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Discrete Fixed Point Theorem Reconsidered*

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Abstract

The aim of this note is to indicate an example that demonstrates the incorrectness of Iimura's discrete fixed point theorem (Iimura 2003) and to present a corrected statement using the concept of integrally convex sets.

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1 Introduction

Iimura (2003) gave a discrete fixed point theorem for set-valued correspondences on discrete sets. The theorem claims that a discretely convex-valued direction-preserving correspondence defined on a contiguously convex set has a fixed point (see Section 2 for the precise statement). In this note we indicate an example that demonstrates the incorrectness of this statement, and rectify the statement using the concept of integrally convex sets introduced by Favati and Tardella (1990).

2 Iimura's statement and a counterexample

Let $\mathbf{R}$ and $\mathbf{Z}$ denote the sets of all reals and all integers, respectively. Given a positive integer n, we denote by $\mathbf{Z}^n$ the set of all integer vectors $x = (x_i \in \mathbf{Z} :$

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$i = 1, \dots, n$). A finite set $X \subseteq \mathbf{Z}^n$ is called *discretely convex* (or *hole free*, Murota (2003)) if

$$X = \overline{X} \cap \mathbf{Z}^n, \tag{1}$$

where $\overline{X}$ denotes the convex hull of X. A finite set $X \subseteq \mathbf{Z}^n$ is said to be *contiguously convex* if

$$\forall y \in \overline{X},\ \exists x \in X : ||x - y||_\infty < 1, \tag{2}$$

where $||x - y||_\infty = \max\{|x_i - y_i| \mid i = 1, \dots, n\}$. By the definitions, a contiguously convex set is discretely convex. For two integer vectors x and x', we define a relation $x \simeq x'$ as

$$x \simeq x' \Leftrightarrow ||x - x'||_\infty \leq 1. \tag{3}$$

Let X be a nonempty finite subset of $\mathbf{Z}^n$ and $\Gamma : X \to\!\!\to X$ be a nonempty-valued correspondence. A point $x \in X$ is said to be a *fixed point* if $x \in \Gamma(x)$. For any $x \in \mathbf{Z}^n$, let $\pi_\Gamma(x)$ denote the projection of x onto $\overline{\Gamma(x)}$, i.e.,

$$||\pi_\Gamma(x) - x||_2 = \min_{y \in \overline{\Gamma(x)}} ||y - x||_2, \tag{4}$$

where $||y - x||_2 = \left(\sum_{i=1}^n (y_i - x_i)^2\right)^{1/2}$. We denote $\pi_\Gamma(x) - x$ by $\tau(x)$, and define

$$\sigma(x) = (\mathrm{sign}(\tau_i(x)) \in \{+1, 0, -1\} : i = 1, \dots, n), \tag{5}$$

where $\tau_i(x)$ denotes the ith component of $\tau(x)$.

According to Iimura (2003), a correspondence $\Gamma : X \to\!\!\to X$ is said to be *direction preserving*[1] if for all $x, x' \in X$ with $x \simeq x'$,

$$\sigma_i(x) > 0 \Longrightarrow \sigma_i(x') \geq 0 \qquad (i = 1, \dots, n), \tag{6}$$

where $\sigma_i(x)$ denotes the ith component of $\sigma(x)$. The condition is equivalent to

$$\sigma_i(x) < 0 \Longrightarrow \sigma_i(x') \leq 0 \qquad (i = 1, \dots, n). \tag{7}$$

Iimura (2003) made the following statement.

Iimura's Statement: Let X be a finite contiguously convex subset of $\mathbf{Z}^n$. If $\Gamma : X \to\!\!\to X$ is a nonempty- and discretely convex-valued direction preserving correspondence, then Γ has a fixed point.

A counterexample exists to the above statement. We consider the finite set $X \subseteq \mathbf{Z}^3$ defined as

$$X = \{a = (0, 1, 0), b = (1, 0, 0), c = (2, 0, 0), d = (3, 0, 0), e = (4, 0, 1)\} \tag{8}$$

and the correspondence $\Gamma : X \to\!\!\to X$ defined as

$$\Gamma(a) = \Gamma(b) = \{e\},\ \Gamma(c) = \{a, e\},\ \Gamma(d) = \Gamma(e) = \{a\}. \tag{9}$$

[1] We note that "direction preserving" can also be defined in terms of τ as: if for all $x, x' \in X$ with $x \simeq x'$, $\tau_i(x) > 0 \Longrightarrow \tau_i(x') \geq 0$ for all $i = 1, \dots, n$.

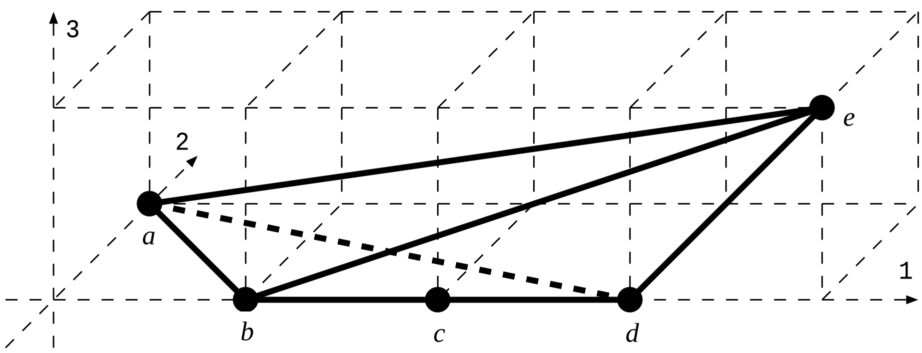

Figure 1: A contiguously convex set $X = \{a = (0,1,0), b = (1,0,0), c = (2,0,0), d = (3,0,0), e = (4,0,1)\}$ for the counterexample with $\Gamma(a) = \Gamma(b) = \{e\}$, $\Gamma(c) = \{a, e\}$, $\Gamma(d) = \Gamma(e) = \{a\}$.

Figure 1 shows that X is a contiguously convex set and Γ is a nonempty- and discretely convex-valued correspondence. Furthermore, Γ is direction preserving, because

$$a \simeq b,\ b \simeq c,\ c \simeq d,\ d \simeq e, \tag{10}$$

the other pairs of distinct points are not in the relation $\simeq$, and τ and σ are calculated as

$$\left.\begin{array}{rcr} \tau(a) & = & (\ 4,\ -1,\ \ 1) \\ \tau(b) & = & (\ 3,\ \ 0,\ \ 1) \\ \tau(c) & = & (\ 0,\ 1/2,\ 1/2) \\ \tau(d) & = & (-3,\ \ 1,\ \ 0) \\ \tau(e) & = & (-4,\ \ 1,\ -1) \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcr} \sigma(a) & = & (+1,\ -1,\ +1) \\ \sigma(b) & = & (+1,\ \ 0,\ +1) \\ \sigma(c) & = & (\ 0,\ +1,\ +1) \\ \sigma(d) & = & (-1,\ +1,\ \ 0) \\ \sigma(e) & = & (-1,\ +1,\ -1). \end{array}\right. \tag{11}$$

Obviously, Γ has no fixed point.

3 Theorem for integrally convex sets

In this section, we give a discrete fixed point theorem for integrally convex sets.

For $x \in \mathbf{R}^n$, we define a neighborhood $N(x)$ of x by

$$N(x) = \{y \in \mathbf{Z}^n \mid \|x - y\|_\infty < 1\}. \tag{12}$$

A finite set of integer points $X \subseteq \mathbf{Z}^n$ is said to be *integrally convex* if it satisfies

$$x \in \overline{X} \Longrightarrow x \in \overline{X \cap N(x)} \qquad (\forall x \in \mathbf{R}^n) \tag{13}$$

(see Murota (2003), Section 3.4). By the definitions, an integrally convex set is contiguously convex.

Lemma 1 *For any finite integrally convex set $X \subset \mathbf{Z}^n$, there exists a simplicial decomposition[2] $\mathcal{S}$ of $\overline{X}$ such that for any $y \in \overline{X}$, the vertices of the smallest simplex*

[2] $\mathcal{S}$ satisfies (a) $\overline{X} = \bigcup_{S \in \mathcal{S}} S$, (b) $S \in \mathcal{S}$, S': a face of $S \Longrightarrow S' \in \mathcal{S}$, and (c) $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 \neq \emptyset \Longrightarrow S_1 \cap S_2$: a face of S_1 and S_2.

$S(y) \in \mathcal{S}$ *containing* y *belong to* $N(y)$*. Therefore, we have* $y \in \overline{S(y) \cap N(y)}$ *for all* $y \in \overline{X}$ *and* $\{x\} \in \mathcal{S}$ *for all* $x \in X$*.*

Proof. The proof is given at the end of this section. ∎

Lemma 2 *Suppose that* $\Gamma : X \to\to X$ *is a nonempty- and discretely convex-valued correspondence. For* $x \in X$ *we have* $x \in \Gamma(x)$ *if and only if* $\tau(x) = \mathbf{0}$*.*

Proof. By the definition of τ, we have $\tau(x) = \mathbf{0}$ if and only if $x \in \overline{\Gamma(x)}$. The latter condition is equivalent to $x \in \Gamma(x)$, since $\Gamma(x) = \overline{\Gamma(x)} \cap \mathbf{Z}^n$. Hence, $\tau(x) = \mathbf{0}$ if and only if $x \in \Gamma(x)$. ∎

Theorem 3 *Let* $X \subset \mathbf{Z}^n$ *be a nonempty finite integrally convex set. If* $\Gamma : X \to\to X$ *is a nonempty- and discretely convex-valued direction preserving correspondence, then* Γ *has a fixed point, that is, there exists* $x \in X$ *such that* $x \in \Gamma(x)$*.*

Proof. We make use of Brouwer's fixed point theorem, which says that every continuous mapping from a compact convex set of $\mathbf{R}^n$ to itself has a fixed point. We define a continuous mapping γ from $\overline{X}$ to $\overline{X}$. For any point $x \in X$, we define $\gamma(x) = \pi_\Gamma(x)$. Since X is a finite integrally convex set, there exists a simplicial decomposition $\mathcal{S}$ of $\overline{X}$ satisfying conditions in Lemma 1. Let y be an arbitrary point in $\overline{X}$. By Lemma 1, we have $y \in \overline{S(y) \cap N(y)}$. Let

$$y = \sum_{z \in S(y) \cap N(y)} \lambda_z z, \quad \sum \lambda_z = 1, \quad \lambda_z \geq 0, \tag{14}$$

be the uniquely determined convex combination. That is, $(\lambda_z \mid z \in S(y) \cap N(y))$ is the barycentric coordinate of y in $\overline{S(y) \cap N(y)}$. Then, we define $\gamma(y)$ by

$$\gamma(y) = \sum_{z \in S(y) \cap N(y)} \lambda_z \pi_\Gamma(z). \tag{15}$$

Since $\pi_\Gamma(z) \in \overline{X}$ for all $z \in X$, we have $\gamma(y) \in \overline{X}$ for all $y \in \overline{X}$. Moreover, since $\mathcal{S}$ is a simplicial decomposition, γ is continuous. By Brouwer's fixed point theorem, γ has a fixed point, say, $y \in \overline{X}$.

We next show that γ has an integral fixed point. We have

$$\sum_{z \in S(y) \cap N(y)} \lambda_z z = y = \gamma(y) = \sum_{z \in S(y) \cap N(y)} \lambda_z \pi_\Gamma(z). \tag{16}$$

This says that

$$\sum_{z \in S(y) \cap N(y)} \lambda_z (\pi_\Gamma(z) - z) = \sum_{z \in S(y) \cap N(y)} \lambda_z \tau(z) = \mathbf{0}. \tag{17}$$

Since Γ is direction preserving, we have $\tau(z) = \mathbf{0}$ if $\lambda_z > 0$. Therefore, there exists at least one $z \in S(y) \cap N(y)$ with $\tau(z) = \mathbf{0}$. Such z is a fixed point of Γ by Lemma 2. ∎

We finally show Lemma 1. Before giving a proof, we define the integral convexity of functions. For $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$, its *convex closure* $\overline{f} : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$\overline{f}(x) = \sup \left\{ \langle p, x \rangle + \gamma \;\middle|\; \begin{array}{l} p \in \mathbf{R}^n,\ \gamma \in \mathbf{R} \\ \langle p, y \rangle + \gamma \le f(y)\ (\forall y \in \mathbf{Z}^n) \end{array} \right\} \quad (\forall x \in \mathbf{R}^n). \tag{18}$$

The *local convex extension* $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = \sup \left\{ \langle p, x \rangle + \gamma \;\middle|\; \begin{array}{l} p \in \mathbf{R}^n,\ \gamma \in \mathbf{R} \\ \langle p, y \rangle + \gamma \le f(y)\ (\forall y \in N(x)) \end{array} \right\} \quad (\forall x \in \mathbf{R}^n). \tag{19}$$

A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *integrally convex* if $\overline{f} = \tilde{f}$ (Favati and Tardella (1990)). We note that a set $X \subseteq \mathbf{Z}^n$ is integrally convex if and only if its indicator function δ_X is integrally convex, where δ_X is defined as $\delta_X(x) = 0$ if $x \in X$; otherwise $\delta_X(x) = +\infty$.

Proof of Lemma 1. Let δ_X be the indicator function of X and d be an integer vector such that if $x \neq y$ then $\langle d, x \rangle \neq \langle d, y \rangle$ for all $x, y \in X$ (there exists such d because X is a finite set). We consider the function $h_\epsilon : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ with parameter ϵ defined as

$$h_\epsilon(x) = \delta_X(x) + \sum_{i=1}^{n} (x(i))^2 + \epsilon \exp\{\langle d, x \rangle\} \qquad (\forall x \in \mathbf{Z}^n). \tag{20}$$

We first show that h_ϵ is integrally convex for a sufficiently small positive number ϵ. It is known that $g(x) = \delta_X(x) + \sum_{i=1}^{n} (x(i))^2$ is integrally convex (e.g., see Proposition 3.24 in Murota (2003)). Let us consider the convex closure $\overline{g}$ of g, which is a piecewise linear function. Since the second term of g is a separable quadratic function, we have that for any $p \in \mathbf{R}^n$, $\arg\min \overline{g}[-p]$ is in the intersection of $\overline{X}$ with a hypercube $\{x \in \mathbf{R}^n \mid z \le x \le z + \mathbf{1}\}$ for some $z \in \mathbf{Z}^n$. Since X is a finite set and $\arg\min \overline{g}[-p]$ is included in a hypercube for any $p \in \mathbf{R}^n$, there exists a sufficiently small positive number ϵ such that $\arg\min \overline{h_\epsilon}[-p]$ is also included in the hypercube. This says that $\arg\min h_\epsilon[-p]$ is an integrally convex set for any $p \in \mathbf{R}^n$. It is known that a function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ with a nonempty bounded effective domain is integrally convex if and only if $\arg\min f[-p]$ is an integrally convex set for each $p \in \mathbf{R}^n$ (see Theorem 3.29 in Murota (2003)). Thus, h_ϵ is integrally convex for a sufficiently small positive number ϵ.

This integrally convex function h_ϵ gives a decomposition $\mathcal{S}$ of $\overline{X}$ such that (a) $S \in \mathcal{S}$ if and only if there exists $p \in \mathbf{R}^n$ with $S = \arg\min \overline{h_\epsilon}[-p]$, (b) if $S \in \mathcal{S}$ and S' is a face of S then $S' \in \mathcal{S}$, (c) if $S_1, S_2 \in \mathcal{S}$ and $S_1 \cap S_2 \neq \emptyset$ then $S_1 \cap S_2 \in \mathcal{S}$, (d) the vertices of S belong to X for any $S \in \mathcal{S}$, and (e) $x \in \overline{S \cap N(x)}$ for any $x \in \overline{X}$ and $S \in \mathcal{S}$ with $x \in S$, where (b) and (c) follow from (a), (d) follows from the fact that $\overline{h_\epsilon}$ is the convex closure of h_ϵ, and (e) follows from the integral convexity of h_ϵ. By (c), there exists the smallest simplex $S(y) \in \mathcal{S}$ for any $y \in \overline{X}$. Suppose to the contrary that there exists a vertex x of $S(y)$ with $x \notin N(y)$. Since $S(y)$ is included in a hypercube and $x \in X$, there exists a proper face of $S(y)$ including y. However, this together with (b) contradicts the minimality of $S(y)$. Hence, the vertices of $S(y)$ belong to $N(y)$.

We next show that $\arg\min \overline{h_\epsilon}[-p]$ is a simplex. Suppose to the contrary that $\arg\min \overline{h_\epsilon}[-p]$ is not a simplex. Let $z \in \mathbf{Z}^n$ be such that $\arg\min \overline{h_\epsilon}[-p] \subseteq \{x \in \mathbf{R}^n \mid z \leq x \leq z + \mathbf{1}\}$. Then, there exist disjoint families $\mathcal{I}$ and $\mathcal{J}$ of subsets of $\{1, \ldots, n\}$ and families $\{\lambda_I \mid I \in \mathcal{I}\}$ and $\{\lambda_J \mid J \in \mathcal{J}\}$ of positive rational numbers such that

$$\sum_{I \in \mathcal{I}} \lambda_I \chi_I = \sum_{J \in \mathcal{J}} \lambda_J \chi_J, \tag{21}$$

$$\sum_{I \in \mathcal{I}} \lambda_I h_\epsilon(z + \chi_I) = \sum_{J \in \mathcal{J}} \lambda_J h_\epsilon(z + \chi_J), \tag{22}$$

$$\sum_{I \in \mathcal{I}} \lambda_I = \sum_{J \in \mathcal{J}} \lambda_J = 1, \tag{23}$$

$$z + \chi_I, z + \chi_J \in \arg\min h_\epsilon[-p] \qquad (\forall I \in \mathcal{I},\ \forall J \in \mathcal{J}), \tag{24}$$

where χ_I denotes the characteristic vector of $I \subseteq \{1, \ldots, n\}$. Since $g(x)$ is an integer for each $x \in X$ and ϵ is sufficiently small, we have

$$\sum_{I \in \mathcal{I}} \lambda_I \exp\{\langle d, \chi_I \rangle\} = \sum_{J \in \mathcal{J}} \lambda_J \exp\{\langle d, \chi_J \rangle\}. \tag{25}$$

However, this contradicts that the base of the natural logarithm is a transcendental number, because $\mathcal{I}$ and $\mathcal{J}$ are disjoint, λ_I and λ_J are rational, and $\langle d, \chi_S \rangle$ are mutually distinct integers for all $S \in \mathcal{I} \cup \mathcal{J}$. Thus $\arg\min \overline{h_\epsilon}[-p]$ must be a simplex. ∎

Concluding remark

After the completion of the manuscript, the authors learned that a Russian group of V. Danilov and G. Koshevoy also noticed the incorrectness of Iimura's proof in Iimura (2003).

References

[1] Favati, P. and Tardella, F., 1990, Convexity in nonlinear integer programming, Ricerca Operativa 53, 3–44.

[2] Iimura, T., 2003, A discrete fixed point theorem and its applications, Journal of Mathematical Economics 39, 725–742.

[3] Murota, K., 2003, Discrete Convex Analysis, (Society for Industrial and Applied Mathematics, Philadelphia).