have

$$w_2 - w_1 = \left(v + \alpha D_\tau(\alpha, 1)\frac{l^2 - 1}{2c}\varepsilon^2\right) - l^2 v = -\frac{l^2 - 1}{k}v[k] \in L_\theta.$$

Hence $\Phi_{l,0}$ induces a non-trivial surjective endomorphism on the quotient space Y. $\square$

7. Hopf surfaces

In this section, we shall prove the following:

Theorem 7.1. *A Hopf surface admits a non-trivial surjective endomorphism if and only if it has at least two elliptic curves.*

We set W to be the open set $\mathbb{C}^2 \setminus \{(0,0)\}$ and (z_1, z_2) to be a coordinate system of $\mathbb{C}^2$. A Hopf surface is a compact complex surface whose universal covering space is biholomorphic to W by definition. We write the function $\exp(2\pi\sqrt{-1}z)$ by $\mathrm{e}(z)$.

First, we treat the case of elliptic Hopf surfaces with at most two singular fibers.

Let m_1, m_2, n be positive integers such that $\gcd(m_1, m_2) = \gcd(n, m_1) = \gcd(n, m_2) = 1$ and let τ be a complex number in $\mathbb{H}$. Let $Y = Y(\tau, m_1, m_2, n)$ be the quotient space of W by the following two actions:

$$A\colon (z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2), \quad B\colon (z_1, z_2) \mapsto (\varepsilon_1 z_1, \varepsilon_2 z_2),$$

where $\alpha_i = \mathrm{e}(m_i\tau)$, $\varepsilon_i = \mathrm{e}(m_i/n)$, for $i = 1$, 2. Then Y is an elliptic Hopf surface over $\mathbb{P}^1$ and smooth over $\mathbb{P}^1 \setminus \{0, \infty\}$ by the morphism $(z_1, z_2) \mapsto (z_1^{m_2} : z_2^{m_1})$. The multiplicities of the fibers over $0 = (1 : 0)$ and $\infty = (0 : 1)$ are m_1 and m_2, respectively. Conversely, we know the following result by Kodaira (cf. Lemma 8 of [10, II]):

Fact 7.2. *Let $Y \to \mathbb{P}^1$ be an elliptic Hopf surface smooth outside $\{0, \infty\}$. Let m_1 and m_2 be the multiplicities of the fibers over 0 and ∞, respectively. Suppose that* $\gcd(m_1, m_2) = 1$. *Then $Y \simeq Y(\tau, m_1, m_2, n)$ for some τ and n.*

In particular, if $Y \to \mathbb{P}^1$ is smooth, then $Y \simeq Y(\tau, 1, 1, n)$, which is obtained by the actions

$$A\colon (z_1, z_2) \mapsto \rho(z_1, z_2) = (\rho z_1, \rho z_2), \quad B\colon (z_1, z_2) \mapsto \mathrm{e}(1/n)(z_1, z_2),$$

for $\rho = \mathrm{e}(\tau)$. We write $Y(\tau, 1, 1, n)$ by $Y(\rho, n)$.

Proposition 7.3. *Let $\pi\colon X \to T$ be an elliptic Hopf surface with at most two singular fibers. Then X admits a non-trivial surjective endomorphism.*

Proof. We may assume that π is smooth outside $\{0, \infty\} \subset \mathbb{P}^1 = T$. Let m_1 and m_2 be the multiplicities of the fibers over 0 and ∞, respectively. Let $\Gamma = \mathbb{P}^1 \to T = \mathbb{P}^1$ be the cyclic covering of degree $k = \gcd(m_1, m_2)$ branched at $\{0, \infty\}$. Then the normalization Y of $X \times_T \Gamma$ is an elliptic Hopf surface étale over X. Moreover, $Y \simeq Y(\tau, m_1/k, m_2/k, n)$ for some τ and n by Fact 7.2. A generator of the cyclic Galois group acts on $\Gamma = \mathbb{P}^1$ by $(t_1 : t_2) \mapsto (t_1 : \mathrm{e}(1/k)t_2)$. This lifts to an automorphism of W written by

$$C\colon (z_1, z_2) \mapsto (u(z_1, z_2)^{m_1} z_1,\ u(z_1, z_2)^{m_2}\, \mathrm{e}(l/m_1)\, \mathrm{e}(1/km_1) z_2)$$

for a unit function $u\colon \mathbb{C}^2 \to \mathbb{C}^\star$ and for an integer l. We shall show that u is constant. Since it induces an automorphism of Y, there is an integer q such that

$$\begin{aligned} u(\alpha z_1, \alpha z_2)^{m_1} \alpha_1 &= \mathrm{e}(m_1/n)^q \alpha_1^{\pm} u(z_1, z_2)^{m_1}, \\ u(\alpha z_1, \alpha z_2)^{m_2} \alpha_2 &= \mathrm{e}(m_2/n)^q \alpha_2^{\pm} u(z_1, z_2)^{m_2}, \end{aligned}$$

for any $(z_1, z_2) \in W$. Substituting $(z_1, z_2) = (0, 0)$, we have

$$u(\alpha z_1, \alpha z_2) = \mathrm{e}(q/n) u(z_1, z_2).$$

Then u is constant by

$$|u(z)| = \lim_{p \to \infty} |u(\alpha^p z_1, \alpha^p z_2)| = |u(0, 0)|.$$

Let $\Phi\colon (z_1, z_2) \mapsto (z_1^d, z_2^d)$ be an endomorphism of W for $d > 1$. Then $\Phi \circ A = A^d \circ \Phi$, $\Phi \circ B = B^d \circ \Phi$, and $\Phi \circ C = C^d \circ \Phi$. Hence Φ induces non-trivial surjective endomorphisms on Y and on X. $\square$

Secondly, we treat the case of elliptic Hopf surfaces with at least three multiple fibers. Let $G \subset \mathrm{PGL}(2, \mathbb{C}) \simeq \mathrm{Aut}(\mathbb{P}^1)$ be a finite subgroup and let $\widetilde{G} \subset \mathrm{SL}(2, \mathbb{C})$ be the pull-back by $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PGL}(2, \mathbb{C})$. We denote by $A(g)$ the matrix in $\mathrm{SL}(2, \mathbb{C})$ corresponding to $g \in \widetilde{G}$. We also denote by 1 the unit element of $\widetilde{G}$ and by -1 the element corresponding to the minus of the unit matrix. Note that G is a cyclic group or one of the regular polyhedral groups. We choose $\tau \in \mathbb{H}$ such that $\rho = \mathrm{e}(\tau)$. Let $\chi_i\colon \widetilde{G} \to \mathbb{C}^\star$ be group homomorphisms (characters) for $i = 0$, 1. Let us choose $\psi_i(g) \in \mathbb{Q}$ satisfying $\mathrm{e}(\psi_i(g)) = \chi_i(g)$. We define

$$\varphi(g) := \mathrm{e}(\psi_1(g)\tau + \psi_0(g)(1/m)).$$

An action of $\widetilde{G}$ on $Y(\rho, m)$ is well-defined by the maps

$$(z_1, z_2) \mapsto \varphi(g)(z_1, z_2)\, {}^{\mathrm{t}}\!A(g)$$

for $g \in \widetilde{G}$. Thus, an extension $\widetilde{G}_{m,\chi}$ of the finite group $\widetilde{G}$ by $\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ acts on W. For the action of $g \in \widetilde{G}$ on $Y(\rho, m)$, it has a fixed point if and only if $\varphi(g)\rho^k \,\mathrm{e}(i/m)$ is an eigenvalue of $A(g)$ for some k and i. Equivalently, $\chi_1(g) = 1$ and $\chi_0(g)$ is an eigenvalue of $A(g)^m$. In particular, $g = -1$ acts trivially on $Y(\rho, m)$ if and only if

$$(\chi_1(-1), \chi_0(-1)) = (1, (-1)^m). \tag{7.1}$$

We assume this equality holds for χ_1 and χ_0. Then G acts on $Y(\rho, m)$ and the image $G_{m,\chi}$ of the homomorphism $\widetilde{G}_{m,\chi} \to \mathrm{GL}(2, \mathbb{C})$ given by the action on W is an extension of G by $\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. We also assume that the action of G on $Y(\rho, m)$ is free. This is equivalent to:

$$\chi_1(g) \neq 1 \quad \text{or} \quad \chi_0(g) \text{ is not an eigenvalue of } A(g^m) \tag{7.2}$$

for $g \in \widetilde{G} \setminus \{\pm 1\}$. Then the quotient space $X(\rho, m, G, \chi) := G\backslash Y(\rho, m) = G_{m,\chi}\backslash W$ is an elliptic Hopf surface over $G\backslash\mathbb{P}^1$.

Lemma 7.4. *Let X be a Hopf surface with an elliptic fibration $X \to T$ that has at least three singular fibers. Then X is obtained as the free quotient $X(\rho, m, G, \chi)$ above for some ρ, m, G, χ.*

Proof. Let $\pi\colon X \to T$ be the elliptic fibration. By the argument in the proof of Lemma 4.4, there is a Galois covering $\tau\colon \mathbb{P}^1 \simeq \Gamma \to T$ such that the normalization Y of $X \times_T \Gamma$ is smooth over Γ and étale over X. Then $Y \simeq Y(\rho, m)$ for some ρ and m by Fact 7.2. The universal covering map $W \to X$ is the composite of $W \to Y$ and $Y \to X$. The action of G on Γ lifts to that on Y. For $g \in \widetilde{G}$, a lift of the action of g on Y to W is written by

$$z = (z_1, z_2) \mapsto u(z, g) \cdot (z_1, z_2)\, {}^{\mathrm{t}}\!A(g)$$

for a holomorphic function $u\colon W \times \widetilde{G} \to \mathbb{C}^\star$. The description of the universal covering map $W \to Y = Y(\rho, m)$ implies that, for g, there exist k, $i \in \mathbb{Z}$ such that $u(\rho z, g) = \rho^k \,\mathrm{e}(i/m) u(z, g)$. Since u extends as $\mathbb{C}^2 \times \widetilde{G} \to \mathbb{C}^\star$, we have $\rho^k \,\mathrm{e}(i/m) = 1$ by substituting $z = (0, 0)$. Therefore, u descends to $W/\langle\rho\rangle \times \widetilde{G} \to \mathbb{C}^\star$ which is constant by the compactness of the quotient $W/\langle\rho\rangle$. Hence we may write $u(g) = u(z, g) \in \mathbb{C}^\star$. Therefore, for any g_1, g_2, there exist k and i with $u(g_1 g_2) = \rho^k \,\mathrm{e}(i/m) u(g_1) u(g_2)$. Hence $u(g) = \varphi(g)$ above for some characters χ_1 and χ_0. Thus X is isomorphic to the quotient space of W by $G_{m,\chi}$ above and the action of $G_{m,\chi}$ is free since the action of G on Y is free. □

Lemma 7.5. *$X(\rho, m, G, \chi)$ admits a non-trivial surjective endomorphism if there exists a $\widetilde{G}$-semi-invariant homogeneous polynomial $H(z_1, z_2)$ of degree $d > 2$ such that*

(1) *$H(z_1, z_2)$ has only simple zeros over $\mathbb{P}^1$,*

(2) $\chi_1(g)^{d-2} = \delta(g)^m \chi_0(g)^{d-2} = 1$ *for the character* δ *determined by*

$$H((z_1, z_2)\,{}^{\mathrm{t}}A(g)) = \delta(g) H(z_1, z_2).$$

Proof. (cf. [19]) Let $F_1(z_1, z_2) = -\partial H(z_1, z_2)/\partial z_2$ and $F_2(z_1, z_2) = \partial H(z_1, z_2)/\partial z_1$. Then the morphism $\Phi\colon W \ni (z_1, z_2) \mapsto (F_1(z_1, z_2), F_2(z_1, z_2)) \in W$ is well-defined and

$$\Big(F_1((z_1, z_2)\,{}^{\mathrm{t}}A(g)), F_2((z_1, z_2)\,{}^{\mathrm{t}}A(g))\Big) = \delta(g)\,(F_1(z_1, z_2), F_2(z_1, z_2))\,{}^{\mathrm{t}}A(g)$$

for any g. Thus Φ is $G_{m,\chi}$-equivariant by the condition (2). Hence Φ induces a non-trivial surjective endomorphism of $X(\rho, m, G, \chi)$ since Φ induces an endomorphism of $\mathbb{P}^1$ of degree $d - 1 > 1$. $\square$

Proposition 7.6. *The elliptic Hopf surface $X(\rho, m, G, \chi)$ admits non-trivial surjective endomorphisms.*

Proof. If G is a cyclic group of order of n, then $\widetilde{G}$ is conjugate to the cyclic group generated by

$$A = \begin{pmatrix} \mathrm{e}(1/2n) & 0 \\ 0 & \mathrm{e}(-1/2n) \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbb{C})$. Then the elliptic surface $X(\rho, m, G, \chi) \to G\backslash\mathbb{P}^1$ has at most two singular fibers. Hence the existence of non-trivial surjective endomorphisms on $X(\rho, m, G, \chi)$ for a cyclic group G follows from Proposition 7.3.

Thus we assume G is not cyclic. It is enough to construct H satisfying the condition of Lemma 7.5 in the following cases $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.

Case $(2, 2, n)$: G is the dihedral group D_n of order $2n \geq 4$. We may assume $\widetilde{G}$ is generated by

$$Q := \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} \mathrm{e}(1/2n) & 0 \\ 0 & \mathrm{e}(-1/2n) \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbb{C})$. Then $Q^2 = A^n = -1$ and $QAQ^{-1} = A^{-1}$. In particular, $A^2 \in [\widetilde{G}, \widetilde{G}]$. Thus $\widetilde{G}/[\widetilde{G}, \widetilde{G}]$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ for n odd and to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for n even.

If n is even, then m is even by (7.1) since $Q^2 = -1$.

Let us consider the homogeneous polynomial

$$H(z_1, z_2) = z_1^{2n} - z_2^{2n}$$

of degree $d = 2n$. This has only simple zeros over $\mathbb{P}^1$ and is $\widetilde{G}$-invariant for n odd and $\widetilde{G}$-semi-invariant for n even. Note that $d-2$ is even and moreover $d-2 = 2(n-1) \equiv 0 \mod 4$ for n odd. Thus H satisfies the condition of Lemma 7.5 since $\chi_1^{d-2} = \chi_0^{d-2} = \delta^m = 1$.

Case $(2, 3, 3)$: G is the tetrahedral group isomorphic to the alternating group $\mathfrak{A}_4$. We may assume that $\widetilde{G}$ is generated by

$$A = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{2}}\,\mathrm{e}(1/8)\begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix},$$

where we regard $\sqrt{-1}$ as $\mathrm{e}(1/4)$. Then $A^2 = B^3 = -1$ and $(AB)^3 = 1$. Here, $\widetilde{G}/[\widetilde{G}, \widetilde{G}] \simeq \mathbb{Z}/3\mathbb{Z}$. In particular, $\chi_1^3 = \chi_0^3 = 1$. Let us consider the homogeneous polynomial

$$H(z_1, z_2) = z_1^8 + z_2^8 + 14z_1^4z_2^4$$

of degree $d = 8$. Then this has only simple zeros over $\mathbb{P}^1$ and is $\widetilde{G}$-invariant. Thus H satisfies the condition of Lemma 7.5 since $d - 2 \equiv 0 \mod 3$.

Case $(2, 3, 4)$: G is the octahedral group isomorphic to the symmetric group $\mathfrak{S}_4$. We may assume that $\widetilde{G}$ is generated by

$$A = \begin{pmatrix} \mathrm{e}(1/8) & 0 \\ 0 & \mathrm{e}(-1/8) \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{2}}\mathrm{e}(1/8)\begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix}.$$

Then $A^4 = B^3 = (AB)^2 = -1$. Here, $\widetilde{G}/[\widetilde{G}, \widetilde{G}] \simeq \mathbb{Z}/2\mathbb{Z}$. In particular, the square of any character is trivial. Here m is even by (7.1) since $A^4 = -1$. Let us consider the homogeneous polynomial

$$H(z_1, z_2) = z_1z_2(z_1^4 - z_2^4)$$

of degree $d = 6$. Then this has only simple zeros over $\mathbb{P}^1$ and is $\widetilde{G}$-semi-invariant. Thus H satisfies the condition of Lemma 7.5 since m and $d - 2$ are even.

Case $(2, 3, 5)$: G is the icosahedral group isomorphic to the alternating group $\mathfrak{A}_5$. We may assume that $\widetilde{G}$ is generated by

$$A = -\begin{pmatrix} \beta^{-2} & 0 \\ 0 & \beta^2 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{5}}\begin{pmatrix} -(\beta - \beta^{-1}) & \beta^2 - \beta^{-2} \\ \beta^2 - \beta^{-2} & \beta - \beta^{-1} \end{pmatrix},$$

where $\beta = \mathrm{e}(1/5)$. Then $A^5 = B^2 = -1$ and $(AB)^3 = 1$. Here, $\widetilde{G}$ has no non-trivial characters. Hence the $\widetilde{G}$-invariant polynomial

$$H(z_1, z_2) := z_1z_2(z_1^{10} + 11z_1z_2 - z_2^{10})$$

satisfies the condition of Lemma 7.5. □

Finally, we treat the case of non-elliptic Hopf surfaces. By Theorem 32 of [10, II], a non-elliptic Hopf surface X is obtained as the quotient of W by the following action of $\mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}$: A generator of $\mathbb{Z}$ acts as

$$(z_1, z_2) \mapsto (\alpha_1z_1 + \lambda z_2^m, \alpha_2z_2),$$

where m is a positive integer, α_1, α_2, λ are complex numbers with $0 < |\alpha_1| \le |\alpha_2| < 1$ and $(\alpha_1 - \alpha_2^m)\lambda = 0$. If $\lambda = 0$, then $\alpha_1^p \ne \alpha_2^q$ for any positive integers p, q; A generator $\mathbb{Z}/l\mathbb{Z}$ acts as

$$(z_1, z_2) \mapsto (\varepsilon_1z_1, \varepsilon_2z_2)$$

for primitive l-the roots ε_1, ε_2 of unity with $(\varepsilon_1 - \varepsilon_2^m)\lambda = 0$.

The equation $z_2 = 0$ defines an elliptic curve on X. If $\lambda \neq 0$, then it is a unique curve of X. If $\lambda = 0$, then the equation $z_1 = 0$ defines another elliptic curve and there are no other curves contained in X.

If $\lambda = 0$, then $(z_1, z_2) \mapsto (z_1^d, z_2^d)$ for $d > 1$ gives a non-trivial surjective endomorphism of X. Therefore, the proof of Theorem 7.1 is reduced to the following:

PROPOSITION 7.7. *If $\lambda \neq 0$, then X admits no non-trivial surjective endomorphisms.*

Proof. We write $\alpha = \alpha_2$ and $\varepsilon = \varepsilon_2$. Then $\alpha_1 = \alpha^m$, $\varepsilon_1 = \varepsilon^m$, and $(k, j) \in \mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}$ acts on W by

$$\varphi_{k,j} \colon (z_1, z_2) \mapsto \left(\varepsilon^{jm}(\alpha^{km} z_1 + k\lambda\alpha^{(k-1)m} z_2^m), \varepsilon^j \alpha^k z_2\right).$$

Note that $\varphi_{k,j}$ for $k > 0$ is a *contraction* (cf. Section 10 of [10, II]) in the sense that $\varphi_{k,j}^n(B)$ converges to $(0,0)$ for $n \to +\infty$ for the ball $B = \{|z_1|^2 + |z_2|^2 \leq 1\}$. Suppose that there is an endomorphism $f \colon X \to X$. Let $\Phi \colon W \to W$ be a lift, which is written by

$$\Phi \colon (z_1, z_2) \mapsto (F(z_1, z_2), G(z_1, z_2))$$

for holomorphic functions F, G defined on $\mathbb{C}^2$. Here, $\Phi \circ \varphi_{1,0} = \varphi_{p,q} \circ \Phi$ for some integers p and q. Hence the following functional equations hold:

$$F(\alpha^m z_1 + \lambda z_2^m, \alpha z_2) = \varepsilon^{qm}\left(\alpha^{pm} F(z_1, z_2) + p\lambda\alpha^{(p-1)m} G(z_1, z_2)^m\right), \tag{7.3}$$

$$G(\alpha^m z_1 + \lambda z_2^m, \alpha z_2) = \varepsilon^q \alpha^p G(z_1, z_2). \tag{7.4}$$

Here, we have $p > 0$ by (7.4); otherwise,

$$|G(z_1, z_2)| = |\alpha^{-pk} G(\varphi_{k,0}(z_1, z_2))| \to 0 \quad \text{as} \quad k \to +\infty$$

for $p < 0$ and $G(z_1, z_2)$ is constant for $p = 0$. Moreover, $F(0,0) = G(0,0) = 0$ by

$$\Phi \circ \varphi_{k,0}(z_1, z_2) = \varphi_{p,q}^k \circ \Phi(z_1, z_2) \to \Phi(0,0) = (0,0) \quad \text{as} \quad k \to +\infty.$$

We insert here the following:

LEMMA 7.8. *Let $G(z_1, z_2)$ be an entire holomorphic function satisfying* (7.4). *Then $G(z_1, z_2) = cz_2^p$ for a constant c. If $c \neq 0$, then $\varepsilon^q = 1$.*

Proof. We follow the argument of Kodaira in the proof of Theorem 31 of [10, II]. We may assume that G is not identically zero. We set $G_{(\nu)}(z_1, z_2) := \partial^\nu G(z_1, z_2)/\partial z_1^\nu$ for $\nu \geq 1$. Then

$$\alpha^{m\nu - p}\varepsilon^{-q} G_{(\nu)}(\varphi_{1,0}(z_1, z_2)) = G_{(\nu)}(z_1, z_2)$$

by (7.4). If $m\nu > p$, then $G_{(\nu)}(z_1, z_2) \equiv 0$ by

$$G_{(\nu)}(z_1, z_2) = \alpha^{k(m\nu - p)}\varepsilon^{-qk} G_{(\nu)}(\varphi_{k,0}(z_1, z_2)) \to 0 \quad \text{for} \quad k \to +\infty.$$

Hence we can write

$$G(z_1, z_2) = \sum_{i=0}^{N} G_i(z_2) z_1^i$$

for entire holomorphic functions $G_i(z_2)$ and for an integer $0 \leq N \leq p/m$ such that G_N is not identically zero. By comparing the coefficients of z_1^i on both sides of (7.4), we have

$$\varepsilon^q \alpha^p G_i(z_2) = \alpha^{mi} \sum_{l=i}^{N} \binom{l}{i} G_l(\alpha z_2) \lambda^{l-i} z_2^{m(l-i)}, \tag{7.5}$$

for $0 \leq i \leq N$. In particular, $G_N(\alpha z_2) = \varepsilon^q \alpha^{p-mN} G_N(z_2)$. Hence $\varepsilon^q = 1$, and $G_N(z_2) = c z_2^{p-mN}$ for a constant $c \neq 0$. Suppose that $N \neq 0$. By (7.5) in the case $i = N-1$, we have:

$$\alpha^p G_{N-1}(z_2) = \alpha^{m(N-1)} G_{N-1}(\alpha z_2) + cN\lambda \alpha^{p-m} z_2^{p-mN+m}. \tag{7.6}$$

By comparing the coefficients of z_2^{k-mN+m} on both sides of equation (7.6), we derive a contradiction to $N \neq 0$. Therefore, $N = 0$ and $G(z_1, z_2) = c z_2^p$ for some $c \neq 0$. □

Proof of Proposition 7.7 continued. We have $\varepsilon^q = 1$ and $G(z_1, z_2) = c z_2^p$ for a constant $c \neq 0$ by Lemma 7.8. Thus the equation (7.3) is written by

$$F(\alpha^m z_1 + \lambda z_2^m, \alpha z_2) = \alpha^{pm} F(z_1, z_2) + p\lambda \alpha^{(p-1)m} c^m z_2^{pm}. \tag{7.7}$$

Hence, $F_{(1)} := \partial F / \partial z_1$ satisfies a functional equation

$$F_{(1)}(\alpha^m z_1 + \lambda z_2^m, \alpha z_2) = \alpha^{(p-1)m} F_{(1)}(z_1, z_2)$$

similar to (7.4). Thus $F_{(1)}(z_1, z_2) = c_1 z_2^{(p-1)m}$ for a constant c_1 by Lemma 7.8. Then $F(z_1, z_2) = c_1 z_1 z_2^{(p-1)m} + H(z_2)$ for a holomorphic function $H(z_2)$. By (7.7), we have

$$c_1 \lambda \alpha^{(p-1)m} z_2^{pm} + H(\alpha z_2) = \alpha^{pm} H(z_2) + p\lambda \alpha^{(p-1)m} c^m z_2^{pm}$$

and hence $H(z_2) = \delta z_2^{pm}$ for a constant δ and $c_1 = pc^m$. Thus we obtain:

$$F(z_1, z_2) = pc^m z_1 z_2^{(p-1)m} + \delta z_2^{pm}, \quad G(z_1, z_2) = c z_2^p.$$

If $p \geq 2$, then $F(z_1, 0) \equiv G(z_1, 0) \equiv 0$, which contradicts the assumption that F and G have no common zeros except $(z_1, z_2) = (0, 0)$. Hence $p = 1$ and

$$F(z_1, z_2) = c^m z_1 + \delta z_2^m, \quad G(z_1, z_2) = c z_2$$

for constants $c \neq 0$ and δ. Thus the endomorphism $f \colon X \to X$ is an isomorphism. □

8. Inoue surfaces without curves

In the paper [5], Inoue constructed examples of compact complex surface of class VII_0 with $b_2 = 0$ having no curves. These are called Inoue surfaces and are denoted by S_M, $S^{(+)}_{N,p,q,r;t}$, and $S^{(-)}_{N,p,q,r}$. Moreover Inoue showed in the same paper that if there is an invertible sheaf $\mathcal{L}$ satisfying

$$\mathrm{H}^0(S, \Omega^1_S \otimes \mathcal{L}) \neq 0$$

on a surface S with $b_1(S) - 1 = b_2(S) = 0$ having no curves, then S is one of the surfaces above. By the works [11], [21], we can remove the assumption on the existence of $\mathcal{L}$

above; These Inoue surfaces are characterized as the surfaces with $b_1 = 1$, $b_2 = 0$ having no curves.

Lemma 8.1. *Let $f\colon X \to X$ be an étale endomorphism of a surface of class* VII_0 *with $\kappa(X) = -\infty$. Then $f^*\colon \mathrm{H}^1(X,\mathbb{Z}) \to \mathrm{H}^1(X,\mathbb{Z})$ is identical.*

Proof. Assume the contrary. Then f^* is the multiplication map by an integer $d \neq 1$. We have the isomorphism $\mathrm{H}^1(X,\mathbb{C}^\star) \simeq \mathrm{H}^1(X,\mathcal{O}_X^\star)$ from the exponential sequence on X. Thus $K_X \sim f^*K_X$ implies that $\mathcal{O}_X((d-1)mK_X) \simeq \mathcal{O}_X$ for the order m of the torsion part of $\mathrm{H}_1(X,\mathbb{Z})$. In particular, $\kappa(X) = 0$, a contradiction. □

The Inoue surface S_M is defined as follows: Let M be a matrix in $\mathrm{SL}(3,\mathbb{Z})$ with eigenvalues α, β, $\overline{\beta}$ such that $\alpha > 1$ and $\beta \notin \mathbb{R}$. Here, $\alpha \notin \mathbb{Q}$. Let ${}^{\mathrm{t}}(a_1,a_2,a_3)$ be a real eigenvector with α as the eigenvalue and let ${}^{\mathrm{t}}(b_1,b_2,b_3)$ be an eigenvector with β as the eigenvalue. Then three vectors (a_1,b_1), (a_2,b_2), (a_3,b_3) are $\mathbb{R}$-linearly independent and satisfy

$$(\alpha a_i, \beta b_i) = \sum_{j=1}^{3} m_{ij}(a_j,b_j), \quad \text{where} \quad M = (m_{ij}) \in \mathrm{SL}(3,\mathbb{Z}).$$

Let G_M be the group of automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0&\colon (w,z) \mapsto (\alpha w, \beta z),\\ g_i&\colon (w,z) \mapsto (w + a_i, z + b_i) \quad \text{for} \quad i = 1,\ 2,\ 3. \end{aligned}$$

The action of G_M on $\mathbb{H} \times \mathbb{C}$ is properly discontinuous and free. The surface S_M is defined as the quotient surface of $\mathbb{H} \times \mathbb{C}$ by G_M. The generators g_i satisfy the following relations:

$$g_i g_j = g_j g_i, \quad g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}}, \quad \text{for} \quad 1 \leq i,j \leq 3.$$

Proposition 8.2. *The Inoue surface S_M admits a non-trivial surjective endomorphism.*

Proof. Let Φ be the automorphism of $\mathbb{H} \times \mathbb{C}$ given by $(w,z) \mapsto (nw,nz)$ for an integer $n > 1$. Then $\Phi \circ g_0 = g_0 \circ \Phi$ and $\Phi \circ g_i = g_i^n \circ \Phi$ for $1 \leq i \leq 3$. Thus an endomorphism $f\colon S_M \to S_M$ is defined by Φ. Here $f_*\colon \pi_1(S_M) \to \pi_1(S_M)$ is isomorphic to the homomorphism $G_M \to G_M$ given by $G_M \ni g \mapsto \Phi \circ g \circ \Phi^{-1}$. Thus f is non-trivial. □

The Inoue surface $S^{(+)}_{N,p,q,r;t}$ is defined for a matrix N in $\mathrm{SL}(2,\mathbb{Z})$ with $n := \operatorname{tr} N > 2$, integers p, q, r with $r \neq 0$, and for a complex number t as follows: Let α be an eigenvalue with $\alpha > 1$. Let $\boldsymbol{a} = {}^{\mathrm{t}}(a_1,a_2)$ and $\boldsymbol{b} = {}^{\mathrm{t}}(b_1,b_2)$ be non-zero real column vectors such that $N\boldsymbol{a} = \alpha\boldsymbol{a}$ and $N\boldsymbol{b} = \alpha^{-1}\boldsymbol{b}$. Note that a_i and b_i are non-zero and a_2/a_1 and b_2/b_1 are irrational. We set $\theta := \det(\boldsymbol{a},\boldsymbol{b}) = a_1b_2 - a_2b_1$. For a pair (l_1,l_2) of integers, we set

$$e(l_1,l_2) := \frac{l_1(l_1-1)}{2}b_1a_1 + \frac{l_2(l_2-1)}{2}b_2a_2 + l_1l_2b_1a_2.$$

We define $e_1 := e(n_{11}, n_{12})$ and $e_2 := e(n_{21}, n_{22})$ for the matrix $N = (n_{ij})$. We also define a real column vector $\boldsymbol{c} = {}^{\mathrm{t}}(c_1, c_2)$ by

$$(N - I)\boldsymbol{c} + {}^{\mathrm{t}}(e_1, e_2) - (\theta/r)\,{}^{\mathrm{t}}(p, q) = 0, \tag{8.1}$$

where I denotes the unit matrix. Let $G^{(+)} = G^{(+)}_{N,p,q,r;t}$ be the group of automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0 &: (w, z) \mapsto (\alpha w, z + t), \\ g_i &: (w, z) \mapsto (w + a_i, z + b_i w + c_i) \quad \text{for} \quad i = 1,\, 2, \\ g_3 &: (w, z) \mapsto (w, z - \theta/r)\,. \end{aligned}$$

Then g_3 commutes with g_i for $0 \le i \le 2$. Moreover, we have:

$$g_1 g_2 = g_2 g_1 g_3^r, \quad g_0 g_1 g_0^{-1} = g_1^{n_{11}} g_2^{n_{12}} g_3^p, \quad g_0 g_2 g_0^{-1} = g_1^{n_{21}} g_2^{n_{22}} g_3^q. \tag{8.2}$$

These relations determine the group structure of $G^{(+)}$. The subgroup $\Gamma = \Gamma^{(+)}_r \subset G^{(+)}$ generated by g_1, g_2, and g_3 is normal and the quotient $G^{(+)}/\Gamma$ is a free abelian group of rank one generated by the class of g_0. The center of $G^{(+)}$ is generated by g_3 and contains $[\Gamma, \Gamma]$. The quotient group of Γ by the center is a free abelian group of rank two generated by the classes of g_1 and g_2. The action of $G^{(+)}$ on $\mathbb{H} \times \mathbb{C}$ is properly discontinuous and free. The surface $S^{(+)}_{N,p,q,r;t}$ is defined as the quotient space. More precisely, we denote it by $S^{(+)}_{N,p,q,r;t}(\boldsymbol{a}, \boldsymbol{b})$.

Definition 8.3. (1) Let T_3 denote the subgroup of $T_3(\mathbb{C})$ consisting of $T(x_1, x_2, x_3)$ with $x_i \in \mathbb{R}$.
(2) Let Δ_3 denote the subgroup of $\Delta_3(\mathbb{C})$ consisting of $((x_1, x_2), y)$ with x_1, x_2, $y \in \mathbb{R}$.
(3) Let $\mathbb{D} \colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ be the skew symmetric form defined by

$$\mathbb{D}((l_1, l_2), (l'_1, l'_2)) := l_1 l'_2 - l_2 l'_1.$$

In other expressions, $\mathbb{D}(\xi, \xi') = \det({}^{\mathrm{t}}\xi,\, {}^{\mathrm{t}}\xi')$ for row vectors ξ, $\xi' \in \mathbb{Z}^2$.
(4) For an integer $r \neq 0$, let $\mathbb{Z}[r/2] = \mathbb{Z} + \mathbb{Z}(r/2) \subset \mathbb{Q}$ and let Γ_r be the following group structure defined on $\mathbb{Z}^2 \times \mathbb{Z}[r/2]$:

$$(\xi, y) * (\xi', y') := (\xi + \xi', y + y' + (r/2)\mathbb{D}(\xi, \xi'))\,.$$

An element of Γ_r is denoted by (ξ, y) for a row vector $\xi \in \mathbb{Z}^2$ and $y \in \mathbb{Z}[r/2]$.

The group T_3 acts on $(w, z) \in \mathbb{H} \times \mathbb{C}$ by the multiplication map

$$T(x_1, x_2, x_3)\,{}^{\mathrm{t}}(1, w, z) = {}^{\mathrm{t}}(1, w + x_1, z + x_2 w + x_3).$$

The group homomorphism $\Gamma = \Gamma^{(+)}_r \to T_3$ given by

$$g_1 \mapsto T(a_1, b_1, c_1), \quad g_2 \mapsto T(a_2, b_2, c_2), \quad g_3 \mapsto T(0, 0, -\theta/r)$$

is compatible with the actions on $\mathbb{H} \times \mathbb{C}$. The homomorphism is written explicitly as follows:

$$g_1^{l_1} g_2^{l_2} g_3^{l_3} \mapsto T\left((l_1, l_2)\boldsymbol{a}, (l_1, l_2)\boldsymbol{b}, (l_1, l_2)\boldsymbol{c} - (\theta/r)l_3 + e(l_1, l_2)\right),$$

where $(l_1, l_2)\boldsymbol{a} = l_1a_1 + l_2a_2$, $(l_1, l_2)\boldsymbol{b} = l_1b_1 + l_2b_2$, and $(l_1, l_2)\boldsymbol{c} = l_1c_1 + l_2c_2$. An isomorphism $T_3 \to \Delta_3$ is induced from (6.2). There is a homomorphism $\Gamma \to \Gamma_r$ given by

$$g_1^{l_1} g_2^{l_2} g_3^{l_3} \mapsto ((l_1, l_2), l_3 + (r/2)l_1l_2).$$

Then an element $((l_1, l_2), \lambda) \in \Gamma_r$ comes from Γ if and only if $\lambda - (r/2)l_1l_2 \in \mathbb{Z}$. There is also a homomorphism $\Gamma_r \to \Delta_3$ given by

$$(\xi, y) \mapsto (\xi(\boldsymbol{a}, \boldsymbol{b}), \xi\boldsymbol{c}' - (\theta/r)y), \quad \text{where} \quad \boldsymbol{c}' := \boldsymbol{c} - (1/2)\,{}^{\mathrm{t}}(a_1b_1, a_2b_2).$$

Then we infer that the diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma_r \\ \downarrow & & \downarrow \\ T_3 & \xrightarrow{\simeq} & \Delta_3 \end{array}$$

of injective homomorphisms is commutative. The action g_0 on $\mathbb{H} \times \mathbb{C}$ corresponds to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ t & 0 & 1 \end{pmatrix}. \tag{8.3}$$

For the choice of $\boldsymbol{c}$, the relation (8.1) is equivalent to the last two equalities in (8.2). This is also equivalent to

$$(N - I)\boldsymbol{c}' = (\theta/r)\boldsymbol{p}', \quad \text{where} \quad \boldsymbol{p}' = {}^{\mathrm{t}}(p + (r/2)n_{11}n_{12}, q + (r/2)n_{21}n_{22}). \tag{8.4}$$

In particular, $G^{(+)}$ is a isomorphic to the subgroup of $\mathrm{GL}(3, \mathbb{C})$ generated by the image of $\Gamma \to \mathrm{GL}(3, \mathbb{R}) \to \mathrm{GL}(3, \mathbb{C})$ and by the matrix A.

Lemma 8.4. (1) *An endomorphism φ of Γ_r, i.e., a group homomorphism $\varphi\colon \Gamma_r \to \Gamma_r$, is written as*

$$\Gamma_r \ni (\xi, y) \mapsto \varphi(\xi, y) = (\xi M,\ \xi\boldsymbol{v} + (\det M)y)$$

for a matrix $M \in \mathtt{M}_2(\mathbb{Z})$ and a column vector $\boldsymbol{v} \in \mathbb{Z}[r/2]^2$.

(2) *The semigroup $\mathrm{End}(\Gamma_r)$ of endomorphisms of Γ_r is anti-isomorphic to the following semigroup structure on $\mathtt{M}_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$:*

$$(M_1, \boldsymbol{v}_1) \star (M_2, \boldsymbol{v}_2) = (M_1M_2, M_1 \cdot \boldsymbol{v}_2 + (\det M_2)\boldsymbol{v}_1).$$

(3) *An endomorphism of Γ lifts to Γ_r. A pair $(M, \boldsymbol{v}) \in \mathtt{M}_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$ is induced from an endomorphism of Γ if and only if $v_1 - (r/2)m_{11}m_{12}$, $v_2 - (r/2)m_{21}m_{22} \in \mathbb{Z}$, where $M = (m_{ij})$, $\boldsymbol{v} = {}^{\mathrm{t}}(v_1, v_2)$.*

(4) *The automorphism $\gamma \mapsto g_0\gamma g_0^{-1}$ of Γ corresponds to $(N, \boldsymbol{p}')$.*

(5) *An endomorphism of* $G^{(+)}$ *inducing identity on* $G^{(+)}/\Gamma$ *is given by an endomorphism* $(M, \boldsymbol{v})$ *of* Γ *and integers* l_1, l_2, l_3 *satisfying*

$$MN = NM, \quad \text{and} \quad (M - (\det M)I)\boldsymbol{p}' - (N - I)\boldsymbol{v} = rM\,{}^{\mathrm{t}}(l_2, -l_1). \tag{8.5}$$

Here, g_0 *is mapped to* $g_0 g_1^{l_1} g_2^{l_2} g_3^{l_3}$.

Proof. For an endomorphism φ of Γ_r, we attach $M = (m_{ij})$ and $\boldsymbol{v} = {}^{\mathrm{t}}(v_1, v_2)$ by

$$\varphi((1,0),0) = ((1,0)M, v_1) \quad \text{and} \quad \varphi((0,1),0) = ((0,1)M, v_2).$$

Then (1) and (2) follow from simple calculations. For (3), it is enough to show that the endomorphism lifts. This is because Γ_r is generated by Γ and an element $((0,0), r/2)$ commuting with Γ. (4) follows from the relations (8.2). Let ρ be the endomorphism of (5) and let φ be the induced endomorphism of Γ. Then $\rho(g_0) = g_0\eta$ for some $\Gamma \ni \eta = g_1^{l_1} g_2^{l_2} g_3^{l_3}$. Let $\iota(\eta)$ denote the automorphism $\gamma \mapsto \eta\gamma\eta^{-1}$ for $\gamma \in \Gamma$ and let ν denote another automorphism $\gamma \mapsto g_0\gamma g_0^{-1}$. Then ρ maps $g_0\gamma g_0^{-1} = \nu(\gamma)$ to $g_0\eta\varphi(\gamma)\eta^{-1}g_0^{-1} = \varphi(\nu(\gamma))$. Therefore,

$$\nu \circ \iota(\eta) \circ \varphi = \varphi \circ \nu. \tag{8.6}$$

Conversely, if the relation (8.6) holds, then $\iota(\eta)$ and φ define an endomorphism ρ on $G^{(+)}$. Let $(M, \boldsymbol{v}) \in \mathtt{M}_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$ correspond to φ. We infer that $(I, r\,{}^{\mathrm{t}}(-l_2, l_1))$ corresponds to $\iota(\eta)$ by (8.2). Thus (8.6) is equivalent to (8.5). $\square$

Proposition 8.5. *Let* $f\colon X \to X$ *be a surjective endomorphism of the surface* $X = S^{(+)}_{N,p,q,r;t}(\boldsymbol{a}, \boldsymbol{b})$. *Then* f *is induced from the automorphism*

$$\Phi\colon (w, z) \mapsto \left(cw - \frac{\alpha}{\alpha - 1}(l_1, l_2)\boldsymbol{a},\ (\det M)z + \frac{c}{\alpha - 1}((l_1, l_2)\boldsymbol{b})w + \delta\right),$$

for a matrix $M \in \mathtt{M}_2(\mathbb{Z})$ *with a positive eigenvalue* c, *and for integers* l_1, l_2, *and a complex number* δ, *in which the following conditions are satisfied*:

(1) $\det M \neq 0$, $MN = NM$, *and*

$$(\det M - 1)t + \frac{\theta}{2(n-2)}(l_1, l_2)N\begin{pmatrix} l_2 \\ -l_1 \end{pmatrix} - (l_1, l_2)\boldsymbol{c}' + (\theta/2)l_1 l_2 \in (\theta/r)\mathbb{Z}.$$

(2) *Let* $\boldsymbol{v} = {}^{\mathrm{t}}(v_1, v_2)$ *be the solution of the equation*

$$(M - (\det M)I)\boldsymbol{p}' - (N - I)\boldsymbol{v} = rM\begin{pmatrix} l_2 \\ -l_1 \end{pmatrix}.$$

Then $v_i - (r/2)m_{i1}m_{12} \in \mathbb{Z}$ *for* $i = 1, 2$, *where* $M = (m_{ij})$.

Conversely, if M *and* (l_1, l_2) *satisfy the conditions* (1), (2), *then the automorphism* Φ *above induces an endomorphism on* X *of degree* $(\det M)^2$.

Proof. The space $\mathrm{H}^0(X, \Theta_X)$ of global holomorphic vector fields on X is one-dimensional and is generated by the vector field $\partial/\partial z$ by Proposition 3 of [5]. Let

$$\Phi\colon \mathbb{H} \times \mathbb{C} \ni (w, z) \mapsto (\Phi_1(w, z), \Phi_2(w, z)) \in \mathbb{H} \times \mathbb{C}$$

be a lift of the endomorphism f for some holomorphic functions Φ_i. The lift is an automorphism since f is étale. Note that Φ_1 depends only on w since any holomorphic mapping $\mathbb{C} \to \mathbb{H}$ is constant. Thus $\Phi_1 = F(w)$ for a holomorphic function F on $\mathbb{H}$. The formula $\Phi_*(\partial/\partial z) = \partial\Phi_2/\partial z(\partial/\partial z)$ implies that $\Phi_2 = \varepsilon z + G(w)$ for a constant $\varepsilon \neq 0$ and a holomorphic function G on $\mathbb{H}$.

The injective endomorphism $f_*\colon \pi_1(X) \to \pi_1(X)$ is given by $\pi_1(X) \ni g \mapsto \Phi \circ g \circ \Phi^{-1}$. This defines an element $(M, \boldsymbol{v}) \in \mathtt{M}_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$ and integers l_1, l_2, l_3 by Lemma 8.1 and Lemma 8.4. Here the condition (8.5) is satisfied and $v_i - (r/2)m_{i1}m_{i2} \in \mathbb{Z}$ for $i = 1, 2$, for $M = (m_{ij})$ and $\boldsymbol{v} = {}^{\mathrm{t}}(v_1, v_2)$. Note that $(\xi, y) \in \Gamma_r$ acts on $\in \mathbb{H} \times \mathbb{C}$ by

$$(w, z) \mapsto (w + \xi\boldsymbol{a}, z + (\xi\boldsymbol{b})w + \xi\boldsymbol{c}' - (\theta/r)y + (1/2)(\xi\boldsymbol{a})(\xi\boldsymbol{b}))\,.$$

Hence $\Phi \circ (\xi, y) \circ \Phi^{-1} = (\xi M, \xi\boldsymbol{v} + (\det M)y)$ is equivalent to

$$F(w + \xi\boldsymbol{a}) = F(w) + \xi M\boldsymbol{a}, \quad \text{and} \tag{8.7}$$

$$\begin{aligned} &\varepsilon\,((\xi\boldsymbol{b})w + \xi\boldsymbol{c}' - (\theta/r)y + (1/2)(\xi\boldsymbol{a})(\xi\boldsymbol{b})) + G(w + \xi\boldsymbol{a}) - G(w) \\ &= (\xi M\boldsymbol{b})F(w) + \xi M\boldsymbol{c}' - (\theta/r)(\xi\boldsymbol{v} + (\det M)y) + (1/2)(\xi M\boldsymbol{a})(\xi M\boldsymbol{b}). \end{aligned} \tag{8.8}$$

Similarly, $\Phi \circ g_0 \circ \Phi^{-1} = g_0 g_1^{l_1} g_2^{l_2} g_3^{l_3}$ is equivalent to

$$F(\alpha w) = \alpha(F(w) + \zeta\boldsymbol{a}), \quad \text{and} \tag{8.9}$$

$$(\varepsilon - 1)t + G(\alpha w) - G(w) = (\zeta\boldsymbol{b})F(w) + \zeta\boldsymbol{c}' - (\theta/r)l' + (1/2)(\zeta\boldsymbol{a})(\zeta\boldsymbol{b}), \tag{8.10}$$

where $\zeta = (l_1, l_2)$ and $l' := l_3 + (r/2)l_1 l_2$. Then $F'(w)$ has two periods a_1, a_2 by (8.7). Since a_1/a_2 is irrational, $\mathbb{Z}a_1 + \mathbb{Z}a_2 \subset \mathbb{R}$ is dense, which implies that $F'(w)$ is constant. Then $G''(w)$ has also periods a_1, a_2 by (8.8) and hence $G''(w)$ is constant. Moreover $G''(w) = 0$ by (8.10). We can write

$$F(w) = cw - \frac{\alpha}{\alpha - 1}(l_1, l_2)\boldsymbol{a}$$

for a constant c with $M\boldsymbol{a} = c\boldsymbol{a}$ by (8.7) and (8.9). Note that $c > 0$ since $\operatorname{Im} F(w) = c \operatorname{Im} w > 0$. Let $c^\sharp$ be the conjugate of the algebraic integer c over $\mathbb{Q}$. Then $M\boldsymbol{b} = c^\sharp\boldsymbol{b}$. Similarly from (8.10) and (8.8), we have

$$G(w) = \frac{c}{\alpha - 1}((l_1, l_2)\boldsymbol{b})w + \delta, \quad cM\boldsymbol{b} = \varepsilon\boldsymbol{b}$$

for some $\delta \in \mathbb{C}$. Thus $\varepsilon = cc^\sharp = \det M$. We note that

$$\theta = \frac{a_2 b_2}{n_{21}}(\alpha - \alpha^{-1}), \quad 1/2 - \alpha/(\alpha - 1) = -\frac{\alpha + 1}{2(\alpha - 1)} = -\frac{1}{2(n-2)}(\alpha - \alpha^{-1}),$$

$$((l_1, l_2)\boldsymbol{a})((l_1, l_2)\boldsymbol{b}) = \frac{a_2 b_2}{n_{21}}(l_1, l_2) N \begin{pmatrix} l_2 \\ -l_1 \end{pmatrix}.$$

Thus (8.10) is written by

$$\begin{aligned}(\det M - 1)t &= (1/2 - \alpha/(\alpha - 1))(\zeta \boldsymbol{a})(\zeta \boldsymbol{b}) + \zeta \boldsymbol{c}' - (\theta/r)l' \\ &= -\frac{\theta}{2(n-2)}(l_1, l_2) N \begin{pmatrix} l_2 \\ -l_1 \end{pmatrix} - (l_1, l_2)\boldsymbol{c}' - (\theta/2) l_1 l_2 - (\theta/r) l_3.\end{aligned}$$

Hence the conditions (1) and (2) required for M and (l_1, l_2) are satisfied. Conversely, suppose that the conditions are satisfied. The condition (8.8) for any $(\xi, y) \in \Gamma_r$ is equivalent to

$$(M - (\det M)I)\boldsymbol{c}' = \frac{c}{\alpha - 1}((l_1, l_2)\boldsymbol{b})\boldsymbol{a} + \frac{\alpha c^\sharp}{\alpha - 1}((l_1, l_2)\boldsymbol{a})\boldsymbol{b} + (\theta/r)\boldsymbol{v}.$$

By (8.4) and (8.5), it is also equivalent to

$$\theta M \begin{pmatrix} l_2 \\ -l_1 \end{pmatrix} = c((l_1, l_2)\boldsymbol{b})\boldsymbol{a} - c^\sharp((l_1, l_2)\boldsymbol{a})\boldsymbol{b}.$$

In other words, $Z\,{}^{\mathrm{t}}(l_2, -l_1) = 0$ for the matrix

$$Z := \theta M - c\boldsymbol{a}(b_2, -b_1) + c^\sharp \boldsymbol{b}(a_2, -a_1).$$

However, $Z\boldsymbol{a} = Z\boldsymbol{b} = 0$ by a direct calculation. Hence $Z = 0$. Therefore, $\Phi \pi_1(X) \Phi^{-1} \subset \pi_1(X)$ for the automorphism Φ. Thus an endomorphism of X is induced. □

Theorem 8.6. *$S^{(+)}_{N,p,q,r;t}(\boldsymbol{a}, \boldsymbol{b})$ admits a non-trivial surjective endomorphism if and only if $t \in \mathbb{Q}\theta$, where $\theta = \det(\boldsymbol{a}, \boldsymbol{b})$.*

Proof. If the endomorphism exists, then $t \in \mathbb{Q}\theta$ by Proposition 8.5-(1). Conversely suppose that $t \in \mathbb{Q}\theta$. We consider a matrix $M = (m_{ij}) = kN + I$ for an even integer $k > 0$. Then M has a positive eigenvalue $c = k\alpha + 1$ and $\det M = k^2 + kn + 1 > 1$. It is enough to show that, M satisfies the conditions (1) and (2) of Proposition 8.5 for $(l_1, l_2) = (0, 0)$ for some $k > 0$. By assumption,

$$(\det M - 1)t = k(k+n)t \in \mathbb{Z}(\theta/r)$$

for some k. Let $\boldsymbol{v}$ be the solution of $(M - (\det M)I)\boldsymbol{p}' = (N - I)\boldsymbol{v}$. Since $m_{11} m_{12} = k(kn_{11} + 1)n_{12}$ and $m_{21} m_{22} = kn_{21}(kn_{22} + 1)$ are even, we have only to show that $\boldsymbol{v} \in \mathbb{Z}^2$. We note that $(N - I)^{-1} = (2 - n)^{-1}(N^{-1} - I)$ and $M - (\det M)I = k(N - (k + n)I)$. Thus if k is divisible by $n - 2$, then $\boldsymbol{v} \in \mathbb{Z}^2$. □

The Inoue surface $S^{(-)}_{M,p,q,r}$ is defined in [5, §4] for a matrix $M \in \mathtt{M}_2(\mathbb{Z})$ with $\det M = -1$, $\operatorname{tr} M > 0$ and for integers p, q, $r \neq 0$. The surface $S^{(-)} = S^{(-)}_{M,p,q,r}$ has an Inoue surface $S^{(+)} = S^{(+)}_{N,p_1,q_1,r;0}$ as an unramified double covering for $N = M^2$ and for suitable integers p_1, q_1. The involution of $S^{(+)}$ generating the Galois group is induced from $\Psi\colon (w, z) \mapsto (\beta w, -z)$ for the positive eigenvalue $\sqrt{\alpha} = \beta$ of M.

Theorem 8.7. *$S^{(-)}_{M,p,q,r}$ admits a non-trivial surjective endomorphism.*

Proof. We consider an endomorphism of $S^{(+)}$ given by

$$\Phi\colon (w, z) \mapsto ((k\alpha + 1)w, (k^2 + kn + 1)z)$$

for a suitable integer $k > 0$ as in Theorem 8.6. Then $\Psi \circ \Phi = \Phi \circ \Psi$. Thus Φ also gives a non-trivial surjective endomorphism of $S^{(-)}$. □

9. Inoue surfaces with curves

A parabolic Inoue surface, a hyperbolic Inoue surface, and a half Inoue surface are the first examples of surfaces X of class VII_0 with $a(X) = 0$, $b_2(X) > 0$. Different descriptions from [7] of these surfaces are given in [20] by the theory of torus embeddings.

A parabolic Inoue surface $X_{\lambda,n}$ for a complex number λ with $0 < |\lambda| < 1$ and for a positive integer n is given as the quotient space of a toric variety $\mathbb{T}_{\mathsf{N}}(\Sigma)$ by an automorphism g_λ^n of infinite order which are defined as follows: N is a free abelian group of rank two with basis e_1, e_2 and the fan Σ consists of the cones

$$\{0\}, \quad \mathbb{R}_{\geq 0}e_2, \quad \mathbb{R}_{\geq 0}(e_1 + \nu e_2), \quad \mathbb{R}_{\geq 0}(e_1 + \nu e_2) + \mathbb{R}_{\geq 0}(e_1 + (\nu - 1)e_2)$$

for all $\nu \in \mathbb{Z}$. Let g_λ be the automorphism of the open orbit $\mathbb{T}_{\mathsf{N}} = \mathsf{N} \otimes \mathbb{C}^\star$ given by

$$(z, z') \mapsto (\lambda z, zz'),$$

where $(z, z') \in (\mathbb{C}^\star)^2$ corresponds to $z \otimes e_1 + z' \otimes e_2$. Then g_λ extends holomorphically to an automorphism of $\mathbb{T}_{\mathsf{N}}(\Sigma)$. Note that g_λ^n is given by

$$(z, z') \mapsto (\lambda^n z, \lambda^{\frac{n(n-1)}{2}} z^n z').$$

The surface $X_{\lambda,n}$ is of class VII_0 with $b_2(X) = n$. It contains an elliptic curve E with $E^2 = -n$ and a cycle D of rational curves consisting of n irreducible components with $D^2 = 0$. Here, E is the quotient curve of the orbit corresponding to $\mathbb{R}_{\geq 0}e_2$ and an irreducible component of D is the quotient of the orbit corresponding to $\mathbb{R}_{\geq 0}(e_1 + \nu e_2)$ for some ν.

Proposition 9.1. *Parabolic Inoue surfaces $X_{\lambda,n}$ admit non-trivial surjective endomorphisms.*

Proof. For an integer $k > 1$, let h_k be the following endomorphism of $\mathbb{T}_{\mathsf{N}}$:

$$(z, z') \mapsto \left(z^k,\, z^{\frac{k(k-1)}{2}} z'^{k^2}\right).$$

Then h_k extends to an endomorphism of $\mathbb{T}_{\mathsf{N}}(\Sigma)$ and $g_\lambda^k \circ h_k = h_k \circ g_\lambda$. Thus h_k induces a non-trivial surjective endomorphism on $X_{\lambda,n}$. □

A hyperbolic Inoue surface $X_{\mathfrak{K},\mathsf{N}}$ and a half Inoue surface $\widehat{X}_{\mathfrak{K},\mathsf{N}}$ are defined as follows for a real quadratic field $\mathfrak{K}$ and for a free abelian subgroup $\mathsf{N} \subset \mathfrak{K}$ of rank two generating $\mathfrak{K}$ over $\mathbb{Q}$: Let $\mathfrak{K} \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}^2$ be the isomorphism given by $\xi \mapsto (\xi, \xi^\sharp)$ for $\xi \in \mathfrak{K}$ and for the conjugate $\xi^\sharp$ over $\mathbb{Q}$. We set

$$\Gamma_{\mathsf{N}} = \{u \in \mathcal{O}_{\mathfrak{K}}^\times \mid u > 0,\, u\mathsf{N} = \mathsf{N}\} \quad \text{and} \quad \Gamma_{\mathsf{N}}^+ = \{u \in \Gamma_{\mathsf{N}} \mid u^\sharp > 0\},$$

where $\mathcal{O}_{\mathfrak{K}}^\times$ is the unit group of the ring $\mathcal{O}_{\mathfrak{K}}$ of integers of $\mathfrak{K}$. Then $\Gamma_{\mathsf{N}} \simeq \mathbb{Z}$ and Γ_{N}^+ is a subgroup of index at most two. Let Θ_{N} and Θ'_{N} be the convex hulls of $\mathsf{N} \cap (\mathbb{R}_{>0} \times \mathbb{R}_{>0})$ and $\mathsf{N} \cap (\mathbb{R}_{>0} \times \mathbb{R}_{<0})$, respectively. Let Σ_{N} be the fan of $\mathsf{N} \otimes \mathbb{R} = \mathbb{R}^2$ corresponding to the decomposition of $\mathbb{R}_{>0} \times (\mathbb{R} \setminus \{0\})$ into sectors by rays joining 0 and a point of

$$\mathsf{N} \cap (\partial\Theta_{\mathsf{N}} \cup \partial\Theta'_{\mathsf{N}}).$$

Then Γ_{N}^+ acts on the toric variety $\mathbb{T}_{\mathsf{N}}(\Sigma)$ by $u\times \colon \mathsf{N} \to \mathsf{N}$. If Γ_{N}^+ is of index two in Γ_{N}, then Γ_{N} also acts on the toric variety. Let $\mathtt{Mc}_{\mathsf{N}}(\Sigma)$ be the topological quotient space of $\mathbb{T}_{\mathsf{N}}(\Sigma)$ by the compact torus $\mathsf{N} \otimes \mathrm{U}(1) \subset \mathbb{T}_{\mathsf{N}} = \mathsf{N} \otimes \mathbb{C}^\star$, where $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $\mathrm{ord}_{\mathsf{N}} \colon \mathbb{T}_{\mathsf{N}}(\Sigma) \to \mathtt{Mc}_{\mathsf{N}}(\Sigma)$ be the quotient map. Its restriction to $\mathbb{T}_N$ is described as the composite

$$\mathrm{ord}_{\mathsf{N}} \colon \mathsf{N} \otimes \mathbb{C}^\star \xrightarrow{\mathrm{id}\otimes|\cdot|} \mathsf{N} \otimes \mathbb{R}_{>0} \xrightarrow[\simeq]{\mathrm{id}\otimes(-\log)} \mathsf{N} \otimes \mathbb{R},$$

in which the first arrow is induced from the norm map $z \mapsto |z|$ and the second from $0 < r \mapsto -\log r$. Let V_{N} be the pull-back by $\mathrm{ord}_{\mathsf{N}}$ of the open subset

$$(\mathbb{R}_{>0} \times \mathbb{R}) \cup (\mathtt{Mc}_{\mathsf{N}}(\Sigma) \setminus \mathsf{N} \otimes \mathbb{R}).$$

Then the hyperbolic Inoue surface $X_{\mathfrak{K},\mathsf{N}}$ is defined as the quotient space of V_{N} by the action of Γ_{N}^+. The half Inoue surface $\widehat{X}_{\mathfrak{K},\mathsf{N}}$ is defined in the case $[\Gamma_{\mathsf{N}} \colon \Gamma_{\mathsf{N}}^+] = 2$ as the quotient space V_{N} by the action of Γ_{N}.

Proposition 9.2. *Hyperbolic Inoue surfaces and half Inoue surfaces admit non-trivial surjective endomorphisms.*

Proof. For a positive integer $l > 1$, the multiplication $\mathsf{N} \to \mathsf{N}$ by l defines an endomorphism of $\mathbb{T}_{\mathsf{N}}(\Sigma)$ of degree $l^2 > 1$. This preserves V_{N} and commutes with the action of $\Gamma_{\mathsf{N}}^{(+)}$ or Γ_{N}. Thus a non-trivial surjective endomorphism of degree l^2 is induced. □

Corollary 9.3. *Let X be a successive blowups of an Inoue surface with curves whose centers are nodes of curves. Then X admits a non-trivial surjective endomorphisms.*

Proof. Let Y be an Inoue surface with curves and let $f\colon Y \to Y$ be a non-trivial surjective endomorphism. By replacing f by some power f^k, if necessary, we may assume that $f^{-1}(C) = C$ for any curve C. Then $f^{-1}(P) = P$ for any node of the union $\bigcup C$ of all curves. Let $Y_1 \to Y$ be the blowup at a node P. Then f induces a non-trivial surjective endomorphism $f_1\colon Y_1 \to Y_1$ which also preserves any curve on Y_1. In particular, $f_1^{-1}(P_1) = P_1$ for any node P_1 of the union of all the curves of Y_1. Therefore, if $X \to Y$ is a succession of blowups whose centers are nodes of curves, then a non-trivial surjective endomorphism on X is induced from f. $\square$

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