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A Note on the Equivalence Between Substitutability and $\mathrm{M}^\natural$-convexity

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Abstract

The property of "substitutability" plays a key role in guaranteeing the existence of a stable solution in the stable marriage problem and its generalizations. On the other hand, the concept of $\mathrm{M}^\natural$-convexity, introduced by Murota–Shioura (1999) for functions defined over the integer lattice, enjoys a number of nice properties that are expected of "discrete convexity" and provides with a natural model of utility functions. In this note, we show that $\mathrm{M}^\natural$-convexity is characterized by two variants of substitutability.

Keywords: stable marriage, discrete optimization, convex function, matroid, submodular function

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1 Introduction

Since the pioneering work on the stable marriage problem by Gale–Shapley [7], various generalizations and extensions of the stable marriage model have been proposed in the literature (see [1, 2, 3, 4, 6, 14, 15], etc.), where the property of "substitutability" for preferences plays a key role in guaranteeing the existence of a stable solution. On the other hand, the concept of M-convexity, introduced by Murota [8, 9] for functions defined over the integer lattice, enjoys a number of nice properties that are expected of "discrete convexity;" subsequently, its variant called M$^\natural$-convexity was introduced by Murota–Shioura [11]. Whereas M$^\natural$-convex functions are conceptually equivalent to M-convex functions, the class of M$^\natural$-convex functions is strictly larger than that of M-convex functions. Furthermore, M$^\natural$-concave functions provide with a natural model of utility functions [10, 13, 16]. In particular, it is known that M$^\natural$-concavity is equivalent to the gross substitutes property, and that M$^\natural$-concavity implies submodularity. In this note, we discuss the close relationship between substitutability and M$^\natural$-convexity/M$^\natural$-concavity.

Recently, Eguchi–Fujishige–Tamura [3] extended the stable marriage model to the framework with preferences represented by M$^\natural$-concave utility functions, and showed the existence of a stable solution in their model (see also [2]). Their proof is based on the fact that M$^\natural$-convex functions $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfy the following properties:

(SC^1) $\forall z_1, z_2 \in \mathbf{Z}^V$ with $z_1 \geq z_2$ and $\arg\min\{f(x') \mid x' \leq z_2\} \neq \emptyset$,
$\forall x_1 \in \arg\min\{f(x') \mid x' \leq z_1\}$, $\exists x_2 \in \arg\min\{f(x') \mid x' \leq z_2\}$ such that $z_2 \wedge x_1 \leq x_2$,

(SC^2) $\forall z_1, z_2 \in \mathbf{Z}^V$ with $z_1 \geq z_2$ and $\arg\min\{f(x') \mid x' \leq z_1\} \neq \emptyset$,
$\forall x_2 \in \arg\min\{f(x') \mid x' \leq z_2\}$, $\exists x_1 \in \arg\min\{f(x') \mid x' \leq z_1\}$ such that $z_2 \wedge x_1 \leq x_2$,

where for $x, y \in \mathbf{R}^V$ the vector $x \wedge y \in \mathbf{R}^V$ is given by $(x \wedge y)(w) = \min\{x(w), y(w)\}$ $(w \in V)$. These properties can be regarded as substitutability for utility functions f; indeed, (SC^1) and (SC^2) can be seen as generalizations of substitutability (persistence) in the sense of Alkan–Gale [1] for the choice function $C(z) = \arg\min\{f(y) \mid y \leq z\}$.

Following the work by Eguchi–Fujishige–Tamura [3], Fujishige–Tamura [6] presented a common generalization of the stable marriage model and the assignment game model with M$^\natural$-concave utility functions. It is shown in [6] that the following properties of M$^\natural$-convex functions

(SC$^1_\mathrm{G}$) $\forall p \in \mathbf{R}^V$, $f[p]$ satisfies (SC^1),
(SC$^2_\mathrm{G}$) $\forall p \in \mathbf{R}^V$, $f[p]$ satisfies (SC^2),

which can be seen as stronger versions of substitutability (SC^1) and (SC^2), play a key role in the proof of the existence of a stable solution in this model, where for $p \in \mathbf{R}^V$ the function $f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$f[p](x) = f(x) + \sum_{w \in V} p(w)x(w) \qquad (x \in \mathbf{Z}^V).$$

The main aim of this note is to prove that each of (SC$^1_\mathrm{G}$) and (SC$^2_\mathrm{G}$) characterizes M$^\natural$-convexity of a function.

Theorem 1.1. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function such that the effective domain* $\operatorname{dom} f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$ *is bounded. Then,*

$$f \text{ is } M^\natural\text{-convex} \iff (\mathrm{SC}^1_\mathrm{G}) \iff (\mathrm{SC}^2_\mathrm{G}).$$

This theorem shows that M$^\natural$-concavity of utility functions is an essential assumption in the model of Fujishige–Tamura [6]. Combining Theorem 1.1 and the previous result [13, Theorem 11] clarifies the relationship between substitutability and the gross substitute property for utility functions. The equivalence in Theorem 1.1 was proven by Farooq–Tamura [5] in the special case where $\operatorname{dom} f \subseteq \{0, 1\}^V$, i.e., f is a set function. In this note, we give a proof for a more general case where $\operatorname{dom} f$ is bounded.

2 Preliminaries on M$^\natural$-convexity

In this section, we review the definition and fundamental properties of M$^\natural$-convex functions.

Throughout this paper, we assume that V is a nonempty finite set. The sets of reals and integers are denoted by $\mathbf{R}$ and by $\mathbf{Z}$, respectively. For a vector $x = (x(w) \mid w \in V) \in \mathbf{Z}^V$, we define

$$\operatorname{supp}^+(x) = \{w \in V \mid x(w) > 0\}, \quad \operatorname{supp}^-(x) = \{w \in V \mid x(w) < 0\},$$
$$\operatorname{supp}(x) = \{w \in V \mid x(w) \neq 0\},$$
$$\langle p, x\rangle = \sum_{w \in V} p(w)x(w) \quad (p \in \mathbf{R}^V), \qquad x(S) = \sum_{w \in S} x(w) \quad (S \subseteq V).$$

For any $u \in V$, the characteristic vector of u is denoted by χ_u ($\in \{0, 1\}^V$), i.e., $\chi_u(w) = 1$ if $w = u$ and $\chi_u(w) = 0$ otherwise. We also denote by χ_0 the zero vector. For $x, y \in \mathbf{Z}^V$ with $x \leq y$, we denote $[x, y]_\mathbf{Z} = \{z \in \mathbf{Z}^V \mid x \leq z \leq y\}$.

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. We denote the set of minimizers of f by $\arg\min f = \{x \in \mathbf{Z}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V)\}$, which can be the empty set. For a vector $z \in \mathbf{Z}^V$, we denote

$$X^*(f, z) = \arg\min\{f(x) \mid x \leq z\} \ (= \{x \in \mathbf{Z}^V \mid x \leq z,\ f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V \text{ with } y \leq z)\}).$$

We call a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *M$^\natural$-convex* if it satisfies $\operatorname{dom} f \neq \emptyset$ and (M$^\natural$-EXC):

(M$^\natural$-EXC) $\forall x, y \in \operatorname{dom} f$, $\forall u \in \operatorname{supp}^+(x-y)$, $\exists v \in \operatorname{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

See [11] for the original definition.

We also define the set version of M$^\natural$-convexity. A nonempty set $B \subseteq \mathbf{Z}^V$ is said to be *M$^\natural$-convex* if its indicator function $\delta_B : \mathbf{Z}^V \to \{0, +\infty\}$ defined by

$$\delta_B(x) = \begin{cases} 0 & \text{if } x \in B, \\ +\infty & \text{otherwise} \end{cases}$$

is M$^\natural$-convex. Equivalently, an M$^\natural$-convex set is defined as a nonempty set satisfying the exchange property (B$^\natural$-EXC$_\pm$):

(B$^\natural$-EXC$_\pm$) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v, w \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_w \in B$.

Theorem 2.1 ([11, 17]). *A nonempty set $B \subseteq \mathbf{Z}^V$ is M$^\natural$-convex if and only if it satisfies* (B$^\natural$-EXC$_\pm$).

An M$^\natural$-convex function with bounded effective domain can be characterized by the sets of minimizers.

Theorem 2.2 ([10, Theorem 6.30]). *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that* $\mathrm{dom}\, f$ *is bounded. Then, f is M$^\natural$-convex if and only if for each $p \in \mathbf{R}^V$ the set* $\arg\min f[p]$ *is M$^\natural$-convex.*

3 Proofs

The implications "f is M$^\natural$-convex $\Longrightarrow$ (SC$^1_\mathrm{G}$)" and "f is M$^\natural$-convex $\Longrightarrow$ (SC$^2_\mathrm{G}$)" are shown in [3, 5, 6] (see also Section 4).

Theorem 3.1. *An M$^\natural$-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies* (SC$^1_\mathrm{G}$) *and* (SC$^2_\mathrm{G}$).

In this section, we prove the implications "(SC$^2_\mathrm{G}$) $\Longrightarrow$ (SC$^1_\mathrm{G}$)" and "(SC$^1_\mathrm{G}$) $\Longrightarrow$ f is M$^\natural$-convex."

Theorem 3.2. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$.*
(i) *If f satisfies* (SC$^2_\mathrm{G}$)*, then f also satisfies* (SC$^1_\mathrm{G}$).
(ii) *Suppose that* $\mathrm{dom}\, f$ *is bounded. If f satisfies* (SC$^1_\mathrm{G}$)*, then f is M$^\natural$-convex.*

Combining Theorems 3.1 and 3.2 yields Theorem 1.1, our main result.

3.1 Proof of "(SC$^2_\mathrm{G}$) $\Longrightarrow$ (SC$^1_\mathrm{G}$)"

We prove Theorem 3.2 (i).

Suppose that f satisfies (SC$^2_\mathrm{G}$). Let $p \in \mathbf{R}^V$, $z_1, z_2 \in \mathbf{Z}^V$ be any vectors satisfying $z_1 \geq z_2$ and $X^*(f[p], z_2) \neq \emptyset$, and $x_1^* \in X^*(f[p], z_1)$. Also, let $x_2^* \in X^*(f[p], z_2)$ be a vector minimizing the cardinality of the set $\mathrm{supp}^+(x_2^* - x_1^*)$, and put $S^+ = \mathrm{supp}^+(x_2^* - x_1^*)$. We assume that x_2^* maximizes the value $x_2^*(V \setminus S^+)$ among all vectors $y \in X^*(f[p], z_2)$ with $\mathrm{supp}^+(y - x_1^*) = S^+$. We show that x_2^* satisfies the inequality $z_2 \wedge x_1^* \leq x_2^*$.

For $w \in S^+$, we have $\min\{z_2(w), x_1^*(w)\} = x_1^*(w) < x_2^*(w)$ since $x_1^*(w) < x_2^*(w) \leq z_2(w)$. Hence, it suffices to prove that

$$\min\{z_2(w), x_1^*(w)\} \leq x_2^*(w) \qquad (w \in V \setminus S^+). \tag{3.1}$$

To show this, we define $\widetilde{z}_1, \widetilde{z}_2 \in \mathbf{Z}^V$ by

$$\widetilde{z}_1 = x_1^* \vee x_2^*, \qquad \widetilde{z}_2 = (x_1^* \vee x_2^*) \wedge z_2.$$

For $i = 1, 2$, $x_i^* \in X^*(f[p], \widetilde{z}_i) \subseteq X^*(f[p], z_i)$ holds since $x_i^* \leq \widetilde{z}_i \leq z_i$. As shown below, there exists a vector $q \in \mathbf{R}^V$ satisfying the following conditions:

$$X^*(f[q], \widetilde{z}_1) \neq \emptyset, \text{ and } x(w) = x_1^*(w) \ (w \in V \setminus S^+) \text{ for all } x \in X^*(f[q], \widetilde{z}_1), \tag{3.2}$$

$$x_2^* \in X^*(f[q], \widetilde{z}_2). \tag{3.3}$$

Then, it follows from ($\mathrm{SC}^2_{\mathrm{G}}$) that there exists some $x \in X^*(f[q], \widetilde{z}_1)$ such that $x \wedge \widetilde{z}_2 \leq x_2^*$, implying

$$\min\{x_1^*(w), z_2(w)\} = \min\{x(w), \widetilde{z}_2(w)\} \leq x_2^*(w) \qquad (w \in V \setminus S^+),$$

where the equality is by (3.2) and the definition of $\widetilde{z}_2$. Hence, we have the desired inequality (3.1).

We now show that there exists a vector $q \in \mathbf{R}^V$ satisfying (3.2) and (3.3). Let k be a sufficiently large positive number such that $k > \widetilde{z}_1(w) - x_1^*(w)$ $(w \in S^+)$. Define $d \in \mathbf{R}^V$ by

$$d(w) = \begin{cases} \dfrac{1}{k|S^+|} & (w \in S^+), \\ 1 & (w \in V \setminus S^+). \end{cases}$$

For $i = 1, 2$, we define a value $\eta_i \in \mathbf{R}$ by

$$\eta_i \ = \ \max\{\langle d, x\rangle \mid x \in X^*(f[p], \widetilde{z}_i)\}.$$

Since the set $\widehat{Y}_i = \{y \in \mathbf{Z}^V \mid \langle d, y\rangle > \eta_i,\ y \leq \widetilde{z}_i\}$ is finite and satisfies $f[p](y) > f[p](x_i^*)$ $(y \in \widehat{Y}_i)$, we have

$$X^*(f[q], \widetilde{z}_i) = \{x \mid x \in X^*(f[p], \widetilde{z}_i),\ \langle d, x\rangle = \eta_i\} \qquad (i = 1, 2) \tag{3.4}$$

by putting $q = p - \varepsilon d$ with a sufficiently small positive number ε.

To show that the condition (3.2) holds, let $x \in X^*(f[q], \widetilde{z}_1)$. For $w \in V \setminus S^+$, we have $x(w) \leq \widetilde{z}_1(w) = x_1^*(w)$, implying $x(V \setminus S^+) - x_1^*(V \setminus S^+) \leq 0$. By (3.4), we have

$$\begin{aligned} 0 \ \leq \ \langle d, x\rangle - \langle d, x_1^*\rangle \ &= \ \frac{1}{k|S^+|} \sum_{w \in S^+} \{x(w) - x_1^*(w)\} + x(V \setminus S^+) - x_1^*(V \setminus S^+) \\ &\leq \ \frac{1}{k|S^+|} \sum_{w \in S^+} \{\widetilde{z}_1(w) - x_1^*(w)\} + x(V \setminus S^+) - x_1^*(V \setminus S^+). \end{aligned}$$

Since $(1/k|S^+|) \sum_{w \in S^+} \{\widetilde{z}_1(w) - x_1^*(w)\} < 1$ and $x(V \setminus S^+) - x_1^*(V \setminus S^+)$ is a nonpositive integer, we have $x(V \setminus S^+) - x_1^*(V \setminus S^+) = 0$, implying (3.2).

We next prove that the condition (3.3) holds. It suffices to show that $\langle d, y\rangle \leq \langle d, x_2^*\rangle$ for all $y \in X^*(f[p], \widetilde{z}_2)$. By the definition of $\widetilde{z}_2$, we have $y(S^+) \leq \widetilde{z}_2(S^+) = x_2^*(S^+)$ and $y(w) \leq \widetilde{z}_2(w) \leq x_1^*(w)$ $(w \in V \setminus S^+)$, where the latter implies $\mathrm{supp}^+(y - x_1^*) \subseteq S^+$. By the choice of x_2^*, it holds that $\mathrm{supp}^+(y - x_1^*) = S^+$ and $y(V \setminus S^+) \leq x_2^*(V \setminus S^+)$. Therefore,

$$\langle d, y\rangle - \langle d, x_2^*\rangle = \frac{y(S^+) - x_2^*(S^+)}{k|S^+|} + \{y(V \setminus S^+) - x_2^*(V \setminus S^+)\} \leq 0.$$

This concludes the proof of Theorem 3.2 (i).

3.2 Proof of “($\mathbf{SC}^1_{\mathbf{G}}$) $\Longrightarrow$ f is $\mathbf{M}^\natural$-convex”

We prove Theorem 3.2 (ii).

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that $\mathrm{dom}\, f$ is bounded, and suppose that f satisfies ($\mathrm{SC}^1_{\mathrm{G}}$). We prove the M$^\natural$-convexity of f by using Theorem 2.2, a characterization of M$^\natural$-convex functions by the sets of minimizers. Since $f[p]$ satisfies ($\mathrm{SC}^1_{\mathrm{G}}$) for all $p \in \mathbf{R}^V$, it suffices to show that $\arg\min f$ is an M$^\natural$-convex set. To prove the M$^\natural$-convexity of $\arg\min f$, we use Theorem 2.1; we first consider the case where $x \leq y$ or $x \geq y$ (Lemma 3.3), then the case where $x - y = \chi_s + \chi_u - \chi_r - \chi_t$ for some $r, s, t, u \in V \cup \{0\}$ (Lemmas 3.4, 3.6, 3.7), and finally the general case (Lemma 3.9).

Lemma 3.3. *For any $x, y \in \arg\min f$ with $x \leq y$, we have $[x, y]_{\mathbf{Z}} \subseteq \arg\min f$.*

Proof. We show that any $\widetilde{x} \in [x, y]_{\mathbf{Z}}$ is contained in $\arg\min f$. Since $y \in X^*(f, y)$ and $\widetilde{x} \leq y$, ($\mathrm{SC}^1_{\mathrm{G}}$) implies that there exists some $x_2 \in X^*(f, \widetilde{x})$ ($\subseteq \arg\min f$) such that $\widetilde{x} = \widetilde{x} \wedge y \leq x_2 \leq \widetilde{x}$, i.e., $x_2 = \widetilde{x}$. □

Lemma 3.4. *For any $x, y \in \arg\min f$ with $x - y = 2\chi_u - \chi_v$ for some distinct $u, v \in V$, we have $x - \chi_u, x - \chi_u + \chi_v \in \arg\min f$.*

Proof. We firstly prove that $x - \chi_u + \chi_v \in \arg\min f$. If $x + \chi_v \in \arg\min f$, then Lemma 3.3 implies $x - \chi_u + \chi_v \in \arg\min f$ since $x - \chi_u + \chi_v \in [y, x + \chi_v]_{\mathbf{Z}}$. Hence, we assume $x + \chi_v \not\in \arg\min f$. Let M be a sufficiently large positive number, and ε be a sufficiently small positive number. We define $p \in \mathbf{R}^V$ by

$$p(w) = \begin{cases} -2\varepsilon & \text{if } w = u, \\ -3\varepsilon & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Assume, to the contrary, that $x - \chi_u + \chi_v \not\in \arg\min f$. Then, we have $X^*(f[p], x - \chi_u + \chi_v) = \{y\}$ and $X^*(f[p], x + \chi_v) = \{x\}$. Since $x - \chi_u + \chi_v \leq x + \chi_v$, it follows from ($\mathrm{SC}^1_{\mathrm{G}}$) that $x - \chi_u = (x - \chi_u + \chi_v) \wedge x \leq y$, a contradiction since $x(u) - 1 > y(u)$. Hence, $x - \chi_u + \chi_v \in \arg\min f$ holds.

We then prove that $x - \chi_u \in \arg\min f$. If there exists some $x' \in \arg\min f$ with $x' \leq x - \chi_u$, then Lemma 3.3 implies $x - \chi_u \in \arg\min f$ since $x - \chi_u \in [x', x]_{\mathbf{Z}}$. Hence, we assume that there exists no such $x' \in \arg\min f$, and derive a contradiction. Put $x_* = x + \chi_v - \alpha_* \chi_v$ and $y_* = x + \chi_v - \beta_* \chi_u$, where

$$\alpha_* = \max\{\alpha \mid x + \chi_v - \alpha \chi_v \in \arg\min f\}, \qquad \beta_* = \max\{\beta \mid x + \chi_v - \beta \chi_u \in \arg\min f\}.$$

We define $\widehat{p} \in \mathbf{R}^V$ by

$$\widehat{p}(w) = \begin{cases} \varepsilon \alpha_* & \text{if } w = u, \\ \varepsilon(\beta_* + 1) & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Then, we have $X^*(f[\widehat{p}], x + \chi_v) = \{x_*\}$ and $X^*(f[\widehat{p}], x - \chi_u + \chi_v) = \{y_*\}$. By ($\mathrm{SC}^1_{\mathrm{G}}$), we have $x_* - \chi_u = (x - \chi_u + \chi_v) \wedge x_* \leq y_*$, a contradiction since $x_*(u) - 1 = x(u) - 1 > y(u) \geq y_*(u)$. □

Lemma 3.5. *Let $x, y \in \arg\min f$ be any distinct vectors with $x(V) \geq y(V)$. Suppose that there exists no $z \in \arg\min f$ satisfying $z \leq x \vee y$, $\mathrm{supp}(x - z) \subseteq \mathrm{supp}(x - y)$, and $z(V) > x(V)$. Then, for any $u \in \mathrm{supp}^+(x - y)$ there exists $v \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that $x - \chi_u + \chi_v \in \arg\min f$.*

Proof. Let $u \in \mathrm{supp}^+(x-y)$. Since $x \in X^*(f, x \vee y)$, it follows from ($\mathrm{SC}^1_{\mathrm{G}}$) that there exists some $x_2 \in X^*(f, (x \vee y) - \chi_u)$ ($\subseteq \arg\min f$) such that $((x \vee y) - \chi_u) \wedge x \leq x_2$. This inequality implies

$$\begin{aligned} & x_2(u) = x(u) - 1, \quad x_2(w) = x(w) \ (w \in V \setminus [\mathrm{supp}^-(x - y) \cup \{u\}]), \\ & x_2(w) \geq x(w) \ (w \in \mathrm{supp}^-(x - y)), \end{aligned}$$

from which follows $x(V) \geq x_2(V) \geq x(V) - 1$. Hence, $x_2 = x - \chi_u + \chi_v$ holds for some $v \in \mathrm{supp}^-(x - y) \cup \{0\}$. □

Lemma 3.6. *For any $x, y \in \arg\min f$ with $x - y = \chi_s + \chi_u - \chi_v$ for some distinct $s, u, v \in V$, we have $x - \chi_s + \chi_v, x - \chi_u \in \arg\min f$ or $x - \chi_u + \chi_v, x - \chi_s \in \arg\min f$ (or both).*

Proof. It suffices to show the following claims hold:

(a) $x - \chi_u + \chi_v \in \arg\min f$ or $x - \chi_u \in \arg\min f$,
(b) $x - \chi_s + \chi_v \in \arg\min f$ or $x - \chi_s \in \arg\min f$,
(c) $x - \chi_s + \chi_v \in \arg\min f$ or $x - \chi_u + \chi_v \in \arg\min f$,
(d) $x - \chi_s \in \arg\min f$ or $x - \chi_u \in \arg\min f$.

We firstly prove the claims (a) and (b). If $x + \chi_v \in \arg\min f$, then Lemma 3.3 implies $\{x - \chi_u + \chi_v, x - \chi_s + \chi_v\} \subseteq [y, x + \chi_v]_{\mathbf{Z}} \subseteq \arg\min f$. If $x + \chi_v \notin \arg\min f$, then Lemma 3.5 for x and y implies (a) and (b) since $\mathrm{supp}^-(x - y) = \{v\}$.

We then prove (c). Assume, to the contrary, that neither $x - \chi_s + \chi_v$ nor $x - \chi_u + \chi_v$ is in $\arg\min f$. Then, we have $x - \chi_u \in \arg\min f$ by (a). Since $x - \chi_u \leq x - \chi_u + \chi_v \leq x + \chi_v$, Lemma 3.3 implies $x + \chi_v \notin \arg\min f$. Put $z_1 = x + \chi_v$ and $z_2 = x - \chi_u + \chi_v$. Let M be a sufficiently large positive number, and ε be a sufficiently small positive number. We define $p \in \mathbf{R}^V$ by

$$p(w) = \begin{cases} -2\varepsilon & \text{if } w \in \{s, u\}, \\ -3\varepsilon & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Then, $X^*(f[p], z_1) = \{x\}$. By ($\mathrm{SC}^1_{\mathrm{G}}$), there exists some $x_2 \in X^*(f[p], z_2)$ with $x - \chi_u = z_2 \wedge x \leq x_2 \leq x - \chi_u + \chi_v$, i.e., x_2 is either $x - \chi_u$ or $x - \chi_u + \chi_v$. However, we have

$$\begin{aligned} f[p](x - \chi_u) - f[p](y) &= \varepsilon + f(x - \chi_u) - f(y) > 0, \\ f[p](x - \chi_u + \chi_v) - f[p](y) &= -2\varepsilon + f(x - \chi_u + \chi_v) - f(y) > 0 \end{aligned}$$

since $y \in \arg\min f$ and $x - \chi_u + \chi_v \notin \arg\min f$. This shows that $x_2 \notin X^*(f[p], z_2)$, a contradiction. Hence, the claim (c) holds.

We finally prove (d). Assume, to the contrary, that neither $x - \chi_s$ nor $x - \chi_u$ is in $\arg\min f$. Since $\{x, x - \chi_u + \chi_v, x - \chi_s + \chi_v\} \subseteq \arg\min f$ by (a) and (b), Lemma 3.4 implies $x - 2\chi_u + \chi_v, x - 2\chi_s + \chi_v, x - \chi_v \notin \arg\min f$. By Lemma 3.3, if $x' \in \mathbf{Z}^V$ satisfies at least one of the inequalities $x' \le x - \chi_u$, $x' \le x - \chi_s$, $x' \le x - \chi_v$, $x' \le x - 2\chi_u + \chi_v$, and $x' \le x - 2\chi_s + \chi_v$, then $x' \notin \arg\min f$. This shows that $\arg\min f \cap \{x' \mid x' \le z_1\} \subseteq \{x, y, x - \chi_u + \chi_v, x - \chi_s + \chi_v, x + \chi_v\}$, where $z_1 = x + \chi_v$. We define $\widehat{p} \in \mathbf{R}^V$ by

$$\widehat{p}(w) = \begin{cases} \varepsilon & \text{if } w \in \{s, u\}, \\ 3\varepsilon & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Then, we have $X^*(f[\widehat{p}], z_1) = \{x\}$ and $X^*(f[\widehat{p}], z_2) = \{y\}$, where $z_2 = x - \chi_u + \chi_v$. By ($\mathrm{SC}^1_{\mathrm{G}}$), we have $x - \chi_u = z_2 \wedge x \le y$, a contradiction since $x(s) > y(s)$. Hence, the claim (d) holds. □

Lemma 3.7. *Let $x, y \in \mathrm{dom}\, f$ be any vectors satisfying $||x - y||_1 = 4$ and $x(V) = y(V)$, and $u \in \mathrm{supp}^+(x - y)$. Then, there exist $v, w \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that $x - \chi_u + \chi_v, y + \chi_u - \chi_w \in \arg\min f$.*

Proof. Suppose that $y = x - \chi_s - \chi_u + \chi_r + \chi_t$ for some $r, s, t, u \in V$ with $\{s, u\} \cap \{r, t\} = \emptyset$. We show that $x - \chi_u + \chi_v \in \arg\min f$ and $y + \chi_u - \chi_w \in \arg\min f$ hold for some $v, w \in \{r, t, 0\}$.

We firstly consider the case where there exists some $z \in \arg\min f$ satisfying

$$z \le x \vee y, \qquad \mathrm{supp}(x - z) \subseteq \mathrm{supp}(x - y), \qquad z(V) > x(V). \tag{3.5}$$

This assumption implies

$$\{x + \chi_r,\ x + \chi_t,\ x + \chi_r + \chi_t,\ y + \chi_s,\ y + \chi_u\} \cap \arg\min f \neq \emptyset.$$

We first claim that $x + \chi_r \in \arg\min f$ or $x + \chi_t \in \arg\min f$ holds. If $x + \chi_r + \chi_t \in \arg\min f$, then Lemma 3.3 implies $\{x + \chi_r, x + \chi_t\} \subseteq \arg\min f$. If $y + \chi_u \in \arg\min f$, then Lemmas 3.4 and 3.6 for $y + \chi_u = x - \chi_s + \chi_r + \chi_t$ and x imply $x + \chi_r \in \arg\min f$ or $x + \chi_t \in \arg\min f$. The case where $y + \chi_s \in \arg\min f$ can be dealt with similarly.

We, w.l.o.g., assume that $x + \chi_r \in \arg\min f$. Lemmas 3.4 and 3.6 for $x + \chi_r = y + \chi_u + \chi_s - \chi_t$ and y imply $\{y + \chi_u, y + \chi_s - \chi_t\} \subseteq \arg\min f$ or $\{y + \chi_s, y + \chi_u - \chi_t\} \subseteq \arg\min f$. If the former holds, then we are done since $y + \chi_s - \chi_t = x - \chi_u + \chi_r$. If the latter holds, then we can apply Lemmas 3.4 and 3.6 to $y + \chi_s = x - \chi_u + \chi_r + \chi_t$ and x to obtain $x - \chi_u + \chi_r \in \arg\min f$ or $x - \chi_u + \chi_t \in \arg\min f$.

We then consider the case where there exists no $z \in \arg\min f$ satisfying (3.5). By Lemma 3.5, we have $x - \chi_u + \chi_v \in \arg\min f$ and $x - \chi_s + \chi_{v'} \in \arg\min f$ for some $v, v' \in \{r, t, 0\}$. If $v' \neq 0$, then we have $x - \chi_s + \chi_{v'} = y + \chi_u - \chi_w$ for some $w \in \{r, t\}$. If $v' = 0$, then we can apply Lemmas 3.4 and 3.6 to y and $x - \chi_s$ to obtain $y + \chi_u - \chi_r \in \arg\min f$ or $y + \chi_u - \chi_t \in \arg\min f$. □

Lemma 3.8. *Let $x, y, z \in \mathbf{Z}^V$ be any distinct vectors with $z \le x \vee y$ and $z(V) > \max\{x(V), y(V)\}$. Then, we have $||z - x||_1 < ||x - y||_1$ and $||z - y||_1 < ||x - y||_1$.*

Proof. We prove $||z - x||_1 < ||x - y||_1$ only. Put $S^+ = \mathrm{supp}^+(x - y)$, $C = \mathrm{supp}^-(x - z)$ ($\subseteq \mathrm{supp}^-(x - y)$), $D = \mathrm{supp}^-(x - y) \setminus C$, and $E = V \setminus \mathrm{supp}(x - y)$. Then,

$$\begin{aligned} ||x - y||_1 - ||x - z||_1 &= z(S^+ \cup D \cup E) + y(C \cup D) - y(S^+) - z(C) - 2x(D) - x(E) \\ &> 2[y(C) - z(C)] + 2[y(D) - x(D)] \ \ge\ 0, \end{aligned}$$

where the first inequality is by $z(V) > y(V)$ and $y(E) = x(E)$, and the second by $y(C) \geq z(C)$ and $y(D) \geq x(D)$. ☐

Lemma 3.9. $\arg\min f$ *satisfies* (B$^\natural$-EXC$_\pm$)*, i.e.,* $\arg\min f$ *is an M$^\natural$-convex set if it is nonempty.*

Proof. Let $x, y \in \arg\min f$ and $u \in \mathrm{supp}^+(x - y)$. We show by induction on $||x - y||_1$ that

$$x - \chi_u + \chi_v \in \arg\min f \qquad (\exists v \in \mathrm{supp}^-(x - y) \cup \{0\}), \tag{3.6}$$

$$y + \chi_u - \chi_w \in \arg\min f \qquad (\exists w \in \mathrm{supp}^-(x - y) \cup \{0\}). \tag{3.7}$$

By Lemmas 3.3, 3.4, and 3.6, we may assume $\mathrm{supp}^+(x-y) \neq \emptyset$, $\mathrm{supp}^-(x-y) \neq \emptyset$, and $||x-y||_1 \geq 4$.

We first claim that the following (3.8) or (3.9) holds:

$$x' = x - \chi_s + \chi_t \in \arg\min f \qquad (\exists s \in \mathrm{supp}^+(x - y),\ \exists t \in \mathrm{supp}^-(x - y) \cup \{0\}), \tag{3.8}$$

$$y' = y + \chi_i - \chi_j \in \arg\min f \qquad (\exists i \in \mathrm{supp}^+(x - y) \cup \{0\},\ \exists j \in \mathrm{supp}^-(x - y)). \tag{3.9}$$

If there exists no $z \in \arg\min f$ satisfying $z \leq x \vee y$, $\mathrm{supp}(x - z) \subseteq \mathrm{supp}(x - y)$, and $z(V) > \max\{x(V), y(V)\}$, then Lemma 3.5 implies (3.8) or (3.9) according as $x(V) \geq y(V)$ or $x(V) < y(V)$. Hence, we assume that such $z \in \arg\min f$ exists. We may also assume $z \neq x \vee y$, since otherwise $(x \vee y) - \chi_w \in \arg\min f$ ($\forall w \in \mathrm{supp}(x - y)$) holds by Lemma 3.3. Therefore, we have $\mathrm{supp}^+(x-z) \cap \mathrm{supp}^+(x-y) \neq \emptyset$ or $\mathrm{supp}^-(z-y) \cap \mathrm{supp}^-(x-y) \neq \emptyset$. Note that $||x-z||_1 < ||x-y||_1$ and $||y - z||_1 < ||x - y||_1$ by Lemma 3.8. If $\mathrm{supp}^+(x - z) \cap \mathrm{supp}^+(x - y) \neq \emptyset$, then the induction hypothesis for x and z implies $x - \chi_s + \chi_t \in \arg\min f$ for some $s \in \mathrm{supp}^+(x - z) \cap \mathrm{supp}^+(x - y)$ and $t \in \mathrm{supp}^-(x - z) \cup \{0\} \subseteq \mathrm{supp}^-(x - y) \cup \{0\}$, i.e., (3.8) holds. Similarly, (3.9) holds if $\mathrm{supp}^-(z - y) \cap \mathrm{supp}^-(x - y) \neq \emptyset$.

In the following, we assume that (3.8) holds; the case where (3.9) holds can be dealt with similarly and therefore the proof is omitted.

(Case 1: $\mathrm{supp}^+(x' - y) = \emptyset$) We have $\mathrm{supp}^+(x - y) = \{u\}$, implying $x' = x - \chi_u + \chi_t$ ($\exists t \in \mathrm{supp}^-(x-y) \cup \{0\}$), i.e., (3.6) holds. Since $x' \leq y$, it follows from Lemma 3.3 that $y - \chi_j \in \arg\min f$ for $j \in \mathrm{supp}^-(x' - y) \subseteq \mathrm{supp}^-(x - y)$. Since $||x - (y - \chi_j)||_1 < ||x - y||_1$ and $\mathrm{supp}^+(x - (y - \chi_j)) = \{u\}$, the induction hypothesis implies $(y - \chi_j) + \chi_u - \chi_h \in \arg\min f$ for some $h \in \mathrm{supp}^-(x - (y - \chi_j)) \cup \{0\} \subseteq \mathrm{supp}^-(x - y) \cup \{0\}$. If $h \neq 0$ then we apply Lemma 3.4 or 3.6 to $y - \chi_j + \chi_u - \chi_h$ and y to obtain $\{y + \chi_u - \chi_j, y + \chi_u - \chi_h\} \cap \arg\min f \neq \emptyset$, i.e., (3.7) holds.

(Case 2: $\mathrm{supp}^+(x'-y) \neq \emptyset$, $u \notin \mathrm{supp}^+(x'-y)$) Since $u \in \mathrm{supp}^+(x-y)$, we have $x' = x-\chi_u+\chi_t$ for some $t \in \mathrm{supp}^-(x - y) \cup \{0\}$, i.e., (3.6) holds. Since $||x' - y||_1 < ||x - y||_1$, the induction hypothesis for x' and y implies $\widetilde{y} = y + \chi_i - \chi_j \in \arg\min f$ for some $i \in \mathrm{supp}^+(x' - y) \subseteq \mathrm{supp}^+(x - y) \setminus \{u\}$ and $j \in \mathrm{supp}^-(x' - y) \cup \{0\} \subseteq \mathrm{supp}^-(x - y) \cup \{0\}$. Since $||x - \widetilde{y}||_1 < ||x - y||_1$, the induction hypothesis for x, $\widetilde{y}$, and $u \in \mathrm{supp}^+(x - \widetilde{y})$ implies $\widetilde{y} + \chi_u - \chi_h \in \arg\min f$ for some $h \in \mathrm{supp}^-(x - \widetilde{y}) \cup \{0\} \subseteq \mathrm{supp}^-(x - y) \cup \{0\}$. Applying Lemma 3.3, 3.4, 3.6, or 3.7 to $\widetilde{y} + \chi_u - \chi_h = y + \chi_i + \chi_u - \chi_j - \chi_h$ and y, we have $\{y + \chi_u - \chi_j, y + \chi_u - \chi_h\} \cap \arg\min f \neq \emptyset$, i.e., (3.7) holds.

(Case 3: $u \in \mathrm{supp}^+(x' - y)$) Since $||x' - y||_1 < ||x - y||_1$, the induction hypothesis for x', y, and $u \in \mathrm{supp}^+(x' - y)$ implies $y + \chi_u - \chi_w \in \arg\min f$ for some $w \in \mathrm{supp}^-(x' - y) \cup \{0\} \subseteq \mathrm{supp}^-(x - y) \cup \{0\}$, i.e., (3.7) holds. By using this fact we can show (3.6) in a similar way as in Case 2. ☐

4 Concluding Remarks

It is shown in [3, 5, 6] that M$^\natural$-convexity of a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ implies the properties (SC^1) and (SC^2). Theorem 3.1 is an immediate consequence of this fact since $f[p]$ is M$^\natural$-convex for any $p \in \mathbf{R}^V$ if f is M$^\natural$-convex. In fact, the properties (SC^1) and (SC^2) hold true under a weaker assumption than M$^\natural$-convexity. We call a function f *semistrictly quasi M$^\natural$-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies (SSQM$^\natural$):

(SSQM$^\natural$) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y) \cup \{0\}$:
(i) $f(x - \chi_u + \chi_v) \geq f(x) \Longrightarrow f(y + \chi_u - \chi_v) \leq f(y)$, and
(ii) $f(y + \chi_u - \chi_v) \geq f(y) \Longrightarrow f(x - \chi_u + \chi_v) \leq f(x)$.

It is easy to see that any M$^\natural$-convex function satisfies (SSQM$^\natural$). See [12] for more accounts on semistrictly quasi M$^\natural$-convex functions.

Theorem 4.1. *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with* (SSQM$^\natural$) *satisfies* (SC^1) *and* (SC^2).

Proof. We prove (SC^1) only; (SC^2) can be shown similarly and the proof is omitted.

Let $z_1, z_2 \in \mathbf{Z}^V$ be any vectors with $z_1 \geq z_2$ and $X^*(f, z_2) \neq \emptyset$. Also, let $x_1 \in X^*(f, z_1)$. We choose $x_2 \in X^*(f, z_2)$ minimizing the value $\sum\{x_1(w) - x_2(w) \mid w \in \mathrm{supp}^+((x_1 \wedge z_2) - x_2)\}$. Assume, to the contrary, that $\mathrm{supp}^+((x_1 \wedge z_2) - x_2) \neq \emptyset$. Let $u \in \mathrm{supp}^+((x_1 \wedge z_2) - x_2)$ ($\subseteq \mathrm{supp}^+(x_1 - x_2)$). By (SSQM$^\natural$), there exists $v \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that if $f(x_1 - \chi_u + \chi_v) \geq f(x_1)$ then $f(x_2 + \chi_u - \chi_v) \leq f(x_2)$. Since $x_1 - \chi_u + \chi_v \leq x_1 \vee x_2 \leq z_1$, we have $f(x_1 - \chi_u + \chi_v) \geq f(x_1)$. Hence, $f(x_2 + \chi_u - \chi_v) \leq f(x_2)$ follows. By the choice of u we have $x_2 + \chi_u - \chi_v \leq z_2$. This implies that $x_2 + \chi_u - \chi_v \in X^*(f, z_2)$, which contradicts the choice of x_2. Hence we have $x_1 \wedge z_2 \leq x_2$. $\square$

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