Algebra of sectors

Katsunori Kawamura
Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-8502, Japan

The set Sect$\mathcal{A}$ of all unitary equivalence classes of unital $*$-endomorphisms of a unital C*-algebra $\mathcal{A}$ is called the sector of $\mathcal{A}$. We show that there is an exotic algebraic structure on Sect$\mathcal{A}$ when $\mathcal{A}$ includes a Cuntz algebra as a C*-subalgebra with common unit. Next we explain that the set BSpec$\mathcal{A}$ of all unitary equivalence classes of unital $*$-representations of $\mathcal{A}$ is a right module of Sect$\mathcal{A}$. An essential algebraic formulation of branching laws of representations is given by submodules of BSpec$\mathcal{A}$. As application, we show that the action of Sect$\mathcal{A}$ on BSpec$\mathcal{A}$ distinguishes elements of Sect$\mathcal{A}$.

1. Introduction

For a unital $*$-algebra $\mathcal{A}$, the set Sect$\mathcal{A}$ of all unitary equivalence classes of unital $*$-endomorphisms of $\mathcal{A}$ is called the *sector* of $\mathcal{A}$. An element of Sect$\mathcal{A}$ is called a sector of $\mathcal{A}$, too. Sectors are studied in fields of quantum field theory([**4**, **10**, **12**, **24**]) and subfactors([**13**, **14**, **15**, **23**]) for formulation of super selection theory and index theory of subalgebras, respectively. According to each standpoint, their mathematical definitions of sectors are different in general. A definition of sector which is a set of some equivalence classes of representations of an observable algebra is interpreted to our definition through a relation between representations and endomorphisms under several assumptions. It is well-known that there are operations on Sect$\mathcal{A}$(or subsets of Sect$\mathcal{A}$) which are similar to direct sum and tensor product among representations of a group. Under these operations and some assumptions, a commutative algebra which consists of some sectors is called a *fusion rule algebra*([**10**]). We consider that the essential assumption for the existence of such sum is coming from Borchers property which states the existence of sufficient isometries in an observable algebra.

Without a special assumption, Sect$\mathcal{A}$ is always a semigroup by composition of sectors which is not abelian in general. In this paper, we show that there is a completely symmetric N-ary operation on Sect$\mathcal{A}$ which seems

e-mail:kawamura@kurims.kyoto-u.ac.jp.

"sum" with respect to the product of sectors when there is a unital $*$-embedding of the Cuntz algebra $\mathcal{O}_N$ into $\mathcal{A}$.

In this paper, $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ is the set of all unital $*$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ for each unital $*$-algebras $\mathcal{A}$ and $\mathcal{B}$.

Theorem 1.1. *Let* $\mathrm{Sect}\mathcal{A}$ *be the sector semigroup of a unital* C^**-algebra* $\mathcal{A}$.

(i) *For* $\mathcal{A}$*, assume that*

$$ {}^{\exists}N \geq 2 \quad s.t. \quad \mathrm{Hom}(\mathcal{O}_N,\mathcal{A}) \neq \emptyset. \tag{1.1}$$

Then there is an N*-ary operation* p *on* $\mathrm{Sect}\mathcal{A}$ *such that*

$$p(x_{\sigma(1)},\ldots,x_{\sigma(N)}) = p(x_1,\ldots,x_N), \quad p\circ(id^{N-1}\times p) = p\circ(p\times id^{N-1}),$$

$$yp(x_1,\ldots,x_N) = p(yx_1,\ldots,yx_N), \quad p(x_1,\ldots,x_N)y = p(x_1y,\ldots,x_Ny)$$

for $\sigma\in\mathfrak{S}_N$ *and* $x_1,\ldots,x_N,y\in\mathrm{Sect}\mathcal{A}$ *where* $\mathfrak{S}_N$ *is the group of all permutations of numbers* $1,\ldots,N$ *and id is the identity map on* $\mathrm{Sect}\mathcal{A}$.

(ii) *For each* $N\geq 3$*, there is a* C^**-algebra* $\mathcal{A}$ *which satisfies* $\mathrm{Hom}(\mathcal{O}_N,\mathcal{A})\neq\emptyset$ *and* $\mathrm{Hom}(\mathcal{O}_{N-1},\mathcal{A})=\emptyset$.

(iii) *If* $\mathcal{A}$ *is a von Neumann algebra, then either of the followings holds:* (a) $\mathrm{Hom}(\mathcal{O}_N,\mathcal{A})\neq\emptyset$ $({}^{\forall}N\geq 2)$, (b) $\mathrm{Hom}(\mathcal{O}_N,\mathcal{A})=\emptyset$ $({}^{\forall}N\geq 2)$.

If we simply denote p by "+", then we see that

$$\begin{cases} x_{\sigma(1)}+\cdots+x_{\sigma(N)} = x_1+\cdots+x_N \quad ({}^{\forall}\sigma\in\mathfrak{S}_N), \\[2ex] (x_1+\cdots+x_N)+x_{N+1}+\cdots+x_{2N-1} \\ \quad = x_1+(x_2+\cdots+x_{N+1})+x_{N+2}+\cdots+x_{2N-1} \\ \quad = \cdots = x_1+\cdots+x_{N-1}+(x_N+\cdots+x_{2N-1}), \\[2ex] y(x_1+\cdots+x_N) = yx_1+\cdots+yx_N, \\[2ex] (x_1+\cdots+x_N)y = x_1y+\cdots+x_Ny. \end{cases} \tag{1.2}$$

We see that (1.2) is interpreted as commutativity, associativity of + and distributive law among + and $\cdot$. In this way, + can be considered as "*N-ary sum*" on $\mathrm{Sect}\mathcal{A}$ and $\mathrm{Sect}\mathcal{A}$ is an algebra with the N-ary sum and ordinary binary product without inverse operation of this sum. An algebra with such unusual operation is known as universal algebra([**7**, **11**]). When $N=2$, $\mathrm{Sect}\mathcal{A}$ is an ordinary algebra without inverse operation of the sum. By Theorem 1.1 (ii), (iii), $\mathrm{Sect}\mathcal{A}$ has a non binary sum only if $\mathcal{A}$ is not a von Neumann algebra.

Furthermore, this algebraic structure of $\mathrm{Sect}\mathcal{A}$ has applications of branching laws of representations by embeddings and endomorphisms. We introduce a module of $\mathrm{Sect}\mathcal{A}$ which is naturally arising from an algebraic formulation of branching laws of representations of $\mathcal{A}$ by sectors.

Theorem 1.2. *Let* $\mathrm{BSpec}\mathcal{A}$ *be the abelian semigroup of all unitary equivalence classes of unital* $*$*-representations of a unital* C^**-algebra* $\mathcal{A}$ *by direct sum* $\oplus$*. Then there is a right action* R *of the semigroup* $\mathrm{Sect}\mathcal{A}$ *on* $\mathrm{BSpec}\mathcal{A}$*. Furthermore if* $\mathcal{A}$ *satisfies (1.1), then* $(\mathrm{BSpec}\mathcal{A}, R)$ *is a unital right module of the algebra* $\mathrm{Sect}\mathcal{A}$ *which satisfies (1.2), that is,*

$$(v \oplus w)R_x = vR_x \oplus wR_x, \quad (vR_x)R_y = vR_{xy},$$

$$vR_{x_1+\cdots+x_N} = vR_{x_1} \oplus \cdots \oplus vR_{x_N}$$

for each $x, y, x_1, \ldots, x_N \in \mathrm{Sect}\mathcal{A}$ *and* $v, w \in \mathrm{BSpec}\mathcal{A}$*.*

For example, our studies in [**20**, **21**, **22**], branching laws of representations of $\mathcal{O}_N$ are smartly explained by $\mathrm{Sect}\mathcal{O}_N$ and $\mathrm{BSpec}\mathcal{O}_N$. As application of this action, we can distinguish sectors and inclusions of C*-subalgebras by comparing their branching laws. We show concrete sectors of $\mathcal{O}_N$ which are defined by polynomials of the canonical generators $s_1, \ldots, s_N$ of $\mathcal{O}_N$ and their conjugates and branching laws of representations of the CAR algebra which are associated with endomorphisms of $\mathcal{O}_N$ in [**2**].

Theorem 1.3. *Define* $\rho, \bar{\rho}, \eta \in \mathrm{End}\mathcal{O}_2$ *by*

$$\rho(s_1) \equiv s_{12,1} + s_{11,2}, \quad \bar{\rho}(s_1) \equiv s_{21,1} + s_{12,2}, \quad \eta(s_1) \equiv s_{22,1} + s_{11,2},$$

$$\rho(s_2) \equiv s_2, \quad \bar{\rho}(s_2) \equiv s_{11,1} + s_{12,2}, \quad \eta(s_2) \equiv s_{21,1} + s_{12,2}$$

where $s_{ij,k} \equiv s_i s_j s_k^*$ *for* $i, j, k = 1, 2$*. Denote elements in* $\mathrm{Sect}\mathcal{O}_2$ *which are associated with* $\rho, \bar{\rho}, \eta$ *by* $[\rho], [\bar{\rho}], [\eta]$*, respectively.*

(i) $\rho, \bar{\rho}, \eta$ *are not surjective and the following is a set of mutually different irreducible sectors of* $\mathcal{O}_2$*:* $\{[\bar{\rho}]^n[\eta][\rho], [\eta], [\bar{\rho}]^n, [\rho], [\rho]^2 : n \geq 1\}$*.*

(ii) *The following equations in* $\mathrm{Sect}\mathcal{O}_2$ *hold:*

$$[\bar{\rho}][\rho] = [\iota] + [\alpha],\ [\rho][\bar{\rho}] = [\iota] + [\beta_1],\ [\bar{\rho}]^2[\rho]^2 = [\iota] + [\alpha] + [\eta],\ [\bar{\rho}][\alpha][\rho] = [\eta]$$

where ι *is the identity map on* $\mathcal{O}_2$ *and* α*,* β_1*,* β_2 *are in* $\mathrm{End}\mathcal{O}_2 \cap \mathrm{Aut}\mathcal{O}_2$ *which are defined by the following transpositions, respectively:* $s_1 \leftrightarrow s_2$*,* $s_1 \leftrightarrow -s_1$*,* $s_2 \leftrightarrow -s_2$*.*

(iii) *The statistical dimension* d_{ρ^n} *of* ρ^n *is* $2^{n/2}$ *for* $n \geq 1$*.*

By Theorem 1.3, $[\rho]$ and $[\bar{\rho}]$ does not commute, but it seems that they are conjugate.

In § 2, we define the sector as a homomorphism class space and introduce the algebra of sectors of a unital $*$-algebra. In § 3, we consider Theorem 1.2 and its application. In § 4, we introduce sectors of $\mathcal{O}_N$ arising from permutations and their spectrum modules. Branching laws of these representations of $\mathcal{O}_N$ are explained by submodules of these modules. In § 5, we treat sectors of $\mathcal{O}_N$ and their fusion rules more concretely. In § 6, we consider sectors which are arising from inclusions among $\mathcal{O}_N$ and UHF_N.

2. An algebraic structure on the sector

We show an exotic algebraic structure of Sect$\mathcal{A}$ under $\mathcal{O}_N$-including condition of a unital $*$-algebra $\mathcal{A}$. For this aim, we prepare several conditions about the "size" of $\mathcal{A}$. Next, we introduce an N-ary operation on the homomorphism space.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be unital $*$-algebras and we do not assume that any algebra is equipped with a topology in this section if there is no special assumption. In this paper, any representation, homomorphism and endomorphism of algebras are assumed unital and $*$-preserving.

2.1. Algebraic embeddings of the Cuntz algebras. We start to consider homomorphisms among the Cuntz algebras and algebras.

Lemma 2.1. *Let $M, N \geq 2$. Then $\mathrm{Hom}(\mathcal{O}_M, \mathcal{O}_N) \neq \emptyset$ if and only if there is a positive integer k such that $M = (N-1)k + 1$.*

Proof. If $M = (N-1)k+1$, then $\mathrm{Hom}(\mathcal{O}_M, \mathcal{O}_N) \neq \emptyset$ by (6.1) in § 6. Assume that $\varphi \in \mathrm{Hom}(\mathcal{O}_M, \mathcal{O}_N)$. By K-theory([**5**]), φ arises a homomorphism $\hat{\varphi}$ from $K_0(\mathcal{O}_M)$ to $K_0(\mathcal{O}_N)$, $K_0(\mathcal{O}_N) \cong \mathbf{Z}_{N-1}$ and the class $[I_N]$ of the unit of $\mathcal{O}_N$ is a generator of $K_0(\mathcal{O}_N)$. Because $\hat{\varphi}([I_M]) = [I_N]$ is a generator of $\mathbf{Z}_{N-1}$, $\hat{\varphi}(K_0(\mathcal{O}_M)) = K_0(\mathcal{O}_N)$. This shows that there is a surjective homomorphism from $\mathbf{Z}_{M-1}$ to $\mathbf{Z}_{N-1}$. In consequence, $M-1 \geq N-1$ and $M-1$ must be divided by $N-1$. Hence the statement holds. □

By Lemma 2.1, Theorem 1.1 (ii) is proved. In order to define algebraic operations on sectors, the Cuntz algebra is used as "glue" among sectors.

Definition 2.2. *For $N \geq 2$, $(t_1, \ldots, t_N)$ is a system of $\mathcal{O}_N$-generators in $\mathcal{A}$ if $t_1, \ldots, t_N \in \mathcal{A}$ satisfy the following relations:*

$$t_i^* t_j = \delta_{ij} I \quad (i,j = 1, \ldots, N), \quad t_1 t_1^* + \ldots + t_N t_N^* = I.$$

We denote $H_N\mathcal{A}$ the set of all systems of $\mathcal{O}_N$-generators in $\mathcal{A}$.

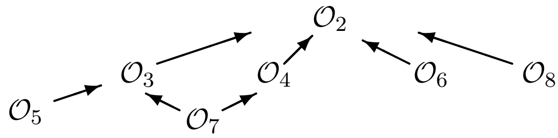
If $\mathcal{A}$ is a C*-algebra, then an element in $H_N\mathcal{A}$ is in one-to-one correspondence with that of $\mathrm{Hom}(\mathcal{O}_N, \mathcal{A})$ by $(t_i)_{i=1}^N \leftrightarrow \varphi(s_i) \equiv t_i$ for $i = 1, \ldots, N$. Therefore $H_N\mathcal{A} \neq \emptyset$ if and only if $\mathrm{Hom}(\mathcal{O}_N, \mathcal{A}) \neq \emptyset$.

Lemma 2.3. (i) *If $H_N\mathcal{A} \neq \emptyset$, then $H_N(\mathcal{A} \otimes \mathcal{B}) \neq \emptyset$ for each $\mathcal{B}$.*
(ii) *If $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \neq \emptyset$ and $H_N\mathcal{A} \neq \emptyset$, then $H_N\mathcal{B} \neq \emptyset$.*
(iii) *If $\mathcal{A} \subset \mathcal{B}$ is a unital inclusion such that $H_N\mathcal{A} \neq \emptyset$, then $H_N\mathcal{B} \neq \emptyset$.*
(iv) *If $H_N\mathcal{A} \neq \emptyset$, then $H_{(N-1)k+1}\mathcal{A} \neq \emptyset$ for each $k \geq 1$. Specially, if $H_2\mathcal{A} \neq \emptyset$, then $H_N\mathcal{A} \neq \emptyset$ for each $N \geq 2$.*
(v) *$H_N(\mathcal{A} \oplus \mathcal{B}) \neq \emptyset$ if and only if $H_N\mathcal{A} \neq \emptyset$ and $H_N\mathcal{B} \neq \emptyset$.*

Proof. (i) $(t_1, \ldots, t_N) \in H_N\mathcal{A}$ implies $(t_1 \otimes I, \ldots, t_N \otimes I) \in H_N(\mathcal{A} \otimes \mathcal{B})$.
(ii) If $\varphi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $(t_1, \ldots, t_N) \in H_N\mathcal{A}$, then $(\varphi(t_1), \ldots, \varphi(t_N)) \in H_N\mathcal{B}$.

(iii) Because the inclusion map of $\mathcal{A}$ into $\mathcal{B}$ is in $\mathrm{Hom}(\mathcal{A},\mathcal{B})$, the statement holds by (ii).
(iv) By Lemma 2.1 and $H_N\mathcal{O}_N \neq \emptyset$, it holds by (iii).
(v) Assume that $(t_1,\ldots,t_N) \in H_N(\mathcal{A}\oplus\mathcal{B})$ and denote $I_\mathcal{A}$ and $I_\mathcal{B}$ are units of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $(I_\mathcal{A}t_1,\ldots,I_\mathcal{A}t_N) \in H_N\mathcal{A}$ and $(I_\mathcal{B}t_1,\ldots,I_\mathcal{B}t_N) \in H_N\mathcal{B}$. On the other hand, if $(v_i)_{i=1}^N \in H_N\mathcal{A}$ and $(u_i)_{i=1}^N \in H_N\mathcal{B}$, then put $t_i \equiv v_i + u_i \in \mathcal{A}\oplus\mathcal{B}$. Then we see that $(t_i)_{i=1}^N \in H_N(\mathcal{A}\oplus\mathcal{B})$. □

Proposition 2.4. (i) *We have the following inclusions among the Cuntz algebras* $\mathcal{O}_2,\ldots,\mathcal{O}_8$*:*



For $2 \leq N < M \leq 8$*, there is no homomorphism from* $\mathcal{O}_M$ *to* $\mathcal{O}_N$ *if there is no oriented path from* $\mathcal{O}_M$ *to* $\mathcal{O}_N$ *in this illustration. Specially,* $H_2\mathcal{O}_3 = \emptyset$*,* $H_3\mathcal{O}_3 \neq \emptyset$*,* $H_4\mathcal{O}_3 = \emptyset$*,* $H_N\mathcal{O}_2 \neq \emptyset$ *for each* $N \geq 2$*.*

(ii) *If* $\mathcal{R}$ *is a von Neumann algebra, then* $H_2\mathcal{R} \neq \emptyset$ *or* $H_N\mathcal{R} = \emptyset$ *for any* $N \geq 2$

Proof. (i) By Lemma 2.1, it follows.
(ii) Assume that $\mathcal{R}$ is a von Neumann algebra. If $\mathcal{R}$ is finite, then $H_2\mathcal{R} = \emptyset$. If $\mathcal{R}$ is properly infinite, then $H_2\mathcal{R} \neq \emptyset$. Assume that $\mathcal{R}$ satisfies $H_N\mathcal{R} \neq \emptyset$ for some $N \geq 2$ and $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is the canonical decomposition such that $\mathcal{R}_1$ is finite or $\{0\}$, and $\mathcal{R}_2$ is properly infinite or $\{0\}$. Then $H_N\mathcal{R} \neq \emptyset$ only when $\mathcal{R}_1 = \{0\}$ by Lemma 2.3 (v). In consequence, if $H_N\mathcal{R} \neq \emptyset$ for some $N \geq 2$, then $\mathcal{R}$ is properly infinite and $H_2\mathcal{R} \neq \emptyset$. Therefore the statement holds. □

Theorem 1.1 (iii) is proved. In this way, the property of $\mathcal{A}$ about $H_N\mathcal{A}$ is different in whether $\mathcal{A}$ is a von Neumann algebra or not.

Lemma 2.5. *Let* α *be an action of* $\mathbf{Z}_N$ *of* $\mathcal{O}_N$ *by cyclic permutation of the canonical generators and* $\mathcal{O}_N^{\mathbf{Z}_N}$ *be the fixed point subalgebra of* $\mathcal{O}_N$ *by* α*. Then we have the followings:* (i) $H_2\mathcal{O}_2^{\mathbf{Z}_2} \neq \emptyset$*.* (ii) $H_2\mathcal{O}_3^{\mathbf{Z}_3} = \emptyset$*,* $H_3\mathcal{O}_3^{\mathbf{Z}_3} \neq \emptyset$*.*

Proof. (i) Let $\rho \in \mathrm{End}\mathcal{O}_2$ by $\rho(s_1) \equiv s_1s_2s_1^* + s_2s_1s_2^*$, $\rho(s_2) \equiv s_1s_1s_1^* + s_2s_2s_2^*$. Then $\rho(\mathcal{O}_2) \subset \mathcal{O}_2^{\mathbf{Z}_2}$ and the statement holds.
(ii) Because $\mathcal{O}_3^{\mathbf{Z}_3}$ is a subalgebra of $\mathcal{O}_3$, the first statement follows by Lemma 2.3 (iii). Let $\rho_\nu \in \mathrm{End}\mathcal{O}_3$ by

$$\left\{ \begin{array}{l} \rho_\nu(s_1) \equiv s_{12,3} + s_{23,1} + s_{31,2}, \quad \rho_\nu(s_2) \equiv s_{21,3} + s_{32,1} + s_{13,2}, \\ \\ \qquad\qquad \rho_\nu(s_3) \equiv s_{11,1} + s_{22,2} + s_{33,3} \end{array} \right. \tag{2.1}$$

where $s_{ij,k} \equiv s_i s_j s_k^*$ for $i,j,k = 1,2,3$. Then $\rho_\nu(\mathcal{O}_3) \subset \mathcal{O}_3^{\mathbf{Z}_3}$. □

We show that $H_N\mathcal{O}_N^{\mathbf{Z}_N} \neq \emptyset$ for each $N \geq 4$ in Example 5.7.

2.2. An N-ary operation on the homomorphism space. For $\varphi, \varphi' \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$, let $\varphi + \varphi'$ be the sum of two linear maps from $\mathcal{A}$ to $\mathcal{B}$. Then $\varphi + \varphi' \notin \mathrm{Hom}(\mathcal{A},\mathcal{B})$ because $(\varphi+\varphi')(I) = 2I \neq I$. Therefore $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ is not closed under such sum. In stead of $\varphi + \varphi'$, we define a new operation on $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ and $\mathrm{End}\mathcal{A}$. For a unital $*$-algebra $\mathcal{A}$, $u \in \mathcal{A}$ is an *isometry* if $u^*u = I$. $u \in \mathcal{A}$ is a *unitary* if $u^*u = uu^* = I$.

Definition 2.6. (i) *$\varphi_1, \varphi_2 \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ are equivalent if there is a unitary u in $\mathcal{B}$ such that $u\varphi_1(a)u^* = \varphi_2(a)$ for each $a \in \mathcal{A}$. In this case, we denote $\varphi_1 \sim \varphi_2$.*
(ii) *$\varphi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ is proper if φ is not surjective.*
(iii) *$\varphi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ is irreducible if $\varphi(\mathcal{A})' \cap \mathcal{B} = \mathbf{C}I$ where $\varphi(\mathcal{A})' \cap \mathcal{B} \equiv \{b \in \mathcal{B} : \varphi(a)b = b\varphi(a)\ {}^\forall a \in \mathcal{A}\}$.*

Irreducible proper endomorphism is important for the study of endomorphisms in comparison with that of automorphisms.

If $\varphi_1, \varphi_2 \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ satisfy $\varphi_1 \sim \varphi_2$, then φ_1 is proper if and only if φ_2 is, φ_1 is irreducible if and only if φ_2 is. For $\varphi_1 \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ and $\varphi_2 \in \mathrm{Hom}(\mathcal{B},\mathcal{C})$, $\varphi_2 \circ \varphi_1 \in \mathrm{Hom}(\mathcal{A},\mathcal{C})$. Specially $\mathrm{End}\mathcal{A} = \mathrm{Hom}(\mathcal{A},\mathcal{A})$ is a unital semigroup with respect to composition of endomorphisms. Immediately, we see the following:

Lemma 2.7. (i) *If $\varphi_1, \varphi_1' \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ and $\varphi_2, \varphi_2' \in \mathrm{Hom}(\mathcal{B},\mathcal{C})$ satisfy $\varphi_1 \sim \varphi_1'$ and $\varphi_2 \sim \varphi_2'$, then $\varphi_2 \circ \varphi_1 \sim \varphi_2' \circ \varphi_1'$.*
(ii) *For $\varphi_1 \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$, $\varphi_2 \in \mathrm{Hom}(\mathcal{B},\mathcal{C})$, if φ_2 and $\varphi_2 \circ \varphi_1$ are irreducible and φ_2 is injective, then φ_1 is irreducible.*

(i) If $\mathrm{Ad}u_i \circ \varphi_i = \varphi_i'$ for $i = 1,2$, then $\mathrm{Ad}(u_2\varphi_2(u_1)) \circ (\varphi_2 \circ \varphi_1) = \varphi_2' \circ \varphi_1'$. (ii) By assumption, $\mathbf{C}I = \{(\varphi_2 \circ \varphi_1)(\mathcal{A})\}' \cap \mathcal{C} \supset \{(\varphi_2 \circ \varphi_1)(\mathcal{A})\}' \cap \varphi_2(\mathcal{B}) = \varphi_2(\varphi_1(\mathcal{A})' \cap \mathcal{B}) \supset \mathbf{C}I$. Hence $\mathbf{C}I = \varphi_2(\varphi_1(\mathcal{A})' \cap \mathcal{B})$. Because φ_2 is injective, $\mathbf{C}I = \varphi_1(\mathcal{A})' \cap \mathcal{B}$ and φ_1 is irreducible. □

For $N \geq 2$, let $\mathrm{Hom}(\mathcal{A},\mathcal{B};N) \equiv \{(\varphi_i)_{i=1}^N : \varphi_i \in \mathrm{Hom}(\mathcal{A},\mathcal{B}),\ i = 1,\ldots,N\}$. For $\Phi = (\varphi_i)_{i=1}^N, \Psi = (\psi_i)_{i=1}^N \in \mathrm{Hom}(\mathcal{A},\mathcal{B};N)$, we denote $\Phi \sim \Psi$ if $\phi_i \sim \psi_i$ for each $i = 1,\ldots,N$.

Lemma 2.8. *Assume that $H_N\mathcal{B} \neq \emptyset$. For $\xi = (t_i)_{i=1}^N \in H_N\mathcal{B}$ and $\Phi = (\varphi_i)_{i=1}^N \in \mathrm{Hom}(\mathcal{A},\mathcal{B};N)$, define a linear map $<\xi|\Phi>$ from $\mathcal{A}$ to $\mathcal{B}$ by*

$$<\xi|\Phi> \equiv \mathrm{Ad}t_1 \circ \varphi_1 + \cdots + \mathrm{Ad}t_N \circ \varphi_N \tag{2.2}$$

where $\mathrm{Ad}t_i \circ \varphi_i \equiv t_i\varphi_i(\cdot)t_i^*$ for $i = 1,\ldots,N$. Then the followings hold:

(i) $<\xi|\Phi>\in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ *for* $\xi\in H_N\mathcal{B}$ *and* $\Phi\in \mathrm{Hom}(\mathcal{A},\mathcal{B};N)$.

(ii) *If* $\Phi,\Psi\in \mathrm{Hom}(\mathcal{A},\mathcal{B};N)$ *satisfy* $\Phi\sim\Psi$, *then* $<\xi|\Phi>\sim<\eta|\Psi>$ *for* $\xi,\eta\in H_N\mathcal{B}$.

(iii) *For any permutation* $\sigma\in\mathfrak{S}_N$, $<\xi|\Phi>\sim<\xi|\Phi^\sigma>$ *where* $\Phi^\sigma\equiv(\varphi_{\sigma(1)},\ldots,\varphi_{\sigma(N)})$.

(iv) *For* $(\varphi_i)_{i=1}^{2N-1}\in\mathrm{Hom}(\mathcal{A},\mathcal{B};2N-1)$ *and* $\xi,\eta,\xi^{'},\eta^{'}\in H_N\mathcal{B}$, *let* $\Phi_{1,N},\Phi_{N,2N-1}$, $\Phi^{(2)},\Phi^{(3)}\in H_N\mathcal{B}$ *by*

$$\Phi_{1,N}\equiv(\varphi_i)_{i=1}^N,\quad \Phi_{N,2N-1}\equiv(\varphi_i)_{i=N}^{2N-1},$$

$$\Phi^{(2)}\equiv(\varphi_a,\varphi_{N+1},\ldots,\varphi_{2N-1}),\quad \Phi^{(3)}\equiv(\varphi_1,\ldots,\varphi_{N-1},\varphi_b),$$

where $\varphi_a\equiv<\xi|\Phi_{1,N}>$ *and* $\varphi_b\equiv<\xi^{'}|\Phi_{N,2N-1}>$. *Then* $<\eta|\Phi^{(2)}>\sim<\eta^{'}|\Phi^{(3)}>$.

Proof. Assume that $\xi=(t_i)_{i=1}^N$, $\eta=(u_i)_{i=1}^N$, $\Phi=(\varphi_i)_{i=1}^N$ and $\Psi=(\psi_i)_{i=1}^N\in\mathrm{Hom}(\mathcal{A},\mathcal{B};N)$.
(i) By direct computation, the statement follows.
(ii) Assume that there are unitaries $v_1,\ldots,v_N\in\mathcal{B}$ such that $\mathrm{Ad}v_i\circ\psi_i=\varphi_i$ for $i=1,\ldots,N$. Let $T\equiv u_1v_1t_1^*+\cdots+u_Nv_Nt_N^*$. Then $\mathrm{Ad}T\circ<\xi|\Phi>=<\eta|\Psi>$.
(iii) Let $\xi^{\sigma^{-1}}\equiv(t_{\sigma^{-1}(1)},\ldots,t_{\sigma^{-1}(N)})\in H_N\mathcal{B}$. Then $<\xi|\Phi^\sigma>=<\xi^{\sigma^{-1}}|\Phi>\sim<\xi|\Phi>$ by (ii).
(iv) Assume that $\xi^{'}=(t_i^{'})_{i=1}^N,\eta^{'}=(u_i^{'})_{i=1}^N$. Then

$$<\eta|\Phi^{(2)}>=\ \sum_{j=1}^N\mathrm{Ad}(u_1t_j)\circ\varphi_j+\sum_{i=2}^N\mathrm{Ad}u_i\circ\varphi_{N+i-1},$$

$$<\eta^{'}|\Phi^{(3)}>=\ \sum_{i=1}^{N-1}\mathrm{Ad}u_i^{'}\circ\varphi_i+\sum_{j=1}^N\mathrm{Ad}(u_N^{'}t_j^{'})\circ\varphi_{j+N-1}.$$

Let $T\equiv u_1^{'}t_1^*u_1^*+\cdots+u_{N-1}^{'}t_{N-1}^*u_1^*+u_N^{'}t_1^{'}t_N^*u_1^*+u_N^{'}t_2^{'}u_2^*+\cdots+u_N^{'}t_N^{'}u_N^*$.
Then $\mathrm{Ad}T\circ<\eta|\Phi^{(2)}>=<\eta^{'}|\Phi^{(3)}>$. □

By Lemma 2.8, we see that $<\xi|\cdot>$ is an N-ary operation on $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ for each $\xi\in H_N\mathcal{B}$.

Lemma 2.9. *If* $\varphi=<\xi|\Phi>$ *for* $\xi\in H_N\mathcal{B}$ *and* $\Phi\in\mathrm{Hom}(\mathcal{A},\mathcal{B};N)$, *then* φ *is not irreducible.*

Proof. Assume that $\xi=(u_i)_{i=1}^N$ and $\Phi=(\varphi_i)_{i=1}^N$. Then $U\equiv u_1u_1^*-u_2u_2^*-\cdots-u_Nu_N^*$ satisfies $U\varphi(x)=\varphi(x)U$ for each $x\in\mathcal{A}$. Hence $U\in\varphi^{'}(\mathcal{A})\cap\mathcal{B}$ and $U\notin\mathbf{C}I$. Therefore the statement holds. □

2.3. Operations on the sector. For unital $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, define

$$\mathrm{Sect}(\mathcal{A},\mathcal{B})\equiv\mathrm{Hom}(\mathcal{A},\mathcal{B})/\sim.$$

$\mathrm{Sect}(\mathcal{A},\mathcal{B})$ is often defined by $\mathrm{Hom}(\mathcal{A},\mathcal{B})/\mathrm{Inn}\mathcal{B}$ where $\mathrm{Inn}\mathcal{B}$ is the inner automorphism group of $\mathcal{B}$. The space $\mathrm{Sect}(\mathcal{A},\mathcal{B})$ of homomorphism classes is called the *sector* from $\mathcal{A}$ to $\mathcal{B}$. An element of $\mathrm{Sect}(\mathcal{A},\mathcal{B})$ is called a sector from $\mathcal{A}$ to $\mathcal{B}$, too. Remark that the symbol $\mathrm{Sect}(\mathcal{A},\mathcal{B})$ in [**15**] and ours are different in the position of $\mathcal{A}$ and $\mathcal{B}$, and the former is a subset of the latter in general. Specially, we denote $\mathrm{Sect}\mathcal{A} \equiv \mathrm{Sect}(\mathcal{A},\mathcal{A})$. Denote $[\varphi] \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$ by $[\varphi] \equiv \{\varphi^{'} \in \mathrm{Hom}(\mathcal{A},\mathcal{B}) : \varphi^{'} \sim \varphi\}$. $[\varphi]$ is *proper* if φ is. $[\varphi]$ is *irreducible* if φ is.

If α is an isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$, then a map L_α from $\mathrm{Sect}(\mathcal{B},\mathcal{A}_1)$ to $\mathrm{Sect}(\mathcal{B},\mathcal{A}_2)$ which is defined by $L_\alpha[\varphi] \equiv [\alpha\circ\varphi]$ is bijective. A map R_α from $\mathrm{Sect}(\mathcal{A}_1,\mathcal{B})$ to $\mathrm{Sect}(\mathcal{A}_2,\mathcal{B})$ which is defined by $[\varphi]R_\alpha \equiv [\varphi\circ\alpha]$ is bijective, too.

For $[\varphi_1] \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$ and $[\varphi_2] \in \mathrm{Sect}(\mathcal{B},\mathcal{C})$,

$$[\varphi_2][\varphi_1] \equiv [\varphi_2\circ\varphi_1] \in \mathrm{Sect}(\mathcal{A},\mathcal{C}) \tag{2.3}$$

is well-defined by Lemma 2.7. Furthermore we see that $x(yz) = (xy)z$ for $x \in \mathrm{Sect}(\mathcal{C},\mathcal{D})$, $y \in \mathrm{Sect}(\mathcal{B},\mathcal{C})$ and $z \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$. (2.3) is called the *sector product*. Specially, $\mathrm{Sect}\mathcal{A}$ is a unital semigroup with unit $[\iota]$ where ι is the identity map on $\mathcal{A}$. $\mathrm{Sect}\mathcal{A}$ is non abelian in general. The outer automorphism group $\mathrm{Out}\mathcal{A} \equiv \{[\alpha] : \alpha \in \mathrm{Aut}\mathcal{A}\}$ of $\mathcal{A}$ is a subgroup of $\mathrm{Sect}\mathcal{A}$. If $x \in \mathrm{Out}\mathcal{A} \cap \mathrm{Sect}\mathcal{A}$, then x is irreducible. For a unital $*$-algebra $\mathcal{A}$, $\mathrm{Sect}\mathcal{A}$ is called the *sector semigroup* of $\mathcal{A}$.

Lemma 2.10. *Assume that $\mathcal{B}$ is simple. For $y \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$ and $x \in \mathrm{Sect}(\mathcal{B},\mathcal{C})$, if both x and xy are irreducible, then y is irreducible.*

Proof. Assume that $x = [\varphi_2]$ and $y = [\varphi_1]$. Because $\mathcal{B}$ is simple, any element in $\mathrm{Hom}(\mathcal{B},\mathcal{C})$ is injective. Hence φ_2 is injective. By Lemma 2.7 (ii), φ_1 is irreducible. Hence the statement holds. □

Under assumption $H_N\mathcal{A} \neq \emptyset$ for $\mathcal{A}$ in Definition 2.2, we can consider the following "N-ary additive structure" on $\mathrm{Sect}\mathcal{A}$.

Lemma 2.11. *Assume that $H_N\mathcal{B} \neq \emptyset$. For $[\varphi_1],\ldots,[\varphi_N] \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$, define*

$$p([\varphi_1],\ldots,[\varphi_N]) \equiv [< \xi|\Phi >] \tag{2.4}$$

where $\Phi = (\varphi_1,\ldots,\varphi_N)$, $\xi \in H_N\mathcal{B}$ and $< \cdot|\cdot >$ is in (2.2). Then the followings hold:

(i) *p is well-defined as an N-ary operation on $\mathrm{Sect}(\mathcal{A},\mathcal{B})$, that is, the lhs in (2.4) is independent of the choice of both ξ and representatives $\varphi_1,\ldots,\varphi_N$.*
(ii) *p is completely symmetric, that is,* $p(x_{\sigma(1)},\ldots,x_{\sigma(N)}) = p(x_1,\ldots,x_N)$ *for each $\sigma \in \mathfrak{S}_N$ and $x_1,\ldots,x_N \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$.*

(iii) $p \circ (p \times id^{N-1}) = p \circ (id^j \times p \times id^{N-1-j})$ *for* $j = 1, \ldots, N-1$ *where* id *is the identity map on* $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$.

Proof. By Lemma 2.8, (i) and (ii) hold. (iii) is verified by Lemma 2.8 (iv) and similar discussion. □

Definition 2.12. *When* $H_N\mathcal{B} \neq \emptyset$, p *in* (2.4) *is called the* N*-ary sector sum on* $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$. (2.3) *and* p *are called sector operations.*

We denote $x_1 + \cdots + x_N \equiv p(x_1, \ldots, x_N)$ for $x_1, \ldots, x_N \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$. Then we see that

$$x_{\sigma(1)} + \cdots + x_{\sigma(N)} = x_1 + \cdots + x_N,$$

$$x_1 + \cdots + x_{N-1} + (x_N + \cdots + x_{2N-1}) = (x_1 + \cdots + x_N) + x_{N+1} + \cdots + x_{2N-1}$$

for $x_1, \ldots, x_{2N-1} \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ and $\sigma \in \mathfrak{S}_N$. In this way, the notation $x_1 + \cdots + x_N$ is reasonable as a kind of sum. Because the notation " + " means a binary operation usually, it may give rise to a misunderstanding. In stead of this weak side, " + "(or which is denoted by $\oplus$)is often used in convenience([**3**, **14**]).

In consequence, we have the followings:

Proposition 2.13. *Assume that* $H_N\mathcal{B} \neq \emptyset$.

(i) $(\mathrm{Sect}(\mathcal{A}, \mathcal{B}), +)$ *becomes an abelian (=completely symmetric)*N*-ary semigroup. Specially,* $(\mathrm{Sect}(\mathcal{A}, \mathcal{B}), +)$ *is an ordinary abelian semigroup when* $N = 2$.

(ii) *If* $H_N\mathcal{B}' \neq \emptyset$ *and* $\phi \in \mathrm{Hom}(\mathcal{B}, \mathcal{B}')$, *then a map* L_ϕ *from* $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$ *to* $\mathrm{Sect}(\mathcal{A}, \mathcal{B}')$ *which is defined by* $L_\phi[\varphi] \equiv [\phi \circ \varphi]$ *is an* N*-ary semigroup homomorphism, that is,* $L_\phi(x_1 + \cdots + x_N) = L_\phi(x_1) + \cdots + L_\phi(x_N)$.

(iii) *If* $\phi \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, *then a map* R_ϕ *from* $\mathrm{Sect}(\mathcal{A}_2, \mathcal{B})$ *to* $\mathrm{Sect}(\mathcal{A}_1, \mathcal{B})$ *which is defined by* $[\varphi]R_\phi \equiv [\varphi \circ \phi]$ *is an* N*-ary semigroup homomorphism. Specially, if* $\mathcal{A}_1 \cong \mathcal{A}_2$, *then* $\mathrm{Sect}(\mathcal{A}_2, \mathcal{B})$ *and* $\mathrm{Sect}(\mathcal{A}_1, \mathcal{B})$ *are isomorphic as an* N*-ary semigroup.*

Furthermore we can check the followings:

$$x(y_1 + \cdots + y_N) = xy_1 + \cdots + xy_N, \quad (y_1 + \cdots + y_N)z = y_1 z + \cdots + y_N z$$

for any $x \in \mathrm{Sect}(\mathcal{B}, \mathcal{C})$, $y_1, \ldots, y_N \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ and $z \in \mathrm{Sect}(\mathcal{D}, \mathcal{A})$.

Theorem 2.14. *Assume that* $H_N\mathcal{A} \neq \emptyset$.

(i) $\mathrm{Sect}\mathcal{A}$ *is a unital* N*-ary algebra with respect to the sector product and the sector sum.*

(ii) *If* $\mathcal{A} \cong \mathcal{B}$, *then* $\mathrm{Sect}\mathcal{B}$ *has an* N*-ary algebraic structure and* $\mathrm{Sect}\mathcal{A} \cong \mathrm{Sect}\mathcal{B}$ *as an* N*-ary algebra.*

Proof. (i) By Proposition 2.13 and discussion in the above, the statement holds.
(ii) Let α be an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Then $H_N\mathcal{B} \neq \emptyset$ by Lemma 2.3 (ii). Hence Sect$\mathcal{B}$ is a unital N-ary algebra with respect to sector operations. We see that a map F from Sect$\mathcal{A}$ to Sect$\mathcal{B}$ defined by $F([\rho]) \equiv [\alpha \circ \rho \circ \alpha^{-1}]$ for $[\rho] \in$ Sect$\mathcal{A}$ is a unital isomorphism from Sect$\mathcal{A}$ to Sect$\mathcal{B}$. □

When $N = 2$, we call 2-ary(=binary)algebra by algebra simply.

If we denote $Nx \equiv p(x, \ldots, x)$ for $x \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$, then we see that $Mx \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ is well-defined for each $x \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ and $M \in \mathbf{N}_N \equiv \{(N-1)k + 1 : k = 0, 1, 2, \ldots\}$. Therefore we have a map from $\mathbf{N}_N \times \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ to $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$. Because $\mathbf{N}_N$ itself is a commutative N-ary algebra, it seems that $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$ is a "module" of $\mathbf{N}_N$ and Sect$\mathcal{A}$ is an algebra with "the coefficient ring" $\mathbf{N}_N$.

The following is well-known as an empirical rule in the theory of subfactors:

Corollary 2.15. *If $\mathcal{R}$ is a properly infinite von Neumann algebra, then* Sect$\mathcal{R}$ *is always an algebra.*

Proof. By Proposition 2.4 (ii), it holds. □

By Corollary 2.15 and Proposition 2.4 (ii), an exotic algebraic structure of Sect$\mathcal{A}$ does not appear when $\mathcal{A}$ is a von Neumann algebra. We see that the difference of operator topology has much effect on the algebraic structure of the sector.

Definition 2.16. (i) *For a unital $*$-algebra $\mathcal{A}$ which satisfies $H_N\mathcal{A} \neq \emptyset$,* Sect$\mathcal{A}$ *which is attained with sector operations is called the sector algebra of $\mathcal{A}$.*

(ii) *$\mathcal{S}$ is a sector algebra if $\mathcal{S}$ is an N-ary subalgebra of* Sect$\mathcal{A}$ *for some unital $*$-algebra $\mathcal{A}$ which satisfies $H_N\mathcal{A} \neq \emptyset$.*

By Grothendieck construction, we can obtain an abelian group from an abelian semigroup $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$ when $H_2\mathcal{B} \neq \emptyset$. By Proposition 2.4, we have a non trivial ternary sum on Sect$\mathcal{O}_3$. In the same way, we see the non-triviality of the N-ary sum on Sect$\mathcal{O}_N$ for each $N \geq 2$. These systems are already considered as *universal algebras*([**7**, **11**]) in only a purely theoretical framework. We give an exact formulation of our system as a universal algebra in Appendix A. In this point of view, we see that sector algebras are essentially new and exotic examples of universal algebra with non binary sum. The sector is a new kind of *number*.

When $N = 2$, it may be that Sect$\mathcal{A}$ should be called the *ring of sectors*. According to the terminology of universal algebra, we call Sect$\mathcal{A}$ by the algebra of sectors in this article. This exotic algebraic structure of Sect$\mathcal{A}$

is compatible to both the algebraic structure of fusion rule algebra and branching laws of representations of C*-algebras. Examples are shown in § 4, § 5, § 6.

Proposition 2.17. *Assume that* $H_N\mathcal{A}_i \neq \emptyset$ *for* $i = 1, 2$.

(i) *Denote* $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \oplus \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2) \equiv \mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \times \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$ *and* $x \oplus y \equiv (x, y) \in \mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \times \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$. *Then* $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \oplus \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$ *is an* N*-ary semigroup by an* N*-ary operation* $x_1 \oplus y_1 + \cdots + x_N \oplus y_N \equiv (x_1 + \cdots + x_N) \oplus (y_1 + \cdots + y_N)$.

(ii) *Define a map* F *from* $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \oplus \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$ *to* $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1 \oplus \mathcal{A}_2)$ *by*

$$F([\varphi_1] \oplus [\varphi_2]) \equiv [\varphi_1 \oplus \varphi_2] \quad (\, [\varphi_1] \oplus [\varphi_2] \in \mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \oplus \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)\,),$$

$$(\varphi_1 \oplus \varphi_2)(a) \equiv \varphi_1(a) \oplus \varphi_2(a) \quad (a \in \mathcal{B}).$$

Then F *is an* N*-ary semigroup isomorphism.*

Proof. (i) The N-ary associativity of the operation $+$ on $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \oplus \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$ follows from that of sector sums of $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1)$ and $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$, respectively.
(ii) For $(\varphi_1, \varphi_2) \in \mathrm{Hom}(\mathcal{B}, \mathcal{A}_1) \times \mathrm{Hom}(\mathcal{B}, \mathcal{A}_2)$, we see that $\varphi_1 \oplus \varphi_2 \in \mathrm{Hom}(\mathcal{B}, \mathcal{A}_1 \oplus \mathcal{A}_2)$. Furthermore $[\varphi_1 \oplus \varphi_2]$ is uniquely defined for $[\varphi_1] \oplus [\varphi_2]$. Therefore F is well-defined on $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1) \oplus \mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$. Finally, we easily can check that F is bijective and $F(\,([\varphi_1] + \cdots + [\varphi_N]) \oplus ([\psi_1] + \cdots + [\psi_N])\,) = F([\varphi_1] \oplus [\psi_1]) + \cdots + F([\varphi_N] \oplus [\psi_N])$. □

In consequence, if $H_N\mathcal{A}_i \neq \emptyset$ for $i = 1, \ldots, m$, then the following abelian N-ary semigroup isomorphism holds:

$$\mathrm{Sect}(\mathcal{B}, \oplus_{i=1}^m \mathcal{A}_i) \cong \oplus_{i=1}^m \mathrm{Sect}(\mathcal{B}, \mathcal{A}_i).$$

Assume that $H_N\mathcal{A}_1 \neq \emptyset$. If $\varphi \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, then we have an N-ary semigroup homomorphism L_φ from $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_1)$ to $\mathrm{Sect}(\mathcal{B}, \mathcal{A}_2)$. Even if φ is injective, L_φ is not injective in general. For example, put $\mathcal{A}_1 \equiv \mathcal{O}_2 \oplus \mathcal{O}_2$ $\mathcal{A}_2 \equiv M_2(\mathbf{C}) \otimes \mathcal{O}_2 = M_2(\mathcal{O}_2)$ and a map ι from $\mathcal{A}_1$ to $\mathcal{A}_2$ by $\iota(A, B) \equiv \mathrm{diag}(A, B) \in \mathcal{A}_2$. Put $\varphi_1, \varphi_2 \in \mathrm{Hom}(\mathcal{B}, \mathcal{O}_2)$ such that $\varphi_1 \not\sim \varphi_2$. Put $\varphi \equiv \varphi_1 \oplus \varphi_2, \varphi' \equiv \varphi_2 \oplus \varphi_1 \in \mathrm{Hom}(\mathcal{B}, \mathcal{A}_1)$. Then $\varphi \not\sim \varphi'$ in $\mathrm{Hom}(\mathcal{B}, \mathcal{A}_1)$. On the other hand, $\iota \circ \varphi \sim \iota \circ \varphi' \in \mathrm{Hom}(\mathcal{B}, \mathcal{A}_2)$. Therefore $L_\iota([\varphi]) = L_\iota([\varphi'])$ but $[\varphi] \neq [\varphi']$. Hence L_ι is not injective.

2.4. Fusion rules, conjugate sectors and the canonical sector. We introduce general definitions of fusion rule and conjugate sector, and show the existence of the canonical sector. Their examples are treated in § 5.

Definition 2.18. *When* $H_N\mathcal{B} \neq \emptyset$, $x \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ *is decomposable if there is* $x_1, \ldots, x_N \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ *such that* $x = x_1 + \cdots + x_N$. *If* x *is not decomposable,* x *is called indecomposable.*

If x is irreducible, then x is indecomposable by Lemma 2.9. If x is decomposable, then x is proper. We do not know whether the indecomposability implies the irreducibility or not.

Assume that $H_N\mathcal{C} \neq \emptyset$ for $N \geq 2$. For $x \in \mathrm{Sect}(\mathcal{B},\mathcal{C})$ and $y \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$, if there are $z_1,\ldots,z_N \in \mathrm{Sect}(\mathcal{A},\mathcal{C})$ such that the following holds:

$$xy = z_1 + \cdots + z_N,$$

then this equation is called the *fusion rule* of x and y. In § 3, we show that fusion rules are useful to compute branching laws of representation arising from x and y. Assume that there is a set $\mathcal{S} = \{x_\lambda \in \mathrm{Sect}\mathcal{A} : \lambda \in \Lambda\}$ which satisfies that for each $\mu,\nu \in \Lambda$, there is $n_{\mu\nu,\lambda} \in \{0\} \cup \{(N-1)l+1 \in \mathbf{N} : l \geq 0\}$ for each $\lambda \in \Lambda$ such that

$$x_\mu x_\nu = \sum_{\lambda\in\Lambda} n_{\mu\nu,\lambda} x_\lambda. \tag{2.5}$$

Furthermore if $n_{\mu\nu,\lambda} = n_{\nu\mu,\lambda}$, then $< \mathcal{S} >$ is a commutative fusion rule algebra([**10**]) where $< \mathcal{S} >$ is the smallest subset of $\mathrm{Sect}\mathcal{A}$ which is closed under both N-ary sector sum and sector product for $\mathcal{S} \subset \mathrm{Sect}\mathcal{A}$. So-called fusion rule algebra is a subalgebra of $\mathrm{Sect}\mathcal{A}$ with assumption $H_2\mathcal{A} \neq \emptyset$.

Definition 2.19. *Assume that $H_N\mathcal{A} \neq \emptyset$ and $H_M\mathcal{B} \neq \emptyset$ for some $N, M \geq 2$. For $x \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$, $\bar{x} \in \mathrm{Sect}(\mathcal{B},\mathcal{A})$ is a left(resp.right)conjugate of x if there are $y_1,\ldots,y_{N-1} \in \mathrm{Sect}\mathcal{A}$(resp. $z_1,\ldots,z_{M-1} \in \mathrm{Sect}\mathcal{B}$) such that*

$$\bar{x}x = [\iota_\mathcal{A}] + y_1 + \cdots + y_{N-1} \quad (\textit{resp.} \quad x\bar{x} = [\iota_\mathcal{B}] + z_1 + \cdots + z_{M-1})$$

where $\iota_\mathcal{A}$ and $\iota_\mathcal{B}$ are identity maps on $\mathcal{A}$ and $\mathcal{B}$, respectively.

About conjugate sector and related topics in quantum field theory and index theory, see [**3**, **4**, **10**, **12**, **13**, **14**, **15**].

Assume that $H_N\mathcal{A} \neq \emptyset$. $< \mathrm{Out}\mathcal{A} >$ is the free N-ary algebra generated by the group $\mathrm{Out}\mathcal{A}$. Specially, when $N = 2$, $< \mathrm{Out}\mathcal{A} >$ is the ordinary free algebra of $\mathrm{Out}\mathcal{A}$ without inverse of sum. From this, if $x = [\alpha_1] + \cdots + [\alpha_{(N-1)k+1}]$, $\alpha_1,\ldots,\alpha_{(N-1)k+1} \in \mathrm{Aut}\mathcal{A}$, then $y' = [\alpha_1^{-1}] + z_1 + \cdots + z_{(N-1)l}$ is the left and right conjugate of x for each $z_1 + \cdots + z_{(N-1)l} \in \mathrm{Sect}\mathcal{A}$ and $l \geq 0$. In this way, the conjugate sector in Definition 2.19 is not unique in general. We show an example of sectors $x, y \in \mathrm{Sect}\mathcal{O}_2$ which are proper, irreducible and mutually conjugate but $xy \neq yx$ in § 4.

We denote the identity map on $\mathcal{A}$ by ι. If $H_N\mathcal{A} \neq \emptyset$, then a sector

$$c_N \equiv \underbrace{[\iota] + \cdots + [\iota]}_{N} \in \mathrm{Sect}\mathcal{A}$$

is called the *N-ary canonical sector* of $\mathcal{A}$. For $\xi = (u_1,\ldots,u_N) \in H_N\mathcal{A}$, let $\rho_\xi(x) \equiv u_1xu_1^* + \cdots + u_Nxu_N^*$ for $x \in \mathcal{A}$. By definition, $[\rho_\xi] = c_N$. We see that the canonical sector of $\mathcal{O}_N$ coincides the sector associated with the canonical endomorphism. The following trivial fusion rules hold:

$$(c_N)^l = N^{l-1}c_N, \quad c_N x = x c_N = N x$$

for each $l \geq 1$ and $x \in \mathrm{Sect}\mathcal{A}$.

3. Spectrum modules

3.1. Definition. Let $\mathrm{Rep}\mathcal{A}$(*resp.* $\mathrm{IrrRep}\mathcal{A}$) be the set of all unital(*resp.* irreducible) $*$-representations of a unital C*-algebra $\mathcal{A}$. We simply denote π for $(\mathcal{H}, \pi) \in \mathrm{Rep}\mathcal{A}$. Let $\mathrm{BSpec}\mathcal{A}$ (*resp.* $\mathrm{Spec}\mathcal{A}$)be the set of all unitary equivalence classes of unital(*resp.* irreducible)$*$-representations of $\mathcal{A}$. Then $\mathrm{BSpec}\mathcal{A}$ is an abelian semigroup with respect to direct sum:

$$\mathrm{BSpec}\mathcal{A} \times \mathrm{BSpec}\mathcal{A} \ni ([\pi], [\pi^{'}]) \mapsto [\pi] \oplus [\pi^{'}] \equiv [\pi \oplus \pi^{'}] \in \mathrm{BSpec}\mathcal{A}.$$

We call $\mathrm{BSpec}\mathcal{A}$ the *spectrum semigroup of* $\mathcal{A}$. For $[\varphi] \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$, define

$$[\pi]R_{[\varphi]} \equiv [\pi \circ \varphi] \quad ([\pi] \in \mathrm{BSpec}\mathcal{B}). \tag{3.1}$$

We see that $[\pi]R_{[\varphi]}$ is well-defined in $\mathrm{BSpec}\mathcal{A}$ and $R_{[\varphi]}$ is a map from $\mathrm{BSpec}\mathcal{B}$ to $\mathrm{BSpec}\mathcal{A}$. Furthermore it is possible to show that

$$(v \oplus w)R_x = vR_x \oplus wR_x, \quad (vR_x)R_y = vR_{xy} \tag{3.2}$$

for $v, w \in \mathrm{BSpec}\mathcal{B}$, $x \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ and $y \in \mathrm{Sect}(\mathcal{C}, \mathcal{A})$. Hence R_x is a homomorphism between two semigroups. Hence R is a realization of a set $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$ in $\mathrm{Hom}(\mathrm{BSpec}\mathcal{B}, \mathrm{BSpec}\mathcal{A})$. Specially, R is a unital right action of a semigroup $\mathrm{Sect}\mathcal{A}$ on $\mathrm{BSpec}\mathcal{A}$ such that $R_{[\iota]} = I$. Therefore $(\mathrm{BSpec}\mathcal{A}, R)$ is a right module of the sector semigroup $\mathrm{Sect}\mathcal{A}$ without inverse of sum.

Definition 3.1. (i) *A map R in (*3.1*) is called the spectrum realization of* $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$.
(ii) $(\mathrm{Sect}\mathcal{A}, R)$ *is called the spectrum module of the sector semigroup* $\mathrm{Sect}\mathcal{A}$.
(iii) $\mathcal{S}$ *is a submodule of* $(\mathrm{Sect}\mathcal{A}, R)$ *if* $\mathcal{S}$ *is a subsemigroup of* $\mathrm{BSpec}\mathcal{A}$ *which is closed under the action of* $\mathrm{Sect}\mathcal{A}$.

Assume that $H_N\mathcal{B} \neq \emptyset$. Then we can verify that

$$vR_{x_1 + \cdots + x_N} = vR_{x_1} \oplus \cdots \oplus vR_{x_N} \tag{3.3}$$

for $x_1, \ldots, x_N \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ and $v \in \mathrm{BSpec}\mathcal{B}$. Hence R is a homomorphism of the N-ary semigroup $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$ to $\mathrm{Hom}(\mathrm{BSpec}\mathcal{B}, \mathrm{BSpec}\mathcal{A})$.

Definition 3.2. (i) *When $H_N\mathcal{B} \neq \emptyset$, a map R is called the spectrum homomorphism from* $\mathrm{Sect}(\mathcal{A}, \mathcal{B})$ *to* $\mathrm{Hom}(\mathrm{BSpec}\mathcal{B}, \mathrm{BSpec}\mathcal{A})$.
(ii) *When $H_N\mathcal{A} \neq \emptyset$,* $(\mathrm{BSpec}\mathcal{A}, R)$ *is called the spectrum module of the N-ary algebra* $\mathrm{Sect}\mathcal{A}$.

(iii) *When* $H_N\mathcal{A} \neq \emptyset$*,* $\mathcal{S}$ *is a submodule of* $(\mathrm{BSpec}\mathcal{A}, R)$ *if* $\mathcal{S}$ *is a subsemigroup of* $\mathrm{BSpec}\mathcal{A}$ *which is closed under the action of the* N*-ary algebra* $\mathrm{Sect}\mathcal{A}$*.*

Theorem 3.3. (i) *Assume that* $\mathcal{B}$ *is simple. For* $x \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$*, if there is* $v \in \mathrm{Spec}B$ *such that* $vR_x \in \mathrm{Spec}\mathcal{A}$*, then* x *is irreducible.*

(ii) *Assume that* $\mathcal{A}$ *is simple. For* $x \in \mathrm{Sect}\mathcal{A}$*, if there is* $v \in \mathrm{Spec}\mathcal{A}$ *such that* $vR_x = v$*, then* x^n *is irreducible for each* $n \geq 1$*.*

(iii) *For* $[\varphi_1], [\varphi_2] \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$*, if there is* $\pi \in \mathrm{Rep}\mathcal{B}$ *such that* $\pi \circ \varphi_1 \not\sim \pi \circ \varphi_2$*, then* $[\varphi_1] \neq [\varphi_2]$*.*

(iv) *For* $x \in \mathrm{Sect}\mathcal{A}$*, if there is* $v \in \mathrm{Spec}\mathcal{A}$ *such that* $vR_x \notin \mathrm{Spec}\mathcal{A}$*, then* x *is proper.*

(v) *Assume that* $N \geq 2$ *is minimal with respect to* $H_N\mathcal{B} \neq \emptyset$*. For* $x \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$*, if there is* $v \in \mathrm{Spec}\mathcal{B}$ *such that the totality of irreducible components of* vR_x *is less than* N*, then* x *is indecomposable.*

Proof. (i) By assumption, there are irreducible representations $(\mathcal{H}, \pi)$ of $\mathcal{B}$ and $(\mathcal{H}', \pi')$ of $\mathcal{A}$ such that $v = [(\mathcal{H}, \pi)]$ and $vR_x = [(\mathcal{H}', \pi')]$. Because both $[\pi] \in \mathrm{Sect}(\mathcal{B}, \mathcal{L}(\mathcal{H}))$ and $[\pi'] \in \mathrm{Sect}(\mathcal{A}, \mathcal{L}(\mathcal{H}'))$ are irreducible, and $[\pi]x = vR_x = [\pi']$. Therefore x is irreducible by Lemma 2.10.
(ii) We can assume that $\pi \sim \pi \circ \rho$. From this, $\pi \sim \pi \circ \rho^n$ for each $n \geq 1$. By (i), $[\rho^n] = [\rho]^n$ is irreducible for each $n \geq 1$.
(iii) We see that if $\varphi_1 \sim \varphi_2$, then $\pi \circ \varphi_1 \sim \pi \circ \varphi_2$ for any $\pi \in \mathrm{Rep}\mathcal{B}$. Hence the statement holds.
(iv) If $\varphi \in \mathrm{Aut}\mathcal{A}$, then $\pi \circ \varphi$ is irreducible for any irreducible representation π of $\mathcal{A}$. Hence the statement holds.
(v) If x is decomposable, then there are $x_1, \ldots, x_N \in \mathrm{Sect}(\mathcal{A}, \mathcal{B})$ such that $x = x_1 + \cdots + x_N$. From this, $vR_x = vR_{x_1} \oplus \cdots \oplus vR_{x_N}$. Therefore the totality of irreducible components of vR_x is greater than equal N. From this, the statement holds. □

3.2. Branching laws and spectrum modules. For $\mathcal{S} \subset \mathrm{BSpec}\mathcal{A}$, let $< \mathcal{S} >$ be the set of all finite direct sums of elements in $\mathcal{S}$, $< \mathcal{S} >_\infty$ be the set of all countably infinite direct sums of elements in $\mathcal{S}$ and $< \mathcal{S} >_\int$ be the set of all direct integrals of elements in $\mathcal{S}$. Then $< \mathcal{S} >, < \mathcal{S} >_\infty, < \mathcal{S} >_\int$ are subsemigroups of $\mathrm{BSpec}\mathcal{A}$ and $< \mathcal{S} > \subset < \mathcal{S} >_\infty \subset < \mathcal{S} >_\int$.

Definition 3.4. *Let* $\mathcal{T}$ *be a subsemigroup of the sector semigroup* $\mathrm{Sect}\mathcal{A}$*.*

(i) $(\mathrm{BSpec}\mathcal{A}, R|_\mathcal{T})$ *is called the (right)spectrum module of* $\mathcal{T}$*.*

(ii) V *is a* $\mathcal{T}$*-submodule of* $\mathrm{BSpec}\mathcal{A}$ *if* V *is a subsemigroup of* $\mathrm{BSpec}\mathcal{A}$ *and* $VR_x \subset V$ *for each* $x \in \mathcal{T}$*.*

Assume that $\mathcal{A} \subset \mathcal{B}$ is a unital inclusion of C*-algebra and denote ι_0 this inclusion map. Then the restriction $\pi|_\mathcal{A}$ of $\pi \in \mathrm{IrrRep}\mathcal{B}$ on $\mathcal{A}$ is not irreducible in general. If there are a family $\{\pi_\lambda\}_{\lambda\in\Lambda} \subset \mathrm{IrrRep}\mathcal{A}$ and a family

$\{a_\lambda \in \{0,1,2,\ldots,\aleph_0,\aleph_1,\ldots,\} : \lambda \in \Lambda\}$ of multiplicities such that

$$\pi|_{\mathcal{A}} \sim \bigoplus_{\lambda\in\Lambda} a_\lambda \pi_\lambda, \tag{3.4}$$

then (3.4) is called the *branching law* of π which is arising from the restriction on $\mathcal{A}$. In general, (3.4) is described by direct integral. (3.4) is equivalent to an equation

$$vR_x = \bigoplus_{\lambda\in\Lambda} a_\lambda w_\lambda \tag{3.5}$$

where $x \equiv [\iota_0]$, $v \equiv [\pi]$ and $w_\lambda \equiv [\pi_\lambda]$ for $\lambda \in \Lambda$. The branching law of $v \in \mathcal{S}$ by $x \in \mathrm{Sect}(\mathcal{A},\mathcal{B})$ is given by (3.5). In order to classify endomorphisms of $\mathcal{A}$, we introduced a graph from branching laws of an endomorphism in [**21**]. This classification is just that of sectors.

On the other hand, when we compute the branching law in (3.5), the information about $x = x_1 + \cdots + x_N$ or $x = yz$ for some other sectors $x_1,\ldots,x_N,y,z$ is often useful by (3.2) and (3.3). These examples are shown in § 4, § 5, § 6.

4. Permutative sectors of $\mathcal{O}_N$ and their spectrum modules

Sectors are interested in quantum field theory and subfactor theory. Therefore $\mathrm{Sect}\mathcal{A}$ is mainly treated when $\mathcal{A}$ is a local observable algebra or a factor. In this purpose, $\mathrm{Sect}\mathcal{O}_N$ is often considered to make examples of inclusions of algebras with non trivial indices. However it seems that the structure of $\mathrm{Sect}\mathcal{O}_N$ itself is not well-known.

In general, there is no uniqueness of irreducible decomposition for representations of a C*-algebra and the branching law for them make no sense. We introduce nice classes of representations and endomorphisms of the Cuntz algebras which satisfy the uniqueness of irreducible decomposition ([**6**, **8**, **9**, **16**, **17**, **19**]). On these representations, we show branching laws arising from endomorphisms([**20**, **21**, **22**]). We explain sectors and their spectrum modules associated with these representations and endomorphisms.

4.1. Permutative representations and endomorphisms. We introduce several sets of multiindices which consist of numbers $1,\ldots,N$ for $N \geq 2$ in order to describe invariants of representations of $\mathcal{O}_N$.

Put $\{1,\ldots,N\}^0 \equiv \{0\}$, $\{1,\ldots,N\}^k \equiv \{(j_l)_{l=1}^k : j_l = 1,\ldots,N,\, l = 1,\ldots,k\}$ for $k \geq 1$ and $\{1,\ldots,N\}^\infty \equiv \{(j_n)_{n\in\mathbf{N}} : j_n \in \{1,\ldots,N\},\, n \in \mathbf{N}\}$. Denote $\{1,\ldots,N\}^* \equiv \coprod_{k\geq 0}\{1,\ldots,N\}^k$, $\{1,\ldots,N\}^*_1 \equiv \coprod_{k\geq 1}\{1,\ldots,N\}^k$, $\{1,\ldots,N\}^\# \equiv \{1,\ldots,N\}^*_1 \sqcup \{1,\ldots,N\}^\infty$. For $J \in \{1,\ldots,N\}^\#$, the *length* $|J|$ of J is defined by $|J| \equiv k$ when $J \in \{1,\ldots,N\}^k$. For $J_1, J_2 \in \{1,\ldots,N\}^*$ and $J_3 \in \{1,\ldots,N\}^\infty$ $J_1 \cup J_2 \equiv (j_1,\ldots,j_k,j_1^{'},\ldots,j_l^{'})$, $J_1 \cup J_3 \equiv (j_1,\ldots,j_k,j_1^{''},j_2^{''},\ldots)$

when $J_1 = (j_1, \ldots, j_k)$, $J_2 = (j_1^{'}, \ldots, j_l^{'})$ and $J_3 = (j_n^{''})_{n\in\mathbf{N}}$. Specially, we define $J \cup \{0\} = \{0\} \cup J = J$ for $J \in \{1, \ldots, N\}^{\#}$ and $(i, J) \equiv (i) \cup J$ for convenience. For $J \in \{1, \ldots, N\}^{*}$ and $k \geq 2$, $J^k \equiv \underbrace{J \cup \cdots \cup J}_{k}$. For $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$ and $\tau \in \mathbf{Z}_k$, denote $\tau(J) = (j_{\tau(1)}, \ldots, j_{\tau(k)})$.

$J \in \{1, \ldots, N\}^{*}_{1}$ is *periodic* if there are $m \geq 2$ and $J_0 \in \{1, \ldots, N\}^{*}_{1}$ such that $J = J_0^m$. For $J_1, J_2 \in \{1, \ldots, N\}^{*}_{1}$, $J_1 \sim J_2$ if there are $k \geq 1$ and $\tau \in \mathbf{Z}_k$ such that $|J_1| = |J_2| = k$ and $\tau(J_1) = J_2$. For $(J, z), (J^{'}, z^{'}) \in \{1, \ldots, N\}^{*}_{1} \times U(1)$, $(J, z) \sim (J^{'}, z^{'})$ if $J \sim J^{'}$ and $z = z^{'}$ where $U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$. For $J_1 = (j_1, \ldots, j_k), J_2 = (j_1^{'}, \ldots, j_k^{'}) \in \{1, \ldots, N\}^k$, $k \geq 1$, $J_1 \prec J_2$ if $\sum_{l=1}^{k}(j_l^{'} - j_l)N^{k-l} \geq 0$. $J \in \{1, \ldots, N\}^{*}_{1}$ is *minimal* if $J \prec J^{'}$ for each $J^{'} \in \{1, \ldots, N\}^{*}_{1}$ such that $J \sim J^{'}$. Specially, any element in $\{1, \ldots, N\}$ is non periodic and minimal. $J \in \{1, \ldots, N\}^{\infty}$ is *eventually periodic* if there are $J_0, J_1 \in \{1, \ldots, N\}^{*}_{1}$ such that $J = J_0 \cup J_1^{\infty}$. For $J_1, J_2 \in \{1, \ldots, N\}^{\infty}$, $J_1 \sim J_2$ if there are $J_3, J_4 \in \{1, \ldots, N\}^{*}$ and $J_5 \in \{1, \ldots, N\}^{\infty}$ such that $J_1 = J_3 \cup J_5$ and $J_2 = J_4 \cup J_5$.

Put $[1, \ldots, N]^{*} \equiv \{J \in \{1, \ldots, N\}^{*}_{1} : J \text{ is minimal and non periodic}\}$. Then $[1, \ldots, N]^{*}$ is in one-to-one correspondence with the set of all equivalence classes of non periodic elements in $\{1, \ldots, N\}^{*}_{1}$. Put $[1, \ldots, N]^{\infty}$ the set of all equivalence classes of non eventually periodic elements in $\{1, \ldots, N\}^{\infty}$ and $[1, \ldots, N]^{\#} \equiv [1, \ldots, N]^{*} \sqcup [1, \ldots, N]^{\infty}$.

Put α an action of a unitary group $U(N)$ on $\mathcal{O}_N$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^{N} g_{ji}s_j$ for $i = 1, \ldots, N$ and $g = (g_{ij})_{i,j=1}^{N} \in U(N)$. Specially we denote $\gamma_w \equiv \alpha_{g(w)}$ when $g(w) = w \cdot I \subset U(N)$ for $w \in U(1)$. For $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, we denote $s_J = s_{j_1} \cdots s_{j_k}$ and $s_J^{*} = s_{j_k}^{*} \cdots s_{j_1}^{*}$.

A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ is *permutative* if there is a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ which satisfies $^{\forall}(n, i) \in \Lambda \times \in \{1, \ldots, N\}$, $^{\exists}m \in \Lambda$ s.t. $\pi(s_i)e_n = e_m$. $(\mathcal{H}, \pi, \Omega)$ is a *generalized permutative(=GP) representation of $\mathcal{O}_N$ with cycle* by $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $k \geq 1$ and a phase $z \in U(1)$ if $\Omega \in \mathcal{H}$ is a cyclic unit vector such that $\pi(s_J)\Omega = z\Omega$ and $\{\pi(s_{j_l} \cdots s_{j_k})\Omega : l = 1, \ldots, k\}$ is an orthonormal family in $\mathcal{H}$. We denote $P(J; z) = (\mathcal{H}, \pi, \Omega)$ and $P(J) \equiv P(J; 1)$ simply. $(\mathcal{H}, \pi, \Omega)$ is a *GP representation of $\mathcal{O}_N$ with chain* by $J \in \{1, \ldots, N\}^{\infty}$ if $\Omega \in \mathcal{H}$ is a cyclic unit vector such that $\{\pi(s_{J_n})^{*}\Omega\}_{n\in\mathbf{N}}$ is an orthonormal family where $J_n \equiv (j_1, \ldots, j_n)$ when $J = (j_m)_{m\in\mathbf{N}}$. We denote $P(J) = (\mathcal{H}, \pi, \Omega)$ simply.

Any permutative representation is completely reducible. Any cyclic(*resp.* irreducible)permutative representation is equivalent to $P(J)$ for some $J \in \{1, \ldots, N\}^{\#}$(*resp.* some $J \in [1, \ldots, N]^{*}$ or some non eventually periodic $J \in \{1, \ldots, N\}^{\infty}$). For each $J \in \{1, \ldots, N\}^{\#}$, $P(J)$ exists uniquely up to unitary equivalences. $P(J)$ is equivalent to a cyclic permutative representation. For $\rho \in \mathrm{End}\mathcal{O}_N$ and $P(J; z) = (\mathcal{H}, \pi, \Omega)$, we denote $P(J; z) \circ \rho = (\mathcal{H}, \pi \circ \rho, \Omega)$.

Theorem 4.1. (i) *For $J \in \{1,\ldots,N\}_1^*$ and $z \in U(1)$, $P(J;z)$ is irreducible if and only if J is non periodic. For $J \in \{1,\ldots,N\}^\infty$, $P(J)$ is irreducible if and only if J is non eventually periodic.*

(ii) *For $J_1, J_2 \in \{1,\ldots,N\}_1^*$ and $z_1, z_2 \in U(1)$, $P(J_1;z_1) \sim P(J_2;z_2)$ if and only if $(J_1,z_1) \sim (J_2,z_2)$ where $P(J_1;z_1) \sim P(J_2;z_2)$ means the unitary equivalence of two representations which satisfy the condition $P(J_1;z_1)$ and $P(J_2;z_2)$, respectively. For $J_1, J_2 \in \{1,\ldots,N\}^\infty$, $P(J_1) \sim P(J_2)$ if and only if $J_1 \sim J_2$.*

(iii) *If $J \in \{1,\ldots,N\}^k$, $k \geq 1$ and $z \in U(1)$, then $P(J;1)\circ\gamma_z = P(J;z^k)$. If $J \in \{1,\ldots,N\}^\infty$ and $z \in U(1)$, then $P(J)\circ\gamma_z = P(J)$.*

(iv) *For $J \in \{1,\ldots,N\}_1^*$, $z \in U(1)$ and $l \geq 1$,*

$$P(J^l;z) = \bigoplus_{j=1}^{l} P(J;\xi^{j-1}z^{1/l}) \tag{4.1}$$

where $\xi \equiv e^{2\pi\sqrt{-1}/l}$. This decomposition is unique up to unitary equivalences. Specially we have $P(J^l;1) = \bigoplus_{j=1}^{l} P(J;\xi^{j-1})$.

Proof. See Theorem 2.12 in [**21**]. □

We omit the decomposition of chain in this article(see [**19**]). In consequence, we have the following:

Theorem 4.2. (i) *A set $\{P(J;z) : J \in \{1,\ldots,N\}_1^*, z \in U(1)\}$ of representations of $\mathcal{O}_N$ is closed with respect to irreducible decomposition, and the number of components of decomposition is always finite.*

(ii) *$[1,\ldots,N]^\#$ is in one-to-one correspondence with the set of all equivalence classes of irreducible permutative representations of $\mathcal{O}_N$.*

We review endomorphisms of $\mathcal{O}_N$ arising from permutations in [**20**, **21**, **22**]. Put $\mathfrak{S}_{N,l}$ the set of all bijections on a set $\{1,\ldots,N\}^l$ for $l \geq 1$ and $\mathfrak{S}_{N,*} \equiv \bigcup_{l\geq 1}\mathfrak{S}_{N,l}$. For $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, $\psi_\sigma \in \mathrm{End}\mathcal{O}_N$ is defined by

$$\psi_\sigma(s_i) \equiv u_\sigma s_i \quad (i=1,\ldots,N), \quad u_\sigma \equiv \sum_{J\in\{1,\ldots,N\}^l} s_{\sigma(J)} s_J^*.$$

ψ_σ is called the *permutative endomorphism* of $\mathcal{O}_N$ by σ. Put the following sets:

$$E_{N,*} \equiv \bigcup_{l\geq 1} E_{N,l}, \quad E_{N,l} \equiv \{\psi_\sigma \in \mathrm{End}\mathcal{O}_N : \sigma \in \mathfrak{S}_{N,l}\} \quad (l \geq 1). \tag{4.2}$$

If $\sigma \in \mathfrak{S}_N$, then ψ_σ is an automorphism of $\mathcal{O}_N$ which satisfies $\psi_\sigma(s_i) = s_{\sigma(i)}$ for $i = 1,\ldots,N$. Specially, ψ_{id} is the identity map on $\mathcal{O}_N$. If $\sigma \in \mathfrak{S}_{N,2}$ is defined by $\sigma(i,j) \equiv (j,i)$ for $i,j = 1,\ldots,N$, then ψ_σ is just the canonical endomorphism of $\mathcal{O}_N$. If $\rho \in E_{N,l}$ and $\rho^{'} \in E_{N,l'}$, then $\rho\circ\rho^{'} \in E_{N,l+l'-1}$ for each $l, l^{'} \geq 1$(see Proposition 3.3 in [**21**]). Remark that $\psi_\sigma \circ \psi_\eta \neq$

$\psi_{\sigma\circ\eta}$ in general. $E_{N,*}$ is a subsemigroup of End$\mathcal{O}_N$. Put $\mathrm{End}_{U(1)}\mathcal{O}_N \equiv \{\rho \in \mathrm{End}\mathcal{O}_N : {}^{\forall}z \in U(1) \quad \rho\circ\gamma_z = \gamma_z\circ\rho\}$. For any $\rho \in \mathrm{End}_{U(1)}\mathcal{O}_N$, $\rho|_{UHF_N} \in \mathrm{End}UHF_N$ where we denote $UHF_N \equiv \mathcal{O}_N^{U(1)}$. We see that $E_{N,*} \subset \mathrm{End}_{U(1)}\mathcal{O}_N$ and $\psi_\sigma|_{UHF_N} \in \mathrm{End}UHF_N$.

Theorem 4.3. (i) *If ρ is a permutative endomorphism and $(\mathcal{H},\pi)$ is a permutative representation of $\mathcal{O}_N$, then $(\mathcal{H},\pi\circ\rho)$ is a permutative representation, too.*

(ii) *If $(\mathcal{H},\pi)$ is $P(J)$ for $J \in \{1,\ldots,N\}_1^*$ and $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, then there are $1 \leq m \leq N^{l-1}$, a family $\{J_i\}_{i=1}^m \subset \{1,\ldots,N\}_1^*$ and a family $\{(\mathcal{H}_i,\pi_i)\}_{i=1}^m$ of subrepresentations of $(\mathcal{H},\pi\circ\psi_\sigma)$ such that*

$$(\mathcal{H},\pi\circ\psi_\sigma) = (\mathcal{H}_1,\pi_1)\oplus\cdots\oplus(\mathcal{H}_m,\pi_m) \tag{4.3}$$

and $(\mathcal{H}_i,\pi_i)$ is $P(J_i)$ for $i=1,\ldots,m$. Furthermore if $|J|=k$, $k\geq 1$, then $|J_i| \in \{ak : a=1,\ldots,N^{l-1}\}$.

(iii) *The rhs in (4.3) is unique up to unitary equivalences.*

Proof. See Theorem 3.4 in [**21**]. □

(4.3) is called the *branching law* of $(\mathcal{H},\pi)$ by ψ_σ. By uniqueness of $P(J)$ and Theorem 4.3 (iii), we can simply denote (4.3) as

$$P(J)\circ\psi_\sigma = P(J_1)\oplus\cdots\oplus P(J_m). \tag{4.4}$$

For a given J, $J_1,\ldots,J_m$ in (4.4) are computed by a Mealy machine associated with σ in [**22**].

We define an N-ary operation on $\mathfrak{S}_{N,*}$.

Lemma 4.4. (i) *Let $\sigma \in \mathfrak{S}_{N,l+l'}$ and $l,l' \geq 1$. If there is $\eta \in \mathfrak{S}_{N,l}$ such that $\sigma(J,K) = (\eta(J),K)$ for each $J \in \{1,\ldots,N\}^l$ and $K \in \{1,\ldots,N\}^{l'}$, then $\psi_\sigma = \psi_\eta \in E_{N,l}$.*

(ii) *For a family $\{\sigma^{(i)}\}_{i=1}^N \subset \mathfrak{S}_{N,*}$, put $l_i \in \mathbf{N}$ by $\sigma^{(i)} \in \mathfrak{S}_{N,l_i}$ for $i=1,\ldots,N$ and $l \equiv \max\{l_i : i=1,\ldots,N\}$. Define $\hat{\sigma} \in \mathfrak{S}_{N,l+2}$ by*

$$\hat{\sigma}(i,j,J) \equiv (\,j,\ (\sigma^{(j)})'(i,J)\,)$$

for $i,j \in \{1,\ldots,N\}$ and $J \in \{1,\ldots,N\}^l$ where $(\sigma^{(i)})' \equiv \sigma^{(i)}\times id^{l-l_i+1}$ for $i=1,\ldots,N$. Then we have $\psi_{\hat{\sigma}} = <\xi|\Phi> \in E_{N,l+1}$ where $\Phi \equiv (\psi_{\sigma^{(i)}})_{i=1}^N$ and $\xi \equiv (s_i)_{i=1}^N \in H_N\mathcal{O}_N$.

Proof. (i) follows from direct computation. Let $\Phi' \equiv (\psi_{(\sigma^{(i)})'})_{i=1}^N$. Then we see that $\psi_{\hat{\sigma}} = <\xi|\Phi'>$. From this $\psi_{\hat{\sigma}} = \mathrm{Ad}s_1\circ\psi_{(\sigma^{(1)})'}+\cdots+\mathrm{Ad}s_N\circ\psi_{(\sigma^{(N)})'} = \mathrm{Ad}s_1\circ\psi_{\sigma^{(1)}}+\cdots+\mathrm{Ad}s_N\circ\psi_{\sigma^{(N)}} = <\xi|\Phi>$ by (i). Put $(\sigma^{(i)})'' \equiv \sigma^{(i)}\times id^{l-l_i}$ when $l_i \neq l$, and $(\sigma^{(i)})'' \equiv \sigma^{(i)}$ when $l_i = l$. Then we see that $\psi_{\hat{\sigma}} = \mathrm{Ad}s_1\circ\psi_{(\sigma^{(1)})''}+\cdots+\mathrm{Ad}s_N\circ\psi_{(\sigma^{(N)})''} \in E_{N,l+1}$. Hence the statement holds. □

4.2. Permutative sectors and their spectrum modules. Let

$$SE_{N,*} \equiv \bigcup_{l\geq 1} SE_{N,l}, \quad SE_{N,l} \equiv \{[\psi_\sigma] \in \mathrm{Sect}\mathcal{O}_N : \sigma \in \mathfrak{S}_{N,l}\}. \tag{4.5}$$

Theorem 4.5. (i) *$SE_{N,*}$ is an N-ary subalgebra of* $\mathrm{Sect}\mathcal{O}_N$.

(ii) *If $x_i \in SE_{N,l_i}$ for $i = 1,\ldots,N$, then $x_1 + \cdots + x_N \in SE_{N,l+1}$ for $l \equiv \max\{l_i : i = 1,\ldots,N\}$.*

(iii) *If $x \in SE_{N,l}$ and $y \in SE_{N,l'}$, then $xy \in SE_{N,l+l'-1}$.*

(iv) *Under an identification UHF_N with $\mathcal{O}_N^{U(1)}$, $SE_{N,*}|_{UHF_N} \equiv \{[\rho|_{UHF_N}] : \rho \in E_{N,*}\}$ is a subsemigroup of* $\mathrm{Sect}UHF_N$.

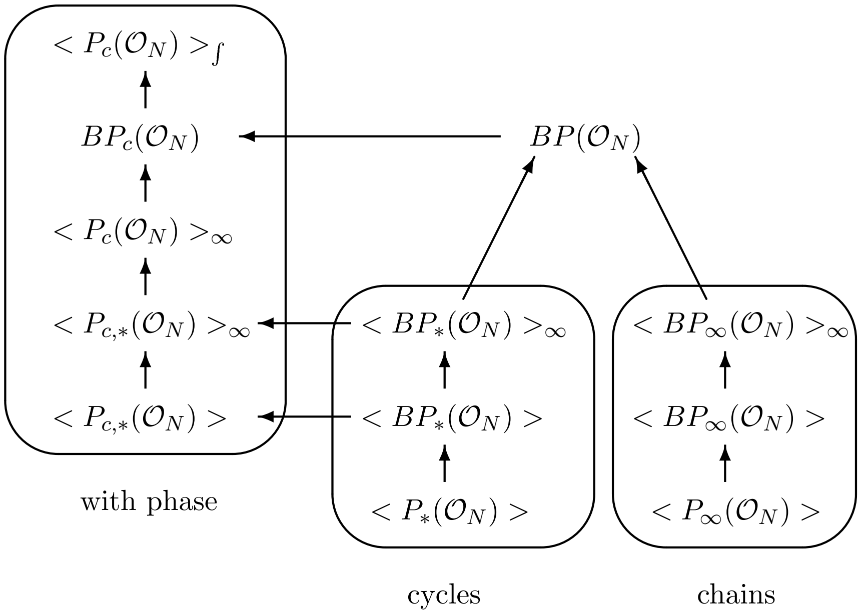
Proof. (i) For $\{[\psi_{\sigma^{(i)}}]\}_{i=1}^N \subset SE_{N,*}$, we see that $[\psi_{\sigma^{(1)}}]+\cdots+[\psi_{\sigma^{(N)}}] = [< \xi|\Phi >] = [\psi_{\hat{\sigma}}] \in SE_{N,*}$ where ξ, Φ are taken in Lemma 4.4 (ii). Then $SE_{N,*}$ is closed under the sector sum on $\mathrm{Sect}\mathcal{O}_N$. Because $E_{N,*}$ in (4.2) is closed under product, $SE_{N,*}$ is an N-ary subalgebra of $\mathrm{Sect}\mathcal{O}_N$.
(ii) This follows from Lemma 4.4 (ii).
(iii) and (iv) follow from the paragraph before Theorem 4.3. □

We call $SE_{N,*}$ the *permutative sector algebra* of $\mathcal{O}_N$ and an element in $SE_{N,*}$ is called a *permutative sector* of $\mathcal{O}_N$. Remark that even if $l \neq l'$, $SE_{N,l} \cap SE_{N,l'} \neq \emptyset$ in general by Lemma 4.4 (i). Since $\#SE_{N,l} \leq \#E_{N,l} = N^l! < \infty$, an N-ary subalgebra $< SE_{N,l} >$ of $SE_{N,*}$ is finitely generated and noncommutative. We see that a sector algebra $< SE_{N,l} >$ is a nice example in a study of sector algebras of $\mathcal{O}_N$. $< SE_{N,1} >$ is shown in the Example 4.6. We treat the first non trivial example $< SE_{2,2} >$ and sectors which does not belong into $< SE_{N,l} >$ in § 5.3.

Example 4.6. For each $N \geq 2$, α_g is outer when $g \in U(N)$, $g \neq I$. Therefore $< \{[\alpha_g] : g \in U(N)\} > \cong < U(N) >$ where $< U(N) >$ is a free N-ary algebra generated by $U(N)$. Therefore $\mathrm{Sect}\mathcal{O}_N$ has an N-ary subalgebra $< U(N) >$. Specially, $< \mathfrak{S}_N > = < SE_{N,1} >$ is an N-ary subalgebra of $SE_{N,*}$.

We introduce several subsemigroups of $\mathrm{BSpec}\mathcal{O}_N$. In this section, we identify a representation and its unitary equivalence class. Let $BP(\mathcal{O}_N)$ (*resp.* $P(\mathcal{O}_N)$) be the set of all unitary equivalence classes of (*resp.* irreducible) permutative representations of $\mathcal{O}_N$, and $BP_*(\mathcal{O}_N)$(*resp.* $BP_\infty(\mathcal{O}_N)$) the subset of $BP(\mathcal{O}_N)$ which consists of all cyclic permutative representations with a cycle (*resp.* a chain). Put $BP_c(\mathcal{O}_N) \equiv \{vR_{[\gamma_z]} : v \in BP(\mathcal{O}_N),\ z \in U(1)\}$, $BP_\#(\mathcal{O}_N) \equiv BP_*(\mathcal{O}_N) \sqcup BP_\infty(\mathcal{O}_N)$, $P_{c,*}(\mathcal{O}_N) \equiv \{P(J;z) : J \in [1,\ldots,N]^*,\ z \in U(1)\}$, $P_*(\mathcal{O}_N) \equiv BP_*(\mathcal{O}_N) \cap P(\mathcal{O}_N)$, $P_\infty(\mathcal{O}_N) \equiv BP_\infty(\mathcal{O}_N) \cap P(\mathcal{O}_N)$, $P_c(\mathcal{O}_N) \equiv P_{c,*}(\mathcal{O}_N) \sqcup P_\infty(\mathcal{O}_N)$.

We see that $P(\mathcal{O}_N) = \{P(J) : J \in [1,\ldots,N]^{\#}\}$, $P_\infty(\mathcal{O}_N) = \{P(J) : J \in [1,\ldots,N]^\infty\}$, $P_*(\mathcal{O}_N) = \{P(J) : J \in [1,\ldots,N]^*\}$ by Theorem 4.1. In [**21**], we show the following inclusions of abelian semigroups:



where any inclusion is proper. These inclusions show relations among classes of representations. For example, $<BP_*(\mathcal{O}_N)>\subset<P_{c,*}(\mathcal{O}_N)>$ means that any element in $BP_*(\mathcal{O}_N)$ can be expressed as a finite direct sum of elements in $P_{c,*}(\mathcal{O}_N)$. Since $P_{c,*}(\mathcal{O}_N)$ is the set of unitary equivalence classes of irreducible representations, this inclusion shows irreducible decomposition of elements in $BP_*(\mathcal{O}_N)$ with finite multiplicity and finite components. Furthermore the following holds: $<BP_{\#}(\mathcal{O}_N)>_\infty = BP(\mathcal{O}_N) = <BP(\mathcal{O}_N)>_\infty$. Put $BP_k(\mathcal{O}_N) \equiv \{P(J) : J \in \{1,\ldots,N\}^k_{min}\}$, $P_{c,l}(\mathcal{O}_N) \equiv \{P(J)\circ\gamma_z \in P_{c,*} : |J| = l\}$ where $\{1,\ldots,N\}^k_{min}$ is the set of all minimal elements in $\{1,\ldots,N\}^k$ for $k \geq 1$. Then the following holds by Theorem 4.1 (iv):

$$<BP_k(\mathcal{O}_N)>\subset \bigoplus_{l\in D(k)} <P_{c,l}(\mathcal{O}_N)>$$

where $D(k)$ is the set of all divisors of k. By Proposition 5.2 in [**21**], we have the following:

Proposition 4.7. *Let* $(\mathrm{BSpec}\mathcal{O}_N, R)$ *be the spectrum module of* $\mathrm{Spec}\mathcal{O}_N$.

(i) *The followings are $SE_{N,*}$-submodules of* $(\mathrm{BSpec}\mathcal{O}_N,\ R|_{SE_{N,*}})$*:*
$<BP_\infty(\mathcal{O}_N)>$, $<BP_\infty(\mathcal{O}_N)>_\infty$, $BP(\mathcal{O}_N)$, $<P_c(\mathcal{O}_N)>_\int$,
$<BP_*(\mathcal{O}_N)>_\infty$, $<P_{c,*}(\mathcal{O}_N)>_\infty$, $<BP_*(\mathcal{O}_N)>$, $<P_{c,*}(\mathcal{O}_N)>$.

(ii) *Let* $\widehat{SE}_{N,*}\equiv\{x[\gamma_z]:x\in SE_{N,*},\,z\in U(1)\}$*. The followings are* $\widehat{SE}_{N,*}$*-submodules of* $(\mathrm{BSpec}\mathcal{O}_N, R|_{\widehat{SE}_{N,*}})$*:* $< BP_\infty(\mathcal{O}_N) >$, $< BP_\infty(\mathcal{O}_N) >_\infty$, $< P_c(\mathcal{O}_N) >_\int$, $< P_{c,*}(\mathcal{O}_N) >$, $< P_{c,*}(\mathcal{O}_N) >_\infty$.

(iii) *For the following grading*

$$< BP_*(\mathcal{O}_N) >=\bigoplus_{k\geq 1} < BP_k(\mathcal{O}_N) >,$$

we have

$$< BP_k(\mathcal{O}_N) > R_x \subset \bigoplus_{a=1}^{N^{l-1}} < BP_{ak}(\mathcal{O}_N) >$$

when $x\in SE_{N,l}$, $l\geq 1$.

5. Examples of fusion rule and branching law

We treat polynomial endomorphisms of $\mathcal{O}_N$ and elements in $\mathrm{Sect}\mathcal{O}_N$ associated with them.

5.1. $SE_{2,2}$**.** Recall $SE_{N,l}$ in (4.5). $SE_{2,2}$ includes sufficiently nontrivial elements. By § 4 in [**21**], we see the following:

Table 5.1.

$$SE_{2,2}=\left\{[\psi_\sigma]\in \mathrm{Sect}\mathcal{O}_2:\sigma=\begin{array}{l} id,(12),(13),(14),(23),(24),(34),\\ (123),(132),(124),(142),(143),(234),\\ (1243),(1342),(12)(34)\end{array}\right\}$$

ψ_σ	$\psi_\sigma(s_1)$	$\psi_\sigma(s_2)$	property
ψ_{id}	s_1	s_2	*inn.aut*
ψ_{12}	$s_{12,1}+s_{11,2}$	s_2	*irr.end*
ψ_{13}	$s_{21,1}+s_{12,2}$	$s_{11,1}+s_{22,2}$	*irr.end*
ψ_{14}	$s_{22,1}+s_{12,2}$	$s_{21,1}+s_{11,2}$	*red.end*
ψ_{23}	$s_{11,1}+s_{21,2}$	$s_{12,1}+s_{22,2}$	*red.end*
ψ_{24}	$s_{11,1}+s_{22,2}$	$s_{21,1}+s_{12,2}$	*irr.end*
ψ_{34}	s_1	$s_{22,1}+s_{21,2}$	*irr.end*
ψ_{123}	$s_{12,1}+s_{21,2}$	$s_{11,1}+s_{22,2}$	*red.end*
ψ_{124}	$s_{12,1}+s_{22,2}$	$s_{21,1}+s_{11,2}$	*red.end*
ψ_{132}	$s_{21,1}+s_{11,2}$	$s_{12,1}+s_{22,2}$	*red.end*
ψ_{142}	$s_{22,1}+s_{11,2}$	$s_{21,1}+s_{12,2}$	*irr.end*
ψ_{143}	$s_{22,1}+s_{12,2}$	$s_{11,1}+s_{21,2}$	*red.end*
ψ_{234}	$s_{11,1}+s_{21,2}$	$s_{22,1}+s_{12,2}$	*red.end*
ψ_{1243}	$s_{12,1}+s_{22,2}$	$s_{11,1}+s_{21,2}$	*red.end*
ψ_{1342}	$s_{21,1}+s_{11,2}$	$s_{22,1}+s_{12,2}$	*red.end*
$\psi_{(12)(34)}$	$s_{12,1}+s_{11,2}$	$s_{22,1}+s_{21,2}$	*out.aut*

where "inn.aut", "out.aut", "irr.end" and "red.end" mean an inner automorphism, an outer automorphism, a proper irreducible endomorphism and a proper reducible endomorphism, respectively, and $s_{ij,k} \equiv s_i s_j s_k^*$ for $i,j,k = 1,2$.

Theorem 5.2. *In 16 elements in $SE_{2,2}$, 2 in 16 are not proper. One of them is inner and other is outer. In 14 proper sectors in $SE_{2,2}$, there are 5 irreducible sectors. The last 9 non irreducible sectors are sums of two non proper sectors:*

$$(5.1)\quad \begin{cases} [\psi_{14}] = [\alpha] + [-\alpha], & [\psi_{23}] = [\iota] + [\iota], & [\psi_{123}] = [\iota] + [\alpha], \\ [\psi_{124}] = [\alpha] + [\alpha\beta_2], & [\psi_{132}] = [\iota] + [\beta_1], & [\psi_{143}] = [\alpha] + [\alpha\beta_1], \\ [\psi_{234}] = [\iota] + [\beta_2], & [\psi_{1243}] = [\alpha] + [\alpha], & [\psi_{1342}] = [\iota] + [-\iota] \end{cases}$$

where ι is the identity map on $\mathcal{O}_2$ and $\alpha, \beta_1, \beta_2 \in \mathrm{Aut}\mathcal{O}_2$ are defined by transpositions $\alpha : s_1 \leftrightarrow s_2$, $\beta_1 : s_1 \leftrightarrow -s_1$, $\beta_2 : s_2 \leftrightarrow -s_2$, $-\iota \equiv \iota\theta$, $-\alpha \equiv \alpha\theta$, $\theta : s_i \leftrightarrow -s_i$ for $i = 1,2$.

Proof. Let $\xi \equiv (s_1, s_2), \xi' \equiv (2^{-1/2}(s_1 - s_2), 2^{-1/2}(s_1 + s_2)) \in H_2\mathcal{O}_2$. Denote $\varphi_1 +_\zeta \varphi_2 \equiv <\zeta|(\varphi_1,\varphi_2)>$ for $\zeta = \xi, \xi'$ and $\varphi_1, \varphi_2 = \iota, \alpha, \beta_1, \beta_2$. Then the following equations hold: $\psi_{123} = \iota +_\xi \alpha$, $\psi_{14} = \alpha +_{\xi'} \alpha$, $\psi_{124} = \alpha +_{\xi'} \alpha\beta_2$, $\psi_{132} = \iota +_{\xi'} \beta_1$, $\psi_{143} = \alpha +_{\xi'} \alpha\beta_1$, $\psi_{234} = \iota +_{\xi'} \beta_2$, $\psi_{1342} = \iota +_{\xi'} \theta$, $\psi_{23} = \iota +_\xi \iota$, $\psi_{1243} = \alpha +_\xi \alpha$. Therefore the statements hold. □

By Theorem 5.2, branching laws of sectors in (5.1) are more easily computed than direct computations in [**20**, **21**, **22**].

Note that $\psi_{12}, \psi_{13}, \psi_{24}, \psi_{34}, \psi_{143}$ are irreducible and proper.

Proof of Theorem 1.3. We see that $\rho = \psi_{12}$, $\bar{\rho} = \psi_{13}$ and $\eta = \psi_{142}$.
(i) We can verify the following relations: $\bar{\rho} \circ \rho = \psi_{123} = \alpha +_\xi \iota$, $\rho \circ \bar{\rho} = \psi_{132} = \iota +_{\xi'} \beta_1$ by Theorem 5.2.
(ii) This follows from (i) and Theorem 3.3.
(iii) For $\rho_1, \rho_2 \in \mathrm{End}\mathcal{O}_2$, denote (ρ_1, ρ_2) be the set of intertwiners between ρ_1 and ρ_2. We see that $S \equiv 2^{-1/2}(s_1 + s_2) \in (\iota, \rho \circ \bar{\rho})$ and $R \equiv s_2 \in (\iota, \bar{\rho} \circ \rho)$. Hence $\bar{\rho}(S^*)R = 2^{-1/2}I$. By the definition of d_{ρ^n}(see [**4**]), the statistical dimension of ρ is $\sqrt{2}$. In the same way, we see that $d_{\rho^n} = 2^{n/2}$ for each $n \geq 1$. □

Remark 5.3. By Theorem 1.3, it seems that $[\bar{\rho}]$ is the conjugate sector of $[\rho]$ but we do not know exact relation with them. We see that $P(1)R_{[\rho][\bar{\rho}]} = P(11)$ and $P(1)R_{[\bar{\rho}][\rho]} = P(1) \oplus P(2)$. By Theorem 1.3 and comparison of branching laws, we see that

$$[\rho][\bar{\rho}] \neq [\bar{\rho}][\rho].$$

The irreducibility of $[\rho]^n$, $n \geq 3$ and the structure of noncommutative subalgebra $< \{[\rho], [\bar{\rho}]\} > \subset \mathrm{Sect}\mathcal{O}_2$ are unknown yet.

Proposition 5.4. (i) $[\eta]^2 = ([\iota] + [\theta])([\iota] + [\alpha])$. *That is,* η *is self conjugate.*

(ii) *For* $n \geq 2$,

$$[\eta]^n = \begin{cases} 4^{k-1}([\iota] + [\theta])([\iota] + [\alpha]) & \textit{when } n = 2k, \\ \\ 2 \cdot 4^{k-1}[\eta]([\iota] + [\theta]) & \textit{when } n = 2k+1. \end{cases}$$

Proof. We denote $[\rho], [\eta]$ by ρ, η simply.
(i) $\eta^2 = (\bar{\rho}\alpha\rho)(\bar{\rho}\alpha\rho) = \bar{\rho}\alpha(\iota + \beta_1)\alpha\rho = \iota + \alpha + \bar{\rho}\theta\beta_1\rho = \iota + \alpha + \theta(\iota + \alpha)$ where we use $\beta_1\rho = \rho$ and $\bar{\rho}\rho = \iota + \alpha$.
(ii) By (i), we see that $\eta^4 = (\iota + \theta)^2(\iota + \alpha)^2 = 2(\iota + \theta) \cdot 2(\iota + \alpha) = 4(\iota + \theta)(\iota + \alpha) = 4\eta^2$, $\eta^{2n} = (\eta^2)^n = 4^{n-1}\eta^2$. Hence the statement holds. □

Corollary 5.5. (i) *The following isomorphism of commutative algebra holds without inverse of sum:* $< \{[\bar{\rho}]\} > \cong \mathbf{N} = \{1, 2, 3, \ldots\}$.

(ii) $< \{[\eta]\} >$ *is a commutative* 3*-dimensional algebra which is isomorphic to* $\mathbf{Z}_{\geq 0}e_1 \oplus \mathbf{Z}_{\geq 0}e_2 \oplus \mathbf{Z}_{\geq 0}e_3 \setminus \{0\}$ *and* e_1, e_2, e_3 *satisfy the followings:*

$$e_1^2 = e_2, \quad e_1e_2 = e_2e_1 = e_3, \quad e_1e_3 = e_3e_1 = 4e_2, \quad e_2e_3 = e_3e_2 = 4e_3$$

where $\mathbf{Z}_{\geq 0}$ *is the set of non negative integers and*

$$e_1 \equiv [\eta], \quad e_2 \equiv ([\iota] + [\alpha])([\iota] + [\theta]), \quad e_3 \equiv 2[\eta]([\iota] + [\theta]).$$

Problem 5.6. By comparison with $N = 2$, the case $N = 3$ is not so easy because $\#E_{3,2} = 9!$.

(i) Enumerate $\#SE_{3,2}$.
(ii) Enumerate proper and irreducible elements in $SE_{3,2}$.
(iii) Is any non irreducible elements in $SE_{3,2}$ decomposed into the sum of three elements in non proper elements in $\mathrm{Sect}\mathcal{O}_3$?

For example, ρ_ν in (2.1) is proper and irreducible([**20**]). We have the following natural questions: Is there a conjugate endomorphism $\bar{\rho}_\nu$ of ρ_ν ? If $\bar{\rho}_\nu$ exists, then compute fusion rules of $\rho_\nu \circ \bar{\rho}_\nu$ and $\bar{\rho}_\nu \circ \rho_\nu$, and determine the statistical dimension. If $\bar{\rho}_\nu$ exists, then whether is $\bar{\rho}_\nu \in E_{3,2}$ or not ?

5.2. $SE_{N,l}$**.** We show applications of fusion rules for branching laws of representations of $\mathcal{O}_N$ by endomorphisms.

Example 5.7. Assume that $N \geq 3$. Define $\rho \in \mathrm{End}\mathcal{O}_N$ by

$$\begin{cases} \rho(s_1) \equiv \sum_{j=1}^N s_{jj,j}, \quad \rho(s_N) \equiv \sum_{j=1}^N s_{\tau^{j-1}(1)}s_{\tau^{j-2}(1)}s^*_{\tau^j(1)}, \\ \\ \rho(s_i) \equiv \sum_{j=1}^N s_{\tau^{j-1}(1)}s_{\tau^{j+i-2}(1)}s^*_{\tau^{j+i-1}(1)} \quad (i = 2, \ldots, N-1) \end{cases}$$

where $\tau \in \mathbf{Z}_N$ is defined by $\tau(j) \equiv j+1 \quad (j=1,\ldots,N-1), \quad \tau(N) \equiv 1$. Then $\alpha_\sigma \circ \rho = \rho$ for each $\sigma \in \mathbf{Z}_N$. The following branching law holds for each $k = 1, \ldots, N$:

$$P(k) \circ \rho = P(1) \oplus P(N-1, N).$$

Each branching components is irreducible and the branching number is less than N. Therefore ρ is indecomposable by Theorem 3.3. Since ρ is $\mathbf{Z}_N$-invariant, ρ is proper. From this, $\rho \in \mathrm{Hom}(\mathcal{O}_N, \mathcal{O}_N^{\mathbf{Z}_N})$. Hence $H_N\mathcal{O}_N^{\mathbf{Z}_N} \neq \emptyset$.

Example 5.8. We show several formulae of decompositions of sectors and these are used to compute branching laws:

(i) Let $\rho_1 \in \mathrm{End}\mathcal{O}_2$ by

$$\begin{cases} \rho_1(s_1) \equiv s_{112}s_{11}^* + s_{111}s_{12}^* + s_{221}s_{21}^* + s_{212}s_{22}^*, \\ \rho_1(s_2) \equiv s_{12}s_1^* + s_{211}s_{21}^* + s_{222}s_{22}^*. \end{cases}$$

Then $[\rho_1] = [\psi_{12}] + [\psi_{13}]$ where ψ_{12}, ψ_{13} are in Table 5.1.

(ii) Let $\rho_2 \in \mathrm{End}\mathcal{O}_3$ by

$$(5.2) \quad \begin{cases} \rho_2(s_1) \equiv s_{11,1} + s_{21,3} + s_{31,2}, \quad \rho_2(s_2) \equiv s_{12,1} + s_{22,2} + s_{32,3}, \\ \rho_2(s_3) \equiv s_{13,1} + s_{23,2} + s_{33,3}. \end{cases}$$

Then $[\rho_2] \equiv [\iota] + [\iota] + [\beta_1]$ where β_1 is an automorphism of $\mathcal{O}_3$ defined by transposition $s_1 \leftrightarrow -s_1$.

(iii) Let $\rho_3 \in \mathrm{End}\mathcal{O}_N$ by

$$\rho_3(s_i) \equiv s_N s_i s_1^* + s_{N-1} s_i s_N^* + \cdots + s_1 s_i s_2^* \quad (i = 1, \ldots, N).$$

Then $[\rho_3] = [\gamma_{z_1}] + \cdots + [\gamma_{z_N}]$ where $z_i \equiv e^{2\pi\sqrt{-1}(i-1)/N}$ for $i = 1, \ldots, N$.

Proof. (i) We see that $\rho_1 =< (s_1, s_2)|(\psi_{12}, \psi_{13}) >$.
(ii) We see that $\rho_2 \equiv < \xi|(\iota, \iota, \beta_1) >$ where $\xi \equiv (s_1, 2^{-1/2}(s_2+s_3), 2^{-1/2}(s_2 - s_3))$.
(iii) Put $\Phi \equiv (\gamma_{z_1}, \ldots, \gamma_{z_N})$ and $\xi \equiv (t_1, \ldots, t_N) \in H_N\mathcal{O}_N$ by $t_i \equiv N^{-1/2} \sum_{j=1}^N e^{2\pi\sqrt{-1}(i-1)(j-1)/N} s_i$ for $i = 1, \ldots, N$. Then $\rho_3 =< \xi|\Phi >$. □

By (i), we see that $[P(1)]R_{[\rho_1]} = [P(1)]R_{[\psi_{12}]+[\psi_{13}]} = [P(12)] \oplus [P(2)]$. $[\rho_1]^2 = 2[\iota] + [\alpha] + [\beta_1] + [\psi_{12}] + [\psi_{13}]$. Hence ρ_1 is self conjugate.

By (ii), we have $P(1) \circ \rho_2 \sim P(1) \circ \iota \oplus P(1) \circ \iota \oplus P(1) \circ \beta_1 \sim P(1) \oplus P(1) \oplus P(1; -1)$. Hence $P(1) \circ \rho_2 \sim P(1) \oplus P(11)$ by Theorem 4.1 (iii) and (iv).

By (iii), the following holds:

$$(5.3) \quad P(1) \circ \rho_3 \sim P(\underbrace{1 \cdots 1}_{N}).$$

In order to show this directly, we must prepare a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ which is $P(1)$ and check the action $\pi^{'} \equiv \pi\circ\rho_3$ on vectors in $\mathcal{H}$. Because the definition of ρ_3 is long when N is large, the computation of the action of $\pi^{'}(s_i)$ on $\mathcal{H}$ needs much computation. By using decomposition of ρ_3, we have $[P(1)]R_{[\rho_3]} = [P(1)]R_{[\gamma_{z_1}]+\cdots+[\gamma_{z_N}]} = [P(1)]R_{[\gamma_{z_1}]}\oplus\cdots\oplus[P(1)]R_{[\gamma_{z_N}]} = [P(1;z_1)]\oplus\cdots\oplus[P(1;z_N)] = [P(1\cdots 1)]$ by Theorem 4.1 (iii) and (iv). Hence we get (5.3) easier than direct computation.

5.3. Polynomial sectors arising from embeddings. Let $\mathcal{A}$ be a unital $*$-algebra and $M,N\geq 2$. For $\xi=(v_i)_{i=1}^M, \eta=(u_i)_{i=1}^M \in H_M\mathcal{A}$ and $g\in U(M)$, define

$$(\xi|g|\eta)\equiv\sum_{i,j=1}^{M} g_{ij}v_iu_j^*$$

Then $(\xi|g|\eta)$ is a unitary in $\mathcal{A}$. For $\eta,\xi\in H_M\mathcal{O}_N$ and $g\in U(M)$, put

$$\Theta_{\xi,g,\eta}(s_i)\equiv(\xi|g|\eta)\cdot s_i \quad (i=1,\ldots,N).$$

Then $\Theta_{\xi,g,\eta}\in \mathrm{End}\mathcal{O}_N$.

Lemma 5.9. *Let $\xi=(u_i)_{i=1}^M, \eta=(v_i)_{i=1}^M\in H_M\mathcal{O}_N$ and $g\in U(M)$. If*

$$g\in\mathfrak{S}_M \text{ and } u_i,v_i\in\{s_J: J\in\{1,\ldots,N\}_1^*\} \text{ for } i=1,\ldots,M, \tag{5.4}$$

then $\Theta_{\xi,g,\eta}$ transforms permutative representations of $\mathcal{O}_N$ to those, that is, if $(\mathcal{H},\pi)$ is a permutative representation of $\mathcal{O}_N$, then $(\mathcal{H},\pi\circ\Theta_{\xi,g,\eta})$ is a permutative representation, too.

Proof. By assumption, $\Theta_{\xi,g,\eta}(s_i)=\sum_{j=1}^M v_ju_{\sigma(j)}^*s_i$ for some permutation $g=\sigma\in\mathfrak{S}_M$. For a given permutative representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$, we see that $\Theta_{\xi,g,\eta}(s_i)$ transforms the canonical basis of $\mathcal{H}$ to itself for each $i=1,\ldots,N$. Hence the statement holds. □

Put $\mathcal{S}\equiv<\{[\Theta_{\xi,g,\eta}]:\xi,\eta\in H_M\mathcal{O}_N,\, g\in U(M)$ satisfy (5.4)$\}>$. Lemma 5.9 is interpreted that $(BP(\mathcal{O}_N),R|_{\mathcal{S}})$ is a $\mathcal{S}$-module. Examples of ξ,η in Lemma 5.9 are shown in Lemma 6.1.

Example 5.10. Let $\xi_1,\xi_2\in H_3\mathcal{O}_2$ and $g\in U(3)$. Denote $t_i\equiv\Theta_{\xi_1,g,\xi_2}(s_i)$ for $i=1,2$.

(i) In [**2**], we introduced the following examples: Put $\xi_1\equiv(s_1s_1,s_1s_2,s_2)$, $\xi_2\equiv(s_1,s_2s_2,s_2s_1)$ and $g=I$. Then

$$t_1=s_1s_1,\quad t_2=s_1s_2s_2^*+s_2s_1^*. \tag{5.5}$$

We have the following algebraic isomorphism: $<\{[\iota],[\Theta_{\xi_1,g,\xi_2}]\}>\cong \mathbf{N}[x]$ where $\mathbf{N}[x]$ is a set of all polynomials of a variable x with the coefficient set $\mathbf{N}$. Furthermore $\{\,[\Theta_{\xi_1,g,\xi_2}]^n\,\}_{n\geq1}$ is the set of mutually different, proper irreducible sectors.

Replace g by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then we see that

$$t_1 = s_1 s_1, \quad t_2 = s_1 s_2 s_1^* + s_2 s_2^*. \tag{5.6}$$

The branching law of Θ_{ξ_1,g,ξ_2} on a permutative representation with cycle always has infinite branches. Therefore endomorphisms of $\mathcal{O}_2$ associated with (5.5) and (5.6) are inequivalent.

(ii) Let $a, b \in \mathbf{R}$, $a^2 + b^2 = 1$. When

$$\xi_1 = \xi_2 = (s_1, s_2 s_2, s_2 s_1), \quad g = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$t_1 = a s_1 + b s_2 s_2, \quad t_2 = b s_1 s_2^* - a s_2 s_2 s_2^* + s_2 s_1 s_1^*.$$

Specially, when $a = 2/(1+\sqrt{5})$ and $b = \sqrt{a}$, this is the example $\rho \equiv \Theta_{\xi_1,g,\xi_2}$ in [**15**], p 21 which satisfies $[\rho]^2 = [\rho] + [\iota]$ and $[\mathcal{O}_2, \rho(\mathcal{O}_2)] = (3+\sqrt{5})/2$.

Example 5.11. $\rho \in \mathrm{End}\mathcal{O}_N$ in § 3, [**14**] is given as follows: Let G be an abelian group with $\#G = N$, $<\cdot,\cdot>: G \times G \to U(1)$ be a non-degenerate symmetric pairing of G and itself:$<g, h> = <h, g>$ is a character for each variable and $<g, h> = 1$ for all $h \in G$ implies $g = 0$, and U is a representation of G on $\mathcal{O}_N$. $\rho \in \mathrm{End}\mathcal{O}_N$ is given by

$$\rho(s_h) \equiv N^{-1/2} U_h \left(\sum_{k \in G} s_k \right) U_h^*, \quad U_h \equiv \sum_{k \in G} <h, k> s_k s_k^*.$$

Let $\{S_{k_1,k_2}\}_{k_1,k_2 \in G}$ be the set of the canonical generators of $\mathcal{O}_{N^2}$ and $\xi \equiv (u_{k_1,k_2})_{k_1,k_2 \in G}$ $\eta \equiv (v_{k_1,k_2})_{k_1,k_2 \in G} \in H_{N^2}\mathcal{O}_N$ by $u_{k_1,k_2} \equiv (\varphi \circ \alpha)(S_{k_1,k_2})$, $v_{k_1,k_2} \equiv (\varphi \circ \beta)(S_{k_1,k_2})$ where $\varphi \in \mathrm{Hom}(\mathcal{O}_{N^2}, \mathcal{O}_N)$ and $\alpha, \beta \in \mathrm{Aut}\mathcal{O}_{N^2}$ which are defined by

$$\varphi(S_{k,h}) \equiv s_k s_h,\ \alpha(S_{k,h}) \equiv N^{-1/2} \sum_{k' \in G} <k, k'> S_{k',h},\ \beta(S_{k,h}) \equiv <h, k> S_{k,h}.$$

Then we can verify that $\rho = \Theta_{\xi,I,\eta}$.

Example 5.12. $\rho \in \mathrm{End}\mathcal{O}_3$ in § 4, [**14**] is given by

$$\rho(s_1) \equiv 2^{-1}(s_1 + s_2) + 2^{-1/2} s_3 s_3, \quad \rho(s_2) \equiv U \rho(s_1) U^*,$$

$$\rho(s_3) \equiv 2^{-1/2} \bar{w}(s_1 - s_2) s_3^* + w s_3 (s_1 s_1^* - s_2 s_2^*)$$

where $U \equiv s_1 s_1^* + s_2 s_2^* - s_3 s_3^* \in \mathcal{O}_3$ and w is a complex number satisfying $w^3 = 1$. It is known that $[\rho]^2 = [\rho] + [\iota] + [\alpha]$ where $\alpha(s_1) \equiv s_2$, $\alpha(s_2) \equiv s_1$,

$\alpha(s_3) \equiv -s_3$ and ρ has the statistical dimension 2. Put $g = (g_{ij}) \in U(5)$ by

$$g = \begin{pmatrix} 2^{-1} & 2^{-1} & \bar{w}2^{-1/2} & 0 & 0 \\ 2^{-1} & 2^{-1} & -\bar{w}2^{-1/2} & 0 & 0 \\ 2^{-1/2} & -2^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & -w \end{pmatrix}$$

and $\xi \equiv (s_1, s_2, s_3s_3, s_3s_1, s_3s_2), \eta \equiv (s_1, s_2U, s_3s_3, s_3s_1, s_3s_2) \in H_5\mathcal{O}_3$. Then we see that $\rho = \Theta_{\xi,g,\eta}$.

6. Sectors arising from inclusions of C*-algebras

Inclusions of C*-algebras are studied by their indices and group actions([**15**]). We introduce another method to study of inclusions.

Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of unital C*-algebras with the inclusion map ι. If $\iota' : \mathcal{A}' \subset \mathcal{B}'$ is another inclusion of unital C*-algebras such that $\mathcal{A}' \cong \mathcal{A}$ and $\mathcal{B}' \cong \mathcal{B}$, then we see that the difference between ι and ι' is arrived at that of elements in Sect$(\mathcal{A}, \mathcal{B})$. If $x \equiv [\iota]$ $y \equiv [\iota']$ and there is $v \in$ BSpec$\mathcal{B}$ such that $vR_x \neq vR_y$, then $x \neq y$. In this way, the classification of inclusions is checked by comparison of their branching laws. Therefore the spectrum module of Sect$(\mathcal{A}, \mathcal{B})$ is important to consider inclusions of $\mathcal{A}$ to $\mathcal{B}$. We treat Sect$(\mathcal{O}_N, \mathcal{O}_M)$ and Sect$(UHF_N, \mathcal{O}_N)$ in this section.

6.1. Sect$(\mathcal{O}_M, \mathcal{O}_N)$**.** By Lemma 2.1, we see that Sect$(\mathcal{O}_M, \mathcal{O}_N) \neq \emptyset$ if and only if $^\exists l \geq 1$ s.t. $M = (N-1)l+1$. We review results in [**18**]. Let $s_1, \ldots, s_N$ be the canonical generators of $\mathcal{O}_N$. Assume that $M = (N-1)k + 1$, $k \geq 2$. Put

$$(6.1) \qquad \begin{cases} t_i \equiv s_i & (i = 1, \ldots, N-1), \\ t_{(N-1)l+i} \equiv (s_N)^l s_i & \begin{pmatrix} l = 1, \ldots, k-1, \\ i = 1, \ldots, N-1 \end{pmatrix}, \\ t_M \equiv (s_N)^k. \end{cases}$$

Then $(t_1, \ldots, t_M) \in H_M\mathcal{O}_N$. If $u_1, \ldots, u_M$ are the canonical generators of $\mathcal{O}_M$, then $\varphi_{M,N}(u_i) \equiv t_i$ for $i = 1, \ldots, M$ defines a unital embedding of $\mathcal{O}_M$ into $\mathcal{O}_N$. Hence $\varphi_{M,N} \in H_M\mathcal{O}_N$ and $[\varphi_{M,N}] \in$ Sect$(\mathcal{O}_M, \mathcal{O}_N)$.

Lemma 6.1. (i) *The embedding $\varphi_{M,N}$ of $\mathcal{O}_M$ into $\mathcal{O}_N$ in (*6.1*) transforms permutative representations of $\mathcal{O}_N$ to those of $\mathcal{O}_M$, that is, $R_{[\varphi_{M,N}]} \in$* Hom$(BP(\mathcal{O}_N), BP(\mathcal{O}_M))$.

(ii) *If $J \in \{1, \ldots, N\}_1^*$, then there are $1 \leq m < \infty$ and $\{J_i\}_{i=1}^m \subset \{1, \ldots, M\}_1^*$ such that $P(J) \circ \varphi_{M,N} = P(J_1) \oplus \cdots \oplus P(J_m)$, that is, $R_{[\varphi_{M,N}]} \in$* Hom$(< BP_*(\mathcal{O}_N) >, < BP_*(\mathcal{O}_M) >)$.

Proof. We denote $\varphi_{M,N}$ by φ simply. Assume that $s_1, \ldots, s_N$ and $u_1, \ldots, u_M$ are canonical generators of $\mathcal{O}_N$ and $\mathcal{O}_M$, respectively.
(i) Let $(\mathcal{H}, \pi)$ be a permutative representation of $\mathcal{O}_N$. Then we can realize $\mathcal{H} = l_2(\Lambda)$ for some set Λ. By assumption for each $i = 1, \ldots, N$ and $n \in \Lambda$, there is $m \in \Lambda$ such that $\pi(s_i)e_n = e_m$. For $i = 1, \ldots, M$, $\varphi(u_i)$ is a monomial of $s_1, \ldots, s_N$. Hence, for each $i = 1, \ldots, M$ and $n \in \Lambda$, there is $m^{'} \in \Lambda$ such that $(\pi \circ \varphi)(u_i)e_n = e_{m'}$. Therefore $\pi \circ \varphi$ is a permutative representation of $\mathcal{O}_M$.
(ii) Assume that $J \in \{1, \ldots, N\}^k$, $k \geq 1$. By Lemma 2.7 in [**21**], we can choose $P(J) = (l_2(\mathbf{N}), \pi)$ such that $\pi(s_i)e_n = e_{N(n-1)+i}$ for $any i = 1, \ldots, N$ and $n \geq k+1$. Denote $\pi^{'} \equiv \pi \circ \varphi$. Then we see that $\pi^{'}(u_i)e_n \in W \equiv \{e_{n'} : n^{'} \geq k+1\}$ for each $i = 1, \ldots, M$ when $n \geq k+1$. Then $\pi^{'}$ has neither chain nor cycle in $V \equiv \overline{\mathrm{Lin} < W >}$. Because $\dim V^{\perp} < \infty$, $\pi^{'}$ has finite number of cycles in $V^{\perp}$. In consequence $\pi^{'}$ has finite number of cycles in $l_2(\mathbf{N})$. Because of the completely reducibility of permutative representation, the statement holds. $\square$

For example, consider a case $(M, N) = (3, 2)$. Let $\phi_1, \phi_2 \in \mathrm{Hom}(\mathcal{O}_3, \mathcal{O}_2)$ be defined by

$$\phi_1(u_1) \equiv s_1, \quad \phi_1(u_2) \equiv s_2 s_1, \quad \phi_1(u_3) \equiv s_2 s_2,$$
$$\phi_2(u_1) \equiv s_1 s_1, \quad \phi_2(u_2) \equiv s_1 s_2, \quad \phi_2(u_3) \equiv s_2.$$

By Lemma 6.1 (ii) and the similar discussion, we see that for each $J \in \{1, 2\}_1^*$, there are $1 \leq m < \infty$ and $J_1, \ldots, J_m \in \{1, 2, 3\}_1^*$ such that $P(J) \circ \phi_i = P(J_1) \oplus \cdots \oplus P(J_m)$ for $i = 1, 2$.

Let $P(1; z) = (\mathcal{H}, \pi, \Omega)$ be a GP representation of $\mathcal{O}_2$. Let $\pi_i \equiv \pi \circ \phi_i$ for $i = 1, 2$. Then $\pi_i(u_1)\Omega = z^i \Omega$. From this and some discussion, we see that $P(1; z) \circ \phi_i = P(1; z^i)$ for $i = 1, 2$. Therefore $P(1; z) \circ \phi_1 \not\sim P(1; z) \circ \phi_2$. From this, $\phi_1 \not\sim \phi_2$ by Theorem 3.3 (iv). Hence $[\phi_1] \neq [\phi_2]$.

In the same way, we have concrete polynomial embeddings of the Cuntz-Krieger algebra $\mathcal{O}_A$ into $\mathcal{O}_N$ in [**18**]. Specially, $\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}_2) \neq \emptyset$ for any A. Therefore $\mathrm{Sect}(\mathcal{O}_A, \mathcal{O}_2)$ is always a semigroup.

6.2. $\mathrm{Sect}(UHF_N, \mathcal{O}_N)$**.** Under identification $UHF_N = \mathcal{O}_N^{U(1)}$, this canonical inclusion $\varphi_0 : UHF_N \hookrightarrow \mathcal{O}_N$ is in $\mathrm{Hom}(UHF_N, \mathcal{O}_N)$. By composing φ_0 and elements in $\mathrm{End}\mathcal{O}_N$, we have examples in $\mathrm{Hom}(UHF_N, \mathcal{O}_N)$. By branching laws arising from φ_0 in [**6**], elements in $\mathrm{Sect}(UHF_N, \mathcal{O}_N)$ are distinguished. We show explicit branching laws of permutative representations of $\mathcal{O}_N$ which is restricted on UHF_N.

Theorem 6.2. *Let $(\mathcal{H}, \pi, \Omega)$ be $P(J)$ for a non periodic $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$. Then there is the following irreducible decomposition:*

$$(\mathcal{H}, \pi|_{UHF_N}) = (V_1, \pi|_{UHF_N}) \oplus \cdots \oplus (V_k, \pi|_{UHF_N}) \tag{6.2}$$

where $V_i \equiv \overline{V_{i,0}}$, $V_{i,0} \equiv \mathrm{Lin} < \{\pi(s_{J'})e_i \in \mathcal{H} : J' \in \{1,\ldots,N\}^{kl},\, l \geq 0\} >$ *and* $e_i \equiv \pi(s_{j_i}\cdots s_{j_k})\Omega$ *for* $i = 1,\ldots,k$.

Proof. Here we denote $\pi(s_i)$ by s_i simply. We see that $s_{I_0}s^*_{J_0}e_j \in V_{j,0}$ for $j = 1,\ldots,k$ when $I_0, J_0 \in \{1,\ldots,N\}^*$ satisfy $|I_0| = |J_0|$. Therefore V_j is a UHF_N-module. If $x \in V_j$, then there is $J' \in \{1,\ldots,N\}^{kl}$ such that $< s^*_{J'}x|e_j > \neq 0$. We replace x by $s^*_{J'}x$. Then we have a decomposition $x = e_j + y$ where $< y|e_j >= 0$. From this, $T_n x \to e_j$ when $n \to \infty$ where $T_n \equiv s^n_J(s^n_J)^*$ for $n \in \mathbf{N}$ because J is non periodic. Hence $e_j \in UHF_N x$. In this way, V_j is an irreducible UHF_N-module. For $x_i \in V_i$ and $x_j \in V_j$, we can verify that $< s_{J'}e_i|s_{J''}e_j >= \delta_{J'J''} < e_i|e_j >= \delta_{J'J''}\delta_{ij}$. Therefore $V_1,\ldots,V_k$ are mutually orthogonal. Because $P(J)$ is an irreducible permutative representation, $\{s_{J'}e_1 : J \in \{1,\ldots,N\}^*\}$ is a complete orthonormal system of $\mathcal{H}$. If $|J'| = lk + j$, $0 \leq j \leq k-1$, then $s_{J'}s_{i_j,\ldots,i_k}e_1 \in V_j$. Therefore $\mathcal{H} = V_1 \oplus \cdots \oplus V_k$. □

We simply denote (6.2) by

$$P(j_1,\ldots,j_k)|_{UHF_N} = \bigoplus_{\sigma\in\mathbf{Z}_k} P[j_{\sigma(1)},\ldots,j_{\sigma(k)}]$$

where $P[j_{\sigma(1)},\ldots,j_{\sigma(k)}] \equiv (V_{\sigma(1)}, \pi|_{UHF_N})$. In consequence, we see that $R_{[\varphi_0]} \in \mathrm{Hom}(< P_*(\mathcal{O}_N) >, < \mathrm{Spec}UHF_N >)$. Specially $CAR \cong UHF_2 = \mathcal{O}_2^{U(1)}$ is treated in [**1**, **2**]. About $\mathrm{Sect}UHF_N$, see Theorem 4.5 (iv).

Acknowledgement: We would like to thank T. Nozawa for good lectures of super selection theory.

Appendix A. N-ary semigroups, N-ary algebras and their modules

A well-known generalization of semigroup, group, algebra and module is a *universal algebra*([**7**, **11**]). A universal algebra is a set S together with a system of N-ary operations for S; here N may vary. In order to explain an exotic algebraic structure of sector explicitly, we prepare several notions for universal algebra.

For a set S and $N \geq 2$, denote S^N the set of all N-tuples of elements from S. p is an *N-ary operation* on S if p is a map from S^N to S.

Definition A.1. *Let* $N \geq 2$ *and* S *be a non empty set.*

(i) *An N-ary operation p on S is N-arily associative if p satisfies* $p(y_1, x_{N+1},\ldots,x_{2N-1}) = p(x_1, y_2, x_{N+2},\ldots,x_{2N-1}) = \cdots = p(x_1,\ldots, x_{N-1}, y_N)$ *for each* $x_1,\ldots,x_{2N-1} \in S$ *where* $y_i \equiv p(x_i,\ldots,x_{N+i-1})$. *We call "N-arily associative" by "associative" simply.*

(ii) (S,p) *is an* N*-ary semigroup if* p *is an associative* N*-ary operation on* S*.*

(iii) *An* N*-ary semigroup* (S,p) *is* N*-arily commutative if* p *is completely symmetric, that is,* $p(x_{\sigma(1)},\ldots,x_{\sigma(N)}) = p(x_1,\ldots,x_N)$ *for each* $x_1,\ldots,$ $x_N \in S$ *and permutation* $\sigma \in \mathfrak{S}_N$*. We call "*$N$*-arily commutative" by "commutative" or abelian simply.*

We see that the 2-ary associativity is just the ordinary associativity of a binary operation and a 2-ary semigroup is an ordinary semigroup. The 2-ary commutativity is just the ordinary commutativity of a binary operation, too.

Lemma A.2. *If* (S,p) *is an* N*-ary semigroup, then* S *has an associative* $(N-1)k+1$*-ary operation for each* $k \geq 1$*.*

Proof. When $k = 1$, it is trivial. Fix $k \geq 2$. Define $p^{'}$ an $(N-1)k+1$-ary operation on S recursively as follows: For $x_1,\ldots,x_{(N-1)k+1} \in S$, put $y_1 \equiv p(x_1,\ldots,x_N)$, $y_{i+1} \equiv p(y_i, x_{(N-1)i+2},\ldots,x_{(N-1)(i+1)+1})$ for $i = 1,\ldots,k-1$ and $p^{'}(x_1,\ldots,x_{(N-1)k+1}) \equiv y_k$. Then we see that $p^{'}$ is an $(N-1)k+1$-arily associative $(N-1)k+1$-ary operation on S. □

Remark that the inverse of Lemma A.2 does not hold in general.

When (S,p) is a commutative N-ary semigroup, we simply denote

$$x_1 + \cdots + x_N = p(x_1,\ldots,x_N)$$

for $x_1,\ldots,x_N \in S$ for convenience. We see that

$$\begin{gathered}
x_{\sigma(1)} + \cdots + x_{\sigma(N)} = x_1 + \cdots + x_N, \\
(x_1 + \cdots + x_N) + x_{N+1} + \cdots + x_{2N-1} \\
= x_1 + (x_2 + \cdots + x_{N+1}) + x_{N+2} + \cdots + x_{2N-1} \\
= \cdots = x_1 + \cdots + x_{N-1} + (x_N + \cdots + x_{2N-1})
\end{gathered}$$

for $x_1,\ldots,x_{2N-1} \in S$ and $\sigma \in \mathfrak{S}_N$. Furthermore we denote $Nx \equiv p(x,\ldots,x)$ for $x \in S$. We see that if $x \in S$, then $((N-1)k+1)x$ in S for each $k \geq 1$. Therefore the N-ary subsemigroup of S generated by $x \in S$ is $\{x, Nx, (2N-1)x, (3N-2)x, \ldots\}$. We can denote $x_1 + \cdots + x_{(N-1)k+1}$ for $x_1,\ldots,x_{(N-1)k+1} \in S$.

Definition A.3. (i) (S,p,q) *is an* N*-ary prealgebra if* (S,p) *is an abelian* N*-ary semigroup and* (S,q) *is a semigroup such that the followings hold:*

$$\left\{\begin{array}{l}
q(y, p(x_1,\ldots,x_N)) = p(q(y,x_1),\ldots,q(y,x_N)), \\[2ex]
q(p(x_1,\ldots,x_N), y) = p(q(x_1,y),\ldots,q(x_N,y))
\end{array}\right. \tag{A.1}$$

for each $x_1, \ldots, x_N, y \in S$. In this case, p and q are called the sum and the product of S, respectively. We call an N-ary prealgebra by an N-ary algebra simply.

(ii) *An N-ary algebra (S, p, q) in (i) is unital if there is an element $I \in S$ such that I is a unit of (S, q), that is, $q(x, I) = q(I, x) = x$ for each $x \in S$.*

(iii) *S_0 is a subalgebra of an N-ary algebra (S, p, q) if S_0 is a subset of S which is closed under both p and q.*

(iv) *For a subset F of S, a subalgebra S_0 of an N-ary algebra (S, p, q) is generated by F if S_0 is the minimal subalgebra of S which contains F. In this case, we denote S_0 by $< F >$.*

For an N-ary algebra (S, p, q), we denote p by $+$ and $xy \equiv q(x, y)$ for $x, y \in S$ simply. Then we see that

$$y(x_1 + \cdots + x_N) = yx_1 + \cdots + yx_N, \quad (x_1 + \cdots + x_N)y = x_1y + \cdots + x_Ny$$

for each $x_1, \ldots, x_N, y \in S$. In this way, algebraic operations among the sum and the product of an N-ary algebra seem quite similar to those of an ordinary algebra except the following two points: (i) There is no inverse element in S with respect to the sum. When $N = 2$, it is sufficient to consider Grothendiek construction from abelian semigroup. However, it is no idea to consider the inverse in S when $N \geq 3$. (ii) It makes no sense to consider $x + y$ for $x, y \in S$ when $N \geq 3$.

Definition A.4. (i) *For two N-ary semigroups (S, p) and $(S^{'}, p^{'})$, φ is a homomorphism from (S, p) to $(S^{'}, p^{'})$ if φ is a map from S to $S^{'}$ such that $\varphi(p(x_1, \ldots, x_N)) = p^{'}(\varphi(x_1), \ldots, \varphi(x_N))$ for each $x_1, \ldots, x_N \in S$.*

(ii) *For two N-ary algebras (S, p, q) and $(S^{'}, p^{'}, q^{'})$, φ is a homomorphism from (S, p, q) to $(S^{'}, p^{'}, q^{'})$ if φ is an N-ary semigroup homomorphism from (S, p) to $(S^{'}, p^{'})$ and it is a semigroup homomorphism from (S, q) to $(S^{'}, q^{'})$.*

Definition A.5. (i) *(V, R) is a right module of an N-ary semigroup (S, p) if V is an abelian semigroup and there is a map R from $V \times S$ to V such that*

$$R(v + w, x) = R(v, x) + R(w, x),$$

$$R(v, p(x_1, \ldots, x_N)) = R(R(\cdots R(R(v, x_1), x_2), \ldots, x_{N-1}), x_N)$$

for each $x, x_1, \ldots, x_N \in S$ and $v, w \in V$.

(ii) *(V, R) is a right module of an N-ary algebra (S, p, q) if (V, R) is both a right module of an N-ary semigroup (S, p) and that of a semigroup (S, q).*

(iii) *A right module (V, R) of a unital N-ary algebra (S, p, q) is unital if $R(v, I) = v$ for each $v \in V$.*

(iv) *For a right module* (V, R) *of an* N*-ary algebra* (S, p, q)*,* V_0 *is a submodule of* (V, R) *if* V_0 *is a subsemigroup of* V *and* $R(v, x) \in V_0$ *for each* $(v, x) \in V_0 \times S$*.*

References

[1] M.Abe and K.Kawamura, *Recursive Fermion System in Cuntz Algebra. I —Embeddings of Fermion Algebra into Cuntz Algebra—*, Comm. Math. Phys. **228**,85-101 (2002).

[2] M.Abe and K.Kawamura, *Recursive Fermion System in Cuntz Algebra. II —Endomorphism, Automorphism and Branching of Representation—*, preprint RIMS-1362 (2002).

[3] H.Araki, Y.Kawahigashi and H.Kosaki edit., *Subfactors*, World Scientific Publishing Co.Pte.Ltd.(1994).

[4] H.Baumgärtel and M.Wollenberg, *Causal nets of operator algebras*, Akademie Verlag, 1992.

[5] B.Blackadar, *K-theory of operator algebras*, MSRI Publications, vol 5, Springer-Verlag, New York, 1986.

[6] O.Bratteli and P.E.T.Jorgensen, *Iterated function Systems and Permutation Representations of the Cuntz algebra*, Memories Amer. Math. Soc. No.663 (1999).

[7] P.M.Cohn, *Universal Algebra*, Harper and Row, New York, 1965.

[8] K.R.Davidson and D.R.Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras*, Math.Ann. 311, 275-303 (1998).

[9] K.R.Davidson and D.R.Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proc. London Math. Soc. (3) 78 (1999) 401-430.

[10] J.Fröhlich and T.Kerler, *Quantum Groups, Quantum Categories and Quantum Field Theory*, Lecture Note in Math. 1542, Springer-Verlag, Berlin Heidelberg, 1993.

[11] G.Grätzer, *Universal algebra*, 2nd edition, Springer-Verlag, New York, 1979.

[12] R. Haag, *Local Quantum Physics*, 2nd. edition, Springer-Verlag, 1996.

[13] M.Izumi, *The structure of sectors associated with Longo-Rehren inclusions I. general theory*, Comm.Math.Phys. **213** (2000) 127-179.

[14] M.Izumi, *The structure of sectors associated with Longo-Rehren inclusions II. Examples*, Rev.Math.Phys. **13**, No.5(2001)603-674.

[15] M.Izumi, *Inclusions of simple* C^**-algebras*, arXic:math.OA/0106117 v1 13 Jun 2001.

[16] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. I —Generalization of cycle type—*, preprint RIMS-1380 (2002).

[17] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. II —Irreducible decomposition of periodic cycle—*, preprint RIMS-1388 (2002).

[18] K.Kawamura, *Polynomial embedding of Cuntz-Krieger algebra into Cuntz algebra*, preprint RIMS-1391 (2003).

[19] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. III —Generalization of chain type—*, preprint RIMS-1423 (2003).

[20] K.Kawamura, *Polynomial endomorphisms of the Cuntz algebras arising from permutations. I —General theory—*, preprint RIMS-1427 (2003).

[21] K.Kawamura, *Polynomial endomorphisms of the Cuntz algebras arising from permutations. II —Branching laws of endomorphisms—*, preprint RIMS-1433 (2003).

[22] K.Kawamura, *Polynomial endomorphisms of the Cuntz algebras arising from permutations. III —Branching laws and automata—*, preprint RIMS-1442 (2004).

[23] R.Longo and K.H.Rehren, *Nets of subfactors*, Rev.Math.Phys. **7** (1995) 567-597.

[24] I.Ojima, *A unified scheme for generalized sectors based on selection criteria —Order parameters of symmetries and of thermality and physical meanings of adjunctions—*, preprint RIMS-1404 (2003).