Polynomial endomorphisms of the Cuntz algebras arising from permutations. III
—Branching laws and automata—

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In order to compute branching laws of representations of the Cuntz algebras by endomorphisms, we construct automata which are called Mealy machines associated with endomorphisms, and show that outputs from these machines for inputs of information of representations give their branching laws.

1. Introduction

In [**11**, **12**], we introduce a class of endomorphisms of the Cuntz algebra $\mathcal{O}_N$ and show branching laws of permutative representations by them. These branching laws are interesting subjects themselves and they are useful to classify endomorphisms effectively. It is expected that they are computed more smartly and their meanings are well understood clearly. On the other hand, an automaton is a typical object to consider algorithm of computation in the computer science([**5**, **6**, **7**]). An automaton is a machine which changes the internal state by an input. A Mealy machine is a kind of automaton with output. In this paper, we show that the better algorithm of computation of branching law is given by a semi-Mealy machine associated with an endomorphism.

For $N \geq 2$, put $\{1,\ldots,N\}_1^* \equiv \coprod_{k\geq 1}\{1,\ldots,N\}^k$, $\{1,\ldots,N\}^k \equiv \{(j_n)_{n=1}^k : j_n = 1,\ldots,N,\ n = 1,\ldots,k\}$ for $k \geq 1$. For $J \in \{1,\ldots,N\}_1^*$, we have a representation $P(J) = (\mathcal{H},\pi)$ of $\mathcal{O}_N$ in [**11**] which is equivalent to a cyclic permutative representation of $\mathcal{O}_N$ with a cycle in [**1**, **3**, **4**]. Let $\mathfrak{S}_{N,l}$ be the set of all bijections on the set $\{1,\ldots,N\}^l$ for $l \geq 1$. For an element $\sigma \in \mathfrak{S}_{N,l}$, we have an endomorphism ψ_σ of $\mathcal{O}_N$ in [**11**]. We denote $\pi \circ \psi_\sigma$ by $P(J)\circ\psi_\sigma$ in this case. In [**11**], we show that for each J, there are $J_1,\ldots,J_m$, $1 \leq m \leq N^{l-1}$ such that $P(J)\circ\psi_\sigma$ can be always uniquely decomposed into the direct sum of $P(J_1),\ldots,P(J_m)$ up to unitary equivalences:

$$P(J)\circ\psi_\sigma \sim P(J_1)\oplus\cdots\oplus P(J_m). \tag{1.1}$$

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Concrete several branching laws by ψ_σ are already given in [**11**, **12**](precise their definitions are given in § 2). We show an algorithm to seek $J_1, \ldots, J_m$ for J by reducing problem to a semi-Mealy machine as an input($= J$) and outputs($= J_1, \ldots, J_m$).

A *semi-Mealy machine* is a data $(Q, \Sigma, \Delta, \delta, \lambda)$ which consists of non empty finite sets Q, Σ, Δ and two maps δ from $Q \times \Sigma$ to Q, λ from $Q \times \Sigma$ to Δ([**6**, **7**, **13**]). For $q_0 \in Q$, a *Mealy machine* is a data $(Q, \Sigma, \Delta, \delta, \lambda, q_0)$. For an *input* word $x = a_1 \cdots a_\alpha$ which consists of alphabets $a_1, \ldots, a_\alpha$ in Σ, we have an *output* word $y = b_1 \cdots b_\beta$ which consists of alphabets $b_1, \ldots, b_\beta$ in Δ according to rule of δ and λ. Let Σ^* and Δ^* be free semigroups generated by Σ and Δ, respectively. $\hat{\delta}$ is a map from $Q \times \Sigma^*$ to Q and $\hat{\lambda}$ is a map from $Q \times \Sigma^*$ to Δ^* which are defined by $\hat{\delta}(q, wa) \equiv \delta(\hat{\delta}(q, w), a)$, $\hat{\lambda}(q, wa) \equiv \hat{\lambda}(q, w)\lambda(\hat{\delta}(q, w), a)$ for $q \in Q$, $w \in \Sigma^*$ and $a \in \Sigma$. We denote $\hat{\delta}$, $\hat{\lambda}$ by δ, λ simply(further their explanation is given in § 3). For symbols $a_1, \ldots, a_N$, $b_1, \ldots, b_N$, $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $r \geq 1$, we denote $a_J = a_{j_1} \cdots a_{j_k}$, $b_J = b_{j_1} \cdots b_{j_k}$ and $a_J^r \equiv \underbrace{a_J \cdots a_J}_{r}$.

Under these preparations, we have the following result:

Theorem 1.1. *Let $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$. Then there is a semi-Mealy machine $M_\sigma = (Q, \Sigma, \Delta, \delta, \lambda)$ such that $\Sigma = \{a_1, \ldots, a_N\}$, $\Delta = \{b_1, \ldots, b_N\}$ and the followings hold: For each $J \in \{1, \ldots, N\}_1^*$, there are $p_1, \ldots, p_m \in Q$ and $r_1, \ldots, r_m \in \mathbf{N}$ such that $\delta(p_j, a_J^{r_j}) = p_j$ for $j = 1, \ldots, m$ and (1.1) holds where $J_1, \ldots, J_m \in \{1, \ldots, N\}_1^*$ are taken as $b_{J_i} = \lambda(p_i, a_J^{r_i})$ for $i = 1, \ldots, m$.*

In § 2, we review branching function systems, permutative representations and permutative endomorphisms of $\mathcal{O}_N$. In § 3, we review automata, Mealy machine and introduce that arising from a permutation. From these preparations, Theorem 1.1 is proved. In § 4, we show examples of Mealy diagram of a semi-Mealy machine M_σ and branching laws of ψ_σ for concrete $\sigma \in \mathfrak{S}_{N,l}$.

2. Branching of representations of the Cuntz algebras by endomorphisms

We introduce several notions of multi indices which consist of numbers $1, \ldots, N$ for $N \geq 2$ in order to describe invariants of representations of $\mathcal{O}_N$. Recall $\{1, \ldots, N\}_1^*$ in § 1. For $J \in \{1, \ldots, N\}_1^*$, the *length* $|J|$ of J is defined by $|J| \equiv k$ when $J \in \{1, \ldots, N\}^k$, $k \geq 1$. For $J_1 = (j_1, \ldots, j_k), J_2 = (j_1^{'}, \ldots, j_l^{'}) \in \{1, \ldots, N\}_1^*$, put $J_1 \cup J_2 \equiv (j_1, \ldots, j_k, j_1^{'}, \ldots, j_l^{'})$. Specially, we define $(i, J) \equiv (i) \cup J$ for convenience. For $J \in \{1, \ldots, N\}_1^*$ and $k \geq 2$, $J^k \equiv \underbrace{J \cup \cdots \cup J}_{k}$. For $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$ and $\tau \in \mathbf{Z}_k$, denote $\tau(J) = (j_{\tau(1)}, \ldots, j_{\tau(k)})$. $J \in \{1, \ldots, N\}_1^*$ is *periodic* if there are $m \geq 2$ and

$J_0 \in \{1,\ldots,N\}_1^*$ such that $J = J_0^m$. For $J_1, J_2 \in \{1,\ldots,N\}_1^*$, $J_1 \sim J_2$ if there are $k \geq 1$ and $\tau \in \mathbf{Z}_k$ such that $J_1, J_2 \in \{1,\ldots,N\}^k$ and $\tau(J_1) = J_2$. For $J_1 = (j_1,\ldots,j_k), J_2 = (j_1^{'},\ldots,j_k^{'}) \in \{1,\ldots,N\}^k$, $k \geq 1$, $J_1 \prec J_2$ if $\sum_{l=1}^{k}(j_l^{'} - j_l)N^{k-l} \geq 0$. $J \in \{1,\ldots,N\}_1^*$ is *minimal* if $J \prec J^{'}$ for each $J^{'} \in \{1,\ldots,N\}_1^*$ such that $J \sim J^{'}$. Put $[1,\ldots,N]^* \equiv \{J \in \{1,\ldots,N\}_1^* :$ J is minimal and non periodic$\}$. $[1,\ldots,N]^*$ is in one-to-one correspondence with the set of all equivalence classes of non periodic elements in $\{1,\ldots,N\}_1^*$.

2.1. Branching function systems. Let Λ be an infinite set and $N \geq 2$. $f = \{f_i\}_{i=1}^N$ is a *branching function system on* Λ if f_i is an injective transformation on Λ for $i = 1,\ldots,N$ such that a family of their images coincides a partition of Λ. Put $\mathrm{BFS}_N(\Lambda)$ the set of all branching function systems on Λ. For $N \geq 2$, $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_1)$ and $g = \{g_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_2)$ are *equivalent* if there is a bijection φ from Λ_1 to Λ_2 such that $\varphi \circ f_i \circ \varphi^{-1} = g_i$ for $i = 1,\ldots,N$. For $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$, we denote $f_J \equiv f_{j_1} \circ \cdots \circ f_{j_k}$ when $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, $k \geq 1$, and define $f_0 \equiv$ *id*. For $x, y \in \Lambda$, $x \sim y$(with respect to f) if there are $J_1, J_2 \in \{1,\ldots,N\}^*$ and $z \in \Lambda$ such that $f_{J_1}(z) = x$ and $f_{J_2}(z) = y$. For $x \in \Lambda$, denote $A_f(x) \equiv \{y \in \Lambda : x \sim y\}$. Let $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$. f is *cyclic* if there is an element $x \in \Lambda$ such that $\Lambda = A_f(x)$. For $k \geq 1$, $\{n_1,\ldots,n_k\} \subset \Lambda$ is a k-*cycle* of f if $n_l \neq n_{l'}$ when $l \neq l^{'}$ and there is $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$ such that $f_{j_l}(n_l) = n_{l-1}$ for $l = 2,\ldots,k$ and $f_{j_1}(n_l) = n_k$. $\{n_l\}_{l\in\mathbf{N}} \subset \Lambda$ is a *chain* of f if $n_l \neq n_{l'}$ when $l \neq l^{'}$ and there is $\{j_l \in \{1,\ldots,N\} : l \in \mathbf{N}\}$ such that $f_{j_l}^{-1}(n_l) = n_{l+1}$ for each $l \in \mathbf{N} \equiv \{1,2,3,\ldots\}$. f has a k*-cycle(chain)* if there is a k-cycle(*resp.* chain) of f in Λ. Specially, we call simply that f has a cycle if f has a k-cycle for some $k \geq 1$.

Let Ξ be a set and Λ_ω be an infinite set for $\omega \in \Xi$. For $f^{[\omega]} = \{f_i^{[\omega]}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_\omega)$, f is the *direct sum* of $\{f^{[\omega]}\}_{\omega\in\Xi}$ if $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$ for a set $\Lambda \equiv \coprod_{\omega\in\Xi}\Lambda_\omega$ which is defined by $f_i(n) \equiv f_i^{[\omega]}(n)$ when $n \in \Lambda_\omega$ for $i = 1,\ldots,N$ and $\omega \in \Xi$. For $f \in \mathrm{BFS}_N(\Lambda)$, $f = \bigoplus_{\omega\in\Xi} f^{[\omega]}$ is a *decomposition* of f into a family $\{f^{[\omega]}\}_{\omega\in\Xi}$ if there is a family $\{\Lambda_\omega\}_{\omega\in\Xi}$ of subsets of Λ such that f is the direct sum of $\{f^{[\omega]}\}_{\omega\in\Xi}$. For each $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$, there is a decomposition $\Lambda = \coprod_{\omega\in\Xi}\Lambda_\omega$ such that $\#\Lambda_\omega = \infty$, $f|_{\Lambda_\omega} \equiv \{f_i|_{\Lambda_\omega}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_\omega)$ and $f|_{\Lambda_\omega}$ is cyclic for each $\omega \in \Xi$. Assume that f is cyclic. Then there is only one case in the followings: a) f has just one cycle. b) f has just one chain where we identify a chain $\{n_l \in \Lambda\}_{l\in\mathbf{N}}$ with a chain $\{m_l \in \Lambda\}_{l\in\mathbf{N}}$ when there are $M, L \geq 0$ such that $n_{l+L} = m_l$ for each $l > M$(see Proposition 2.5 in [**11**]).

Definition 2.1. (i) *For $J \in \{1,\ldots,N\}^k$, $k \geq 1$, $f \in \mathrm{BFS}_N(\Lambda)$ is $P(J)$ if f is cyclic and has a cycle $\{n_1,\ldots,n_k\}$ such that $f_J(n_k) = n_k$.*

(ii) *For $f \in \mathrm{BFS}_N(\Lambda)$ and $J \in \{1,\ldots,N\}_1^*$, g is a $P(J)$-component of f if g is a direct sum component of f and g is $P(J)$.*

Recall $\mathfrak{S}_{N,l}$ in § 1. For $\sigma \in \mathfrak{S}_{N,l}$ and $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$, put $f^{(\sigma)} = \{f_i^{(\sigma)}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$ by

$$f_i^{(\sigma)} \equiv f_{\sigma(i)} \quad (l=1), \quad f_i^{(\sigma)}(f_J(n)) \equiv f_{\sigma(i,J)}(n) \quad (l \geq 2) \tag{2.1}$$

for $n \in \Lambda$, $i = 1,\ldots,N$ and $J \in \{1,\ldots,N\}^{l-1}$. Let $J \in \{1,\ldots,N\}_1^*$ and $\sigma \in \mathfrak{S}_N = \mathfrak{S}_{N,1}$. If $f \in \mathrm{BFS}_N(\Lambda)$ is $P(J)$, then $f^{(\sigma)}$ is $P(J_{\sigma^{-1}})$ where $J_{\sigma^{-1}} \equiv \left(\sigma^{-1}(j_1),\ldots,\sigma^{-1}(j_k)\right)$ for $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, $k \geq 1$.

Lemma 2.2. *Put $J \in \{1,\ldots,N\}_1^*$. Then the followings hold:*

(i) *There is $f \in \mathrm{BFS}_N(\Lambda)$ for some set Λ such that f is $P(J)$.*
(ii) *For $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, there is $1 \leq m \leq N^{l-1}$ such that $f^{(\sigma)}$ is decomposed into a direct sum of m cycles. Furthermore the length of each cycle is a multiple of the length of J.*

Proof. See Lemma 2.7 in [**12**]. □

2.2. Permutative representations. For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**2**]), that is, it is a C*-algebra with generators $s_1,\ldots,s_N$ which satisfy

$$s_i^* s_j = \delta_{ij} I \quad (i,j = 1,\ldots,N), \quad s_1 s_1^* + \cdots + s_N s_N^* = I. \tag{2.2}$$

In this paper, any representation and endomorphism are assumed unital and $*$-preserving. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1,\ldots,S_N$ on an infinite dimensional Hilbert space which satisfy (2.2) in order to construct a representation of $\mathcal{O}_N$. In the same reason, it is sufficient to define elements $T_1,\ldots,T_N$ in $\mathcal{O}_N$ which satisfy (2.2) in order to construct an endomorphism of $\mathcal{O}_N$. For a multiindex $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, we denote $s_J = s_{j_1}\cdots s_{j_k}$ and $s_J^* = s_{j_k}^* \cdots s_{j_1}^*$.

Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{O}_N$. $(\mathcal{H},\pi)$ is a *permutative representation* of $\mathcal{O}_N$ if there are a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ and $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$ for some infinite set Λ such that $\pi(s_i)e_n = e_{f_i(n)}$ for each $n \in \Lambda$ and $i = 1,\ldots,N$. $(\mathcal{H},\pi,\Omega)$ is a *generalized permutative(=GP) representation* of $\mathcal{O}_N$ with cycle by $J \in \{1,\ldots,N\}^k$, $k \geq 1$ if $\Omega \in \mathcal{H}$ is a cyclic unit vector such that $\pi(s_J)\Omega = \Omega$ and $\{\pi(s_{j_1}\cdots s_{j_l})\Omega : l = 1,\ldots,k\}$ is an orthonormal family in $\mathcal{H}$. We denote $P(J) = (\mathcal{H},\pi,\Omega)$ simply. $(l_2(\Lambda),\pi_f)$ is the *permutative representation of $\mathcal{O}_N$ by $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$* if $\pi_f(s_i)e_n \equiv e_{f_i(n)}$ for $n \in \Lambda$ and $i = 1,\ldots,N$.

Permutative representations were introduced in [**1**, **3**, **4**]. By [**10**], any permutative representation is completely reducible. Any cyclic(*resp.* irreducible)permutative representation with cycle is equivalent to $P(J)$ for some $J \in \{1,\ldots,N\}_1^*$(*resp.* some $J \in [1,\ldots,N]^*$). For each $J \in \{1,\ldots,N\}_1^*$, $P(J)$ exists uniquely up to unitary equivalences. $P(J)$ is equivalent to a cyclic permutative representation with cycle.

Theorem 2.3. (i) *For* $J \in \{1, \ldots, N\}_1^*$, $P(J)$ *is irreducible if and only if* J *is non periodic.*

(ii) *For* $J_1, J_2 \in \{1, \ldots, N\}_1^*$, $P(J_1) \sim P(J_2)$ *if and only if* $J_1 \sim J_2$ *where* $P(J_1) \sim P(J_2)$ *means the unitary equivalence of two representations which satisfy the condition* $P(J_1)$ *and* $P(J_2)$*, respectively.*

(iii) $[1, \ldots, N]^*$ *is in one-to-one correspondence with the set of equivalence classes of irreducible permutative representations of* $\mathcal{O}_N$ *with cycle.*

Proof. See Theorem 2.12 in [**12**]. □

The followings hold by definition of branching function system and $(l_2(\Lambda), \pi_f)$.

Proposition 2.4. *Let* $f \in \mathrm{BFS}_N(\Lambda)$ *for an infinite set* Λ.

(i) *If* $g \in \mathrm{BFS}_N(\Lambda^{'})$ *for an infinite set* $\Lambda^{'}$ *such that* $f \sim g$*, then* $(l_2(\Lambda), \pi_f) \sim (l_2(\Lambda^{'}), \pi_g)$.

(ii) *If* f *is cyclic, then* $(l_2(\Lambda), \pi_f)$ *is cyclic.*

(iii) *For* $J \in \{1, \ldots, N\}_1^*$*, if* f *is* $P(J)$*, then* $(l_2(\Lambda), \pi_f)$ *is* $P(J)$*, too.*

(iv) *If* $f = f^{(1)} \oplus f^{(2)}$ *and* $\Lambda = \Lambda_1 \sqcup \Lambda_2$ *is the associated decomposition of* f*, then* $(l_2(\Lambda), \pi_f) \sim (l_2(\Lambda_1), \pi_{f^{(1)}}) \oplus (l_2(\Lambda_2), \pi_{f^{(2)}})$.

2.3. Permutative endomorphisms. We review endomorphisms of $\mathcal{O}_N$ arising from permutations in [**11**, **12**]. Assume that $\mathrm{End}\mathcal{A}$ is the set of all unital $*$-endomorphisms of a unital $*$-algebra $\mathcal{A}$ and $\rho, \rho^{'} \in \mathrm{End}\mathcal{A}$ in this subsection. ρ is *proper* if $\rho(\mathcal{A}) \neq \mathcal{A}$. ρ is *irreducible* if $\rho(\mathcal{A})^{'} \cap \mathcal{A} = \mathbf{C}I$ where $\rho(\mathcal{A})^{'} \cap \mathcal{A} \equiv \{x \in \mathcal{A} : \rho(a)x = x\rho(a) {}^{\forall}a \in \mathcal{A}\}$. ρ is *reducible* if ρ is not irreducible. ρ and $\rho^{'}$ are *equivalent* if there is a unitary $u \in \mathcal{A}$ such that $\rho^{'} = \mathrm{Ad}u \circ \rho$. In this case, we denote $\rho \sim \rho^{'}$. Let $\mathrm{Rep}\mathcal{A}$(*resp.* $\mathrm{IrrRep}\mathcal{A}$) be the set of all unital (*resp.* irreducible)$*$-representations of $\mathcal{A}$. We simply denote π for $(\mathcal{H}, \pi) \in \mathrm{Rep}\mathcal{A}$.

Lemma 2.5. (i) *If* $\rho, \rho^{'} \in \mathrm{End}\mathcal{A}$ *and* $\pi, \pi^{'} \in \mathrm{Rep}\mathcal{A}$ *satisfy* $\rho \sim \rho^{'}$ *and* $\pi \sim \pi^{'}$*, then* $\pi \circ \rho \sim \pi^{'} \circ \rho^{'}$.

(ii) *Assume that* $\mathcal{A}$ *is simple. If there is* $\pi \in \mathrm{IrrRep}\mathcal{A}$ *such that* $\pi \circ \rho \in \mathrm{IrrRep}\mathcal{A}$*, too, then* ρ *is irreducible.*

(iii) *If there is* $\pi \in \mathrm{Rep}\mathcal{A}$ *such that* $\pi \circ \rho \not\sim \pi \circ \rho^{'}$*, then* $\rho \not\sim \rho^{'}$.

(iv) *If there is* $\pi \in \mathrm{IrrRep}\mathcal{A}$ *such that* $\pi \circ \rho \notin \mathrm{IrrRep}\mathcal{A}$*, then* ρ *is proper.*

For $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, $\psi_\sigma \in \mathrm{End}\mathcal{O}_N$ is defined by

$$\psi_\sigma(s_i) \equiv u_\sigma s_i \quad (i = 1, \ldots, N)$$

where $u_\sigma \equiv \sum_{J \in \{1,\ldots,N\}^l} s_{\sigma(J)} s_J^*$. ψ_σ is called the *permutative endomorphism* of $\mathcal{O}_N$ by σ. Put the following sets:

$$E_{N,l} \equiv \{\psi_\sigma \in \mathrm{End}\mathcal{O}_N : \sigma \in \mathfrak{S}_{N,l}\} \quad (l \geq 1). \tag{2.3}$$

If $\sigma \in \mathfrak{S}_N$, then ψ_σ is an automorphism of $\mathcal{O}_N$ which satisfies $\psi_\sigma(s_i) = s_{\sigma(i)}$ for $i = 1, \ldots, N$. Specially, if $\sigma = id$, then $\psi_{id} = id$. If $\sigma \in \mathfrak{S}_{N,2}$ is defined by $\sigma(i,j) \equiv (j,i)$ for $i,j = 1,\ldots,N$, then ψ_σ is just the canonical endomorphism of $\mathcal{O}_N$. If $\rho \in E_{N,l}$ and $\rho^{'} \in E_{N,l'}$, then $\rho \circ \rho^{'} \in E_{N,l+l'-1}$ for each $l, l^{'} \geq 1$(see Proposition 3.3 in [**12**]).

Theorem 2.6. (i) *Let Λ be an infinite set. For $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, and $f \in \mathrm{BFS}_N(\Lambda)$, let $f^{(\sigma)}$ be in (2.1). Then we see that $\pi_f \circ \psi_\sigma = \pi_{f^{(\sigma)}}$.*

(ii) *If ρ is a permutative endomorphism and $(\mathcal{H},\pi)$ is a permutative representation of $\mathcal{O}_N$, then $\pi \circ \rho$ is a permutative representation, too.*

(iii) *If $(\mathcal{H},\pi)$ is $P(J)$ for $J \in \{1,\ldots,N\}_1^*$ and $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, then there are $1 \leq m \leq N^{l-1}$, a family $\{J_i\}_{i=1}^m \subset \{1,\ldots,N\}_1^*$ and a family $\{(\mathcal{H}_i,\pi_i)\}_{i=1}^m$ of subrepresentations of $(\mathcal{H},\pi\circ\psi_\sigma)$ such that*

$$(\mathcal{H},\pi\circ\psi_\sigma) = \bigoplus_{i=1}^{m}(\mathcal{H}_i,\pi_i) \tag{2.4}$$

and $(\mathcal{H}_i,\pi_i)$ is $P(J_i)$ for $i = 1,\ldots,m$. Furthermore if $J \in \{1,\ldots,N\}^k$, $k \geq 1$, then $\{J_i\}_{i=1}^m \subset \coprod_{a=1}^{N^{l-1}}\{1,\ldots,N\}^{ak}$.

(iv) *The rhs in (2.4) is unique up to unitary equivalences.*

Proof. See Theorem 3.4 in [**12**]. □

(2.4) is called the *branching law* of $(\mathcal{H},\pi)$ by ψ_σ. By uniqueness of $P(J)$ and Theorem 2.6 (iv), we can simply denote (2.4) as

$$P(J)\circ\psi_\sigma = \bigoplus_{i=1}^{m} P(J_i). \tag{2.5}$$

Definition 2.7. (i) *For $J \in \{1,\ldots,N\}_1^*$, a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ has a $P(J)$-component if $(\mathcal{H},\pi)$ has a subrepresentation $(\mathcal{H}_0,\pi|_{\mathcal{H}_0})$ which is $P(J)$. Specially, a component of a representation $P(J)\circ\rho$ of $\mathcal{O}_N$ is a subrepresentation of $(\mathcal{H},\pi)$ which is equivalent to $P(J^{'})$ for some $J^{'} \in \{1,\ldots,N\}_1^*$.*

(ii) *For an endomorphism $\rho \in \mathrm{End}\mathcal{O}_N$, $P(J)\circ\rho$ has a trivial branching if there is some $J^{'} \in \{1,\ldots,N\}_1^*$ such that $P(J)\circ\rho = P(J^{'})$.*

According to (2.5) and the above discussion, we have the following problems:

Problem 2.8. (i) *Computation of branching law:* Find $\{J_i\}_{i=1}^m$ in (2.5) for a given $J \in \{1,\ldots,N\}_1^*$ for $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$. In usual, the determination of $\{J_i\}_{i=1}^m$ is executed by the following step:

a) Prepare a representation $(\mathcal{H},\pi)$ which is $P(J)$. We often take $\mathcal{H} = l_2(\mathbf{N})$ and $\pi = \pi_f$ for some branching function system f on $\mathbf{N}$.
b) Compute $\pi(\psi_\sigma(s_i))e_n$ for each $n \in \mathbf{N}$ and $i = 1,\ldots,N$. By [**12**], we see that it is sufficient to check for $1 \leq n \leq N^{l-1}k$ when $|J| = k$.
c) Find all cycles in $\mathcal{H}$ by using results in (b).

d) Show that cycles in (c) spans the whole of $\mathcal{H}$.
In this way, the direct computation of branching law is too much of a bother because of a great number of calculated amount when N, k, l are large.

(ii) *Classification of* ψ_σ*:* Classify elements in $E_{N,l}$ for each $N, l \geq 1$ under unitary equivalences. If we know branching laws of ψ_σ, then it is useful for the classification by Lemma 2.5 (i). However, the computation of branching law of every element in $E_{N,l}$ is impracticable because $\#E_{N,l} = \#\mathfrak{S}_{N,l} = N^l!$. Therefore it is necessary to find an effective invariant of ψ_σ.

3. Automata and branching laws

3.1. Finite automata and semi-Mealy machines arising from permutations. Automata theory is the study of abstract computing devices, or "machines". We review several basic notions about automata and their variations in this subsection according to [**5**, **6**, **7**]. $M = (Q, \Sigma, \delta)$ is a *(finite)semiautomaton* if Q and Σ are non empty finite sets and δ is a map from $Q \times \Sigma$ to Q. Q, Σ and δ are called the *set of (internal)states*, the *set of input alphabets* and the *transition function*, respectively. Elements of Q and Σ are called a (internal)state and an input of M, respectively. $\hat{M} = (Q, \Sigma, \delta, q_0, F)$ is an *(deterministic finite)automaton* if $M = (Q, \Sigma, \delta)$ is a semiautomaton with $q_0 \in Q$ and a non empty subset F of Q. q_0 and an element of F are called the initial state and a final state of $\hat{M}$, respectively. Recall Σ^* and the extension of δ on $Q \times \Sigma^*$ in § 1. For $x \in \Sigma^*$, define $Q(x) \equiv \{q \in Q : {}^{\exists}n \in \mathbf{N}\, s.t.\, \delta(q, x^n) = q\}$. We see that $Q(x) \neq \emptyset$ for each $x \in \Sigma^*$ by finiteness of Q.

Definition 3.1. *Let* $M = (Q, \Sigma, \delta)$ *be a semiautomaton and* $x = a_{j_1} \cdots a_{j_k} \in \Sigma^*$*.*

(i) *A sequence* $C = (q_1, \ldots, q_k)$ *in* Q *is a cycle in* M *by* x *if* $q_1, \ldots, q_k$ *satisfy that* $\delta(q_t, a_{j_t}) = q_{t+1}$ *for* $t = 1, \ldots, k-1$ *and* $\delta(q_k, a_{j_k}) = q_1$ *when* $k \geq 2$*, and* $\delta(q_1, a_{j_1}) = q_1$ *when* $k = 1$*. We often denote* $C = q_1 \cdots q_k$ *simply.*

(ii) *For* $q \in Q(x)$*, put* $r_x(q) \equiv \min\{n \in \mathbf{N} : \delta(q, x^n) = q\}$*.*

(iii) *For* $q, q' \in Q(x)$*,* $q \sim q'$ *if there is* $n \in \mathbf{N} \cup \{0\}$ *such that* $\delta(q, x^n) = q'$*.*

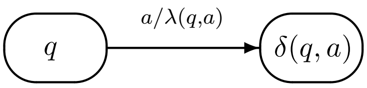
We see that $\sim$ is an equivalence relation in $Q(x)$. Put $R(x) \equiv \{[q] : q \in Q(x)\}$ where $[q] \equiv \{q' \in Q(x) : q \sim q'\}$.

Definition 3.2. *For a semiautomaton* $M = (Q, \Sigma, \delta)$*,* $\{p_1, \ldots, p_m\}$ *is a cyclic basis of* M *for* $x \in \Sigma^*$ *if* $p_1, \ldots, p_m \in Q(x)$ *are mutually inequivalent and* $\#R(x) = m$*.*

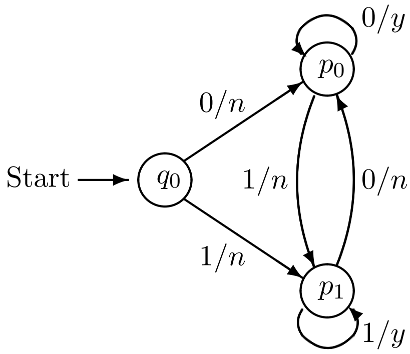
Lemma 3.3. *Let* $M = (Q, \Sigma, \delta)$ *be a semiautomaton and* $x \in \Sigma^*$*. For* $q, q' \in Q(x)$*, if* $q \sim q'$*, then* $r_x(q) = r_x(q')$*.*

Proof. Denote $qD_a \equiv \delta(q, a)$ for $q \in Q$ and $a \in \Sigma$. Then we see that $D_a D_b = D_{ab}$ for each $a, b \in \Sigma$. By assumption, there is $m \in \mathbf{N} \cup \{0\}$ such that $\delta(q, x^m) = q^{'}$. If $n \in \mathbf{N}$ satisfies $\delta(q, x^n) = q$, then $\delta(q^{'}, x^n) = q^{'} D_{x^n} = qD_{x^m} D_{x^n} = qD_{x^{n+m}} = qD_{x^n} D_{x^m} = qD_{x^m} = q^{'}$. Hence we see that $r_x(q) \geq r_x(q^{'})$. By the same argument, we see that $r_x(q^{'}) \geq r_x(q)$, too. Therefore the statement holds. □

Next we consider semi-Mealy machines([**13**]) in order to describe branching law. Recall § 1. For a semi-Mealy machine $(Q, \Sigma, \Delta, \delta, \lambda)$, Δ and λ are called the set of output alphabets and the map of outputs, respectively. When $q_i = \delta(q_{i-1}, a_i)$ for $i = 1, \ldots, n$ and $x = a_1 \cdots a_n \in \Sigma^*$, we see that $\lambda(q_0, x) = \lambda(q_0, a_1)\lambda(q_1, a_2) \cdots \lambda(q_{n-1}, a_n)$. $\lambda(q_0, x)$ is called the output by an input x. A *transition diagram(Mealy diagram)* $\mathcal{D}(M)$ of a semi-Mealy machine $M = (Q, \Sigma, \Delta, \delta, \lambda)$ is a directed graph with labeled edges which has a set Q of vertices and a set $E \equiv \{(q, \delta(q, a), a) \in Q \times Q \times \Sigma : q \in Q,\ a \in \Sigma\}$ of directed edges with labels. The meaning of $(q, \delta(q, a), a)$ is an edge from q to $\delta(q, a)$ with a label $a/\lambda(q, a)$ for $a \in \Sigma$:



We show an example in § 2.7, [**7**]. Let $M = (\{q_0, p_0, p_1\}, \{0, 1\}, \{y, n\}, \delta, \lambda, q_0)$ be a Mealy machine with the following $\mathcal{D}(M)$:



Mealy machine

For an input 01100, the output from M is *nnyny* and the path is $q_0 p_0 p_1 p_1 p_0 p_0$. $C = p_0 p_1$ is a cycle in a semi-Mealy machine $M_0 = (\{q_0, p_0, p_1\}, \{0, 1\}, \{y, n\}, \delta, \lambda)$ by $x = 10$ and $\lambda(p_0, x) = nn$.

Definition 3.4. *For a semi-Mealy machine $M = (Q, \Sigma, \Delta, \delta, \lambda)$, $x \in \Sigma^*$ and $p \in Q(x)$,*

$$\kappa_x(p) \equiv \lambda(p, x^r) \in \Delta \quad (r \equiv r_x(p))$$

is called the principal output of M for x from p.

Finally, we introduce semi-Mealy machines arising from permutations. For $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 2$ and $J \in \{1,\ldots,N\}^l$, we define $\sigma_1(J),\ldots,\sigma_l(J) \in \{1,\ldots,N\}$ by $\sigma(J) = (\sigma_1(J),\ldots,\sigma_l(J))$ and $\sigma_{n,m}(J) \equiv (\sigma_n(J),\ldots,\sigma_m(J))$ for $1 \leq n < m \leq l$. Denote $\{1,\ldots,N\}^0 \equiv \{0\}$ for convenience.

Definition 3.5. *For $N \geq 2$ and $l \geq 1$, $M_\sigma \equiv (Q,\Sigma,\Delta,\delta,\lambda)$ is called the semi-Mealy machine by $\sigma \in \mathfrak{S}_{N,l}$ if*

$$Q \equiv \{q_J : J \in \{1,\ldots,N\}^{l-1}\}, \quad \Sigma \equiv \{a_1,\ldots,a_N\}, \quad \Delta \equiv \{b_1,\ldots,b_N\}$$

and maps $\delta : Q \times \Sigma \to Q$, $\lambda : Q \times \Sigma \to \Delta$ are defined by

$$\delta(q_J,a_i) \equiv \begin{cases} q_0 & (l=1), \\ q_{(\sigma^{-1})_{2,l}(J,i)} & (l \geq 2), \end{cases} \qquad \lambda(q_J,a_i) \equiv \begin{cases} b_{\sigma^{-1}(i)} & (l=1), \\ b_{(\sigma^{-1})_1(J,i)} & (l \geq 2) \end{cases}$$

for $i = 1,\ldots,N$ and $J \in \{1,\ldots,N\}^{l-1}$.

We see that $\hat{M}_{\sigma,J_0} \equiv (Q,\Sigma,\Delta,\delta,\lambda,q_{J_0})$ is a Mealy machine for each $J_0 \in \{1,\ldots,N\}^{l-1}$. By Definition 3.5, there are N^{l-1} states in M_σ for $\sigma \in \mathfrak{S}_{N,l}$. We have a family $\{\hat{M}_{\sigma,J_0} : J_0 \in Q\}$ of Mealy machines associated with $\sigma \in \mathfrak{S}_{N,l}$. We show examples of M_σ in § 4.

3.2. The main theorem. In order to show the main theorem, we prepare several tools and lemmata.

Definition 3.6. *Let $\sigma \in \mathfrak{S}_{N,l}$ and $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, $l \geq 2$, $k \geq 1$.*

(i) *A sequence $\mathcal{I} = (I_i)_{i=1}^\alpha$ in $\{1,\ldots,N\}^{l-1}$ is an intertwining system of σ for J if there are $a \in \mathbf{N}$ and $T = (t_1,\ldots,t_\alpha) \in \{1,\ldots,N\}^\alpha$ such that $\alpha = ka$, $\sigma(t_1,I_1) = (I_\alpha,j_\alpha)$ and $\sigma(t_i,I_i) = (I_{i-1},j_{i-1})$ for each $i = 2,\ldots,\alpha$ where $j_{lk+i} \equiv j_i$ for $l \geq 1$ and $i = 1,\ldots,k$. In this case, we denote $\sigma(T,\mathcal{I}) = (\mathcal{I},J^a)$. We denote $\mathrm{ITS}(\sigma,J)$ the set of all intertwining system of σ for J and put $\mathrm{ITS}(\sigma,J;T,a) \equiv \{\mathcal{I} \in \mathrm{ITS}(\sigma,J) : \sigma(T,\mathcal{I}) = (\mathcal{I},J^a)\}$.*

(ii) *$\mathcal{I}_0 = (I_i^{'})_{i=1}^\beta$ is a subsystem of an intertwining system $\mathcal{I} = (I_i)_{i=1}^\alpha$ of σ for J if $\mathcal{I}_0 \in \mathrm{ITS}(\sigma,J)$ such that $\beta \leq \alpha$ and $I_i^{'} = I_i$ for $i = 1,\ldots,\beta$. In this case, we denote $\mathcal{I}_0 \prec \mathcal{I}$.*

(iii) *$\mathcal{I} \in \mathrm{ITS}(\sigma,J)$ is minimal if $\mathcal{I}$ is minimal with respect to $\prec$.*

(iv) *$\mathcal{I} = (I_i)_{i=1}^\alpha, \mathcal{I}^{'} = (I_i^{'})_{i=1}^{\alpha^{'}} \in \mathrm{ITS}(\sigma,J)$ are equivalent if $\alpha = \alpha^{'}$ and there is $\beta \in \mathbf{N} \cup \{0\}$ such that $(I_1^{'},\ldots,I_\alpha^{'}) = (I_{\beta k+1},\ldots,I_\alpha,I_1,\ldots,I_{\beta k})$. In this case, we denote $\mathcal{I} \sim \mathcal{I}^{'}$.*

Recall a cycle of a branching function system in § 2.1 and put Λ a countably infinite set.

Lemma 3.7. *Let $\sigma \in \mathfrak{S}_{N,l}$, $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $k \geq 1$, $\mathcal{I} = (I_i)_{i=1}^{\alpha} \in \mathrm{ITS}(\sigma, J; T, a)$ and $f \in \mathrm{BFS}_N(\Lambda)$ be $P(J)$ for $T = (t_1, \ldots, t_\alpha) \in \{1, \ldots, N\}^\alpha$ and $a \in \mathbf{N}$. Then the followings hold:*

(i) *Let $\{n_1, \ldots, n_k\}$ be a cycle of f in Λ such that $f_{j_i}(n_i) = n_{i-1}$ for $i = 2, \ldots, k$ and $f_{j_1}(n_1) = n_k$. Put a sequence $(m_1, \ldots, m_\alpha)$ in Λ by*

$$m_1 \equiv f_{I_1}(n_\alpha) \quad m_j \equiv f_{I_j}(n_{j-1}) \quad (j = 2, \ldots, \alpha),$$

$$n_{k\mu+j} \equiv n_j \quad (\mu \geq 1,\ j = 1, \ldots, k).$$

Then it satisfies that $f_{t_i}^{(\sigma)}(m_i) = m_{i-1}$ for $i = 2, \ldots, \alpha$ and $f_{t_1}^{(\sigma)}(m_1) = m_\alpha$. Specially, we have $f_T^{(\sigma)}(m_\alpha) = m_\alpha$.

(ii) *Let $(m_1, \ldots, m_\alpha)$ be in (i). Then $\mathcal{I}$ is minimal if and only if $m_i \neq m_j$ when $i \neq j$ for each $i, j = 1, \ldots, \alpha$.*

Proof. (i) $f_{t_\alpha}^{(\sigma)}(m_\alpha) = f_{\sigma(t_\alpha, I_\alpha)}(n_{k-1}) = f_{(I_{\alpha-1}, j_{\alpha-1})}(n_{k-1}) = f_{I_{\alpha-1}}(n_{k-2})$ $= m_{\alpha-1}$. In the same way, we have the statement.
(ii) By proof of (i), we see that $f_T^{(\sigma)}(m_\alpha) = m_\alpha$. By definition of m_i and the injectivity of f_i, if $m_i = m_j$ for some $i < j$, then there is $\beta \in \mathbf{N}$ such that $\beta < \alpha$ and $m_\beta = m_\alpha$. From this, $f_T(m_\beta) = m_\beta$ and $\mathcal{I}_0 \equiv \{I_i\}_{i=1}^{\beta}$ is a subsystem of $\mathcal{I}$. Hence $\mathcal{I}$ is not minimal. If $\mathcal{I}$ is not minimal, we see that $m_i = m_{i+ka}$ for some $a \geq 1$. Hence the statement holds. □

By Lemma 3.7, we denote $C(\mathcal{I}) = \{m_1, \ldots, m_\alpha\}$ which is obtained by a minimal intertwining system $\mathcal{I}$. $C(\mathcal{I})$ is a cycle of $f^{(\sigma)}$. We denote

$$\Lambda(\mathcal{I}) \equiv \{f_J^{(\sigma)}(m_i) \in \Lambda : J \in \{1, \ldots, N\}_1^*, i = 1, \ldots, \alpha\}.$$

Then $(\Lambda(\mathcal{I}), f^{(\sigma)}|_{\Lambda(\mathcal{I})})$ is a component of $f^{(\sigma)}$. We see that $\Lambda(\mathcal{I}) = \{f_J^{(\sigma)}(f_{I_\alpha}(n_1)) \in \Lambda : J \in \{1, \ldots, N\}_1^*\}$ by cyclicity of $\Lambda(\mathcal{I})$ under $f^{(\sigma)}$.

Lemma 3.8. *Let $f \in \mathrm{BFS}_N(\Lambda)$ be $P(J)$ for $J \in \{1, \ldots, N\}_1^*$ and $\sigma \in \mathfrak{S}_{N,l}$ for $l \geq 2$. Assume that $\mathcal{I}, \mathcal{I}' \in \mathrm{ITS}(\sigma, J)$ are minimal. Then the followings are equivalent:* (i) $\mathcal{I} \sim \mathcal{I}'$. (ii) $\Lambda(\mathcal{I}) = \Lambda(\mathcal{I}')$.
Furthermore the followings are equivalent: (i) $\mathcal{I} \not\sim \mathcal{I}'$. (ii) $\Lambda(\mathcal{I}) \cap \Lambda(\mathcal{I}') = \emptyset$.

Proof. Assume that $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$ for $k \geq 1$. Let $\{n_1, \ldots, n_k\}$ be the cycle of f. By Lemma 3.7, we have two cycles $\{m_1, \ldots, m_\alpha\}$ and $\{m_1', \ldots, m_{\alpha'}'\}$ of $f^{(\sigma)}$ by $\mathcal{I}$ and $\mathcal{I}'$, respectively. Then $\Lambda(\mathcal{I}) = \Lambda(\mathcal{I}')$ if and only if $\{m_1, \ldots, m_\alpha\} = \{m_1', \ldots, m_{\alpha'}'\}$. By definition of m_i and m_i' in Lemma 3.7 and injectivity of f_i, this is equivalent that $\mathcal{I} \sim \mathcal{I}'$. From this, we have the former statement.

By the first half of the statement and its proof, we see that $\mathcal{I} \not\sim \mathcal{I}'$ if and only if $\Lambda(\mathcal{I}) \neq \Lambda(\mathcal{I}')$ if and only if $\{m_1, \ldots, m_\alpha\} \neq \{m_1', \ldots, m_{\alpha'}'\}$. By the uniqueness of a cycle in a cyclic component of a branching function system, we see that $\{m_1,\ldots,m_\alpha\} \neq \{m_1',\ldots,m_{\alpha'}'\}$ if and only if $\Lambda(\mathcal{I}) \cap \Lambda(\mathcal{I}') = \emptyset$. Hence we have the last half of the statement. □

Let $\mathcal{I} = (I_i)_{i=1}^{\alpha} \in \mathrm{ITS}(\sigma, J; T, a)$ for $\sigma \in \mathfrak{S}_{N,l}$, $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, $T \in \{1,\ldots,N\}_1^*$ and $a \geq 1$. Assume that $\mathcal{I}$ is minimal. Then we see that $p_{\mathcal{I}} \equiv q_{I_1} \in Q(x)$ for $x \equiv a_J$. On the other hand, if $p \in Q(x)$, then there are $I_1,\ldots,I_\alpha$ such that $q_{I_1} = p$ and $q_{I_{i+1}} = \delta(q_{I_i}, a_{j_i})$ for $i = 1,\ldots,\alpha-1$. We see that $\mathcal{I}_p \equiv (I_i)_{i=1}^{\alpha} \in \mathrm{ITS}(\sigma, J; T, a)$, $a \equiv r_x(p)$ and it is minimal. In this way, $Q(x) \ni p \mapsto \mathcal{I}_p$ is a one-to-one correspondence.

Lemma 3.9. *For $p, p' \in Q(x)$, $p \sim p'$ if and only if $\mathcal{I}_p \sim \mathcal{I}_{p'}$.*

Proof. Assume that $p = q_{I_1}$ and $p' = q_{I_1'}$. Then we have $q_{I_1},\ldots,q_{I_\alpha}$, $q_{I_1'},\ldots,q_{I_\alpha'} \in Q$ such that $q_{I_{i+1}} = \delta(q_{I_i}, a_{j_i})$ and $q_{I_{i+1}'} = \delta(q_{I_i'}, a_{j_i})$.

Assume that $p \sim p'$. If we denote $\mathcal{I}_p = (I_i)_{i=1}^{\alpha}$ and $\mathcal{I}_{p'} = (I_i')_{i=1}^{\alpha'}$, then we see that $\alpha = \alpha'$ by Lemma 3.3. If $p = p'$, then it is clear. If $p \neq p'$, then there is $n \in \mathbf{N}$ such that $\delta(p, x^n) = p'$. This is equivalent that $(q_{I_1'},\ldots,q_{I_\alpha'}) = (q_{I_{nk+1}},\ldots,q_{I_\alpha},q_{I_1},\ldots,q_{I_{nk}})$. Furthermore this is equivalent that $(I_1',\ldots,I_\alpha') = (I_{nk+1},\ldots,I_\alpha,I_1,\ldots,I_{nk})$. From this, $\mathcal{I}_p \sim \mathcal{I}_{p'}$. We see that this argument shows the inverse direction, too. □

Corollary 3.10. *For $p, p' \in Q(x)$, $p \sim p'$ if and only if $\Lambda(\mathcal{I}_p) = \Lambda(\mathcal{I}_{p'})$. $p \not\sim p'$ if and only if $\Lambda(\mathcal{I}_p) \cap \Lambda(\mathcal{I}_{p'}) = \emptyset$.*

Proof. By Lemma 3.9 and Lemma 3.8, it holds. □

Lemma 3.11. *Let $f \in \mathrm{BFS}_N(\Lambda)$, $\sigma \in \mathfrak{S}_{N,l}$ and $T, J \in \{1,\ldots,N\}_1^*$ for $l \geq 2$. Assume that f is $P(J)$ and $M_\sigma = (Q, \Sigma, \Delta, \delta, \lambda)$ is the semi-Mealy machine by σ. Then the followings are equivalent:*

(i) *There are $a \geq 1$ and $\mathcal{I} \in \mathrm{ITS}(\sigma, J; T, a)$.*
(ii) *There is $\Lambda_0 \subset \Lambda$ such that $(\Lambda_0, f^{(\sigma)}|_{\Lambda_0})$ is $P(T)$.*
(iii) *There is $p \in Q(x)$ such that $b_T = \kappa_x(p)$ for $x \equiv a_J$.*

Proof. See Appendix A. □

Lemma 3.12. *Let $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 2$, $f \in \mathrm{BFS}_N(\Lambda)$ be $P(J)$ for $J \in \{1,\ldots,N\}_1^*$, $M_\sigma = (Q, \Sigma, \Delta, \delta, \lambda)$ be the semi-Mealy machine by σ and $x \equiv a_J \in \Sigma^*$.*

(i) *Assume that $n_0 \in \Lambda$ such that $f_J(n_0) = n_0$, $I \in \{1,\ldots,N\}^{l-1}$ and $T = (t_1,\ldots,t_\alpha) \in \{1,\ldots,N\}^\alpha$, $\alpha \geq 1$ satisfy that $p = q_I \in Q(x)$ and $b_T = \kappa_x(p)$. If $n_p \equiv f_{\sigma(t_1,I)}(n_0)$, then $f_T^{(\sigma)}(n_p) = n_p$.*

(ii) *For* $p \in Q(x)$*, denote* $\Lambda_p \equiv \{f^{(\sigma)}_{J'}(n_p) : J' \in \{1,\ldots,N\}^*_1\}$ *where* $n_p \in \Lambda$ *is in (i). Then* $(\Lambda_p, f^{(\sigma)}|_{\Lambda_p})$ *is* $P(T)$*.*

(iii) *Let* $\{p_1,\ldots,p_m\}$ *be a cyclic basis of* M_σ *for* x*. Then the following decomposition of branching function system holds:*

$$f^{(\sigma)} = f^{[1]} \oplus \cdots \oplus f^{[m]}$$

where $f^{[i]} \equiv f^{(\sigma)}|_{\Lambda_i}$*,* $\Lambda_i \equiv \Lambda_{p_i}$ *for* $i = 1,\ldots,m$*. If* $J_1,\ldots,J_m \in \{1,\ldots,N\}^*_1$ *satisfy* $\kappa_x(p_i) = b_{J_i}$ *for* $i = 1,\ldots,m$*, then* $f^{[i]}$ *is* $P(J_i)$ *for* $i = 1,\ldots,m$*.*

Proof. (i) By Lemma 3.7 (i) and the argument above Lemma 3.9, it holds.
(ii) We see that $\Lambda_p = \Lambda(\mathcal{I}_p)$. By (i), the statement holds.
(iii) By (ii) and Corollary 3.10, we see that $f^{[i]} \in \mathrm{BFS}_N(\Lambda_i)$. The decomposition holds by Lemma 3.11. By (ii), the last statement holds. □

Under these preparations, we show the main theorem.

Theorem 3.13. *Let* $M_\sigma = (Q,\Sigma,\Delta,\lambda,\delta)$ *be the semi-Mealy machine by* $\sigma \in \mathfrak{S}_{N,l}$*,* $l \geq 1$ *and* $J \in \{1,\ldots,N\}^*_1$*. Assume that* $\{p_1,\ldots,p_m\}$ *is a cyclic basis of* M_σ *for* $x \equiv a_J$*. If* $J_1,\ldots,J_m \in \{1,\ldots,N\}^*_1$ *satisfy* $b_{J_i} = \kappa_x(p_i)$ *for* $i = 1,\ldots,m$*, then the following holds:*

$$P(J) \circ \psi_\sigma \sim P(J_1) \oplus \cdots \oplus P(J_m).$$

Proof. Assume that $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$. When $l = 1$, we see that $P(J) \circ \psi_\sigma = P(J_{\sigma^{-1}})$ by Lemma 2.10 in [**11**] where $J_{\sigma^{-1}} \equiv (\sigma^{-1}(j_1),\ldots,\sigma^{-1}(j_k))$. Then we see that $\lambda(q_0, a_J) = b_{J_{\sigma^{-1}}}$. When $l \geq 2$, it holds by Proposition 2.4, Theorem 2.6 (iv) and Lemma 3.12. □

It seems that the statement in Theorem 3.13 depends on the choice of $\{p_1,\ldots,p_m\}$. For $p, p' \in Q(x)$, if $p \sim p'$, then we see that $T \sim T'$ when $b_T = \kappa_x(p)$ and $b_{T'} = \kappa_x(p')$. By this reason and Theorem 2.3 (ii), we see that the result in Theorem 3.13 is unique up to unitary equivalences for any cyclic basis of M_σ.

By Theorem 3.13, Theorem 1.1 is shown and the complexity to compute branching law in Problem 2.8 (i) is reduced, for example, it is not necessary to prepare representation space for calculating a branching law in Theorem 3.13.

Lemma 3.14. *Let* $\mathfrak{S}_{N,l}$*,* $l \geq 1$*. If the Mealy diagram of* M_σ *has* m *connected components, then* $P(J) \circ \psi_\sigma$ *has* m *components of direct sum at least for each* $J \in \{1,\ldots,N\}^*_1$*.*

Proof. Assume that $M_\sigma = \{Q,\Sigma,\Delta,\delta,\lambda\}$ and $J \in \{1,\ldots,N\}^*_1$. Then there is one cycle at each connected component of Q at least. Therefore there are m cycles in Q at least. By Theorem 3.13, $P(J) \circ \psi_\sigma$ has m components of direct sum. $\square$

4. Examples

For concrete permutations, we show examples of semi-Mealy machine and permutative endomorphism of $\mathcal{O}_N$ and compute their branching laws by using Mealy diagrams according to Theorem 3.13. Recall $E_{N,l}$ in (2.3).

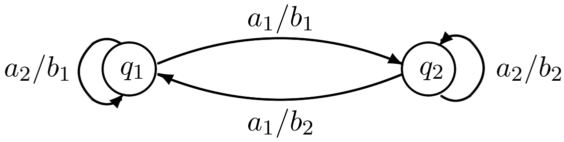
4.1. $E_{2,2}$**.** In [**11**], we show that there are 16 inequivalence classes in $E_{2,2}$ and there are 5 irreducible and proper classes $\mathcal{E}$ in them. We treat 3 elements in $\mathcal{E}$ here. For each $\sigma \in \mathfrak{S}_{2,2}$, $M_\sigma = (Q, \Sigma, \Delta, \delta, \lambda)$ consists of $Q = \{q_1, q_2\}$, $\Sigma = \{a_1, a_2\}$ and $\Delta = \{b_1, b_2\}$.

Let $\sigma \in \mathfrak{S}_{2,2}$ be a transposition by $\sigma(1,1) \equiv (1,2)$. Then $M_\sigma = (\{q_1, q_2\}, \{a_1, a_2\}, \{b_1, b_2\}, \delta, \lambda)$ where δ and λ are given by

p	$\delta(p, a_1)$	$\delta(p, a_2)$	$\lambda(p, a_1)$	$\lambda(p, a_2)$
q_1	q_2	q_1	b_1	b_1
q_2	q_1	q_2	b_2	b_2

ψ_σ and the Mealy diagram $\mathcal{D}(M_\sigma)$ of M_σ are as follows:

$$\begin{cases} \psi_\sigma(s_1) \equiv s_1 s_2 s_1^* + s_1 s_1 s_2^*, \\ \psi_\sigma(s_2) \equiv s_2, \end{cases}$$



where a/b on p to q means $\delta(p, a) = q$ and $\lambda(p, a) = b$ for $p, q \in Q = \{q_1, q_2\}$. ψ_σ is irreducible and proper(Table 5.5 in [**11**]). We denote ψ_σ by ψ_{12} in convenience. We show several branching laws by ψ_{12}:

input	cycles	outputs	branching law
a_1	$q_1 q_2$	$b_1 b_2$	$P(1) \circ \psi_{12} = P(12)$
a_2	q_1, q_2	b_1, b_2	$P(2) \circ \psi_{12} = P(1) \oplus P(2)$
$a_1 a_2$	$q_1 q_2 q_2 q_1$	$b_1 b_2 b_2 b_1$	$P(12) \circ \psi_{12} = P(1122)$
$a_1 a_1 a_2 a_2$	$q_1 q_2 q_1 q_1$, $q_2 q_1 q_2 q_2$	$b_1 b_2 b_1 b_1$, $b_2 b_1 b_2 b_2$	$P(1122) \circ \psi_{12}$ $= P(1112) \oplus P(1222)$

where we use Theorem 2.3 (ii).

Proposition 4.1. *For* $J = (j_1, \ldots, j_k) \in \{1,2\}^k$, $k \geq 1$, *define* $n_1(J) \equiv \#\{j \in J : j = 1\}$. *Then we have* $P(J) \circ \psi_{12}$ *has just two-branching when* $n_1(J)$ *is even and* $P(J) \circ \psi_{12}$ *has no trivial branching when* $n_1(J)$ *is odd.*

Proof. Because such transition of two states q_1 and q_2 in the above diagram happens only when $j = 1$, $n_1(J)$ is the number of changes two states. If $n_1(J)$ is even, then a path from a state always comes back to itself. Therefore just two cycles occur. On the other hand, If $n_1(J)$ is odd, then one cycle occurs. The number of cycles is just that of branching of $P(J)\circ\psi_{12}$. Therefore the assertion holds. □

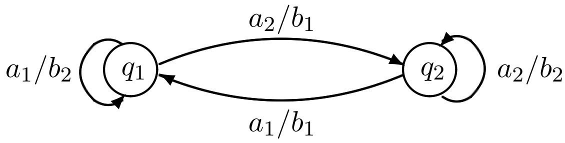
By Proposition 4.1, for each $J\in\{1,2\}_1^*$, there are some J_1, J_2 or J_3 which satisfy the following:

$$P(J)\circ\psi_{12}=\begin{cases} P(J_1)\oplus P(J_2) & (n_1(J)=\text{ even}),\\ P(J_3) & (n_1(J)=\text{ odd}).\end{cases}$$

In this way, it is remarkable that the graph theoretical property of Mealy diagram gives information of branching.

Let $\sigma\in\mathfrak{S}_{2,2}$ be a transposition by $\sigma(1,1)\equiv(2,1)$. Then ψ_σ, $\mathcal{D}(M_\sigma)$ and branching laws of ψ_σ are as follows:

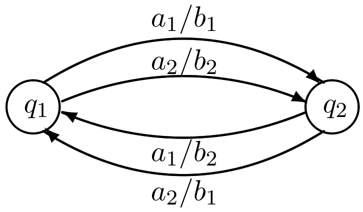
$$\begin{cases} \psi_\sigma(s_1)\equiv s_2s_1s_1^*+s_1s_2s_2^*,\\ \psi_\sigma(s_2)\equiv s_1s_1s_1^*+s_2s_2s_2^*,\end{cases}$$



input	cycles	outputs	branching law
a_1	q_1	b_2	$P(1)\circ\psi_\sigma=P(2)$
a_2	q_2	b_2	$P(2)\circ\psi_\sigma=P(2)$
a_1a_2	q_2q_1	b_1b_2	$P(12)\circ\psi_\sigma=P(11)$
$a_1a_1a_2$	$q_2q_1q_1$	$b_1b_2b_1$	$P(112)\circ\psi_\sigma=P(112)$
$a_1a_2a_2$	$q_2q_1q_2$	$b_1b_1b_2$	$P(122)\circ\psi_\sigma=P(112)$

Let $\sigma\in\mathfrak{S}_{2,2}$ be defined by $\sigma(1,1)\equiv(2,2)$, $\sigma(1,2)\equiv(1,1)$, $\sigma(2,1)\equiv(2,1)$, $\sigma(2,2)\equiv(1,2)$. Then ψ_σ, $\mathcal{D}(M_\sigma)$ and branching laws of ψ_σ are as follows:

$$\begin{cases} \psi_\sigma(s_1)\equiv s_2s_2s_1^*+s_1s_1s_2^*,\\ \psi_\sigma(s_2)\equiv s_2s_1s_1^*+s_1s_2s_2^*,\end{cases}$$



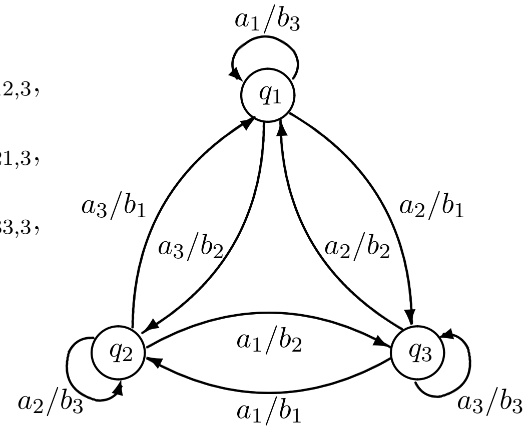
input	cycles	outputs	branching law
a_1	q_1q_2	b_1b_2	$P(1)\circ\psi_\sigma=P(12)$
a_2	q_1q_2	b_2b_1	$P(2)\circ\psi_\sigma=P(12)$
a_1a_2	q_1q_2, q_2q_1	b_1b_1, b_2b_2	$P(12)\circ\psi_\sigma=P(11)\oplus P(22)$

4.2. $E_{3,2}$**.** Note that $\#E_{2,2}=2^2!=24$ and $\#E_{3,2}=3^2!\sim 3.6\times10^5$. Hence it is difficult to classify every element in $E_{3,2}$ by computing its branching laws by comparison with the case $E_{2,2}$. We see that $M_\sigma=(\{q_1,q_2,q_3\},\{a_1,a_2,a_3\},\{b_1,b_2,b_3\},\delta,\lambda)$ for each $\sigma\in\mathfrak{S}_{3,2}$. Put σ_0 a transformation on $\{1,2,3\}^2$ by

$$\sigma_0:\begin{pmatrix}11&12&13\\21&22&23\\31&32&33\end{pmatrix}\mapsto\begin{pmatrix}23&31&12\\32&13&21\\11&22&33\end{pmatrix}. \tag{4.1}$$

Then $\rho_\nu \equiv \psi_{\sigma_0}$ and $\mathcal{D}(M_{\sigma_0})$ are as follows:

$$\begin{cases} \rho_\nu(s_1) \equiv & s_{23,1} + s_{31,2} + s_{12,3}, \\ \rho_\nu(s_2) \equiv & s_{32,1} + s_{13,2} + s_{21,3}, \\ \rho_\nu(s_3) \equiv & s_{11,1} + s_{22,2} + s_{33,3}, \end{cases}$$

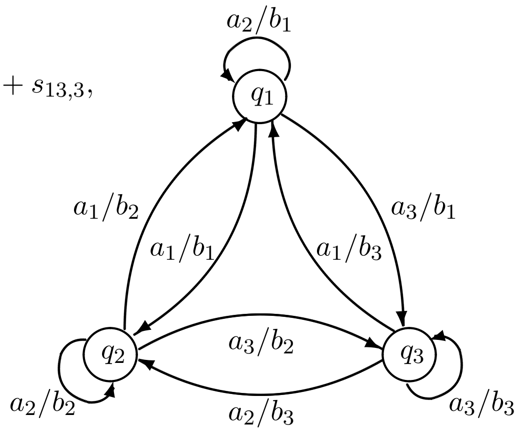


where $s_{ij,k} \equiv s_i s_j s_k^*$ for $i, j, k = 1, 2, 3$. ρ_ν is proper and irreducible by Theorem 1.2 in [**11**]. We show several branching laws by ρ_ν:

input	cycles	outputs	branching law
a_1	q_1, q_2q_3	b_3, b_1b_2	$P(1) \circ \rho_\nu = P(3) \oplus P(12)$
a_1a_2	$q_1q_1q_3q_2q_2q_3$	$b_3b_1b_1b_3b_2b_2$	$P(12) \circ \rho_\nu = P(113223)$
$a_1a_2a_3$	$q_1q_1q_3q_3q_2q_2$, $q_2q_3q_1$	$b_3b_1b_3b_1b_3b_1$, $b_2b_2b_2$	$P(123) \circ \rho_\nu = P(131313) \oplus P(222)$
$a_1a_3a_2$	$q_1q_1q_2q_2q_3q_3$, $q_3q_2q_1$	$b_3b_2b_3b_2b_3b_2$, $b_1b_1b_1$	$P(132) \circ \rho_\nu = P(232323) \oplus P(111)$

Let $\sigma \in \mathfrak{S}_{3,2}$ be a transposition by $\sigma(1,1) \equiv (1,2)$. ψ_σ, $\mathcal{D}(M_\sigma)$ and branching laws of ψ_σ are as follows:

$$\begin{cases} \psi_\sigma(s_1) \equiv & s_{12,1} + s_{11,2} + s_{13,3}, \\ \psi_\sigma(s_2) \equiv & s_2, \\ \psi_\sigma(s_3) \equiv & s_3, \end{cases}$$



input	cycles	outputs	branching law
a_1	q_1q_2	b_1b_2	$P(1)\circ\psi_\sigma = P(12)$
a_2	q_1, q_2	b_1, b_2	$P(2)\circ\psi_\sigma = P(1)\oplus P(2)$
a_3	q_3	b_3	$P(3)\circ\psi_\sigma = P(3)$

From this, we see that ψ_σ^n is proper and irreducible by Lemma 2.5 (ii), (iv) for each $n \geq 1$. By Lemma 2.5 (ii), ψ_σ and ρ_ν are not equivalent.

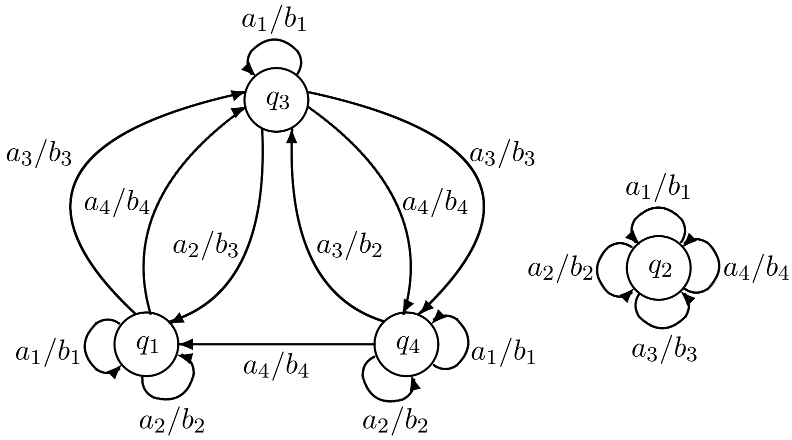
4.3. $E_{4,2}$. Note $\#E_{4,2} = 16!$. Let $\sigma \in \mathfrak{S}_{4,2}$ be defined by:

$$\sigma^{-1} : \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix} \mapsto \begin{pmatrix} 11 & 21 & 33 & 43 \\ 12 & 22 & 32 & 42 \\ 13 & 31 & 34 & 44 \\ 14 & 24 & 23 & 41 \end{pmatrix}.$$

Then ψ_σ and $\mathcal{D}(M_\sigma)$ are as follows:

$$\psi_\sigma(s_1) \equiv s_{11,1} + s_{21,2} + s_{31,3} + s_{41,4}, \quad \psi_\sigma(s_2) \equiv s_{12,1} + s_{22,2} + s_{43,3} + s_{42,4},$$

$$\psi_\sigma(s_3) \equiv s_{32,1} + s_{23,2} + s_{13,3} + s_{33,4}, \quad \psi_\sigma(s_4) \equiv s_{44,1} + s_{24,2} + s_{14,3} + s_{34,4},$$



When $J = (1)$, the transition at each vertex comes back itself by input a_1. Hence there are just four cycles. Furthermore, each output is b_1. Therefore $P(1)\circ\psi_\sigma = P(1)\oplus P(1)\oplus P(1)\oplus P(1)$. In the same way, we have

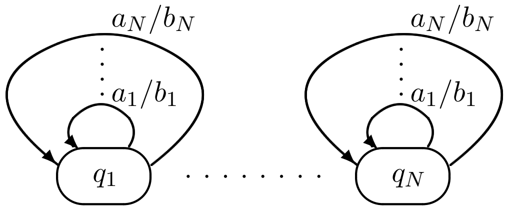
$$P(2)\circ\psi_\sigma = P(2)\oplus P(2)\oplus P(2), \quad P(4)\circ\psi_\sigma = P(4)\oplus P(444).$$

This is an example of Lemma 3.14. Furthermore we have the followings:

input	cycles	outputs	branching law
a_3	q_3q_4, q_2	b_2b_3, b_3	$P(3)\circ\psi_\sigma = P(23)\oplus P(3)$
a_1a_4	$q_1q_1q_3q_3q_4q_4$, q_2q_2	$b_1b_4b_1b_4b_1b_4$, b_1b_4	$P(14)\circ\psi_\sigma$ $= P(141414)\oplus P(14)$
$a_1a_2a_3a_4$	$q_4q_4q_3q_4$, $q_2q_2q_2q_2$	$b_1b_2b_2b_4$, $b_1b_2b_3b_4$	$P(1234)\circ\psi_\sigma$ $= P(1224)\oplus P(1234)$

4.4. Canonical endomorphism. The Mealy diagram of the semi-Mealy machine associated with the canonical endomorphism ρ of $\mathcal{O}_N$(see § 2.3) is given as follows:

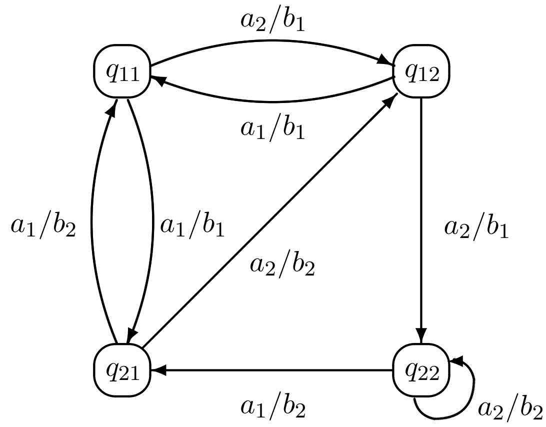
$$\rho(x) \equiv s_1 x s_1^* + \cdots + s_N x s_N^*,$$



In this case, there is no transition among states. We see that $P(J) \circ \psi_\sigma = P(J)^{\oplus N}$ for each $J \in \{1, \ldots, N\}_1^*$ where $P(J)^{\oplus N}$ is the direct sum of N-copies of $P(J)$. In general, $\pi \circ \psi_\sigma = \pi^{\oplus N}$ for any representation π of $\mathcal{O}_N$ by Proposition 4.8 in [**12**].

4.5. $E_{2,3}$**.** Let $\sigma \in \mathfrak{S}_{2,3}$ be a transposition by $\sigma(1,1,1) \equiv (1,2,1)$. Then $\psi_\sigma \in E_{2,3}$, $\mathcal{D}(M_\sigma)$ and branching laws of ψ_σ are as follows:

$$\psi_\sigma(s_1) \equiv s_1 s_2 s_1 s_1^* s_1^* + s_1 s_1 s_2 s_2^* s_1^* + s_1 s_1 s_1 s_1^* s_2^* + s_1 s_2 s_2 s_2^* s_2^*, \quad \psi_\sigma(s_2) \equiv s_2,$$



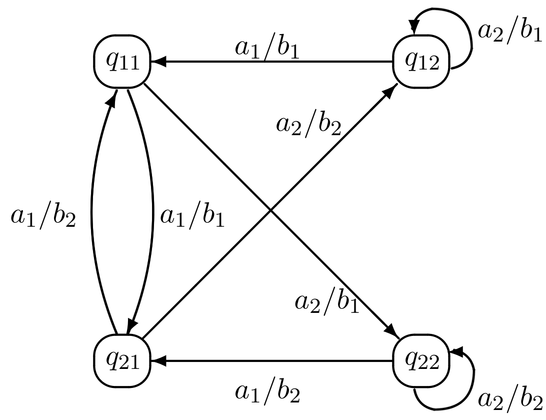
input	cycles	outputs	branching law
a_1	$q_{11}q_{21}$	b_1b_2	$P(1) \circ \psi_\sigma = P(12)$
a_2	q_{22}	b_2	$P(2) \circ \psi_\sigma = P(2)$
a_1a_2	$q_{12}q_{11}$	b_1b_1	$P(12) \circ \psi_\sigma = P(11)$
$a_1a_1a_2$	$q_{12}q_{11}q_{21}$	$b_1b_1b_2$	$P(112) \circ \psi_\sigma = P(112)$

We see that ψ_σ^n is irreducible and proper for each $n \geq 1$ by Lemma 2.5 (ii), (iv).

Let $\sigma \in \mathfrak{S}_{2,3}$ be defined by

$$\sigma : \begin{pmatrix} 111 & 112 & 121 & 122 \\ 211 & 212 & 221 & 222 \end{pmatrix} \mapsto \begin{pmatrix} 121 & 122 & 111 & 112 \\ 211 & 212 & 221 & 222 \end{pmatrix}.$$

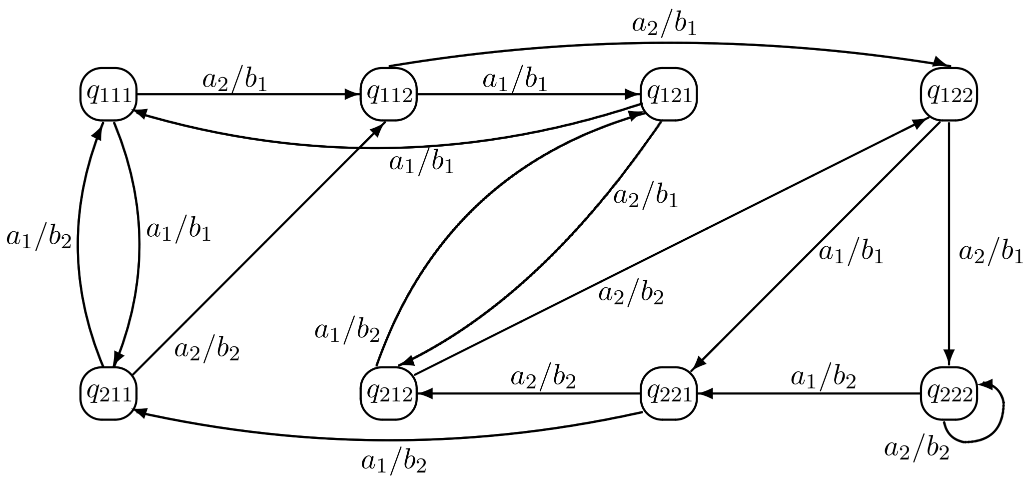
In this case, we see that $\psi_\sigma = \psi_{12} \in E_{2,2}$ in § 4.1. $\mathcal{D}(M_\sigma)$ and branching laws of ψ_σ are as follows:



input	cycles	outputs	branching law
a_1	$q_{11}q_{21}$	b_1b_2	$P(1) \circ \psi_\sigma = P(12)$
a_2	q_{12}, q_{22}	b_1, b_2	$P(2) \circ \psi_\sigma = P(1) \oplus P(2)$
a_1a_2	$q_{12}q_{11}q_{22}q_{21}$	$b_1b_1b_2b_2$	$P(12) \circ \psi_\sigma = P(1122)$

4.6. $E_{2,4}$. Note $\#E_{2,4} = 16!$. Let $\sigma \in \mathfrak{S}_{2,4}$ be a transposition by $\sigma(1,1,1,1) \equiv (1,2,1,1)$. $\psi_\sigma \in E_{2,4}$, $\mathcal{D}(M_\sigma)$ and branching laws of ψ_σ are given as follows:

$$\begin{aligned}\psi(s_1) \equiv\ & s_{1211}s_{111}^* + s_{1112}s_{112}^* + s_{112}s_{12}^* + s_{1111}s_{211}^* + s_{1212}s_{212}^* + s_{122}s_{22}^*,\\ \psi(s_2) \equiv\ & s_2,\end{aligned}$$

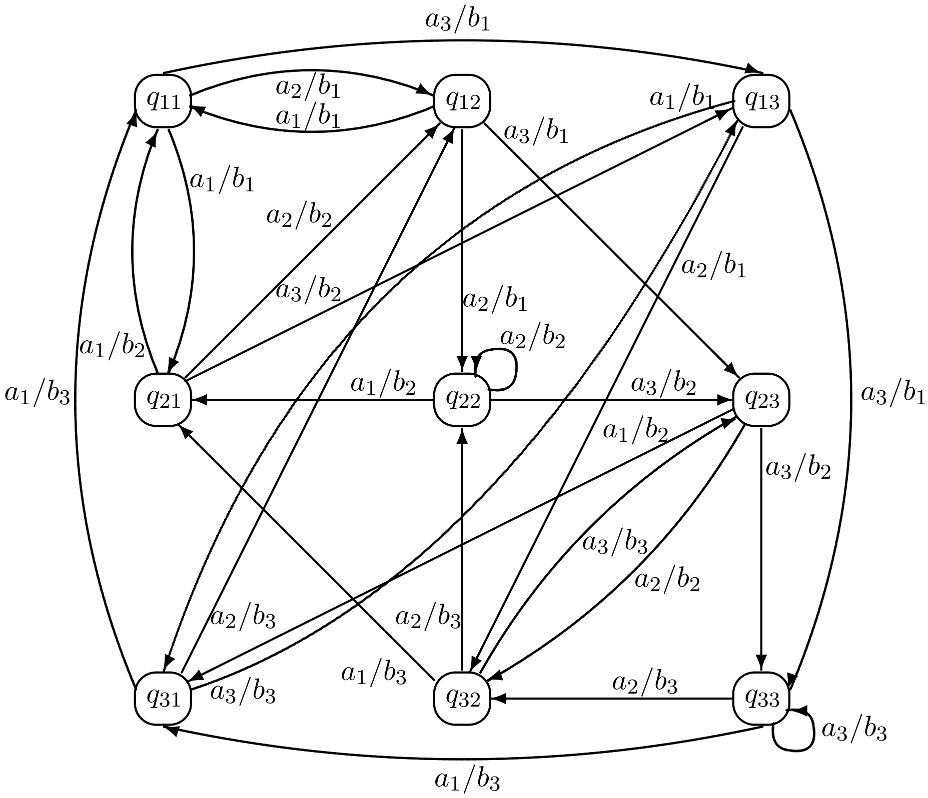


input	cycles	outputs	branching law
a_1	$q_{111}q_{211}$	b_1b_2	$P(1)\circ\psi_\sigma = P(12)$
a_2	q_{222}	b_2	$P(2)\circ\psi_\sigma = P(2)$
a_1a_2	$q_{212}q_{121}$	b_2b_1	$P(12)\circ\psi_\sigma = P(12)$
$a_1a_1a_2$	$q_{112}q_{121}q_{111}$	$b_1b_1b_1$	$P(112)\circ\psi_\sigma = P(111)$

4.7. $E_{3,3}$**.** Let $\sigma \in \mathfrak{S}_{3,3}$ be a transposition by $\sigma(1,1,1) \equiv (1,2,1)$. Then ψ_σ, $\mathcal{D}(M_\sigma)$ and branching laws of ψ_σ are as follows:

$$\psi_\sigma(s_1) \equiv s_{121}s_{11}^* + s_{112}s_{12}^* + s_{113}s_{13}^* + s_{111}s_{21}^* + s_{122}s_{22}^* + s_{123}s_{23}^* + s_{13}s_3^*,$$

$$\psi_\sigma(s_2) \equiv s_2, \quad \psi_\sigma(s_3) \equiv s_3 \quad \text{(where we use notation in § 2.2)},$$



input	cycles	outputs	branching law
a_1	$q_{11}q_{21}$	b_1b_2	$P(1) \circ \psi_\sigma = P(12)$
a_2	q_{22}	b_2	$P(2) \circ \psi_\sigma = P(2)$
a_3	q_{33}	b_3	$P(3) \circ \psi_\sigma = P(3)$
a_1a_2	$q_{12}q_{11}$	b_1b_1	$P(12) \circ \psi_\sigma = P(11)$
a_1a_3	$q_{13}q_{31}$	b_1b_3	$P(13) \circ \psi_\sigma = P(13)$
a_2a_3	$q_{23}q_{32}$	b_2b_3	$P(23) \circ \psi_\sigma = P(23)$
$a_1a_2a_3$	$q_{23}q_{31}q_{12}$	$b_2b_3b_1$	$P(123) \circ \psi_\sigma = P(123)$
$a_1a_3a_2$	$q_{32}q_{21}q_{13}$	$b_3b_2b_1$	$P(132) \circ \psi_\sigma = P(132)$

We see that ψ_σ^n is irreducible and proper for each $n \geq 1$ by Lemma 2.5 (ii), (iv).

Appendix A. Proof of Lemma 3.11

Put Λ a countably infinite set and $N, l \geq 2$.

Lemma A.1. *Let $J \in \{1,\ldots,N\}_1^*$. If $f \in \mathrm{BFS}_N(\Lambda)$ is $P(J)$ and there are $J^{'} \in \{1,\ldots,N\}_1^*$ and $n \in \Lambda$ such that $f_{J'}(n) = n$, then there is $M \geq 1$ such that $J^{'} \sim J^M$.*

Proof. Assume that $J = (j_1,\ldots,j_\alpha)$, $J^{'} = (j_1^{'},\ldots,j_\beta^{'})$ and $C \equiv \{n_1,\ldots,n_\alpha\} \subset \Lambda$ is the cycle of f. Put $n_\beta^{'} \equiv f_{j_\beta^{'}}(n)$, $n_{t-1}^{'} \equiv f_{j_t^{'}}(n_t^{'})$ for $t = 2,\ldots,\beta$ and $C^{'} \equiv \{n_1^{'},\ldots,n_\beta^{'}\} \subset \Lambda$. Because f has only one cycle in Λ, we see that $C^{'} = C$. From this, $\beta \geq \alpha$ and there is $M \geq 1$ such that $\beta = M\alpha$. <u>If $M = 1$</u>, then $\alpha = \beta$ and $(l_2(\Lambda), \pi_f)$ is $P(J)$ and $P(J^{'})$ by Proposition 2.4 (iii). Therefore $P(J) \sim P(J^{'})$. By Theorem 2.3 (ii), $J \sim J^{'}$. <u>If $M \geq 2$</u>, then $J_1 \equiv (j_1^{'},\ldots,j_\alpha^{'})$ satisfies $f_{J_1}(n) = n$ and $J_1^M = J^{'}$. From the case $M = 1$, we see that $J_1 \sim J$ and $J^{'} \sim J^M$. □

Proof of Lemma 3.11. Assume that $T = (t_1,\ldots,t_\alpha) \in \{1,\ldots,N\}^\alpha$, $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$ for $\alpha, k \geq 1$.
<u>(i)⇒(ii)</u>: This is shown in Lemma 3.7.
<u>(ii)⇒ (i)</u>: If there is Λ_0 such that $(\Lambda_0, f^{(\sigma)}|_{\Lambda_0})$ is $P(T)$, then there is $n_0 \in \Lambda_0$ such that $f_T^{(\sigma)}(n_0) = n_0$. Since $\Lambda = \coprod_{J' \in \{1,\ldots,N\}^{l-1}} f_{J'}(\Lambda)$, we can denote $n_0 = f_{J'}(n)$ and $f_T^{(\sigma)}(f_{J'}(n)) = f_{J'}(n)$ for some $J^{'} \in \{1,\ldots,N\}^{l-1}$ and $n \in \Lambda$. By computing $f_T^{(\sigma)}(f_{J'}(n))$, we have

$$f_{J'}(n) = f_{(\sigma(t_1,I_1),\sigma_l(t_2,I_2),\ldots,\sigma_l(t_{\alpha-1},I_{\alpha-1}),\sigma_l(t_\alpha,I_\alpha))}(n)$$

where $I_\alpha \equiv J^{'}$, $I_i \equiv \sigma_{1,l-1}(t_{i+1}, I_{i+1})$ for $i = 1, \ldots, \alpha-1$. Because $f_{J'}$ is injective, we have $n = f_{(\sigma_l(t_1,I_1),\ldots,\sigma_l(t_\alpha,I_\alpha))}(n)$. By Lemma A.1, we see that $(\sigma_l(t_1,I_1),\ldots,\sigma_l(t_\alpha,I_\alpha))$ is equivalent to J^a for some $a \geq 1$ and $I_\alpha = J^{'} = \sigma_{1,l-1}(t_1,I_1)$. Put $T^{'} = (t_{\beta+1},\ldots,t_\alpha,t_1,\ldots,t_\beta)$ for $1 \leq \beta \leq \alpha-1$. Because $(\Lambda_0, f^{(\sigma)}|_{\Lambda_0})$ is $P(T^{'})$ for each $1 \leq \beta \leq \alpha-1$, we can take $t_1,\ldots,t_\alpha$ and $I_1,\ldots,I_\alpha$ such that $(\sigma_l(t_1,I_1),\ldots,\sigma_l(t_\alpha,I_\alpha)) = J^a$. Therefore $\mathcal{I} = (I_i)_{i=1}^{\alpha} \in \mathrm{ITS}(\sigma, J; T, a)$. By Lemma 3.7, $\mathcal{I}$ is minimal. In consequence, (i) is satisfied.
(i)$\Leftrightarrow$ (iii): This is shown in a paragraph above Lemma 3.9. □

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