HOLOMORPHIC CURVES IN ABELIAN VARIETIES AND INTERSECTIONS WITH HIGHER CODIMENSIONAL SUBVARIETIES

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ABSTRACT. The purpose of this paper is to prove the following: If the image of a holomorphic map f from $\mathbb{C}$ to an Abelian variety A is Zariski dense, then the counting function with respect to f and a subvariety Z of A with $\mathrm{codim}(Z, A) \geq 2$ is very small in the sense of Nevanlinna theory. And a similar result holds for the differential of any order of the map f.

As an application we prove the truncated second main theorem: If the image of a holomorphic map f from $\mathbb{C}$ to an Abelian variety A is Zariski dense, then the following inequality

$$T(r, f, D) \leq N^{(1)}(r, f, D) + \epsilon T(r, f, L) \ //$$

holds for any $\epsilon > 0$, effective reduced divisor $D \subset A$ and ample line bundle L on A. Here, $T(r, f, D)$ stands for the height function and $N^{(1)}(r, f, D)$ for the truncated counting function as usual in Nevanlinna theory.

As corollaries, we prove a Bloch-Ochiai type theorem for a ramified covering space of an Abelian variety and an uniqueness theorem for holomorphic maps from $\mathbb{C}$ to Abelian varieties.

0. INTRODUCTION

The purpose of this paper is to consider the value distribution theory of holomorphic curves in Abelian varieties. Our main idea is to estimate the intersection numbers between holomorphic curves in Abelian varieties, or their jet lifts, and subvarieties of *codimension greater than one* in the Zariski closures of these curves. One of our main results asserts that such intersection numbers are very small if the holomorphic curves are non-degenerate (Theorem 2.5.1). It seems that estimation of the intersection numbers between holomorphic curves and higher codimensional subvarieties is an interesting problem in the value distribution theory. And this problem is closely related to the conjectural second main theorem in Nevanlinna theory. In this paper, we treat the case that the target spaces are Abelian varieties, and give a satisfactory answer in this case. Though the general cases are open problems, the cases that the target spaces are algebraic surfaces and holomorphic curves are tangent to foliations of that surfaces were considered by M. McQuillan [M98] and M. Brunella [Br99]. The present paper is inspired by these works.

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As applications of our approach, we treat two topics in Nevanlinna theory of holomorphic curves in Abelian varieties: the truncated second main theorem (Theorem 3.1.1) and the unicity problem (Corollary 3.2.3).

Recently, Y.T. Siu-S.K. Yeung [SiY97] [SiY00], R. Kobayashi [Kob00], McQuillan [M99] and J. Noguchi-J. Winkelmann-K. Yamanoi [NoWY02] established the second main theorem for holomorphic curves into Abelian varieties. (The last paper also treats semi-Abelian cases.) Using the terminology of Nevanlinna theory, this second main theorem states that the counting functions are bounded below by the height functions up to some (very) small functions. Then one of the most interesting and important problems is to truncate counting functions. The above papers [SiY00], [NoWY02] also treat this problem (see also [Kob91b]). One of the advantages of truncating the counting functions is that it enables us to control the ramification of holomorphic curves. As a consequence, we can obtain the second main theorem for the case that the target spaces are ramified covering spaces of Abelian varieties (Corollary 3.1.11), and it gives a generalization of Bloch-Ochiai's theorem (Corollary 3.1.14).

The unicity problem treats how to determine holomorphic curves uniquely by the data of intersections with given divisors. R. Nevanlinna himself considered this problem for meromorphic functions on the complex plane $\mathbb{C}$ as an application of his second main theorem. After his work, there has been many important works to improve this. But as far as the author knows, the unicity problem for the case where the target spaces are Abelian varieties has not been so much considered before, except for Y. Aihara's recent work [A00]. Our result (Corollary 3.2.3) may give some sufficient answer in this case.

Now we briefly mention our method to estimate the intersection numbers between holomorphic curves into Abelian varieties and higher codimensional subvarieties. Our method uses the jet bundle. To consider the value distribution of holomorphic curves by using jet bundles, we need two steps.

(1) To consider the properties of jet lifts of holomorphic curves.
(2) To consider the global structure of jet spaces and combine with above properties of jet lifts to get results for the original holomorphic curves.

In this paper, (1) is considered in Proposition 2.1.1, which is given roughly as follows.

Let V be an algebraic variety and $x \in V$ be a point. Let $f : \Delta \to V$ be a holomorphic curve from some open disc $\Delta \subset \mathbb{C}$ to V. Assume $a \in \Delta$ and $f(a) = x$. Then on the l-th jet space $J_l(V) \overset{p_l}{\to} V$, there is some infinitesimal neighborhood of $p_l^{-1}(x)$ with which the l-th jet lift $j_l(f)$ of f intersects with multiplicity greater than $l+1$ at the point a. And the size of this infinitesimal neighborhood is not so big and is controlled by l. (The precise definitions of these terminologies are given in the section 1.)

In our Proposition 2.1.1, we consider a more relative form of this to apply to results such as the truncated second main theorem.

The step (2) above uses some natural properties of Abelian varieties, such as that the jet spaces of Abelian varieties have splitting, and that the Abelian varieties act on the jet spaces. Also, the structure theorem for the Zariski closures of the jet lifts of holomorphic curves into Abelian varieties (Lemma 2.4.1) makes the situation very simple. This structure theorem is essentially due to Siu-Yeung [SiY96] and Noguchi [No98].

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1. Preliminaries

1.1. **Derivations.** Let k be a ring and let R be a k algebra. Let M be an R module. We call a k linear map $D : R \to M$ to be a k derivation if D satisfies the rule

$$D(ab) = aD(b) + bD(a) \quad \text{for } a, b \in R.$$

In case $M = R$, we call D to be a k derivation of R and the set of k derivation of R is denoted by $\mathrm{Der}_k(R)$.

Let $D \in \mathrm{Der}_k(R)$ and set D^s to be $\underbrace{D \circ D \circ \cdots \circ D}_{s \text{ times}}$. Then D^s satisfies the Leibniz rule

$$D^s(ab) = \sum_{i=0}^{s} \binom{s}{i} D^i(a) D^{s-i}(b) \quad \text{for } a, b \in R, \tag{1.1.1}$$

where we set D^0 to be the identity map of R and $\binom{s}{i}$ denotes the binomial coefficient.

Let $I \subset R$ be an ideal of R and let $a \in R$ be an element of R. We define the index $\mathrm{Ind}_{I,D}(a)$ of a with respect to the ideal I and the derivation D to be

$$\mathrm{Ind}_{I,D}(a) = \begin{cases} \min\{k \in \mathbb{Z}; D^k(a) \notin I\}, & {}^{\exists}k \text{ such that } D^k(a) \notin I \\ +\infty & D^k(a) \in I \text{ for all } k \end{cases}$$

where we set D^{-1}, $D^{-2}, \cdots$ to be the 0 map of R. Then we have $\mathrm{Ind}_{I,D}(a) \geq 0$ and

$$\mathrm{Ind}_{I,D}(a) > 0 \Longleftrightarrow a \in I.$$

We have also $\mathrm{Ind}_{I,D}(0) = +\infty$. Let $J \subset R$ be a subset of R. We define $\mathrm{Ind}_{I,D}(J)$ to be

$$\mathrm{Ind}_{I,D}(J) = \min_{a \in J} \mathrm{Ind}_{I,D}(a).$$

For an integer $s \geq 0$, set $I_s = \{a \in R; \mathrm{Ind}_{I,D}(a) \geq s+1\}$. Then by the Leibniz rule (1.1.1), we have

Lemma 1.1.2. *$I_s \subset R$ is an ideal of R, and form a descending sequence*

$$I = I_0 \supset I_1 \supset \cdots \supset I_s \supset \cdots.$$

Let R be a k algebra and let R' be a k' algebra. Let $\varphi : R \to R'$ be a ring homomorphism. For an ideal $I \subset R$ of R, we define the ideal $\varphi^\sharp(I) \subset R'$ of R' to be the ideal generated by $\varphi(I)$. Let $D \in \mathrm{Der}_k(R)$ and $D' \in \mathrm{Der}_{k'}(R')$ be derivations such that $D' \circ \varphi = \varphi \circ D$. Let $J \subset R$ be an ideal. Then we have

Lemma 1.1.3. $\mathrm{Ind}_{I,D}(J) \leq \mathrm{Ind}_{\varphi^\sharp(I),D'}(\varphi^\sharp(J))$.

Let $k \to k'$ be a ring homomorphism and let $D \in \mathrm{Der}_k(R)$ be a derivation. We denote $D_{k'} \in \mathrm{Der}_{k'}(R \otimes_k k')$ for the derivation obtained by the extension of the constant ring; i.e. $D_{k'} = D \otimes \mathrm{id}_{k'}$.

1.2. **Nevanlinna Theory.** Let X be a smooth projective variety and let $D \subset X$ be an effective divisor. Let $f : \mathbb{C} \to X$ be a holomorphic curve such that $f(\mathbb{C}) \not\subset \mathrm{supp}\, D$. We define the counting function of D by

$$N(r, f, D) = \int_1^r \frac{\deg(f^*D \cap \mathbb{C}(t))}{t} dt,$$

where $\mathbb{C}(t) = \{z \in \mathbb{C}; |z| < t\}$. Note that $f^*D \cap \mathbb{C}(t)$ is a divisor on $\mathbb{C}(t)$ with finite supports, hence its degree $\deg(f^*D \cap \mathbb{C}(t))$ makes sense.

Let $L(D)$ be the associated line bundle for D. Let $||\cdot||$ be a Hermitian metric of $L(D)$ and let s_D be a section of $L(D)$ such that D is the zero divisor for s_D. We define the proximity function of D by

$$m(r, f, D) = \int_0^{2\pi} \log \frac{1}{||s_D \circ f(re^{i\theta})||} \frac{d\theta}{2\pi}.$$

Let L be a line bundle on X and let $||\cdot||$ be a Hermitian metric of L. We define the height function of L by

$$T(r, f, L) = \int_1^r \frac{dt}{t} \int_{\mathbb{C}(t)} f^*\,(curv.(L, ||\cdot||)) + O(1),$$

where $curv.(L, ||\cdot||)$ denotes the curvature form of the metrized line bundle $(L, ||\cdot||)$. We define the height function of D by

$$T(r, f, D) = T(r, f, L(D)) + O(1).$$

Then the equality $T(r, f, D) = N(r, f, D) + m(r, f, D) + O(1)$ for a holomorphic curve with $f(\mathbb{C}) \not\subset \mathrm{supp}\, D$ is fundamental in Nevanlinna theory and called as First Main Theorem [NoO90, p.180]. Since we have $m(r, f, D) + O(1) > 0$, the first main theorem implies $N(r, f, D) < T(r, f, D) + O(1)$ which is called as Nevanlinna inequality [NoO90, p.180]. These notations are standard in Nevanlinna theory. For more information about these functions, see [NoO90].

In this paper, we consider the counting function in the more general case; i.e. for the case that $f : \mathbb{C} \to V$ is a holomorphic curve to an algebraic variety V and $Z \subset V$ is a closed subscheme with $f(\mathbb{C}) \not\subset \mathrm{supp}\, Z$. In this case, we define the counting function $N(r, f, Z)$ as follows.

First let V be an affine algebraic variety and let $Z \subset V$ be a closed subscheme. Let $f : \Delta \to V$ be a holomorphic curve from some open disc $\Delta \subset \mathbb{C}$ to V such that $f(\Delta) \not\subset \mathrm{supp}\, Z$. Let $\mathcal{H}(\Delta)$ be the ring of holomorphic functions on Δ. Then the usual derivation $\frac{d}{dz}$ defines the $\mathbb{C}$-derivation $\frac{d}{dz} \in \mathrm{Der}_{\mathbb{C}}(\mathcal{H}(\Delta))$. Let $I_Z \subset \Gamma(V, \mathcal{O}_V)$ be the defining ideal for $Z \subset V$. The holomorphic map f naturally defines a ring homomorphism

$$f^* : \Gamma(V, \mathcal{O}_V) \to \mathcal{H}(\Delta).$$

Let $a \in \Delta$ be a point and let $(a) \subset \mathcal{H}(\Delta)$ be the ideal associated to the point a; i.e.

$$(a) = \{\varphi \in \mathcal{H}(\Delta) |\ \varphi(a) = 0\}.$$

We define the multiplicity of the intersection of f and Z at a by

$$\mathrm{mult}_a\, f \cdot Z = \mathrm{Ind}_{(a),\, \frac{d}{dz}}((f^*)^\sharp I_Z).$$

Next we consider a general algebraic variety V. Let $f : \Delta \to V$ be a holomorphic curve. To define $\mathrm{mult}_a\, f \cdot Z$, take an affine neighborhood $U \subset V$ of $f(a)$ and take a small disk $a \in \Delta' \subset \Delta$ such that $f(\Delta') \subset U$. We define

$$\mathrm{mult}_a\, f \cdot Z = \mathrm{mult}_a\, (f|_{\Delta'}) \cdot (Z|_U),$$

where $f|_{\Delta'}$ and $Z|_U$ denote the restrictions of f to Δ' and Z to U respectively. Then this definition is independent of the choices of U and Δ'.

Let $Z' \hookrightarrow Z$ be a Zariski open subset of Z; i.e. $Z' \hookrightarrow V$ is a Zariski open subset of a closed subscheme. We define the multiplicity by

$$\mathrm{mult}_a\, f \cdot Z' = \begin{cases} \mathrm{mult}_a\, f \cdot Z, & \text{if } f(a) \in \operatorname{supp} Z' \\ 0, & \text{if } f(a) \notin \operatorname{supp} Z' \end{cases}.$$

Now let $f : \mathbb{C} \to V$ be a holomorphic curve and let $Z' \subset V$ be a Zariski open subset of a closed subscheme Z of V with $f(\mathbb{C}) \not\subset \operatorname{supp} Z$. We define the counting function by

$$N\left(r, f, Z'\right) = \int_1^r \frac{\sum_{a \in \mathbb{C}(t)} \mathrm{mult}_a\, f \cdot Z'}{t} dt.$$

For a positive integer $k > 0$, we define the truncated counting function by

$$N^{(k)}\left(r, f, Z'\right) = \int_1^r \frac{\sum_{a \in \mathbb{C}(t)} \min\left(\mathrm{mult}_a\, f \cdot Z',\ k\right)}{t} dt.$$

In the case V is a smooth projective variety X, an effective divisor $D \subset X$ is naturally equipped with a closed subscheme structure (note that D has one local equation), and the above two definitions of counting functions coincide each other.

1.3. **Jet spaces.** Let V be a smooth algebraic variety and let $x \in V$ be a point of V. Let $\mathbf{f}, \mathbf{g} : (\mathbb{C}, 0) \to (V, x)$ be germs of holomorphic mappings from neighborhoods of the origin $0 \in \mathbb{C}$ into V with $\mathbf{f}(0) = \mathbf{g}(0) = x \in V$. For a positive integer k, we write $\mathbf{f} \overset{k}{\sim} \mathbf{g}$ if $\mathbf{f}(z)$ and $\mathbf{g}(z)$ have the same Taylor expansions in z up to order k for some holomorphic local coordinate system around x. Then it is easily checked that the relation "$\overset{k}{\sim}$" is independent of the choice of the holomorphic local coordinate system around x and defines an equivalence relation on the set $\{\mathbf{f} | \mathbf{f} : (\mathbb{C}, 0) \to (V, x)\}$. Let $j_k(\mathbf{f})$ denotes the equivalence class of $\mathbf{f}$ and set

$$J_k(V)_x = \{j_k(\mathbf{f}) | \mathbf{f} : (\mathbb{C}, 0) \to (V, x)\}.$$

Then $J_k(V)_x$ is naturally equipped with complex structure and is isomorphic to $\mathbb{C}^{k \dim V}$. We define a Zariski open subset $J_k(V)_x^{\mathrm{reg}}$ of $J_k(V)_x$ by

$$J_k(V)_x^{\mathrm{reg}} = \{j_k(\mathbf{f}) | \mathbf{f} : (\mathbb{C}, 0) \to (V, x),\ j_1(\mathbf{f}) \overset{1}{\not\sim} j_1(\text{Constant map to } x)\}.$$

The k-th jet space $J_k(V)$ over V is

$$J_k(V) = \bigcup_{x \in V} J_k(V)_x.$$

We set

$$J_k(V)^{\text{reg}} = \bigcup_{x \in V} J_k(V)_x^{\text{reg}}.$$

Then $J_k(V)$ naturally carries the structure of an algebraic variety and $J_k(V)^{\text{reg}}$ is a Zariski open subvariety of $J_k(V)$. In particular, $J_1(V)$ is the tangent space of V. For convenience sake, set $J_0(V) = V$. For $k \geq l$ we have the natural forgetful morphism $p_{k,l} : J_k(V) \to J_l(V)$. The morphism $p_k = p_{k,0} : J_k(V) \to V$ is an affine morphism. Set $p_{k*}\mathcal{O}_{J_k(V)} = \mathcal{J}_V^k$. Then we have

$$J_k(V) = \mathbf{Spec}\ \mathcal{J}_V^k$$

and the natural inclusions

$$\mathcal{O}_V \subset \mathcal{J}_V^1 \subset \mathcal{J}_V^2 \subset \cdots \subset \mathcal{J}_V^k \subset \cdots$$

of the sheaves of algebras over V by the system of morphisms $p_{k,l} : J_k(V) \to J_l(V)$. Hence $\mathcal{J}_V^\infty = \bigcup_{k \geq 0} \mathcal{J}_V^k$ is also a sheaf of algebras over V.

The sheaf $\mathcal{J}_V^\infty$ has a canonical derivation $d \in \mathrm{Der}_{\mathbb{C}}(\mathcal{J}_V^\infty)$, which is a collection of $\mathbb{C}$ derivatives $d : \mathcal{J}_V^k \to \mathcal{J}_V^{k+1}$, satisfying the following condition: Let $U \subset V$ be an affine open subset and let $f : \Delta \to U$ be a holomorphic curve. By the system of jet lifts $j_k(f) : \Delta \to J_k(U)$, we have a system of ring homomorphisms $j_k(f)^* : \Gamma(U, \mathcal{J}_V^k) \to \mathcal{H}(\Delta)$ which gives a ring homomorphism $j_\infty(f)^* : \Gamma(U, \mathcal{J}_V^\infty) \to \mathcal{H}(\Delta)$. Then we have a following commutative diagram

$$\begin{array}{ccc} \Gamma(U, \mathcal{J}_V^\infty) & \xrightarrow{j_\infty(f)^*} & \mathcal{H}(\Delta) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \frac{d}{dz}} \\ \Gamma(U, \mathcal{J}_V^\infty) & \xrightarrow{j_\infty(f)^*} & \mathcal{H}(\Delta) \end{array} \tag{1.3.1}$$

which means $j_\infty(f)^*$ is a homomorphism of $\mathbb{C}$ derivation algebras.

We state the following Proposition from [NoO90, Lemma 6.3.1].

Proposition 1.3.2. *Let V be a smooth affine variety. If $\varphi_1, \cdots, \varphi_d \in \Gamma(V, \mathcal{O}_V)$ ($d = \dim V$) form a local coordinate system around every point of V; i.e. we have $\Omega_V^1 = \mathcal{O}_V d\varphi_1 \oplus \cdots \oplus \mathcal{O}_V d\varphi_d$, we have*

$$\Gamma(V, \mathcal{J}_V^k) \simeq \Gamma(V, \mathcal{O}_V) \otimes_{\mathbb{C}} \mathbb{C}[d\varphi_1, \cdots, d\varphi_d, \cdots, d^k\varphi_1, \cdots, d^k\varphi_d].$$

Let $Z \subset J_k(V)$ be a closed subscheme. For an integer $s \geq 0$, we define a closed subscheme $Z^{(s)} \subset J_{k+s}(V)$ inductively in the following manner.

Put $Z^{(0)} = Z$. For the step $s \to s+1$, let $\mathcal{I}_{Z^{(s)}} \subset \mathcal{J}_V^{k+s}$ be the associated ideal sheaf of $Z^{(s)} \subset J_{k+s}(V)$. We define the ideal sheaf $\mathcal{I}_{Z^{(s+1)}} \subset \mathcal{J}_V^{k+s+1}$ by

$$\mathcal{I}_{Z^{(s+1)}} = \mathcal{I}_{Z^{(s)}} \cdot \mathcal{J}_V^{k+s+1} + d\mathcal{I}_{Z^{(s)}} \cdot \mathcal{J}_V^{k+s+1}$$

where $d : \mathcal{J}_V^{k+s} \to \mathcal{J}_V^{k+s+1}$ denotes the $\mathbb{C}$ derivative. Let $Z^{(s+1)} \subset J_{k+s+1}(V)$ be the associated closed subscheme of $\mathcal{I}_{Z^{(s+1)}} \subset \mathcal{J}_V^{k+s+1}$. Then we have

Lemma 1.3.3. *For a holomorphic curve $f : \Delta \to V$ and a closed subscheme $Z \subset J_k(V)$ with $j_k(f)(\Delta) \not\subset \operatorname{supp} Z$, we have*

$$\operatorname{mult}_a j_{k+s}(f) \cdot Z^{(s)} = \max\left((\operatorname{mult}_a j_k(f) \cdot Z) - s, 0\right)$$

.

Proof. Put $I_{Z^{(s)}} = \Gamma(U, \mathcal{I}_{Z^{(s)}})$ for a suitable affine open subset $U \subset V$. By the definition of the multiplicity, and the equality

$$\operatorname{Ind}_{(a), \frac{d}{dz}}\left((j_{k+s}(f)^*)^{\sharp} I_{Z^{(s)}}\right) = \operatorname{Ind}_{(a), \frac{d}{dz}}\left(j_\infty(f)^*(I_{Z^{(s)}})\right),$$

we have

$$\begin{aligned}
\operatorname{mult}_a j_{k+s}(f) \cdot Z^{(s)} &= \operatorname{Ind}_{(a), \frac{d}{dz}}\left(j_\infty(f)^*(I_{Z^{(s)}})\right) \\
&= \operatorname{Ind}_{(a), \frac{d}{dz}}\left(j_\infty(f)^*(I_{Z^{(s-1)}} + dI_{Z^{(s-1)}})\right) \\
&= \operatorname{Ind}_{(a), \frac{d}{dz}}\left(j_\infty(f)^*(I_{Z^{(s-1)}}) + j_\infty(f)^*(dI_{Z^{(s-1)}})\right) \\
&= \operatorname{Ind}_{(a), \frac{d}{dz}}\left(j_\infty(f)^*(I_{Z^{(s-1)}}) + \tfrac{d}{dz} j_\infty(f)^*(I_{Z^{(s-1)}})\right) \\
&= \max\left((\operatorname{mult}_a j_{k+s-1}(f) \cdot Z^{(s-1)}) - 1, 0\right),
\end{aligned}$$

where the notation $I_{Z^{(s-1)}} + dI_{Z^{(s-1)}}$ means the subset

$$\{a + b;\ a \in I_{Z^{(s-1)}},\ b \in dI_{Z^{(s-1)}}\} \subset \Gamma(U, \mathcal{J}_V^\infty).$$

Using the above equality inductively, we obtain our lemma. □

By this lemma, we have $\operatorname{mult}_a j_k(f) \cdot Z - s \leq \operatorname{mult}_a j_{k+s}(f) \cdot Z^{(s)}$, so the following corollary is immediate.

Corollary 1.3.4. *Let $f : \mathbb{C} \to V$ be a holomorphic curve with $j_k(f)(\mathbb{C}) \not\subset \operatorname{supp} Z$. Then we have*

$$N(r, j_k(f), Z) - sN^{(1)}(r, j_k(f), Z) \leq N\left(r, j_{k+s}(f), Z^{(s)}\right).$$

2. Main Result

2.1. Let V be a smooth algebraic variety and let $P \in V$ be a point. For non-negative integers k, l, consider the algebraic variety $J_{k+l}(V) \times J_k(V)_P$ as a trivial family

$$J_{k+l}(V) \times J_k(V)_P \to J_k(V)_P$$

over $J_k(V)_P$. By the natural projection $J_{k+l}(V)_P \to J_k(V)_P$ and the closed embedding $J_{k+l}(V)_P \hookrightarrow J_{k+l}(V)$, we have the closed embedding $J_{k+l}(V)_P \hookrightarrow J_{k+l}(V) \times J_k(V)_P$. Let $T \subset J_{k+l}(V) \times J_k(V)_P$ be the image of this embedding. Let $T^{\text{reg}} \subset T$ be the Zariski open subset which is the image of the open immersion $J_{k+l}^{\text{reg}}(V)_P \hookrightarrow J_{k+l}(V)_P \simeq T$. Under these settings, we have the following proposition.

Proposition 2.1.1. *There is a closed subscheme $\mathcal{T} \subset J_{k+l}(V) \times J_k(V)_P$ satisfying the following conditions. Here we view $\mathcal{T}$ as a family of closed subschemes of $J_{k+l}(V)$ over $J_k(V)_P$.*

(1) *$\mathcal{T}$ is supported by T. Hence, there is a natural closed immersion $T \hookrightarrow \mathcal{T}$ by considering the reduced structure of $\mathcal{T}$.*

(2) *For a holomorphic curve $f:\Delta\to V$ with $f(a)=P$ and $a\in\Delta$, the closed subscheme $\mathcal{T}_{j_k(f)(a)}\subset J_{k+l}(V)$ which is a fiber of $\mathcal{T}$ over $j_k(f)(a)\in J_k(V)_P$ satisfies*

$$\mathrm{mult}_a\, j_{k+l}(f)\cdot \mathcal{T}_{j_k(f)(a)}\geq l+1.$$

(3) *Let $\mathcal{I}_T\subset\mathcal{O}_{\mathcal{T}}$ be the coherent sheaf of $\mathcal{O}_{\mathcal{T}}$-ideals defining $T\hookrightarrow\mathcal{T}$. Then there is a filtration of coherent $\mathcal{O}_{\mathcal{T}}$-ideal sheaves*

$$0=\mathcal{I}_l\subset\mathcal{I}_{l-1}\subset\cdots\subset\mathcal{I}_1\subset\mathcal{I}_0=\mathcal{I}_T$$

such that all graded pieces $\mathcal{I}_i/\mathcal{I}_{i+1}$ $(0\leq i\leq l-1)$ have $\mathcal{O}_T\simeq\mathcal{O}_{\mathcal{T}}/\mathcal{I}_T$ module structure; i.e. the action of $\mathcal{I}_T$ is trivial. And $\mathcal{I}_i/\mathcal{I}_{i+1}$ $(0\leq i\leq l-1)$ are finite $\mathcal{O}_T$ modules whose restrictions to T^{reg} are rank 1 locally free $\mathcal{O}_{T^{reg}}$ modules.

Proof. Since the problem is local on V, we can assume V is affine and has a local coordinate system around P; i.e. there exist global sections $x_1,\cdots,x_r\in\Gamma(V,\mathcal{O}_V)\overset{\mathrm{def}}{=}A$ $(r=\dim V)$ such that P is defined by $x_1=\cdots=x_r=0$ and $\Omega^1_V=\mathcal{O}_V dx_1\oplus\cdots\oplus\mathcal{O}_V dx_r$. Then by Proposition 1.3.2, we have

$$A^{(s)}\overset{\mathrm{def}}{=}\Gamma(V,\mathcal{J}^s_V)\simeq A\otimes_{\mathbb{C}}\mathbb{C}[dx_1,\cdots,dx_r,\cdots,d^sx_1,\cdots,d^sx_r]$$

and $J_s(V)=\mathrm{Spec}\ A^{(s)}$. Put $K=\Gamma(J_k(V)_P,\mathcal{O}_{J_k(V)_P})$. Then we have

$$K=\mathbb{C}[\overline{dx_1},\cdots,\overline{dx_r},\cdots,\overline{d^kx_1},\cdots,\overline{d^kx_r}]$$

and

$$B\overset{\mathrm{def}}{=}\Gamma(J_{k+l}(V)\times J_k(V)_P,\mathcal{O}_{J_{k+l}(V)\times J_k(V)_P})=A^{(k+l)}\otimes_{\mathbb{C}}K.$$

Put $A^{(\infty)}=\Gamma(V,\mathcal{J}^\infty_V)=\bigcup_{s\geq 0}A^{(s)}$. Define an ideal $I\subset A^{(\infty)}\otimes_{\mathbb{C}}K$ by

$$I=(x_1,\cdots,x_r,dx_1-\overline{dx_1},\cdots,dx_r-\overline{dx_r},\cdots,d^kx_1-\overline{d^kx_1},\cdots,d^kx_r-\overline{d^kx_r}).$$

Note that since $A^{(\infty)}\otimes_{\mathbb{C}}K=B[d^jx_i]_{1\leq i\leq r,k+l<j}$ is a polynomial ring over B, the ideal $B\cap I$ of B is also generated as

$$B\cap I=(x_1,\cdots,x_r,dx_1-\overline{dx_1},\cdots,dx_r-\overline{dx_r},\cdots,d^kx_1-\overline{d^kx_1},\cdots,d^kx_r-\overline{d^kx_r}).$$

Hence, the ideal $B\cap I$ is the defining ideal of T; i.e. $B\cap I=I_T$.

Consider the natural derivation $d_K=d\otimes\mathrm{id}_K\in\mathrm{Der}_K(A^{(\infty)}\otimes_{\mathbb{C}}K)$ obtained by the extension of the constant ring and define a sequence of ideals

$$A^{(\infty)}\otimes_{\mathbb{C}}K\supset I=I_0\supset I_1\supset I_2\supset\cdots$$

by

$$I_t=\{a\in A^{(\infty)}\otimes_{\mathbb{C}}K; d^s_K(a)\in I \text{ for } 0\leq s\leq t\}=\{a\in A^{(\infty)}\otimes_{\mathbb{C}}K;\mathrm{Ind}_{I,d_K}(a)\geq t+1\}.$$

Note that by the Leibniz rule (1.1.1) (or Lemma 1.1.2), I_t is an ideal of $A^{(\infty)}\otimes_{\mathbb{C}}K$. Now let $\mathcal{T}$ be a closed subscheme of $\mathrm{Spec}\ B=J_{k+l}(V)\times J_k(V)_P$ defined by $B\cap I_l$.

Proof of (1). By the definition of I_l and the Leibniz rule, we have $I^{l+1}\subset I_l$, so $(B\cap I)^{l+1}\subset B\cap I_l$. Note that $B\cap I$ is the defining ideal for T.

Proof of (2). Let $f:\Delta\to V$ be a holomorphic curve with $f(a)=P$. Let $q:K\to K_{j_k(f)(a)}\simeq\mathbb{C}$ be the quotient map obtained by

$$\overline{d^ix_j}\mapsto d^ix_j(j_k(f)(a))\quad(1\leq i\leq k,\ 1\leq j\leq r).$$

Then $\mathrm{Spec}(B \otimes_K K_{j_k(f)(a)})$ can be naturally identified with the fiber of

$$J_{k+l}(V) \times J_k(V)_P \to J_k(V)_P$$

over $j_k(f)(a) \in J_k(V)_P$ and $\mathcal{T}_{j_k(f)(a)} \subset \mathrm{Spec}(B \otimes_K K_{j_k(f)(a)})$ is defined by the ideal $q_B(B \cap I_l)$. Here $q_B : B \to B \otimes_K K_{j_k(f)(a)} \simeq A^{(k+l)}$ is the quotient map obtained by taking tensor product of $A^{(k+l)}$ and $q : K \to K_{j_k(f)(a)} \simeq \mathbb{C}$ over $\mathbb{C}$. Recall that $B = A^{(k+l)} \otimes_{\mathbb{C}} K$.

Let $q' : K \to \mathcal{H}(\Delta)$ be the composition of the map $q : K \to K_{j_k(f)(a)} = \mathbb{C}$ and the natural inclusion $\mathbb{C} \hookrightarrow \mathcal{H}(\Delta)$ (considered as constant functions). Now by the two maps $j_\infty(f)^* : A^{(\infty)} \to \mathcal{H}(\Delta)$ and $q' : K \to \mathcal{H}(\Delta)$, we have a map $j_\infty(f)^* \otimes q' : A^{(\infty)} \otimes K \to \mathcal{H}(\Delta)$. Let $q_A : A^{(\infty)} \otimes K \to A^{(\infty)}$ be a map obtained by taking tensor product of $A^{(\infty)}$ and $q : K \to K_{j_k(f)(a)} \simeq \mathbb{C}$ over $\mathbb{C}$. Then we have a commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\iota'} & A^{(\infty)} \otimes K & \xrightarrow{j_\infty(f)^* \otimes q'} & \mathcal{H}(\Delta) \\ {\scriptstyle q_B}\downarrow & & {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ A^{(k+l)} & \xrightarrow{\iota} & A^{(\infty)} & \xrightarrow{j_\infty(f)^*} & \mathcal{H}(\Delta) \end{array}$$

where ι and ι' are the natural inclusions. This diagram and (1.3.1) imply $(j_\infty(f)^* \otimes q') \circ d_K = \frac{z}{dz} \circ (j_\infty(f)^* \otimes q')$ since morphisms q_A and $j_\infty(f)^*$ commute with derivations. We have also

$$(j_\infty(f)^* \circ \iota)^\sharp q_B(B \cap I_l) \subset (j_\infty(f)^* \otimes q')^\sharp I_l.$$

Hence by using $j_\infty(f)^* \circ \iota = j_{k+l}(f)^*$, we have

$$\begin{aligned} l + 1 \le \mathrm{Ind}_{I, d_K}(I_l) &\le \mathrm{Ind}_{(j_\infty(f)^* \otimes q')^\sharp(I), \frac{d}{dz}}((j_\infty(f)^* \otimes q')^\sharp(I_l)) \quad \text{(Lemma 1.1.3)} \\ &\le \mathrm{Ind}_{(j_\infty(f)^* \otimes q')^\sharp(I), \frac{d}{dz}}((j_\infty(f)^* \circ \iota)^\sharp q_B(B \cap I_l)) \\ &\le \mathrm{Ind}_{(a), \frac{d}{dz}}((j_{k+l}(f)^*)^\sharp q_B(B \cap I_l)) \\ &= \mathrm{mult}_a\, j_{k+l}(f) \cdot \mathcal{T}_{j_k(f)(a)}. \end{aligned}$$

Here we use the fact $(j_\infty(f)^* \otimes q')^\sharp(I) \subset (a) \subset \mathcal{H}(\Delta)$ which follows from $f(a) = P$.

Proof of (3). For integers $0 \le s \le l$, let $\mathcal{I}_s \subset \mathcal{O}_{\mathcal{T}}$ be the coherent sheaves of ideals associated to $(B \cap I_s)/(B \cap I_l)$. Note that $(B \cap I_s)/(B \cap I_{s+1})$ naturally has a

$$B/(B \cap I) \simeq \mathbb{C}[d^i x_j]_{1 \le i \le k+l, 1 \le j \le r} \simeq \Gamma(T, \mathcal{O}_T)$$

module structure. (This is because the action of $B \cap I \subset B$ to $(B \cap I_s)/(B \cap I_{s+1})$ is zero.) Since B is a Noetherian ring, $(B \cap I_s)/(B \cap I_{s+1})$ is a finite B module, hence also a finite $B/(B \cap I)$ module. This proves the finiteness part.

Next, we show that the localization $(B \cap I_s)/(B \cap I_{s+1}) \otimes_{B/(B \cap I)} (B/(B \cap I))_{(dx_j)}$ is a free $(B/(B \cap I))_{(dx_j)}$ module of rank 1 for $1 \le j \le r$, which completes the proof. Here note that $(B/(B \cap I))_{(dx_j)}$ denotes the localization with respect to the multiplicative system generated by dx_j and that

$$\bigcup_{1 \le j \le r} \mathrm{Spec}\ (B/(B \cap I))_{(dx_j)}$$

is a Zariski open covering of T^{reg}.

Note that by the definition of I_s and the Leibniz rule (1.1.1), the composition of morphisms

$$I_s \hookrightarrow A^{(\infty)} \otimes K \overset{d_K^{s+1}}{\to} A^{(\infty)} \otimes K \to (A^{(\infty)} \otimes K)/I$$

is a morphism of $A^{(\infty)} \otimes K$ modules and the kernel of this morphism is I_{s+1}. Hence the composition of morphisms

$$\delta_s : B \cap I_s \hookrightarrow I_s \hookrightarrow A^{(\infty)} \otimes K \overset{d_K^{s+1}}{\to} A^{(\infty)} \otimes K \to (A^{(\infty)} \otimes K)/I$$

is a morphism of B modules and the kernel of δ_s is $B \cap I_{s+1}$.

Claim: For $0 \leq s \leq l-1$, the image $\delta_s(B \cap I_s) \subset (A^{(\infty)} \otimes K)/I$ is contained in the subring $B/(B \cap I) \subset (A^{(\infty)} \otimes K)/I$.

To prove this Claim, we consider a differential operator $e \in \mathrm{Der}_K(A^{(\infty)} \otimes K)$ satisfying

$$e(a) = 0 \quad \text{for } a \in A, \quad e(\overline{d^i x_j}) = 0 \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq r$$

$$e(d^i x_j) = 0 \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq r, \quad e(d^i x_j) = d^{i+1} x_j \quad \text{for } k+1 \leq i, 1 \leq j \leq r.$$

Note that such e uniquely exists. We need one lemma.

Lemma 2.1.2. *Let $d_{K,(s)} \in \mathrm{Der}_K(A^{(\infty)} \otimes K)$ be defined inductively by $d_{K,(1)} = d_K - e$, $d_{K,(s+1)} = d_{K,(s)} e - e d_{K,(s)}$. Then there are a non-commutative polynomial P_s in s variables and constants $\alpha_{s,j}$ $(0 \leq j \leq s-1)$ satisfying*

$$d_K^s = P_s(d_{K,(1)}, \cdots, d_{K,(s)}) + \sum_{j=0}^{s-1} \alpha_{s,j} e^{s-j} d_K^j.$$

Proof of Lemma. We prove this by induction.

The case $s = 1$ follows by $d_K^1 = d_K^1 - e + e = d_{K,(1)} + e$.

Induction step $s \to s+1$. By $d_K^{s+1} = d_K^s \cdot d_K$, we have

$$\text{(2.1.3)} \quad d_K^{s+1} = P_s(d_{K,(1)}, \cdots, d_{K,(s)}) \cdot d_K + \sum_{j=0}^{s-1} \alpha_{s,j} e^{s-j} d_K^{j+1}$$

$$= P_s(d_{K,(1)}, \cdots, d_{K,(s)}) \cdot d_{K,(1)} + P_s(d_{K,(1)}, \cdots, d_{K,(s)}) \cdot e + \sum_{j=0}^{s-1} \alpha_{s,j} e^{s-j} d_K^{j+1}.$$

Note that there is a non-commutative polynomial Q_{s+1} of $s+1$ variables satisfying

$$P_s(d_{K,(1)}, \cdots, d_{K,(s)}) \cdot e = e \cdot P_s(d_{K,(1)}, \cdots, d_{K,(s)}) + Q_{s+1}(d_{K,(1)}, \cdots, d_{K,(s)}, d_{K,(s+1)}).$$

By $e \cdot d_K^s = e \cdot P_s(d_{K,(1)}, \cdots, d_{K,(s)}) + \sum_{j=0}^{s-1} \alpha_{s,j} e^{s-j+1} d_K^j$, we have

$$P_s(d_{K,(1)}, \cdots, d_{K,(s)}) \cdot e = e \cdot d_K^s - \sum_{j=0}^{s-1} \alpha_{s,j} e^{s-j+1} d_K^j + Q_{s+1}(d_{K,(1)}, \cdots, d_{K,(s)}, d_{K,(s+1)}).$$

Hence by combining with (2.1.3), we have

$$d_K^{s+1} = P_s(d_{K,(1)}, \cdots, d_{K,(s)}) \cdot d_{K,(1)} + Q_{s+1}(d_{K,(1)}, \cdots, d_{K,(s)}, d_{K,(s+1)}) + e \cdot d_K^s - \sum_{j=0}^{s-1} \alpha_{s,j} e^{s-j+1} d_K^j + \sum_{j=0}^{s-1} \alpha_{s,j} e^{s-j} d_K^{j+1}.$$

This proves our Lemma. □

By the definition of $e, d_{K,(1)}, d_{K,(2)}, \cdots \in \mathrm{Der}_K(A^{(\infty)} \otimes K)$, we have

$$d_{K,(s)}(d^i x_j) = 0 \text{ for } 1 \le s, k+1 \le i$$

$$d_{K,(s)}(a) \in B = A^{(k+l)} \otimes_{\mathbb{C}} K \text{ for } a \in A^{(k)} \otimes K \subset B, \ 1 \le s \le l.$$

Hence we have $d_{K,(s)}(B) \subset B$ for $1 \le s \le l$, so we have

$$P_s(d_{K,(1)}, \cdots, d_{K,(s)})(B) \subset B \text{ for } 1 \le s \le l. \tag{2.1.4}$$

Note that by the definition of e, we have $e(I) \subset I$. So by combining with Lemma 2.1.2, (2.1.4) and the definitions of I_s and δ_s, we get our Claim.

Now for $0 \le s \le l-1$, we conclude that $(B \cap I_s)/(B \cap I_{s+1}) \simeq \delta_s(B \cap I_s)$ and that $\delta_s(B \cap I_s)$ is an ideal of $B/(B \cap I)$. But note that we have $(dx_j)^{s+1} \in \delta_s(B \cap I_s)$ for $1 \le j \le r$. (This is because $x_j^{s+1} \in B \cap I_s$ and $\delta_s(x_j^{s+1}) = \text{const.} \times (dx_j)^{s+1}$.) These show that

$$((B \cap I_s)/(B \cap I_{s+1})) \otimes_{B/(B\cap I)} (B/(B \cap I))_{(dx_j)} \simeq (B/(B \cap I))_{(dx_j)} \quad \text{for } 1 \le j \le r.$$

This completes the proof of Proposition 2.1.1. □

We consider the case V is an Abelian variety A. In this case, we have the canonical decomposition $J_k(A) = A \times J_k$ and $J_k(A)^{\mathrm{reg}} = A \times J_k^{\mathrm{reg}}$. Here $J_k = \mathbb{C}^{k\dim A}$ and $J_k^{\mathrm{reg}} \subset J_k$ is a Zariski open subset. Let $\mathcal{T}$ be the closed subscheme

$$\mathcal{T} \subset J_{k+l}(A) \times J_k(A)_P = A \times J_{k+l} \times J_k$$

as in Proposition 2.1.1. Let $q : \mathcal{T} \to J_{k+l}$ be the composition of morphisms

$$\mathcal{T} \hookrightarrow A \times J_{k+l} \times J_k \xrightarrow{\text{2nd proj}} J_{k+l}.$$

Lemma 2.1.5. *The morphism $q : \mathcal{T} \to J_{k+l}$ is a finite morphism. And the restriction of the direct image sheaf $q_*\mathcal{O}_{\mathcal{T}}$ to J_{k+l}^{reg} is a rank $l+1$ locally free $\mathcal{O}_{J_{k+l}^{reg}}$ module.*

Proof. The image of $J_{k+l} \hookrightarrow J_{k+l}(A) \times J_k(A)_P$ defined at the beginning of this section is given by $\mathrm{supp}\,\mathcal{T} = T$. And the composition of morphisms

$$J_{k+l} \hookrightarrow \mathcal{T} \xrightarrow{q} J_{k+l}$$

is the identity map of J_{k+l}. Now let

$$0 = \mathcal{I}_l \subset \mathcal{I}_{l-1} \subset \cdots \subset \mathcal{I}_1 \subset \mathcal{I}_0 = \mathcal{I}_T \subset \mathcal{O}_{\mathcal{T}}$$

be the filtration of ideals of $\mathcal{O}_{\mathcal{T}}$ as in Proposition 2.1.1 (3). Then the following quotients of the direct image sheaves

$$(*) \qquad q_*\mathcal{I}_s/q_*\mathcal{I}_{s+1} \text{ for } 0 \le s \le l-1, \quad q_*\mathcal{O}_{\mathcal{T}}/q_*\mathcal{I}_0$$

are all finite $\mathcal{O}_{J_{k+l}}$ modules, and so $q_*\mathcal{O}_{\mathcal{T}}$ is also a finite $\mathcal{O}_{J_{k+l}}$ module. Note that q is affine. Hence q is finite.

Restricting the quotient sheaves $(*)$ to J^{reg}_{k+l}, we get rank 1 locally free $\mathcal{O}_{J^{\mathrm{reg}}_{k+l}}$ modules by the same proposition, which proves Lemma 2.1.5. □

2.2. **Nevanlinna's lemma on logarithmic derivatives.** Let A be an Abelian variety and let $J_k(A)$ $(k \geq 0)$ be its jet space. Then we have the natural decomposition $J_k(A) \simeq A \times \mathbb{C}^{k \dim A}$. Consider the compactification $\mathbb{C}^{k \dim A} \subset \mathbb{P}^{k \dim A}$ and let $H_k \subset \mathbb{P}^{k \dim A}$ be the boundary divisor of this compactification. Let $\alpha_k : J_k(A) \to \mathbb{C}^{k \dim A}$ be the natural projection and let $\overline{\alpha_k} : J_k(A) \to \mathbb{P}^{k \dim A}$ be the composition of $\alpha_k : J_k(A) \to \mathbb{C}^{k \dim A}$ and the inclusion $\mathbb{C}^{k \dim A} \to \mathbb{P}^{k \dim A}$.

Proposition 2.2.1. *Let L be an ample line bundle on A. Let $f : \mathbb{C} \to A$ be a holomorphic curve. Then we have*

$$T\left(r, \overline{\alpha_k} \circ j_k(f), H_k\right) \leq O(\log(rT\left(r, f, L\right)))\ ||,$$

where the symbol $||$ means that the inequality holds for $r > 0$ outside a set of finite linear measure.

Proof. Let $dw_1, \cdots, dw_{\dim A}$ be the basis for $H^0(A, \Omega^1_A)$.
For integers $1 \leq n \leq k$ and $1 \leq m \leq \dim A$, put $\zeta^n_m = d^n w_m \circ j_k(f)$. Then ζ^n_m is a holomorphic function on $\mathbb{C}$. Note that the holomorphic curve $\alpha_k \circ j_k(f) : \mathbb{C} \to \mathbb{C}^{k \dim A}$ is represented as $\{\zeta^n_m\}$ $(1 \leq n \leq k,\ 1 \leq m \leq \dim A)$ by using the coordinate system $\{d^n w_m\}$ on $\mathbb{C}^{k \dim A}$. Since we have

$$m\left(r, \zeta^n_m, \infty\right) \leq O(\log(rT\left(r, f, L\right)))\ ||$$

(see [NoO90, p.230]) and $N\left(r, \zeta^n_m, \infty\right) = 0$, we have

$$\begin{aligned} T\left(r, \zeta^n_m, \infty\right) &= N\left(r, \zeta^n_m, \infty\right) + m\left(r, \zeta^n_m, \infty\right) + O(1) \\ &\leq O(\log(rT\left(r, f, L\right)))\ || \end{aligned}$$

for $1 \leq n \leq k$, $1 \leq m \leq \dim A$. This implies our inequality

$$T\left(r, \overline{\alpha_k} \circ j_k(f), H_k\right) \leq O(\log(rT\left(r, f, L\right)))\ ||$$

(cf. [NoO90, p.185]), which complete the proof of our proposition. □

Remarks. (1) See also [Kob00], where another interesting inequality is obtained using methods of the integral geometry.

(2) Since $\zeta^n_m = \frac{d^{n-1}}{dz^{n-1}} \zeta^1_m$, we have

$$T\left(r, \zeta^n_m, \infty\right) \leq O\left(T\left(r, \zeta^1_m, \infty\right)\right) + O\left(\log\left(rT\left(r, \zeta^1_m, \infty\right)\right)\right)\ || \quad \text{(cf. [NoO90, p227])}.$$

Hence we have

$$T\left(r, \overline{\alpha_k} \circ j_k(f), H_k\right) \leq O(T\left(r, \overline{\alpha_1} \circ j_1(f), H_1\right)) + O(\log(rT\left(r, f, L\right)))\ ||. \tag{2.2.2}$$

(3) By the fact that $T\left(r, f, L\right) > O(r^2)$ (cf. [NoO90, p.187]), we have

$$\log(rT\left(r, f, L\right)) < \epsilon T\left(r, f, L\right)\ ||_\epsilon \quad \text{for all } \epsilon > 0.$$

Hence we have $T\left(r, \overline{\alpha_k} \circ j_k(f), H_k\right) \leq \epsilon T\left(r, f, L\right)\ ||_\epsilon$ for all $\epsilon > 0$. Here the symbol $||_\epsilon$ means that the inequality holds for $r > 0$ outside a set (depending on ϵ) of finite linear measure.

2.3. Let A be an Abelian variety and let V be a quasi-projective variety. Let $f : \mathbb{C} \to A$ and $g : \mathbb{C} \to V$ be holomorphic curves. Let L be an ample line bundle on A.

Definition 2.3.1. Take the Zariski closure V' of $g(\mathbb{C})$ in V. Let $\mathbb{C}(V')$ be the rational function field of V'. For $\xi \in \mathbb{C}(V')$, the composition $\xi \circ g$ defines a meromorphic function on $\mathbb{C}$. Hence it gives the holomorphic curve $\xi \circ g : \mathbb{C} \to \mathbb{P}^1$. We say that g is small with respect to f if the inequality

$$T(r, \xi \circ g, \infty) \leq O(\log(rT(r, f, L))) \ ||$$

holds for all $\xi \in \mathbb{C}(V')$.

Remarks. (1) Assume that $g(\mathbb{C})$ is Zariski dense in V. Let $\hat{V} \to V$ be a smooth model of V and let $\hat{g} : \mathbb{C} \to \hat{V}$ be the lifting of g. Then g is small with respect to f if and only if $\hat{g}$ is small with respect to f.

(2) Let $\iota : V \hookrightarrow S$ be an immersion to a smooth projective variety S. Let M be an ample line bundle on S. Then g is small with respect to f if and only if the inequality

$$T(r, \iota \circ g, M) \leq O(\log(rT(r, f, L))) \ ||$$

holds. (cf. [NoO90, p.185])

Lemma 2.3.2. *Keep the notations A,V,f,g,L. Let M_0 be a line bundle on V. Consider the holomorphic curve $\tilde{f} = f \times g : \mathbb{C} \to A \times V$ and a divisor $F \subset A \times V$ which corresponds to a non-zero global section $H^0(A \times V, \pi_1^* L^{\otimes n} \otimes \pi_2^* M_0)$ for some integer n. Here $\pi_1 : A \times V \to A$ is the first projection and $\pi_2 : A \times V \to V$ is the second projection. Assume that g is small with respect to f and that $\tilde{f}(\mathbb{C}) \not\subset \operatorname{supp} F$. Then we have*

$$N\left(r, \tilde{f}, F\right) \leq nT(r, f, L) + O(\log(rT(r, f, L))) \ ||.$$

Proof. By replacing V by a smooth model V' of the Zariski closure of $g(\mathbb{C})$ in V, F by the pull-back of the natural map $A \times V' \to A \times V$ and g by the induced map $g' : \mathbb{C} \to V'$ from g, we may assume that V is smooth and g has Zariski dense image. We may take a compactification $\overline{V}$ of V such that $\overline{V}$ is smooth. Let $\overline{F} \subset A \times \overline{V}$ be a divisor which is an extension of F. Then $\overline{F}$ corresponds to a non-zero global section $H^0(A \times \overline{V}, \overline{\pi_1}^* L^{\otimes n} \otimes \overline{\pi_2}^* \overline{M_0})$ for a line bundle $\overline{M_0}$ on $\overline{V}$ which is an extension of M_0. Here $\overline{\pi_1}$ is the first projection $A \times \overline{V} \to A$ and $\overline{\pi_2}$ is the second projection $A \times \overline{V} \to \overline{V}$. Let $\tilde{f}' : \mathbb{C} \to A \times \overline{V}$ be the holomorphic curve obtained from $\tilde{f}$ and the inclusion $A \times V \subset A \times \overline{V}$. By Nevanlinna's first main theorem, we have

$$T\left(r, \tilde{f}', \overline{F}\right) = nT(r, f, L) + T\left(r, g, \overline{M_0}\right) + O(1).$$

Hence we have

$$T\left(r, \tilde{f}', \overline{F}\right) \leq nT(r, f, L) + O(\log(rT(r, f, L))) \ ||.$$

Since we have

$$N\left(r, \tilde{f}, F\right) = N\left(r, \tilde{f}', \overline{F}\right) \leq T\left(r, \tilde{f}', \overline{F}\right) + O(1),$$

we obtain our lemma. □

2.4. Let A be an Abelian variety. For an integer $k \geq 0$, A naturally acts on $J_k(A)$. For a Zariski closed subset $V \subset J_k(A)$, we put $\mathrm{St}(V) = \{a \in A; a + V = V\}$. Then $\mathrm{St}(V)$ is the closed subgroup variety of A. Let $\mathrm{St}^0(V)$ be the connected component of $\mathrm{St}(V)$ containing the identity element of A. Then $\mathrm{St}^0(V)$ is the Abelian subvariety of A.

For a holomorphic curve $f : \mathbb{C} \to A$, let $W_k(f) \subset J_k(A)$ be the Zariski closure of the image of the holomorphic curve $j_k(f) : \mathbb{C} \to J_k(A)$. In the following lemma, we use the notations in Proposition 2.2.1.

Lemma 2.4.1. *Let A be an Abelian variety and let $f : \mathbb{C} \to A$ be a holomorphic curve such that the image of f is Zariski dense. For an integer $k \geq 0$, let $\kappa : A \to A/\mathrm{St}^0(W_k(f))$ be the quotient map. Let L be an ample line bundle on A and let M be an ample line bundle on $A/\mathrm{St}^0(W_k(f))$. Then we have*

$$T(r, \kappa \circ f, M) \leq O(T(r, \overline{\alpha_1} \circ j_1(f), H_1)) + O(\log r) \ ||.$$

Hence by Proposition 2.2.1, we have

$$T(r, \kappa \circ f, M) \leq O(\log(rT(r, f, L))) \ ||.$$

Remark. This Lemma is a modification of [No98, Lemma 1.2]. For more discussion about the structure of $W_k(f)$, see [NoW02].

Proof. Put $C = A/\mathrm{St}^0(W_k(f))$ and put

$$Y = W_k(f)/\mathrm{St}^0(W_k(f)) \subset C \times J_{k,A} = J_k(A)/\mathrm{St}^0(W_k(f)).$$

Here $J_{k,A} = \mathbb{C}^{k \dim A}$ and $J_k(A) = A \times J_{k,A}$. Let $g : \mathbb{C} \to C \times J_{k,A}$ be the holomorphic curve obtained from the composition of $j_k(f)$ and the quotient map $J_k(A) \to J_k(A)/\mathrm{St}^0(W_k(f))$. Then Y is the Zariski closure of the image of the holomorphic curve g. Let $Y_l \subset J_l(C \times J_{k,A})$ be the Zariski closure of the holomorphic curve $j_l(g)$. We also consider the closed subscheme $Y^{(l)} \subset J_l(C \times J_{k,A})$. Then we have $Y_l \subset \mathrm{supp}\, Y^{(l)}$. We have

$$J_l(C \times J_{k,A}) = C \times J_{l,C} \times J_l(J_{k,A}).$$

Let $\chi_l^1 : J_l(C \times J_{k,A}) \to J_{l,C}$ be the second projection and let $\chi_l^2 : J_l(C \times J_{k,A}) \to J_l(J_{k,A})$ be the third projection. Put $\chi_l = \chi_l^1 \times \chi_l^2 : J_l(C \times J_{k,A}) \to J_{l,C} \times J_l(J_{k,A})$. Note that χ_l is a proper morphism, hence the image $\chi_l(Y_l) \subset J_{l,C} \times J_l(J_{k,A})$ is a Zariski closed subset.

Put $V_l = Y_l \cap \chi_l^{-1}(\chi_l(j_l(g)(0)))$. Note that we have $\chi_l^{-1}(\chi_l(j_l(g)(0))) \simeq C$. Hence we have a nested sequence of Zariski closed sets,

$$C \supset V_1 \supset V_2 \supset V_3 \supset \cdots$$

that eventually stabilizes at the variety V. Since we have $\kappa \circ f(0) \in V_l$ for all $l \geq 0$, we have $V \neq \emptyset$. Note that C naturally acts on $C \times J_{k,A}$ by considering the trivial action on $J_{k,A}$. For $a \in V$, let $\tilde{g} : \mathbb{C} \to C \times J_{k,A}$ be the holomorphic curve $\tilde{g}(z) = g(z) + a - \kappa \circ f(0)$ which is a translate of g by $a - \kappa \circ f(0) \in C$. Then by the construction of $\tilde{g}$, we have $j_l(\tilde{g})(0) \in Y_l \subset \mathrm{supp}\, Y^{(l)}$ for all $l \geq 0$, hence by Taylor series, we have $\tilde{g}(\mathbb{C}) \subset Y$. Since the image of g is Zariski dense in Y, this shows that $\kappa \circ f(0) - a \in C$ stabilizes Y for all $a \in V$. Hence by the construction of Y, we conclude that $\dim V = 0$. By taking sufficiently large $l > 0$, we have $V_l = V$ and

$\dim V_l = 0$. By considering the induced proper morphism $Y_l \to \chi_l(Y_l)$ from χ_l, we conclude that

$$(2.4.2) \qquad \dim Y_l = \dim \chi_l(Y_l).$$

We have $J_{l,C} \times J_l(J_{k,A}) \simeq \mathbb{C}^{l\,\dim C+(l+1)k\,\dim A}$. Let

$$\mathbb{C}^{l\,\dim C+(l+1)k\,\dim A} \subset \mathbb{P}^{l\dim C+(l+1)k\dim A}$$

be the compactification and let $G \subset \mathbb{P}^{l\dim C+(l+1)k\dim A}$ be the boundary divisor of this compactification. Let $\overline{Y_l}$ be the Zariski closure of Y_l in the compactification

$$J_l(C \times J_{k,A}) = C \times \mathbb{C}^{l\,\dim C+(l+1)k\,\dim A} \subset C \times \mathbb{P}^{l\dim C+(l+1)k\dim A}.$$

Let $\overline{\chi_l}' : \overline{Y_l} \to \mathbb{P}^{l\dim C+(l+1)k\dim A}$ be the morphism obtained by the restriction of the second projection

$$C \times \mathbb{P}^{l\dim C+(l+1)k\dim A} \to \mathbb{P}^{l\dim C+(l+1)k\dim A}$$

to $\overline{Y_l}$. Since G is an ample divisor on $\mathbb{P}^{l\dim C+(l+1)k\dim A}$, using (2.4.2), we know that the divisor $G' = \overline{\chi_l}'^* G$ is a big divisor on $\overline{Y_l}$.

Let Q be an ample line bundle on $\overline{Y_l}$. Let $\pi : \overline{Y_l} \to C$ be the natural projection. We also denote $j_l(g)$ for the holomorphic curve $\mathbb{C} \to \overline{Y_l}$ naturally induced from the holomorphic curve $j_l(g) : \mathbb{C} \to J_l(C \times J_{k,A})$. Then we have $\pi \circ j_l(g) = \kappa \circ f$. Hence we have

$$(2.4.3) \qquad T(r, \kappa \circ f, M) \le O(T(r, j_l(g), Q)) + O(1).$$

Since G' is a big divisor, there are positive integers n and m such that $H^0(\overline{Y_l}, mG' - nQ) \ne 0$. Let F be a divisor on $\overline{Y_l}$ which corresponds to this non-zero global section. Then by Nevanlinna's first main theorem, we have

$$mT(r, j_l(g), G') - nT(r, j_l(g), Q) = T(r, j_l(g), F) + O(1).$$

Since the image of the holomorphic curve $j_l(g)$ is Zariski dense in $\overline{Y_l}$, we have

$$T(r, j_l(g), F) > O(1)$$

by Nevanlinna's inequality. Hence we have

$$(2.4.4) \qquad T(r, j_l(g), Q) \le O(T(r, j_l(g), G')) + O(1).$$

Using (2.4.3), we have

$$(2.4.5) \qquad T(r, \kappa \circ f, M) \le O(T(r, j_l(g), G')) + O(1).$$

Let ζ_m^n $(1 \le n \le k, 1 \le m \le \dim A)$ be the holomorphic functions on $\mathbb{C}$ introduced in the proof of Proposition 2.2.1. Then the holomorphic curve $\chi_l^2 \circ j_l(g) : \mathbb{C} \to J_l(J_{k,A})$ is represented as $\{\frac{d^j}{dz^j}\zeta_m^n\}$ for $0 \le j \le l$, $1 \le n \le k$ and $1 \le m \le \dim A$. Since we have $\zeta_m^n = \frac{d^{n-1}}{dz^{n-1}}\zeta_m^1$, we have

$$(2.4.6) \qquad T\left(r, \frac{d^j}{dz^j}\zeta_m^n, \infty\right) \le O\left(T\left(r, \zeta_m^1, \infty\right)\right) + O\left(\log\left(rT\left(r, \zeta_m^1, \infty\right)\right)\right) \ ||.$$

Let $du_1, \cdots, du_{\dim C}$ be a basis for $H^0(C, \Omega_C^1)$. For integers $1 \le n' \le l$ and $1 \le m' \le \dim C$, put $\tilde{\zeta}_{m'}^{n'} = d^{n'} u_{m'} \circ j_l(\kappa \circ f)$. Then $\tilde{\zeta}_{m'}^{n'}$ is a holomorphic function on $\mathbb{C}$. The holomorphic curve $\chi_l^1 \circ j_l(g) : \mathbb{C} \to J_{l,C}$ is represented as $\{\tilde{\zeta}_{m'}^{n'}\}$ for $1 \leq n' \leq l$ and $1 \leq m' \leq \dim C$. Then we have

$$(2.4.7) \qquad T\left(r, \tilde{\zeta}_{m'}^{n'}, \infty\right) \leq O\left(T\left(r, \tilde{\zeta}_{m'}^{1}, \infty\right)\right) + O\left(\log\left(rT\left(r, \tilde{\zeta}_{m'}^{1}, \infty\right)\right)\right) \ ||.$$

Since there is the natural linear map $J_{1,A} \to J_{1,C}$, there are constants $c_{m,m'} \in \mathbb{C}$ for $1 \leq m \leq \dim A$ and $1 \leq m' \leq \dim C$ such that

$$\tilde{\zeta}_{m'}^{1} = \sum_{m=1}^{\dim A} c_{m,m'} \zeta_m^1.$$

Hence we have

$$T\left(r, \tilde{\zeta}_{m'}^{1}, \infty\right) \leq O\left(\sum_{m=1}^{\dim A} T\left(r, \zeta_m^1, \infty\right)\right) + O(1).$$

Since $T\left(r, \zeta_m^1, \infty\right) \leq O(T\left(r, \overline{\alpha_1} \circ j_1(f), H_1\right)) + O(1)$ for $1 \leq m \leq \dim A$, using (2.4.6) and (2.4.7), we have

$$T\left(r, \overline{\chi_l}' \circ j_l(f), G\right) \leq O(T\left(r, \overline{\alpha_1} \circ j_1(f), H_1\right)) + O(\log(rT\left(r, \overline{\alpha_1} \circ j_1(f), H_1\right))) \ ||.$$

Hence by (2.4.5), we obtain our lemma. □

Since $\mathrm{St}^0(W_k(f)) \supset \mathrm{St}^0(W_{k'}(f))$ for $k \leq k'$, there is a nested sequence of Abelian subvarieties

$$A \supset \mathrm{St}^0(W_1(f)) \supset \mathrm{St}^0(W_2(f)) \supset \mathrm{St}^0(W_3(f)) \supset \cdots$$

that eventually stabilizes at an Abelian subvariety $B_0 \subset A$.

Corollary 2.4.8. *Let $Z \subset J_k(A)$ be a closed subscheme. Let L be an ample line bundle on A. Let $B \subset A$ be an Abelian subvariety such that $B_0 \subset B$. Let $f : \mathbb{C} \to A$ be a holomorphic curve such that the image of f is Zariski dense. Let $\xi_k : J_k(A) \to J_k(A)/B$ be the quotient map. Assume that $\xi_k \circ j_k(f)(\mathbb{C}) \not\subset \xi_k(\mathrm{supp}\, Z)$. Then we have*

$$N\left(r, j_k(f), Z\right) \leq O(\log(rT\left(r, f, L\right))) \ ||.$$

Proof. Note that the morphism ξ_k is proper, hence $\xi_k(\mathrm{supp}\, Z)$ is a Zariski closed subset of $J_k(A)/B$. By the assumption $\xi_k \circ j_k(f)(\mathbb{C}) \not\subset \xi_k(\mathrm{supp}\, Z)$, there is an ample divisor E of $C \times \mathbb{C}^{k \dim A}$ such that $\xi_k(\mathrm{supp}\, Z) \subset E$ and $\xi_k \circ j_k(f)(\mathbb{C}) \not\subset E$.

We have $J_k(A)/B = C \times \mathbb{C}^{k \dim A}$. Let $J_k(A)/B \subset C \times \mathbb{P}^{k \dim A}$ be the compactification and let $\overline{E} \subset C \times \mathbb{P}^{k \dim A}$ be the extension of E. We also denote by $\xi_k \circ j_k(f)$ the composition map $\mathbb{C} \overset{\xi_k \circ j_k(f)}{\to} C \times \mathbb{C}^{k \dim A} \subset C \times \mathbb{P}^{k \dim A}$.

By Proposition 2.2.1 and Lemma 2.4.1, the holomorphic curve $\xi_k \circ j_k(f)$ is small with respect to f. Hence we have

$$\begin{aligned} N\left(r, j_k(f), Z\right) &\leq O\left(N\left(r, \xi_k \circ j_k(f), \xi_k(\mathrm{supp}\, Z)\right)\right) \\ &\leq O\left(N\left(r, \xi_k \circ j_k(f), \overline{E}\right)\right) \leq O\left(T\left(r, \xi_k \circ j_k(f), \overline{E}\right)\right) \leq O(\log(rT\left(r, f, L\right))) \ ||. \end{aligned}$$

This proves our corollary. □

2.5. The following Theorem is the main result of this section.

Theorem 2.5.1. *Let A be an Abelian variety and let E be an ample line bundle on A. Let $f : \mathbb{C} \to A$ be a holomorphic curve such that the image of f is Zariski dense. For an integer $k \geq 0$, let $Z \subset W_k(f)$ be a subvariety which has codimension greater than one;* $\operatorname{codim}(Z, W_k(f)) \geq 2$. *Then we have*

$$N^{(1)}(r, j_k(f), Z) \leq \epsilon T(r, f, E) \ ||_\epsilon \quad \textit{for all } \epsilon > 0.$$

Proof. Note that we have $\mathrm{St}^0(W_n(f)) \subset \mathrm{St}^0(W_{n'}(f))$ for $n' \leq n$, hence we have the nested sequence

$$\mathrm{St}^0(W_0(f)) \supset \mathrm{St}^0(W_1(f)) \supset \mathrm{St}^0(W_2(f)) \supset \cdots,$$

which eventually stabilizes at an Abelian subvariety B of A. Let $\kappa : A \to A/B$ be the natural projection. Put $C = A/B$. Then by Poincaré's complete reducibility theorem, there exists an isogeny $C \to C$ such that the base change of the quotient $A \to C$ by this isogeny is the second projection $B \times C \to C$. Replace A by $B \times C$, f by a lifting $f' : \mathbb{C} \to B \times C$ of f over the induced isogeny $B \times C \to A$ and Z by the pull-back Z' with the induced map $W_k(f') \to W_k(f)$. Define the Abelian subvariety $B' \subset B \times C$ from f' by the same manner for the definition of B. Then we have $B' = B$, $\operatorname{codim}(Z', W_k(f')) = \operatorname{codim}(Z, W_k(f))$ and $N(r, j_k(f), Z) = N(r, j_k(f'), Z')$. Hence, to prove our Theorem, we may assume that A has splitting $A = B \times C$.

We may also assume that Z is irreducible. Hence in the following, we prove our Theorem under these assumptions.

Let $Z^{r.s}$ be a Zariski open subset of Z such that $Z^{r.s}$ is nonsingular. It is sufficient to show

$$N^{(1)}(r, j_k(f), Z^{r.s}) \leq \epsilon T(r, f, E) \ ||_\epsilon \quad \text{for all } \epsilon > 0. \tag{2.5.2}$$

For we have

$$N^{(1)}(r, j_k(f), Z) = N^{(1)}(r, j_k(f), Z^{r.s}) + N^{(1)}(r, j_k(f), Z - Z^{r.s})$$

and the 2nd term of the right hand side is small by induction on codimension. Here note that $\dim Z > \dim(Z - Z^{r.s})$.

To prove (2.5.2), we can assume that the image of the composition of the following natural morphisms

$$Z^{r.s} \to W_k(f) \to W_k(f)/B$$

is Zariski dense, otherwise (2.5.2) is obtained by Corollary 2.4.8. Here note that $W_k(f)/B$ is a Zariski closed subset of $J_k(A)/B$.

We have $W_k(f) = B \times (W_k(f)/B)$ by the assumption $A = B \times C$. Let $\gamma_k : W_k(f) \to B$ be the first projection and let $\pi_k : W_k(f) \to W_k(f)/B$ be the second projection.

Lemma 2.5.3. *There exist an ample line bundle L on B and a sequence of positive integers $n(1), n(2), n(3), \cdots$ such that*

(1) $\frac{n(l)}{l} \to 0$ *when* $l \to \infty$

(2) *there exist an effective Cartier divisor* $F_l \subset W_{k+l}(f)$ *and a line bundle* M_l *on* $W_{k+l}(f)/B$ *with the following properties;* F_l *corresponds to a non-zero global section*

$$H^0(W_{k+l}(f), \gamma_{k+l}^* L^{\otimes n(l)} \otimes (\pi_{k+l})^* M_l),$$

and for every point $a \in \mathbb{C}$ *with* $j_k(f)(a) \in \operatorname{supp} Z^{r.s}$, *we have* $\operatorname{mult}_a j_{k+l}(f) \cdot F_l \geq l+1$.

Proof. Let $r_1 : Z^\dagger \to Z$ be a desingularization of Z such that r_1 gives an isomorphism over $Z^{r.s}$. Put $V_k = W_k(f)/B$. Consider the sequence of morphisms

$$Z^{r.s} \overset{r_0}{\hookrightarrow} Z^\dagger \overset{r_1}{\to} Z \overset{r_2}{\hookrightarrow} W_k(f) \overset{r_3}{\to} V_k. \tag{2.5.4}$$

Here r_0, $r_1 \circ r_0$ are open immersions and r_2 is a closed immersion. And $r_3 = \pi_k$. Put $r = r_3 \circ r_2 \circ r_1 : Z^\dagger \to V_k$. Let V_k^r be a Zariski open subset of V_k such that V_k^r is nonsingular and that the fibers of $r : Z^\dagger \to V_k$ over V_k^r are all of the same dimension $\dim Z^\dagger - \dim V_k$. Then the restriction of the family $r : Z^\dagger \to V_k$ to V_k^r is a flat family.

Consider the pull back of the sequence of morphisms (2.5.4) by the natural projection $B \times V_k \to V_k$:

$$B \times Z^{r.s} \overset{s_0}{\hookrightarrow} B \times Z^\dagger \overset{s_1}{\to} B \times Z \overset{s_2}{\hookrightarrow} B \times W_k(f) \overset{s_3}{\to} B \times V_k.$$

Put $s = s_3 \circ s_2 \circ s_1 : B \times Z^\dagger \to B \times V_k$. Then s maps as

$$B \times Z^\dagger \ni (a, z) \overset{s}{\mapsto} (a, r(z)) \in B \times V_k.$$

Let L be an ample line bundle on B and let $\phi : B \times W_k(f) \to B$ be the morphism

$$B \times W_k(f) \ni (a, w) \mapsto a + \gamma_k(w) \in B.$$

Let $L_1^\dagger$ be the line bundle on $B \times Z^\dagger$ which is the pull back of L by the composition of morphisms

$$B \times Z^\dagger \overset{s_2 \circ s_1}{\to} B \times W_k(f) \overset{\phi}{\to} B.$$

Since the restriction of s to $B \times V_k^r$ (i.e. $s|_{V_k^r} : B \times Z^\dagger|_{V_k^r} \to B \times V_k^r$) is a flat family, the semicontinuity theorem [H77, p.288] implies that there is a Zariski open subset $U_n \subset B \times V_k^r$ $(n > 0)$ such that $H^0(B \times Z^\dagger|_P, L_{1,P}^{\dagger \otimes n})$ are all the same dimensional $\mathbb{C}$ vector spaces for $P \in U_n$, put this dimension as G_n. Here $B \times Z^\dagger|_P$ denotes the fiber of the morphism $s : B \times Z^\dagger \to B \times V_k$ over $P \in B \times V_k$, and $L_{1,P}^{\dagger \otimes n}$ is the induced line bundle. Since the intersection $\bigcap_n U_n$ is non-empty, put $(a, w) \in \bigcap_n U_n$. Replacing L by its pull back by the morphism

$$B \ni x \mapsto x + a \in B,$$

we can assume a to be $0 \in B$; the identity element of B.

Now for a positive integer $l > 0$, let $\mathcal{T}_l \subset J_{k+l}(A) \times J_k(A)_0$ be the closed subscheme obtained in Proposition 2.1.1 by putting $P = 0$. Let $\mathcal{T}_l^\dagger \subset A \times J_{k+l} \times J_k \times C$ be the closed subscheme obtained by taking the base change of $\mathcal{T}_l$ by the morphism:

$$\varpi : A \times J_{k+l} \times J_k \times C \to A \times J_{k+l} \times J_k \simeq J_{k+l}(A) \times J_k(A)_0$$

where ϖ maps

$$A \times J_{k+l} \times J_k \times C \ni (a, v, v', c) \overset{\varpi}{\mapsto} (a - c, v, v') \in A \times J_{k+l} \times J_k.$$

Let $\lambda : \mathcal{T}_l^\dagger \to C \times J_{k+l}$ be the morphism obtained by the composition of the natural inclusion

$$\mathcal{T}_l^\dagger \to A \times J_{k+l} \times J_k \times C$$

and the morphism $A \times J_{k+l} \times J_k \times C \to C \times J_{k+l}$ such that

$$A \times J_{k+l} \times J_k \times C \ni (a, v, v', c) \mapsto (\kappa(a), v) \in C \times J_{k+l}.$$

Then by Lemma 2.1.5, λ has the following properties;

- λ is finite,
- the direct image sheaf $\lambda_* \mathcal{O}_{\mathcal{T}_l^\dagger}$ is locally generated by $l+1$ elements as an $\mathcal{O}_{C \times J_{k+l}}$ module on $C \times J_{k+l}^{\mathrm{reg}}$,
- λ gives an isomorphism of underlying topological spaces of $\mathcal{T}_l^\dagger$ and $C \times J_{k+l}$.

To see these properties, we note the following. Let $\mathcal{T}_l^* \subset A \times J_{k+l} \times J_k \times C$ be the closed subscheme obtained by the pull back of $\mathcal{T}_l$ by the morphism

$$A \times J_{k+l} \times J_k \times C \simeq J_{k+l}(A) \times J_k(A)_0 \times C \to A \times J_{k+l} \times J_k \simeq J_{k+l}(A) \times J_k(A)_0$$

such that

$$A \times J_{k+l} \times J_k \times C \ni (a, v, v', c) \mapsto (a, v, v') \in A \times J_{k+l} \times J_k.$$

Let $\lambda^* : \mathcal{T}_l^* \to C \times J_{k+l}$ be the morphism obtained by the composition of the natural inclusion

$$\mathcal{T}_l^* \subset A \times J_{k+l} \times J_k \times C$$

and the morphism $A \times J_{k+l} \times J_k \times C \to C \times J_{k+l}$ such that

$$A \times J_{k+l} \times J_k \times C \ni (a, v, v', c) \mapsto (c, v) \in C \times J_{k+l}.$$

Then by Lemma 2.1.5, $\mathcal{T}_l^*$ and λ^* satisfy the above properties. Consider the morphism

$$\mu : A \times J_{k+l} \times J_k \times C \to A \times J_{k+l} \times J_k \times C$$

such that

$$A \times J_{k+l} \times J_k \times C \ni (a, v, v', c) \mapsto (a - c, v, v', \kappa(a)) \in A \times J_{k+l} \times J_k \times C.$$

Then this μ is an isomorphism and $\mathcal{T}_l^\dagger$ is obtained by the pull back of $\mathcal{T}_l^*$ by μ. Let $\mu' : \mathcal{T}_l^\dagger \to \mathcal{T}_l^*$ be the induced isomorphism from μ. Then we have $\lambda^* \circ \mu' = \lambda$. This gives the above properties for $\mathcal{T}_l^\dagger$ and λ.

Now V_{k+l} is a Zariski closed subset of $C \times J_{k+l}$. We denote $\sigma_{k+l} : V_{k+l} \to C$ for the composition with the first projection $C \times J_{k+l} \to C$ and $\eta_{k+l} : V_{k+l} \to J_{k+l}$ for the composition with the second projection. We have the closed immersion

$$B \times V_{k+l} \times V_k \subset B \times C \times J_{k+l} \times J_k \times C \simeq A \times J_{k+l} \times J_k \times C \tag{2.5.5}$$

where the first inclusion is given by

$$B \times V_{k+l} \times V_k \ni (b, v, v') \mapsto (b, \sigma_{k+l}(v), \eta_{k+l}(v), \eta_k(v'), \sigma_k(v')) \in B \times C \times J_{k+l} \times J_k \times C$$

and the second identification is given by

$$B \times C \times J_{k+l} \times J_k \times C \ni (b, c, u, u', c') \mapsto ((b, c), u, u', c') \in A \times J_{k+l} \times J_k \times C.$$

Let $\mathcal{S}_l \subset B \times V_{k+l} \times V_k$ be the closed subscheme obtained by the pull-back of $\mathcal{T}_l^\dagger$ by (2.5.5). Let $q : \mathcal{S}_l \to V_{k+l}$ be the composition with the second projection $B \times V_{k+l} \times$

$V_k \to V_{k+l}$. Then by the above properties of λ, we have the corresponding properties for q;

- q is finite,
- the direct image image sheaf $q_* \mathcal{O}_{\mathcal{S}_l}$ is locally generated by $l+1$ elements as an $\mathcal{O}_{V_{k+l}}$ module on V_{k+l}^{reg},
- q gives an isomorphism of underlying topological spaces of $\mathcal{S}_l$ and V_{k+l}.

Here we put $V_{k+l}^{\mathrm{reg}} = V_{k+l} \cap (C \times J_{k+l}^{\mathrm{reg}})$, which is a Zariski open subset of V_{k+l}.

We consider the following commutative diagram (2.5.6) obtained by the base change of (2.5.4) with a sequence of morphisms

$$\mathcal{S}_l \hookrightarrow B \times V_{k+l} \times V_k \to B \times V_k \to V_k.$$

Here $B \times V_{k+l} \times V_k \to B \times V_k$ is the natural projection:

$$B \times V_{k+l} \times V_k \ni (a, w, w') \mapsto (a, w') \in B \times V_k.$$

$$\begin{array}{ccccccc}
\mathcal{Z}_l^{r.s} & \longrightarrow & B \times V_{k+l} \times Z^{r.s} & \longrightarrow & B \times Z^{r.s} & \longrightarrow & Z^{r.s} \\
\Big\downarrow u_0 & & \Big\downarrow t_0 & & \Big\downarrow s_0 & & \Big\downarrow r_0 \\
\mathcal{Z}_l^{\dagger} & \longrightarrow & B \times V_{k+l} \times Z^{\dagger} & \longrightarrow & B \times Z^{\dagger} & \longrightarrow & Z^{\dagger} \\
\Big\downarrow u_1 & & \Big\downarrow t_1 & & \Big\downarrow s_1 & & \Big\downarrow r_1 \\
\mathcal{Z}_l & \xrightarrow{v'} & B \times V_{k+l} \times Z & \longrightarrow & B \times Z & \longrightarrow & Z \\
\Big\downarrow u_2 & & \Big\downarrow t_2 & & \Big\downarrow s_2 & & \Big\downarrow r_2 \\
\cdot & \longrightarrow & B \times V_{k+l} \times W_k(f) & \longrightarrow & B \times W_k(f) & \longrightarrow & W_k(f) \\
\Big\downarrow u_3 & & \Big\downarrow t_3 & & \Big\downarrow s_3 & & \Big\downarrow r_3 \\
\mathcal{S}_l & \xrightarrow{v} & B \times V_{k+l} \times V_k & \longrightarrow & B \times V_k & \longrightarrow & V_k
\end{array} \tag{2.5.6}$$

Let $\mathcal{L}_l^{\dagger}$ be the line bundle on $\mathcal{Z}_l^{\dagger}$ obtained by the pull back of $L_1^{\dagger}$ by the morphisms in the above diagram (2.5.6). Let $\mathcal{S}_{l,n}$ be the non-empty Zariski open subset of $\mathcal{S}_l$ obtained by the inverse image of U_n. Then since $\dim H^0(B \times Z^{\dagger}|_P, L_{1,P}^{\dagger \otimes n}) = G_n$ for $P \in U_n$, the direct image sheaf $s_* L_1^{\dagger \otimes n}$ is a locally free sheaf of rank G_n on U_n and the natural map

$$s_* L_1^{\dagger \otimes n} \otimes \mathbb{C}(P) \to H^0(B \times Z^{\dagger}|_P, L_{1,P}^{\dagger \otimes n})$$

is an isomorphism for $P \in U_n$. This follows by the Theorem of Grauert [H77, p.288] since U_n is reduced and irreducible. Here $s : B \times Z^{\dagger} \to B \times V_k$ is the natural map; i.e. $s = s_3 \circ s_2 \circ s_1$. Let u be the morphism $u : \mathcal{Z}_l^{\dagger} \to \mathcal{S}_l$ obtained by the composition $u = u_3 \circ u_2 \circ u_1$, where u_1,u_2,u_3 are the morphisms in the above diagram (2.5.6). Then the natural map

$$u_* \mathcal{L}_l^{\dagger \otimes n} \otimes \mathbb{C}(P) \to H^0(\mathcal{Z}_l^{\dagger}|_P, \mathcal{L}_{l,P}^{\dagger \otimes n})$$

is also surjective, so an isomorphism on $P \in \mathcal{S}_{l,n}$. This follows by the Theorem of Cohomology and Base Change [H77, p.290]. Hence $u_* \mathcal{L}_l^{\dagger \otimes n}$ is locally generated by

G_n elements as an $\mathcal{O}_{\mathcal{S}_l}$ module on $\mathcal{S}_{l,n} \subset \mathcal{S}_l$. Let $V_{k+l,n} = q(\mathcal{S}_{l,n})$ be a non-empty Zariski open subset of V_{k+l} (note that the underlying topological spaces of $\mathcal{S}_l$ and V_{k+l} are the same).
Then by the above properties of q, the direct image sheaf $(q \circ u)_* \mathcal{L}_l^{\dagger \otimes n}$ is locally generated by $(l+1)G_n$ elements as an $\mathcal{O}_{V_{k+l}}$ module on $V_{k+l,n} \cap V_{k+l}^{\mathrm{reg}}$. Here, note that V_{k+l}^{reg} is non-empty (otherwise f must be constant) and V_{k+l} is irreducible. Hence $V_{k+l,n} \cap V_{k+l}^{\mathrm{reg}}$ is also non-empty.

Now look at the following commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{Z}_l^{r.s} & & & & & & \\
\downarrow{\scriptstyle u_0} & & & & & & \\
\mathcal{Z}_l^{\dagger} & \xrightarrow{t_2 \circ v' \circ u_1} & B \times V_{k+l} \times W_k(f) & \xrightarrow{\psi} & B \times V_{k+l} & \xrightarrow{\rho} & B \\
\downarrow{\scriptstyle q \circ u} & & \downarrow{\scriptstyle \text{2nd proj}} & & \downarrow{\scriptstyle \tau} & & \\
V_{k+l} & = & V_{k+l} & = & V_{k+l} & &
\end{array}
$$

where ρ is the first projection, τ is the second projection and ψ is the morphism

$$\psi : B \times V_{k+l} \times W_k(f) \ni (a, v, w) \mapsto (a + \gamma_k(w), v) \in B \times V_{k+l}.$$

Since $(\rho \circ \psi \circ t_2 \circ v' \circ u_1)^* L = \mathcal{L}_l^{\dagger}$, we have a natural morphism

$$\tau_* \rho^* L^{\otimes n} = H^0(B, L^{\otimes n}) \otimes_{\mathbb{C}} \mathcal{O}_{V_{k+l}} \to (q \circ u)_* \mathcal{L}_l^{\dagger \otimes n}. \tag{2.5.7}$$

Here, note that $\rho \circ \psi = \phi \circ \beta$ where $\beta : B \times V_{k+l} \times W_k(f) \to B \times W_k(f)$ is the morphism in the diagram (2.5.6) and ϕ is the morphism used in the definition of the line bundle $L_1^{\dagger}$.

Put $I_n = \dim_{\mathbb{C}} H^0(B, L^{\otimes n})$. Then there are a positive integer n_0 and positive constants C_1, C_2 such that

$$I_n > C_1 n^{\dim B}, \quad G_n < C_2 n^{\dim B - 2} \quad \text{for } n > n_0.$$

Here note that $G_n = \dim_{\mathbb{C}} H^0(B \times Z^{\dagger}|_P, L_{1,P}^{\dagger \otimes n})$ for $P \in \bigcap_n U_n$, and $B \times Z^{\dagger}|_P = s^{-1}(P)$ has dimension $\leq \dim B - 2$ since we have $\mathrm{codim}(Z, W_k(f)) \geq 2$ and $r_3 \circ r_2 : Z \to V_k$ is dominant. Hence for a positive integer l, we can take a positive integer $n(l)$ such that

$$I_{n(l)} > (l+1) G_{n(l)}, \quad \lim_{l \to \infty} \frac{n(l)}{l} \to 0.$$

Let $\mathcal{F}$ be the kernel of (2.5.7) for $n = n(l)$;

$$0 \to \mathcal{F} \to \tau_* \rho^* L^{\otimes n(l)} \to (q \circ u)_* \mathcal{L}_l^{\dagger \otimes n(l)} \quad \text{(exact)}.$$

Then we have $\mathcal{F} \neq 0$. By taking tensor product of a sufficiently ample line bundle M_l on V_{k+l} and $\mathcal{F}$, we may assume that $H^0(V_{k+l}, \mathcal{F} \otimes M_l) \neq 0$. Since we have

$$
\begin{aligned}
H^0(V_{k+l}, \mathcal{F} \otimes M_l) &\subset H^0(V_{k+l}, (\tau_* \rho^* L^{\otimes n(l)}) \otimes M_l) \\
&= H^0(V_{k+l}, \tau_*(\rho^* L^{\otimes n(l)} \otimes \tau^* M_l)) = H^0(B \times V_{k+l}, \rho^* L^{\otimes n(l)} \otimes \tau^* M_l),
\end{aligned}
$$

we may take a divisor $F_l \subset B \times V_{k+l}$ which corresponds to a non-zero global section of $H^0(V_{k+l}, \mathcal{F} \otimes M_l)$. Then we have

$$\mathcal{Z}_l^{r.s} \subset \psi^* F_l.$$

Here note that $\mathcal{Z}_l^{r.s} \subset \mathcal{Z}_l$ is an open immersion and $\mathcal{Z}_l \overset{t_2 \circ v'}{\hookrightarrow} B \times V_{k+l} \times W_k(f)$ is a closed subscheme.

Using the decomposition $A \simeq B \times C$, we let $f_B : \mathbb{C} \to B$ be the holomorphic curve obtained by the composition of f and the first projection $A \to B$, and let $f_C : \mathbb{C} \to C$ be the holomorphic curve obtained by the composition of f and the second projection $A \to C$.

Now let $a \in \mathbb{C}$ be a point such that $j_k(f)(a) \in Z^{r.s}$. Define $\widetilde{f} : \mathbb{C} \to B \times V_{k+l} \times W_k(f)$ by

$$\widetilde{f}(z) = (f_B(z) - f_B(a), \pi_{k+l} \circ j_{k+l}(f)(z), j_k(f)(a)).$$

Then we have

$$\widetilde{f}(\mathbb{C}) \subset B \times V_{k+l} \times Z, \quad \widetilde{f}(a) \in \operatorname{supp} \mathcal{Z}_l^{r.s}, \quad \psi \circ \widetilde{f} = j_{k+l}(f),$$

where the last equality holds under the identification $B \times V_{k+l} \simeq W_{k+l}(f)$.

Since v' is the base change of v in (2.5.6) and $\widetilde{f}$ factors through t_2, we have

$$\operatorname{mult}_a \widetilde{f} \cdot \mathcal{Z}_l = \operatorname{mult}_a (t_3 \circ \widetilde{f}) \cdot \mathcal{S}_l,$$

hence by the construction of $\mathcal{S}_l$ and Proposition 2.1.1 (2), we have

$$\operatorname{mult}_a \widetilde{f} \cdot \mathcal{Z}_l = \operatorname{mult}_a (j_{k+l}(f) - f(a)) \cdot \mathcal{T}_{l,(j_k(f)-f(a))(a)} \geq l+1.$$

Hence we have

$$\operatorname{mult}_a j_{k+l}(f) \cdot F_l = \operatorname{mult}_a \widetilde{f} \cdot \psi^* F_l \geq \operatorname{mult}_a \widetilde{f} \cdot \mathcal{Z}_l{}^{r.s} = \operatorname{mult}_a \widetilde{f} \cdot \mathcal{Z}_l \geq l+1,$$

which proves our Lemma 2.5.3. (Note that we consider F_l as the divisor on $W_{k+l}(f)$ by the identification $B \times V_{k+l} \simeq W_{k+l}(f)$, and that τ corresponds to π_{k+l} by this identification.) □

Now we go back to the proof of our theorem. By the above lemma, we have

$$N(r, j_{k+l}(f), F_l) \geq (l+1) N^{(1)}(r, j_k(f), Z^{r.s}).$$

But Lemma 2.3.2 implies

$$N(r, j_{k+l}(f), F_l) \leq n(l) T(r, f_B, L) + \epsilon T(r, f_B, L) \ ||_\epsilon \quad \text{for all } \epsilon > 0.$$

Here note that by Proposition 2.2.1 and Lemma 2.4.1, $\pi_{k+l} \circ j_{k+l}(f)$ is small with respect to f_B . Hence we have

$$(l+1) N^{(1)}(r, j_k(f), Z^{r.s}) \leq n(l) T(r, f_B, L) + \epsilon T(r, f_B, L) \ ||_\epsilon \quad \text{for all } \epsilon > 0.$$

We have $\lim_{l \to \infty} \frac{n(l)}{l+1} \to 0$ and $O(T(r, f_B, L)) = O(T(r, f, E))||$ for the ample line bundle E on A (cf. Lemma 2.4.1), which proves (2.5.2) and our Theorem 2.5.1. □

Corollary 2.5.8. *Let A, E and f be the same as Theorem 2.5.1. Let $Z \subset W_k(f)$ be a closed subscheme whose support has codimension greater than* 1*;* $\operatorname{codim}(\operatorname{supp} Z, W_k(f)) \geq 2$*. Then we have*

$$N(r, j_k(f), Z) \leq \epsilon T(r, f, E) \ ||_\epsilon \quad \textit{for all } \epsilon > 0.$$

Proof. Consider Z as a closed subscheme of $J_k(A)$. We consider the closed subscheme $Z^{(s)} \subset J_{k+s}(A)$ for $s \geq 0$. Then by Corollary 1.3.4, we have

$$N(r, j_k(f), Z) - sN^{(1)}(r, j_k(f), Z) \leq N\left(r, j_{k+s}(f), Z^{(s)}\right). \tag{2.5.9}$$

Let $B \subset A$ be the same as in the proof of Theorem 2.5.1. Put $C = A/B$. As in the proof of Theorem 2.5.1, it suffices to prove our corollary in the case $A = B \times C$. Let $\xi_k : J_k(A) \to J_k(A)/B$ be the quotient map.

Claim: There is a positive integer s such that

$$\xi_{k+s} \circ j_{k+s}(f)(0) \notin \xi_{k+s}(\operatorname{supp} Z^{(s)}) \subset J_{k+s}(A)/B.$$

Proof of Claim. Suppose that $\xi_{k+s} \circ j_{k+s}(f)(0) \in \xi_{k+s}(\operatorname{supp} Z^{(s)})$ for all $s \geq 0$. Then we have

$$V_s \overset{\text{def}}{=} \operatorname{supp} Z^{(s)} \cap \xi_{k+s}^{-1}(\xi_{k+s} \circ j_{k+s}(f)(0)) \neq \emptyset$$

for all $s \geq 0$. Note that we have canonically $\xi_{k+s}^{-1}(\xi_{k+s} \circ j_{k+s}(f)(0)) \simeq B$, so V_s is a Zariski closed subset of B. Note also that $V_s \supset V_{s+1}$. Thus we have a nested sequence of closed sets,

$$B \supset V_0 \supset V_1 \supset V_2 \supset V_3 \supset \cdots$$

that eventually stabilizes at the variety V. Since we are assuming $V_s \neq \emptyset$ for all s, we have $V \neq \emptyset$. Let $f_B : \mathbb{C} \to B$ be the holomorphic curve obtained by the composition of f and the first projection $B \times C \to B$. Let $a \in V$, and translate the holomorphic curve f by $a - f_B(0) \in B$ and put $\widetilde{f}(z) = f(z) + a - f_B(0)$. Then by the construction of $\widetilde{f}$, we have

$$j_{k+s}(\widetilde{f})(0) \in \operatorname{supp} Z^{(s)} \quad \text{for all} \quad s \geq 0.$$

Hence by Taylor series, we have $j_k(\widetilde{f})(\mathbb{C}) \subset \operatorname{supp} Z$. This is a contradiction since we are assuming $j_k(f)$ is non-degenerate in $W_k(f)$, which proves our claim.

Now let s be an integer such that the above claim is satisfied. Then by Corollary 2.4.8, we have

$$N\left(r, j_{k+s}(f), Z^{(s)}\right) \leq O\left(\log(rT(r, f, E))\right) \ ||.$$

Hence by (2.5.9) and Theorem 2.5.1, we have Corollary 2.5.8. □

3. Applications

In this section, we consider two topics in the Nevanlinna theory of holomorphic curves into Abelian varieties for applications of our Theorem 2.5.1.

3.1. First we consider the truncation level for the second main theorem.

Theorem 3.1.1. *Let A be an Abelian variety and let $D \subset A$ be a reduced effective divisor. Let L be an ample line bundle on A. Let $f : \mathbb{C} \to A$ be a holomorphic curve such that the image of f is Zariski dense. Then we have*

$$T(r, f, D) \leq N^{(1)}(r, f, D) + \epsilon T(r, f, L) \ ||_\epsilon \qquad \textit{for all } \epsilon > 0.$$

Before going to prove this theorem, we state one result from [Y02, Proposition 4.2.1] without proof.

Lemma 3.1.2. *Let X be a smooth projective variety and $D \subset X$ be a divisor. Let E be an ample line bundle on X. Suppose there are a positive integer $s > 0$, a line bundle L on X and a morphism of fiber bundles $\varrho : J_s(X) \to L$ such that $D^{(s)} \subset \varrho^* o_L$. Here $o_L \subset L$ is the divisor corresponding to the zero section. Then for a holomorphic curve $f : \mathbb{C} \to X$ with the non-degeneracy condition $j_s(f)(\mathbb{C}) \not\subset \operatorname{supp} \varrho^* o_L$, we have*

$$m(r, f, D) \leq T(r, f, L) + O(\log(rT(r, f, E))) \ ||.$$

Remarks. (1) Our definition of $D^{(s)}$ is slightly different to the definition made in [Y02]. There, we denote our $D^{(s)}$ by "$D^{(s)}|_{J_s(X)}$".

(2) The term $N_{\text{"}Ram\text{"}f}(r)$ in [Y02, Proposition 4.2.1] is positive for $r > 1$ by definition. Hence our lemma immediately follows from this proposition.

Proof of Theorem 3.1.1. Put $B = \mathrm{St}^0(W_1(f))$ and $C = A/B$. Let $\varphi_0 : A \to C$ be the natural quotient map. Put $\hat{C} = \mathrm{St}^0(W_1(\varphi_0 \circ f))$ and put $C^\dagger = C/\hat{C}$.

First we reduce to the case D is irreducible. If D is not irreducible, write D as a sum of irreducible components $D = D_1 + \cdots + D_q$. Then

$$\begin{aligned}\sum_{i=1}^{q} T(r, f, D_i) &\leq \sum_{i=1}^{q} N^{(1)}(r, f, D_i) + \epsilon T(r, f, L) \ ||_\epsilon \\ &\leq N^{(1)}(r, f, D) + \sum_{\substack{i,j \\ 1 \leq i,j \leq q \\ i \neq j}} N^{(1)}(r, f, D_i \cap D_j) + \epsilon T(r, f, L) \ ||_\epsilon.\end{aligned}$$

But since we have $\operatorname{codim}(D_i \cap D_j, A) \geq 2$ for $i \neq j$, Theorem 2.5.1 implies

$$N^{(1)}(r, f, D_i \cap D_j) \leq \epsilon T(r, f, L) \ ||_\epsilon,$$

and hence $T(r, f, D) \leq N^{(1)}(r, f, D) + \epsilon T(r, f, L) \ ||_\epsilon$.

Next we reduce to the case $\mathrm{St}^0(D) = 0$. If $\mathrm{St}^0(D) \neq 0$, we replace A by $A/\mathrm{St}^0(D)$, D by $D/\mathrm{St}^0(D)$ and $f : \mathbb{C} \to A$ by $f : \mathbb{C} \to A \to A/\mathrm{St}^0(D)$, respectively.

Finally, we reduce to the case that C has splitting $C = \hat{C} \times C^\dagger$. There is an isogeny $C^\dagger \to C^\dagger$ such that the base change of the quotient map $C \to C^\dagger$ by this isogeny is the second projection $\hat{C} \times C^\dagger \to C^\dagger$. Let A' be the base change of the composition of the quotient maps $A \to C \to C^\dagger$ by the above isogeny $C^\dagger \to C^\dagger$. Let D' be the divisor on A' obtained by the pull back of D by the isogeny $A' \to A$. By replacing A by A', D by D' and f by a lifting $f' : \mathbb{C} \to A'$ of f, we may assume that C has splitting $C = \hat{C} \times C^\dagger$. Note that in this final step, D may become non-irreducible. Write D

as a sum of irreducible components $D = D_1 + \cdots + D_q$. Then by the argument of the first step, it is enough to prove our theorem for each irreducible component D_i. By the second reduction step, we have $\mathrm{St}^0(D_i) = 0$. Hence we may assume that D is irreducible, $\mathrm{St}^0(D) = 0$ and C has splitting $C = \hat{C} \times C^{\dagger}$.

Let $\varphi : J_1(A) \to J_1(A)/B$ be the quotient map. Then φ is a proper map, and the function $\dim\{D^{(1)} \cap \varphi^{-1}(x)\}$ $(x \in J_1(A)/B)$ on $J_1(A)/B$ is upper semi-continuous. Hence the set

$$\Lambda = \{x \in J_1(A)/B;\ \mathrm{codim}(D^{(1)} \cap \varphi^{-1}(x), \varphi^{-1}(x)) \leq 1\} \subset J_1(A)/B$$

is a Zariski closed subset of $J_1(A)/B$.

Claim: $W_1(f)/B \not\subset \Lambda$.

Proof of Claim. In the case $B = A$, Λ is a Zariski closed subset of $J_{1,A}$. And we know that

$$\Lambda = \{(0, \cdots, 0) \in J_{1,A}\}.$$

Hence $W_1(f)/B \not\subset \Lambda$ follows from the fact that f is non-constant, which proves our claim in this case. Hence in the following, we assume that $B \neq A$, that is, C is non-trivial.

Since $J_1(A)$ is the tangent space of A and splits into $J_1(A) \simeq A \times J_{1,A}$ $(J_{1,A} = \mathbb{C}^{\dim A})$, $J_{1,A}$ has the natural structure of a vector space. Since $\varphi_0 : A \to C$ induces the map $J_1(A) \to J_1(C) \simeq C \times J_{1,C}$, there is the natural morphism $\tau : J_{1,A} \to J_{1,C}$ of vector spaces. By the assumption $\mathrm{St}^0(D) = 0$ and [Kaw80, Thm 4], there is a Zariski open subset $C_0 \subset C$ such that for $x \in C_0$, each irreducible component E of $D \cap \varphi_0^{-1}(x)$ satisfies $\mathrm{St}^0(E) = 0$. Here we consider $D \cap \varphi_0^{-1}(x)$ and E as divisors on $\varphi_0^{-1}(x) \simeq B$ where this isomorphism is not canonical. Replacing C_0 by smaller Zariski open subset, if necessary, we may assume that$D \cap \varphi_0^{-1}(x)$ is a reduced divisor on $\varphi_0^{-1}(x) \simeq B$ for $x \in C_0$. To see this, note that we may take a Zariski open subset $C_0 \subset C$ such that

- $D \cap \varphi_0^{-1}(x)$ is a divisor on $\varphi_0^{-1}(x) \simeq B$ for $x \in C_0$,
- the composition of the natural map $D_{\mathrm{smooth}} \to A$ and $\varphi_0 : A \to C$ is smooth over C_0 (cf. [H77, p.272]),
- for $x \in C_0$, each irreducible component E of $D \cap \varphi_0^{-1}(x)$ satisfies that $E \cap D_{\mathrm{smooth}}$ is a non-empty Zariski open subset of E.

Here D_{smooth} is the smooth locus of D. By the above conditions, $D \cap \varphi_0^{-1}(x)$ is a reduced divisor on B for $x \in C_0$.

Since $J_1(A)/B \simeq C \times J_{1,A}$, we consider Λ as a Zariski closed subset of $C \times J_{1,A}$. For $x \in C$, let $\Lambda_x \subset J_{1,A}$ be the fiber of the first projection $\Lambda \to C$ over x, and let $\tau_{\Lambda_x} : \Lambda_x \to J_{1,C}$ be the restriction of τ on Λ_x.

Subclaim 1: $\tau_{\Lambda_x}^{-1}(\tau_{\Lambda_x}(v))$ is finite for $x \in C_0$ and $v \in \Lambda_x$.

Let $\varpi : C \times J_{1,A} \to C \times J_{1,C}$ be the morphism such that

$$C \times J_{1,A} \ni (c, v) \overset{\varpi}{\mapsto} (c, \tau(v)) \in C \times J_{1,C}.$$

Consider the Zariski closed subset $W_1(\varphi_0 \circ f) \subset C \times J_{1,C}$ which is the Zariski closure of the image of the holomorphic curve $j_1(\varphi_0 \circ f) : \mathbb{C} \to C \times J_{1,C}$. Then the restriction of ϖ induces the dominant map $\varpi' : W_1(f)/B \to W_1(\varphi_0 \circ f)$.

Subclaim 2: $\dim W_1(f)/B > \dim W_1(\varphi_0 \circ f)$.

Using these subclaims, we may prove our claim as follows. Assume that $W_1(f)/B \subset \Lambda$. Then by Subclaim 1, the map $\varpi' : W_1(f)/B \to W_1(\varphi_0 \circ f)$ is generically finite and dominant. Hence $\dim W_1(f)/B = \dim W_1(\varphi_0 \circ f)$. This contradicts Subclaim 2. We have $W_1(f)/B \not\subset \Lambda$ which proves our claim.

Next, we prove the subclaims to conclude our proof of the claim.

Proof of Subclaim 1. Let $x \in C_0$. Decompose $D \cap \varphi_0^{-1}(x) = \sum_{i=1}^{s} E_i$ into a sum of irreducible components as a divisor on $\varphi_0^{-1}(x) \simeq B$. For $1 \leq i \leq s$, define the subset $P_{x,i} \subset J_{1,A}$ by

$$P_{x,i} = \left\{ v \in J_{1,A};\ E_i \subset \operatorname{supp}\left(\varphi^{-1}\left((x,v)\right) \cap D^{(1)}\right) \right\}.$$

Then this $P_{x,i}$ is a vector subspace of $J_{1,A}$. By the definition of Λ_x, we have

$$\Lambda_x = \bigcup_{1 \leq i \leq s} P_{x,i}.$$

Note that the restriction of τ on $P_{x,i}$ is injective. Otherwise, there is a non-zero vector $v \in \ker(\tau) \cap P_{x,i}$ whose translations define the vector field $\tilde{v}$ on $\varphi_0^{-1}(x) \simeq B$, tangent to E_i. Here note that $D \cap \varphi_0^{-1}(x)$ is a reduced divisor on B. Hence $\mathrm{St}^0(E_i) \neq 0$, which contradicts the assumption for C_0. This proves that the restriction of τ on $P_{x,i}$ is injective and proves our subclaim. □

Proof of Subclaim 2. We first modify the morphism $\varpi' : W_1(f)/B \to W_1(\varphi_0 \circ f)$ to the morphism $\hat{\varpi} : S \to \hat{C} \times V$ as follows. Here S and V are smooth projective varieties and they are birational to compactifications of $W_1(f)/B$ and $W_1(\varphi_0 \circ f)/\hat{C}$, respectively.

Let $\overline{W_1(\varphi_0 \circ f)/\hat{C}}$ be the compactification of $W_1(\varphi_0 \circ f)/\hat{C}$ in $C^{\dagger} \times \mathbb{P}^{\dim C}$. Here $\mathbb{P}^{\dim C}$ is the compactification of $J_{1,C}$. Let V be a smooth model of $\overline{W_1(\varphi_0 \circ f)/\hat{C}}$. By the assumption made in the beginning of this proof of our theorem, we have $C = \hat{C} \times C^{\dagger}$. Hence we have

$$W_1(\varphi_0 \circ f) = \hat{C} \times (W_1(\varphi_0 \circ f)/\hat{C}).$$

Hence $\hat{C} \times V$ is a smooth model of a compactification of $W_1(\varphi_0 \circ f)$.

Let $\overline{W_1(f)/B}$ be the compactification of $W_1(f)/B$ in $C \times \mathbb{P}^{\dim A}$ where $J_{1,A} \subset \mathbb{P}^{\dim A}$ is the compactification of $J_{1,A}$. Let S be a smooth model of $\overline{W_1(f)/B}$ such that the rational map $\hat{\varpi} : S \dashrightarrow \hat{C} \times V$ induced from ϖ' is holomorphic at every point of S. Hence we get the morphism $\hat{\varpi} : S \to \hat{C} \times V$.

Let $H = \mathbb{P}^{\dim A} \backslash J_{1,A}$ be the boundary divisor. Let $H' \subset S$ be the divisor obtained by the pull back of H by the composition of the natural map $S \to \overline{W_1(f)/B}$ and the second projection $\overline{W_1(f)/B} \to \mathbb{P}^{\dim A}$. By Lemma 2.4.1, using the same notation $\overline{\alpha_1}$ in this lemma, we have

(3.1.3) $$T\left(r, \varphi_0 \circ f, M_C\right) \leq O\left(T\left(r, \overline{\alpha_1} \circ j_1(f), H\right)\right) + O(\log r)\ ||.$$

Here M_C is an ample line bundle on C.

Now we assume that

(3.1.4) $$\dim W_1(f)/B = \dim W_1(\varphi_0 \circ f),$$

and derive a contradiction as follows.

By (3.1.4), we have $\dim S = \dim \hat{C} \times V$. Hence, there is an effective divisor $G \subset \hat{C} \times V$ such that

$$H' \subset \hat{\varpi}^* G \tag{3.1.5}$$

as divisors on S.

Let $g : \mathbb{C} \to \hat{C} \times V$ be the holomorphic curve obtained from $j_1(\varphi_0 \circ f) : \mathbb{C} \to W_1(\varphi_0 \circ f)$. Then by (3.1.5), we get

$$m\,(r, \overline{\alpha_1} \circ j_1(f), H) \leq m\,(r, g, G) + O(1).$$

Since we have

$$T\,(r, \overline{\alpha_1} \circ j_1(f), H) = m\,(r, \overline{\alpha_1} \circ j_1(f), H) + O(1),$$

combining with (3.1.3), we get

$$T\,(r, \varphi_0 \circ f, M_C) \leq O\,(m\,(r, g, G)) + O(\log r)\ ||. \tag{3.1.6}$$

Let $g_1 : \mathbb{C} \to \hat{C}$ be the composition of g and the first projection $\hat{C} \times V \to \hat{C}$. Let $g_2 : \mathbb{C} \to V$ be the composition of g and the second projection $\hat{C} \times V \to V$. Let $M_{\hat{C}}$ be an ample line bundle on $\hat{C}$ and let M_V be an ample line bundle on V. Then by Proposition 2.2.1 and Lemma 2.4.1, we have

$$T\,(r, g_2, M_V) \leq O\,(\log(rT\,(r, g_1, M_{\hat{C}})))\ ||. \tag{3.1.7}$$

Next, we prove the following inequality.

$$m\,(r, g, G) \leq O(\log(rT\,(r, g_1, M_{\hat{C}})))\ ||. \tag{3.1.8}$$

This inequality is a generalization of the second main theorem for holomorphic curves in Abelian varieties (cf. [NoWY02], [Kob00]). The methods of the proofs of this second main theorem also adapt to that of (3.1.8). In the following, we give a proof of (3.1.8) on the line of the method in [Y02].

First $\hat{C}$ naturally acts on $\hat{C} \times V$ by considering the trivial action on V. Hence, $\hat{C}$ also acts on $J_s(\hat{C} \times V)$ for $s \geq 0$. Since we have $J_s(\hat{C} \times V) \simeq J_s(\hat{C}) \times J_s(V)$, we have

$$J_s(\hat{C} \times V)/\hat{C} \simeq J_{s,\hat{C}} \times J_s(V).$$

Let $\beta_s : J_s(\hat{C} \times V) \to J_{s,\hat{C}} \times J_s(V)$ be the quotient map. By a method similar to the proof of the claim in Corollary 2.5.8, there is a positive integer s such that $\beta_s \circ j_s(g)(0) \notin \beta_s(\operatorname{supp} G^{(s)})$. Since β_s is proper, $\beta_s(\operatorname{supp} G^{(s)})$ is Zariski closed. Hence there exists an effective divisor $\Theta \subset J_{s,\hat{C}} \times J_s(V)$ such that $G^{(s)} \subset \beta_s^*(\Theta)$ and $\beta_s \circ j_s(g)(0) \notin \operatorname{supp} \Theta$. There are a line bundle R on V and a morphism $\varrho : J_{s,\hat{C}} \times J_s(V) \to R$ of fiber spaces over V such that $\varrho^* o_R = \Theta$, where o_R is the divisor on R corresponds to the zero section. This is a consequence of $\operatorname{Pic}(J_{s,\hat{C}} \times J_s(V)) = \operatorname{Pic}(V)$. Let R' be the line bundle on $\hat{C} \times V$ obtained by the pull-back of R by the second projection $\hat{C} \times V \to V$. Let $\varrho' : J_s(\hat{C} \times V) \to R'$ be the morphism of fiber spaces over $\hat{C} \times V$ obtained by ϱ. Then we have $G^{(s)} \subset \varrho'^* o_{R'}$. Since we have $\varrho'^* o_{R'} = \beta_s^* \Theta$, we have $j_s(g)(0) \notin \operatorname{supp} \varrho'^* o_{R'}$. Hence by Lemma 3.1.2, we have

$$m\,(r, g, G) \leq T\,(r, g_2, R) + O(\log T\,(r, g_1, M_{\hat{C}})) + O(\log T\,(r, g_2, M_V)) + O(\log r)\ ||.$$

Using (3.1.7), we obtain our inequality (3.1.8).

Since we have

$$T(r, g_1, M_{\hat{C}}) \leq O(T(r, \varphi_0 \circ f, M_C)),$$

using (3.1.6) and (3.1.8), we deduce

$$T(r, \varphi_0 \circ f, M_C) \leq O(\log(rT(r, \varphi_0 \circ f, M_C)))\ ||.$$

This gives a contradiction, since we have $T(r, \varphi_0 \circ f, M_C) \geq O(r^2)$. (Note that we are considering the case that C is non-trivial.) Hence we get $\dim W_1(f)/B > \dim W_1(\varphi_0 \circ f)$, which proves our subclaim and concludes our proof of the claim. □

Now we go back to the proof of our theorem. By the second main theorem for Abelian varieties (cf. [NoWY02]), there is a positive constant ρ, depending on f and D such that

$$T(r, f, D) \leq N^{(\rho)}(r, f, D) + O(\log(rT(r, f, L)))\ ||.$$

By the definition of $N^{(k)}(r, f, D)$, we have

$$N^{(k+2)}(r, f, D) - N^{(k+1)}(r, f, D) \leq N^{(k+1)}(r, f, D) - N^{(k)}(r, f, D)$$

and

$$N^{(2)}(r, f, D) - N^{(1)}(r, f, D) \leq N^{(1)}\left(r, j_1(f), D^{(1)}\right).$$

Hence we obtain

$$N^{(\rho)}(r, f, D) - N^{(1)}(r, f, D) \leq (\rho - 1) N^{(1)}\left(r, j_1(f), D^{(1)}\right)$$

and

$$T(r, f, D) \leq N^{(1)}(r, f, D) + (\rho - 1) N^{(1)}\left(r, j_1(f), D^{(1)}\right) + O(\log(rT(r, f, L)))\ ||.$$

Hence to prove our theorem, it suffices to prove

$$N^{(1)}\left(r, j_1(f), D^{(1)}\right) \leq \epsilon T(r, f, L)\ ||_\epsilon \qquad \text{for all } \epsilon > 0. \tag{3.1.9}$$

Write $\mathrm{supp}(D^{(1)} \cap W_1(f))$ as a sum of irreducible components

$$\mathrm{supp}(D^{(1)} \cap W_1(f)) = Z_1 \cup \cdots \cup Z_q.$$

To prove (3.1.9), it suffices to prove

$$N^{(1)}(r, j_1(f), Z_i) \leq \epsilon T(r, f, L)\ ||_\epsilon \qquad \text{for all } \epsilon > 0$$

for $i = 1, \ldots, q$.

In the case $\varphi(Z_i) = W_1(f)/B$, we have $\mathrm{codim}(Z_i, W_1(f)) \geq 2$ by the above claim. Hence, by Theorem 2.5.1, we have

$$N^{(1)}(r, j_1(f), Z_i) \leq \epsilon T(r, f, L)\ ||_\epsilon \qquad \text{for all } \epsilon > 0. \tag{3.1.10}$$

On the other hand, in the case $\varphi(Z_i) \neq W_1(f)/B$, we have (3.1.10) by Corollary 2.4.8, which proves Theorem 3.1.1. □

Corollary 3.1.11. *Let X be a smooth projective variety and assume that there is a surjective and generically-finite map $\pi : X \to A$ to an Abelian variety A. Let L be an ample line bundle on X and let K_X be the canonical bundle on X. Then for a non-degenerate holomorphic curve $f : \mathbb{C} \to X$, we have*

$$T(r, f, K_X) \leq \epsilon T(r, f, L)\ ||_\epsilon \quad \textit{for all } \epsilon > 0.$$

Proof. By the natural morphism $\mathcal{O}_X = \pi^* K_A \hookrightarrow K_X$, there is an associated divisor $D \subset X$ to this injection. By the first main theorem, we have

$$T(r, f, K_X) = T(r, f, D) + O(1) = N(r, f, D) + m(r, f, D) + O(1),$$

and by the second main theorem for the Abelian variety A, we have

$$m(r, f, D) \leq O(m\left(r, f, \pi^{-1}\pi(D)\right) = O(m(r, \pi \circ f, \pi(D))) \leq O(\log(rT(r, f, L)))\ ||.$$

Hence it suffices to show

$$N(r, f, D) \leq \epsilon T(r, f, L)\ ||_\epsilon \quad \text{for all } \epsilon > 0. \tag{3.1.12}$$

Let $U \subset A$ be a Zariski open subset such that the restriction

$$\pi_U : \pi^{-1}(U) \to U$$

is quasi-finite; i.e. all fibers of π_U are finite. Since π_U is also projective, it is finite. Note that we can take U so that $\operatorname{codim}(A - U, A) \geq 2$.

Now decompose D into a sum of divisors $D = D_1 + D_2$ so that all the irreducible components of D_1 have non-trivial intersections with $\pi^{-1}(U)$ and $\pi(\operatorname{supp} D_2) \subset A - U$. By Corollary 2.5.8, we have

$$N(r, f, D_2) \leq \epsilon T(r, f, L)\ ||_\epsilon \quad \text{for all } \epsilon > 0. \tag{3.1.13}$$

Note that $D_1|_U = D|_U$ is the ramification divisor of π_U. Decompose D_1 to a sum of irreducible components $D_1 = \sum_{i=1}^q a_i E_i$, $a_i > 0$. Put $F_i = \operatorname{supp} \pi(E_i)$. Since π_U ramifies at $E_i|_U$ with the ramification index $a_i + 1$, we have

$$\pi^* F_i = (a_i + 1)E_i + G_i,$$

where G_i is an effective divisor on X. Hence we have

$$(a_i + 1) \operatorname{mult}_z f \cdot E_i \leq \operatorname{mult}_z (\pi \circ f) \cdot F_i \quad \text{for all } z \in \mathbb{C}$$

and

$$a_i \operatorname{mult}_z f \cdot E_i + \min(1, \operatorname{mult}_z (\pi \circ f) \cdot F_i) \leq \operatorname{mult}_z (\pi \circ f) \cdot F_i,$$

and so

$$N(r, f, E_i) \leq N(r, \pi \circ f, F_i) - N^{(1)}(r, \pi \circ f, F_i).$$

Hence by Theorem 3.1.1, we have

$$N(r, f, E_i) \leq \epsilon T(r, f, L)\ ||_\epsilon \quad \text{for all } \epsilon > 0.$$

Combining with (3.1.13), we obtain (3.1.12). This proves our Corollary. □

Corollary 3.1.14. *Let X be a projective variety and assume that*

(1) *X is of general type, and*
(2) $\dim H^0(X, \Omega_X) \geq \dim X$.

Then every holomorphic curve $f : \mathbb{C} \to X$ is algebraically degenerate.

Remark. The case $\dim H^0(X, \Omega_X) > \dim X$ is Bloch-Ochiai's Theorem. Our new part is the case $\dim H^0(X, \Omega_X) = \dim X$. C.G. Grant [Gr86] proved this case when X is a surface and the Albanese variety of X is simple.

Proof. By blowing-up, we can assume that X is smooth. Let A be the Albanese variety of X and let $\alpha : X \to A$ be the standard map. Then by condition (2), we have $\dim X \leq \dim A$. It is well known that the image $\alpha(X)$ is not a proper sub-Abelian

variety. If $\alpha(X)$ is a proper subvariety of A, then Bloch-Ochiai's Theorem implies that $\alpha \circ f(\mathbb{C})$ lies in a proper Zariski closed set of $\alpha(X)$.
If $\alpha(X) = A$, then $\alpha : X \to A$ is a generically-finite map. Suppose that f is algebraically non-degenerate.

Let L be an ample line bundle on X. By assumption (1), there are positive integers n,m such that $H^0(X, mK_X - nL) \neq 0$. Let F be a divisor on X which corresponds to this non-zero global section. Then by the Nevanlinna's first main theorem, we have

$$mT(r, f, K_X) - nT(r, f, L) = T(r, f, F) + O(1).$$

Since we are assuming that f is algebraically non-degenerate, we have $T(r, f, F) > O(1)$ by the Nevanlinna's inequality. Hence we have

$$\frac{n}{m} T(r, f, L) < T(r, f, K_X) + O(1)$$

and Corollary 3.1.11 implies that

$$T(r, f, L) < \epsilon T(r, f, L) \ ||_\epsilon \quad \text{for all } \epsilon > 0.$$

But since L is ample, it is well known that $T(r, f, L) \to +\infty$ when $r \to \infty$, which is a contradiction. Hence we conclude that $f(\mathbb{C})$ lies in a proper Zariski closed subset of X. □

3.2. Next, we consider the unicity theorem.

Theorem 3.2.1. *Let A, A' be Abelian varieties and let $D \subset A$, $D' \subset A'$ be reduced and ample divisors. Let $f : \mathbb{C} \to A$ and $f' : \mathbb{C} \to A'$ be non-degenerate holomorphic curves such that*

$$\operatorname{supp} f^* D = \operatorname{supp} f'^* D'.$$

Then there are decompositions of D and D' in the form

$$D = F(D) + E(D), \quad D' = F(D') + E(D')$$

where every ample irreducible component of D (resp. D') is contained in $F(D)$ (resp. $F(D')$) and there is an isomorphism

$$\alpha : A/\operatorname{St}(F(D)) \xrightarrow{\sim} A'/\operatorname{St}(F(D'))$$

with $\alpha \circ \overline{f} = \overline{f'}$. Here we set $\overline{f}$ (resp. $\overline{f'}$) to be the composition

$$\overline{f} : \mathbb{C} \to A \to A/\operatorname{St}(F(D)) \quad (\textit{resp. } \overline{f'} : \mathbb{C} \to A' \to A'/\operatorname{St}(F(D'))).$$

Proof. We may reduce to the case $f(0) = 0$ (the identity element of A) and $f'(0) = 0'$ (the identity element of A') by considering $f(z) - f(0)$, $f'(z) - f'(0)$, $D - f(0)$ and $D' - f'(0)$.

Let B be the Zariski closure of the image of the holomorphic curve $(f, f') : \mathbb{C} \to A \times A'$. Then B is also an Abelian variety (Bloch-Ochiai's Theorem). Let $p : B \to A$ (resp. $p' : B \to A'$) be the composition of morphisms

$$p : B \hookrightarrow A \times A' \xrightarrow{\text{1st proj}} A \quad (\text{resp. } p' : B \hookrightarrow A \times A' \xrightarrow{\text{2nd proj}} A')$$

and put $g = (f, f') : \mathbb{C} \to B$. Then g is non-degenerate and $f = p \circ g$, $f' = p' \circ g$. By the assumption that f and f' are non-degenerate, the morphisms p, p' are surjective and by the assumption that $f(0) = 0$, $f'(0) = 0'$, the morphisms p, p' are homomorphisms of Abelian varieties. Hence, by letting $B_A = \ker p$ and $B_{A'} = \ker p'$, we have isomorphisms

$$B/B_A \tilde{\to} A \quad \text{and } B/B_{A'} \tilde{\to} A'.$$

Let I be a subset of a set of irreducible components of p^*D such that

$$H \in I \iff T(r, g, H) \leq \epsilon T(r, g, L) \ ||_\epsilon \text{ for all } \epsilon > 0$$

where L is an ample line bundle on B. Let J be the complement of I in the set of irreducible components of p^*D. Define I' and J' similarly but from $p^{'*}D'$.

Let H be an irreducible component of the pull back by p of some ample irreducible component of D. Then we have

$$O(T(r, f, D)) = O(T(r, g, H)). \tag{3.2.2}$$

By the assumption $\operatorname{supp} f^*D = \operatorname{supp} f^{'*}D'$, we have $N^{(1)}(r, f, D) = N^{(1)}(r, f', D')$. Since Theorem 3.1.1 implies

$$N^{(1)}(r, f', D') \geq (1-\epsilon) T(r, f', D') \ ||_\epsilon \text{ for all } \epsilon > 0,$$

we have $O(T(r, f', D')) \leq O(T(r, f, D)) \ ||$ and $T(r, g, L) \leq O(T(r, f, D)) \ ||$. Using (3.2.2), we have

$$O(T(r, g, H)) \geq O(T(r, g, L)) \ ||.$$

By the same argument, we also obtain the inequality for an irreducible component H of the pull back by p' of an ample irreducible component of D'.
Hence by the definition of J (resp. J'), we conclude that all the irreducible components of the pull back by p (resp. p') of all the ample irreducible components of D (resp. D') are contained in J (resp. J').

In the following, we consider I,J,I' and J' as subsets of the set of irreducible divisors on B. We claim that $J = J'$ and $B_A, B_{A'} \subset \operatorname{St}(\sum_{H \in J} H)$.

Suppose $H \in J$ and $H \notin J' \cup I'$. Then we have $\operatorname{codim}(H \cap p^{'*}D', B) \geq 2$. By the assumption $\operatorname{supp} f^*D = \operatorname{supp} f^{'*}D'$, we have

$$N^{(1)}(r, g, H) = N^{(1)}\left(r, g, H \cap p^{'*}D'\right),$$

hence Theorem 2.5.1 implies an inequality

$$N^{(1)}(r, g, H) \leq \epsilon T(r, g, L) \ ||_\epsilon.$$

On the other hand, by Theorem 3.1.1, we have

$$T(r, g, H) \leq N^{(1)}(r, g, H) + \epsilon T(r, g, L) \ ||_\epsilon,$$

which is a contradiction because $H \in J$. Hence $H \in J' \cup I'$ and by the definition of J', we have $H \in J'$. Hence $J \subset J'$. By the same argument, we have $J' \subset J$ and $J = J'$.

Next, to prove $B_A \subset \operatorname{St}(\sum_{H \in J} H)$, it suffices to prove that $J \subset I \cup J$ is stabilized by the action of B_A. Let $b \in B_A$ and $H \in J$. Then since H and $b+H$ are algebraically equivalent divisors, we have

$$T(r, g, H) = T(r, g, b + H) + O(1).$$

Hence by the definition of J, we have $b+H \in J$ and $B_A \subset \mathrm{St}(\sum_{H\in J} H)$. By the same argument, we have $B_{A'} \subset \mathrm{St}(\sum_{H\in J'} H) = \mathrm{St}(\sum_{H\in J} H)$, which proves our claim.

Now put $F(D) = (\sum_{H\in J} H)/B_A$ and $F(D') = (\sum_{H\in J} H)/B_{A'}$. Then we have

$$\mathrm{St}(F(D)) = \mathrm{St}(\sum_{H\in J} H)/B_A, \quad \mathrm{St}(F(D')) = \mathrm{St}(\sum_{H\in J} H)/B_{A'}$$

and isomorphisms

$$A/\,\mathrm{St}(F(D)) \tilde{\to} B/\,\mathrm{St}(\sum_{H\in J} H) \tilde{\to} A'/\,\mathrm{St}(F(D')).$$

Put this composition as $\alpha : A/\,\mathrm{St}(F(D)) \tilde{\to} A'/\,\mathrm{St}(F(D'))$. Then we have $\alpha \circ \overline{f} = \overline{f'}$, which proves our Theorem. □

Corollary 3.2.3. *Let A, A' be Abelian varieties and let $D \subset A$, $D' \subset A'$ be divisors such that all their irreducible components are ample and that $\mathrm{St}(D) = \mathrm{St}(D') = 0$. Let $f : \mathbb{C} \to A$, $f' : \mathbb{C} \to A'$ be non-degenerate holomorphic curves such that $\mathrm{supp}\, f^*D = \mathrm{supp}\, {f'}^*D'$. Then there is an isomorphism $\alpha : A \tilde{\to} A'$ such that $\alpha \circ f = f'$.*

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