Bi-crystals and crystal $(GL(V), GL(W))$ duality

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1 Introduction

Consider the tensor product of finite dimensional vector spaces $V \otimes W$. We have an action of $GL(V)$ on $V \otimes W$ induced by standard action on V. Similarly the action of $GL(W)$ on W gives us an action on $V \otimes W$. These actions of $GL(V)$ and $GL(W)$ on $V \otimes W$ clearly commute with one another, so we have a joint action of $GL(V) \times GL(W)$ on $V \otimes W$.

Let $S^\bullet(V)$ and $\Lambda^\bullet(V)$ denote the symmetric and exterior algebras on V, respectively. It is standard that the action of $GL(V)$ gives rise to an action on $S^\bullet(V)$ and $\Lambda^\bullet(V)$ by algebra homomorphism. We can consider the restriction of the action of $GL(V \otimes W)$ on $S^\bullet(V \otimes W)$ (or $\Lambda^\bullet(V \otimes W)$) to $GL(V) \times GL(W)$. For this action we have explicit decompositions of $S^\bullet(V \otimes W)$ and $\Lambda^\bullet(V \otimes W)$ into irreducible modules (see, for example [5]):

$$S^\bullet(V \otimes W) \cong \sum_\lambda V_\lambda \otimes W_\lambda, \tag{1}$$

$$\Lambda^\bullet(V \otimes W) \cong \sum_\lambda V_\lambda \otimes W_{\lambda'}, \tag{2}$$

where summation is running over all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$ with at most n non-zero parts, λ' denotes the conjugate partition (i.e. the shape of the transpose Young diagram)[1], and V_λ denotes the irreducible representation of $GL(V)$ corresponding to λ.

The isomorphisms (1) and (2) are known as $(GL(V), GL(W))$-*duality*. We are going to present the crystal variants of these isomorphisms.

The notion of crystals was initiated by Kashiwara (see [8] and the literature cited there), which influences a lot in combinatorics and representation

[1]The conjugate partition λ' consists of λ_1 parts and there holds $\lambda'_j = \#\{i : \lambda_i \geq j\}$, $j = 1, \ldots, \lambda$.

theory. In [2] we invented the array model for A-type crystals. The set of integer $n \times m$ arrays[2], denoted by $\mathcal{A}(n,m)$, we endowed with three crystal structures of A_n-type, A_m-type and $A_{n\times m}$-type. The set $\mathcal{A}(n,m)$ as an $A_{n\times m}$-crystal corresponds to the representation $S^\bullet(V\otimes W)$ of $GL(V\otimes W)$. The A_n-type and A_m-type crystal structures on $\mathcal{A}(n,m)$ commute. This provides the bi-crystal structure on $\mathcal{A}(n,m)$ and the corresponding decomposition into irreducible bi-crystals takes the form

$$\mathcal{A}(n,m) \cong \sum_\lambda B_V(\lambda)\hat{\otimes} B_W(\lambda), \tag{3}$$

where $B_V(\lambda)$ denotes a crystal which corresponds to the irreducible representation V_λ of $GL(V)$, correspondingly $B_W(\lambda)$ denotes that for W ($n = \dim V$, $m = \dim W$).

This decomposition is the crystal version of $(GL(V), GL(W))$-duality.

On this bi-crystal way we obtain a bijection between the set of arrays $\mathcal{A}(n,m)$ and the set of pairs of semistandard Young tableaux. This bijection slightly differs from the well-known Robinson-Schensted-Knuth bijection. Specifically, one of the tableau (Q-symbol) has to be replaced by the Schützenberger involution to it. Let us note that in [3] it was established that the combinatorics of the crystal structure corresponding to the representation $V^{\otimes n}$ of $GL(V)$ is served by the Robinson-Schensted correspondence. The reason why, in this case, there is no needs to replace Q-symbol by its Schützenberger involution, is that there is no second crystal structure on the set of $\{0,1\}$-arrays with at most one 1 in each row, or there is no bi-crystal structure on the crystal corresponding to $V^{\otimes n}$. The case of $S^\bullet(V\times W)$ is more subtle, and this forces the modification of the RSK-correspondence.

The array model for A-type crystals allows to imbed normal A_n-crystals into $\mathcal{A}(n,\infty)$. Namely, the irreducible A_n-crystal $B_V(\lambda)$ has infinitely many isomorphic embeddings in $\mathcal{A}(n,\infty)$, and each such an embedding is characterized by a "highest weight vector", which, via our bijection, is identified to a semistandard Young tableau of shape λ in the alphabet $1,\ldots$. Because of this any A_n-type crystal might be imbedded in $\mathcal{A}(n,m)$ for an appropriate m in such a manner that each irreducible summand takes its own highest vector. Thus via such an embedding we can distinguish the isomorphic irreducible components of any normal A_n-crystal. Of course, there exist many isomorphic embeddings of the same crystal. On this way, we get a a tensor category $\mathcal{A}_n$ which is equivalent to the tensor category of normal

[2] We do not call them matrices, since we write them in the usual Cartesian coordinates, and do not add them.

crystals. Our conjecture is that the category $\mathcal{A}_n$ is a braided category indeed. Namely, we define a natural isomorphism $R : B_1 \otimes B_2 \to B_2 \otimes B_1$, B_1, $B_2 \in Ob(\mathcal{A}_n)$, and claim that R satisfies the Yang-Baxter equation. Note that this isomorphism is a kind of a generalization of the Schützenberger involution (see Section 5).

To $\Lambda^{\bullet}(V \otimes W)$ we associate the $A_{n\times m}$-crystal $\mathcal{B}(n, m)$, which is isomorphic (as a set) to the set of Boolean (or, equivalently, $\{0, 1\}$-) arrays. Despite $\mathcal{B}(n, m) \subset \mathcal{A}(n, m)$, the crystal structures, which make the set $\mathcal{B}(n, m)$ a bi-crystal, differ of what structures in $\mathcal{A}(n, m)$. In Section 6 we present details of the construction of commuting A_n- and A_m-crystal structures on $\mathcal{B}(n, m)$. The corresponding bi-crystal decomposition takes the form

$$\mathcal{B}(n, m) \cong \sum_{\lambda} B_V(\lambda) \hat{\otimes} B_W(\lambda'), \tag{4}$$

where λ' denotes the form of the conjugate diagram to λ. On this way, we obtain a bijection between the set $\mathcal{B}(n, m)$ and the set of pair of semi-standard Young tableaux of the conjugate shapes. This correspondence also differs from the Knuth correspondence ([4]) by inverting one of tableau by the Schützenberger involution.

Again any normal A_n-crystal might be imbedded into $\mathcal{B}(n, \infty)$. However, here we obtain a bijection between the highest weight vectors of irreducible crystals of the weight λ and semi-standard Young tableaux of the conjugate shape λ' in the alphabet $1, \ldots,$. This Boolean model provides us with another category $\mathcal{B}_n$ which is equivalent to the category of normal crystals.

"Commutative" versions of (3) and (4) (as well as (1) and (2)), i.e. understanding the isomorphisms as isomorphisms of sets, take the form of Cauchy type formulae, and might be served by the usual RSK- and Knuth correspondences. Bi-crystal isomorphisms force the modifications of the above correspondences.

The categories $\mathcal{A}_n$ and $\mathcal{B}_n$ provide several combinatorial interpretations of the coefficients of the decomposition into irreducibles the tensor product of irreducible crystals, the Littlewood-Richardson coefficients. In particular, we obtain the classical interpretation of LR-coefficients, as semi-standard skew tableaux with lattice reading, and two new characterizations (see Section 8).

The set of real-valued arrays $\mathcal{A}^{\mathbb{R}}(n, m)$ has the structure of continuous bi-crystal. Namely, we introduce the notion of continuous A_n-crystal and define two commuting continuous A_n- and A_m-structures on $\mathcal{A}^{\mathbb{R}}(n, m)$. The irreducible continuous A_n-crystal of the weight λ, $B^c_V(\lambda)$, is isomorphic as a set to the Gelfand-Ceitlin polytope $GC(\lambda)$. The bi-crystal decomposition of $\mathcal{A}^{\mathbb{R}}(n,m)$ into irreducibles takes the form

$$\mathcal{A}^{\mathbb{R}}(n,m) \cong \coprod_{\lambda} B_V^c(\lambda)\hat{\otimes} B_W^c(\lambda), \tag{5}$$

where the union is running over all vectors $\lambda \in W(\mathbb{R}^n)$, $W(\mathbb{R}^n) := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots, x_n\}$.

On this way, we obtain continuous variants of the modified RSK-correspondence and the Schützenberger involution.

An intriguing difference between continuous and usual A_n-crystals is that, in the continuous case, the crystal operations E_i^a, F_i^a, $a \geq 0$, $i = 1,\ldots,n-1$, infinitesimally satisfy the Verma relations, while the crystal operations E_i^a, F_i^a, $a = 1,2,\ldots$, $i = 1,\ldots,n-1$, do not satisfy the Verma relations.

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2 A-type crystals

The notion of crystals is initiated by Kashiwara (see [8] and the literature cited there).

We adopt the definition of A_n-crystals in a slightly different setting corresponding to the weight lattice of the reductive group $GL_n(\mathbb{C})$. In this case the weight lattice is the lattice $\mathbb{Z}^n$ of integer points of $\mathbb{R}^n$, the Weyl group is the symmetric group S_n acting on $\mathbb{R}^n$ by permuting the coordinates.

A_n-crystal is a (finite) set B endowed with operations $E_i : B \mapsto B$, $F_i : B \mapsto B$, $i = 1,\ldots,n-1$, and functions $\epsilon_i : B \to \mathbb{Z}$, $\phi_i : B \to \mathbb{Z}$, $i = 1,\ldots,n-1$, $wt : B \to \mathbb{Z}^n$, such that there holds

$$E_i(b) = b' \Leftrightarrow F_i(b') = b \text{ if } b \neq b'; \tag{6}$$

$$wt(E_i(b)) = wt(b) + e_i - e_{i+1} \text{ if } E_i(b) \neq b; \tag{7}$$

$$wt(b)_i - wt(b)_{i+1} = \phi_i(b) - \epsilon_i(b), \tag{8}$$

where e_i denotes the standard basis vector and $wt(b)_i$ denotes the i-th coordinate of the vector $wt(b)$.

Obviously, crystals due to our definition might be transformed into Kashiwara's crystal.

Crystals form a category with a *morphism* $f : B_1 \to B_2$ of crystals B_1 and B_2 being a mapping which commutes with the action of operations E_i, F_i, $i = 1, \ldots, n-1$, and the weight function, $wt(f(b)) = wt(b)$.

This definition of morphisms is a bit stronger than Kashiwara's one, since due to the Kashiwara definition, we might have $wt(f(b)) - wt(b) = k(1, \ldots, 1)$ with some $k \in \mathbb{Z}$.

An element b of a crystal B is said to be a *highest weight vector* if $E_i(b) = b$ for any $i = 1, \ldots, n-1$. An irreducible crystal contains a unique highest weight vector and the functions ϕ_i and ϵ_i of such a crystal B satisfy $\phi_i(b) = \max\{n : F_i^n(b) \neq F_i^{n-1}(b)$ and $\epsilon_i(b) = \max\{n : E_i^n(b) \neq E_i^{n-1}(b)$ for any $b \in B$ and $i = 1, \ldots, n-1$. *Normal crystals* are direct sums of irreducible crystals.

Crystals are nice combinatorial objects and due to the tradition they were treated using combinatorics of semi-standard Young tableaux [8]. However in [2] we demonstrated that main tools of combinatorics of Young tableaux, such as bumping procedure, jeu de taquin, Schützenberger's involution, plactic relations, might be considered as combinations of some crystal operations. Specifically, the crystal, corresponding to $S^\bullet(V \otimes W)$, has two commuting A_n- and A_m-type crystal structures. Any normal A_n-crystal might be embedded (via a crystal morphism) into this crystal with an appropriate W, and the basic combinatorial operations take forms of products of crystal operations with respect to the second (!) A_m-type crystal structure.

The involution

$$s_i(b) = \begin{cases} E_i^{-(\alpha_i, wt(b))} b & \text{if } (\alpha_i, wt(b)) \leq 0, \\ F_i^{(\alpha_i, wt(b))} b & \text{if } (\alpha_i, wt(b)) \geq 0 \end{cases}$$

defines the Weil group action (here the symmetric group S_n action). Namely, $s_i^2 = 1$, and the Coxeter-Moore relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ for $|i-j| \geq 2$ ([2, 8, 12]).

There are two Hecke algebra $H_n(0)$ actions on crystals. Namely, define the operations

$$\mathbf{E}_i(b) = E_i^{\epsilon_i(b)}(b),$$

$$\mathbf{F}_i(b) = F_i^{\phi_i(b)}(b).$$

Then $\mathbf{E}_i^2 = \mathbf{E}_i$ and the Coxeter-Moore relations hold true, similarly, $\mathbf{F}_i^2 = \mathbf{F}_i$ and the Coxeter-Moore relations hold true. In [8] these facts are proven using a decomposition theorem (Theorem 9.3.1) for crystals, in [2] is presented a pure combinatorial proof using commuting crystal structures.

The category of crystals is endowed with tensor product. Namely, as a set $B_1 \otimes B_2$ equals the set $B_1 \times B_2$, the weight function is equal to the sum of the weight functions, $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$, the operations E_i and F_i are set by the rule

$$E_i^a(b \otimes b') = E_i^{a'}(b) \otimes E_i^{a''}(b'), \tag{9}$$

$a'' = \min\{a, \phi_i(b) - \epsilon_i(b')\}$, $a' = a - a''$;

$$F_i^a(b \otimes b') = F_i^{\tilde{a}}(b) \otimes F_i^{\tilde{\tilde{a}}}(b'), \tag{10}$$

$\tilde{a} = \min\{a, \phi_i(b) - \epsilon_i(b')\}$, $\tilde{\tilde{a}} = a - \tilde{a}$.

3 Bi-crystals

In [2] we proved that two natural crystal structures of A_n-type and A_m-type on $\mathcal{A}(n,m)$ commute. These two structures naturally come via two crystal decompositions

$$\mathcal{A}(n,m) = (\mathcal{A}(n,1))^{\otimes \dim(W)} = (\mathcal{A}(1,m))^{\otimes \dim(V)},$$

where $\mathcal{A}(n,1)$, the set of one-row arrays (of length $n = \dim(V)$), denotes the crystal corresponding to the $GL(V)$ representation $S^\bullet(V)$, correspondingly, $\mathcal{A}(1,m)$ denotes a crystal corresponding to the $GL(W)$ representation $S^\cdot(W)$, that is the set of one-column arrays (the column is of length $\dim(W)$).

Let us endow $\mathcal{A}(n,1) \cong \mathbb{Z}_+^n$ with the crystal structure. For $a \in \mathcal{A}(n,1)$, we set $wt(a) = a$; $\phi_i(a) = a(i)$, $\epsilon_i(a) = a(i+1)$; $L_i(a) - a = e_i - e_{i+1}$ if $a(i+1) \neq 0$ and $L_i(a) = a$, otherwise, and $R_i(a) - a = e_{i+1} - e_i$ if $a(i) \neq 0$ and $R_i(a) = a$, otherwise. It is easy to see, that these operations and mapping endow $\mathcal{A}(n,1)$ with the A_n-crystal structure.

Now, the first decomposition

$$\mathcal{A}(n,m) = (\mathcal{A}(n,1))^{\otimes \dim(W)}$$

endows $\mathcal{A}(n,m)$ with the A_n-type crystal structure of the form of the tensor product of m copies of A_n-crystal $\mathcal{A}(n,1)$. We will precisely define the crystal action a bit later.

It is clear, that the second decomposition

$$\mathcal{A}(n,m) = (\mathcal{A}(1,m))^{\otimes \dim(V)}$$

endows $\mathcal{A}(n,m)$ with the A_m-type crystal structure of the form of the tensor product of n copies of A_m-crystal $\mathcal{A}(1,m)$[3]. The A_m-crystal structure on $\mathcal{A}(1,m)$ is defined using the transposition $t : \mathcal{A}(1,m) \to \mathcal{A}(m,1)$.

In [2] we proved the following

Theorem 1. The two above defined crystal structures commutes.

Here are some comments to the proof.

Firstly, we explicitly define the action of the two types crystal operations (recall, that the column's sums is the weight functions for the A_n-type structure, and the row's sums is that for the A_m-type and the functions ϕ_i and ϵ_i are specified as for the normal crystals). Column-wise operations for A_n-type crystal operations we denote L_i and R_i, $i = 1, \ldots, n-1$, respectively: the action of the operator L_i sends an array a to the array $L_i(a)$ which either differs from a only in two adjacent places (i,j) and $(i+1,j)$, $L_i(a)(i,j) = a(i,j)+1$, $L_i(a)(i+1,j) = a(i+1,j)-1$ (of course, the tensor product entry j is determined by the array a), or $L_i(a) = a$.

Row-wise operations for A_m-type crystal operations, we denote D_j and U_j, $j = 1, \ldots, m-1$, respectively: the action of the operator D_j sends an array a to the array $D_j(a)$ which either differs from a only in two adjacent places (i,j) and $(i,j+1)$, $D_j(a)(i,j) = a(i,j)+1$, $D_j(a)(i,j+1) = a(i,j+1)-1$ (of course, the tensor product entry i is determined by the array a), or $D_j(a) = a$.

Operations D_j and U_j might be defined as $D_j(a) = (L_j(a^t))^t$, where a^t denotes the transposition, $a^t(i,j) = a(j,i)$.

Thus, we will specify the definition of these operation action by defining the operations L_1, R_1 on the two-column arrays. Consider an array a

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,2) & a(2,2) \\ a(1,1) & a(2,1) \end{pmatrix}$$

In order to define the action of L_1, consider the following function $l_a : [m] \to$

[3] The double crystal structure on the set of bi-words introduced by Lascoux ([10]) differs from this structure.

$\mathbb{Z}$,

$$l_a(j) = a(2,1) + \sum_{j'=2}^{j} (a(2,j') - a(1,j'-1)).$$

Denote j^* the smallest element of the set Argmax $l_a(j)$. If $f_a(j^*) = 0$, then $L_1(a) = a$, otherwise $L_1(a)$ takes the form

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,j^*+1) & a(2,j^*+1) \\ a(1,j^*)+1 & a(2,j^*)-1 \\ a(1,j^*-1) & a(2,j^*-1) \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

In order to define the action of R_1, consider a function $r_a : [m] \to \mathbb{Z}$,

$$r_a(j) = a(1,m) + \sum_{j'=j}^{m-1} (a(1,j') - a(2,j'+1)).$$

Denote $\hat{j}$ the smallest element of the set Argmax $r_a(j)$. If $r_a(\hat{j}) = 0$, then $R_1(a) = a$, otherwise $R_1(a)$ takes the form

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,\hat{j}+1) & a(2,\hat{j}+1) \\ a(1,\hat{j})-1 & a(2,\hat{j})+1 \\ a(1,\hat{j}-1) & a(2,\hat{j}-1) \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

These operations endow the set of two columns arrays with the A_2-type crystal structure ([2]).

The following property of the above defined functions play an important role for proving the theorem: firstly, $a(2,j^*) > a(1,j^*-1)$; secondly, if

$a(2,j^*) > a(1,j^*) - 1$, then, for the array

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,j^*+1) & a(2,j^*+1) \\ a(1,j^*) & a(2,j^*)-1 \\ a(1,j^*-1) & a(2,j^*-1)+1 \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

L_1 acts at the j^*-s row; finally, if $a(2,j^*) = a(1,j^*) - 1$, then, for the above array, L_1 acts at the $j^* - 1$ row.

Now we have two decompositions of $\mathcal{A}(n,m)$: the first one consists of irreducible A_n-crystals, i.e., connected orbits under L_i, R_i actions, and the second one consists of A_m-crystals, connected orbits under D_j, U_j actions.

The "highest weight vectors" in $\mathcal{A}(n,m)$ under A_n-crystal structure, i.e. arrays $a \in \mathcal{A}(n,m)$ such that $L_i(a) = a$ for any $i = 1,\dots,n-1$, are called **L**-tight. Correspondingly, "highest weight vectors" in $\mathcal{A}(n,m)$ under A_m-crystal structure, i.e. arrays $a \in \mathcal{A}(n,m)$ such that $D_j(a) = a$ for any $j = 1,\dots,m-1$, are called **D**-tight. For each array a there exists a unique **L**-tight array in the orbit through a under actions L_i, R_i, $i = 1,\dots,n-1$ (correspondingly, a unique **D**-tight array in the D_j, U_j-orbit). This follows from the Coxeter-Moore relations among crystal operations $\mathbf{L}_i := L_i^\infty$, $i = 1,\dots,n-1$ ([2]).

As a corollary of the following proposition, we obtain that an irreducible crystal with the highest weight vector of weight $\lambda \in \mathbb{Z}^n$ as a set is isomorphic to the set of semi-standard Young tableaux of shape λ, as it has to be ([3, 8]).

Proposition 1. 1) There is one-to-one correspondence between the set of **L**-tight arrays in $\mathcal{A}(n,m)$ and the set of semi-standard Young tableaux with at most n rows and filled from an the alphabet on m letters.

2) There is one-to-one correspondence between the set of **D**-tight arrays in $\mathcal{A}(n,m)$ and the set of semi-standard Young tableaux with at most m rows and filled from an n letters alphabet.

For proof see [2].

Here we explain this proposition by an example.

Example.

Consider the following $\mathbf{L}$-tight array in $\mathcal{A}(3,4)$

$$\begin{pmatrix} 3 & 4 & 3 \\ 2 & 0 & 0 \\ 1 & 4 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

To get a semi-standard Young tableaux we have to read this array from bottom to top and from left to right. Reading each column gives us filling of a corresponding row in the Young tableau, the content $a(i,j)$ is exactly the number of repetitions of the letters j in the i-th row of the Young tableaux (we consider the French style of drawing Young diagrams and tableaux, that is the Young diagram for a partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ is a collection of boxes in the grid $\mathbb{N} \times \mathbb{N}$ with north-east corners (i,j) such that $j \leq n$, $i \leq \lambda_j$, and a Young tableau is a filling of the diagram from some alphabet increasingly along each row (from left to right) and strictly increasing from bottom to top). Thus, for the above array, we get

$$\begin{matrix} 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \end{matrix}$$

Given an array a, we denote $\mathbf{L}(a)$ the tight array which is obtained by the rule $\mathbf{L}_{n-1} \ldots (\mathbf{L}_2 \ldots \mathbf{L}_{n-1}(\mathbf{L}_1\mathbf{L}_2 \ldots \mathbf{L}_{n-1}(a)))$. Because of the Coxeter-Moore relations between $\mathbf{L}_i$, $\mathbf{L}(a)$ does not depend on a sequence of applications of the operations L_i, $i = 1, \ldots, n-1$, in order to get an $\mathbf{L}$-tight array from a. We denote $\mathbf{D}(a)$ the $\mathbf{D}$-tight array $\mathbf{D}_{m-1} \ldots (\mathbf{D}_2 \ldots \mathbf{D}_{m-1}(\mathbf{D}_1\mathbf{D}_2 \ldots \mathbf{D}_{m-1}(a)))$.

Because the operations L_i and D_j commute, for any array $a \in \mathcal{A}(n,m)$, the semi-standard Young tableaux, which correspond to the tight arrays $\mathbf{L}(a)$ and $\mathbf{D}(a)$, have the same shapes.

Denote by $\mathbf{L}\mathcal{A}$, $\mathbf{D}\mathcal{A}$ and $\mathbf{P}\mathcal{A}$ the set of $\mathbf{L}$-tight arrays, $\mathbf{D}$-tight arrays and $\mathbf{LD}$-tight (or perfect arrays), respectively. Perfect arrays might have non-zero entries at the diagonal only (they correspond to the Yamanouchi tableau). Thus, we get a mapping

$$(\mathbf{L}, \mathbf{D}) : \mathcal{A} \to \mathbf{L}\mathcal{A} \times_{\mathbf{LD}\mathcal{A}} \mathbf{D}\mathcal{A} \tag{11}$$

which sends the set of arrays $\mathcal{A}$ into the fiber product $\mathbf{L}\mathcal{A}$ and $\mathbf{D}\mathcal{A}$ over $\mathbf{P}\mathcal{A}$ (we have in mind the natural mappings $\mathbf{D} : \mathbf{L}\mathcal{A} \to \mathbf{DL}\mathcal{A}$ $\mathbf{L} : \mathbf{D}\mathcal{A} \to \mathbf{LD}\mathcal{A}$).

The mapping $(\mathbf{L}, \mathbf{D})$ agrees with the crystal operations L_i, D_j, if we set them by the rule. Let (l,d) be a point of the fiber product, i.e. l is an

L-tight array, d is an **D**-tight array and there holds $\mathbf{D}l = \mathbf{L}d$. Then, we set

$$L_i(l, d) = (L_i l, L_i d) = (l, L_i d).$$

In [2], we proved that (11) is a bijection indeed. It is slightly differs from the well-known RSK correspondence (see, for example, [4]). Namely, the semi-standard tableaux for $\mathbf{D}(a)$ coincides with the P-symbol of a, while the semi-standard tableaux for $\mathbf{L}(a)$ is the Schützenberger involution to the Q-symbol of a.

Resume. We want to point out that there is no bi-crystal structure on $\mathcal{A}$, which agrees with the tensor product and corresponds to the classical RSK.

4 Irreducible bi-crystals

Let us consider irreducible bi-crystals of $\mathcal{A}(n, m)$ and when decompose of the set of arrays as a sum of such irreducible ones.

An irreducible bi-crystal takes the form of orbit of an array a under the operations L_i, R_i, $i = 1, \dots, n-1$, and D_j, U_j, $j = 1, \dots, m-1$. Because these pairs of operators commute, this set is determined by the shape of a, that is irreducible bi-crystals of $\mathcal{A}(n, m)$ are in one-to-one correspondence with the set of partitions with at most $\min(n, m)$ non-zero parts.

In fact, since the pairs of the operators L_i, R_i, $i = 1, \dots, n-1$, and D_j, U_j, $j = 1, \dots, m-1$, commute, we can first consider the orbit of a under the pair of operations L_i, R_i, $i = 1, \dots, n-1$, and than to consider orbits of all element of this orbit under the pair of operations D_j, U_j, $j = 1, \dots, m-1$. Obviously, this bi-orbit contains a unique perfect array $\mathbf{D}(\mathbf{L}(a))$ (which is, of course, equals $\mathbf{L}(\mathbf{D}(a))$). Let λ be the diagonal of this perfect array (obviously, λ is a partition). Now, consider the orbit of this perfect array under the action of operations R_i, $i = 1, \dots, n-1$, as a result we get the collection of semi-standard Young tableaux of shape λ filled from the alphabet $1 < 2 < \dots, < n$. Thus, we get the crystal $B_V(\lambda)$ for the irreducible representation of $GL(V)$ of weight λ.

Now, considering the crystal $B_V(\lambda) \subset \mathcal{A}(n, m)$ as a "point", we get that the orbit of this "point" under the action of the operations U_j, $j = 1, \dots, m-1$, is isomorphic to the set of semi-standard Young tableaux of shape λ filled from the alphabet $1 < 2 < \dots, m$, and moreover it is isomorphic to the crystal $B_W(\lambda)$ for the irreducible representation of $GL(W)$ of the highest weight λ.

Thus, the bi-orbit of a, an irreducible bi-crystal, takes the form of the exterior tensor product

$$B_V(\lambda)\hat{\otimes}B_W(\lambda).$$

We get the following bi-crystal multiplicity-free decomposition

$$\mathcal{A}(n,m) \cong \sum_{\lambda} B_V(\lambda)\hat{\otimes}B_W(\lambda). \tag{12}$$

The decomposition (12) is the crystal version of the Howe $GL(V), GL(W)$-duality.

5 Combinatorial R-matrix and Schützenberger's involution

According to Theorem 9.3.1 in [8], any normal A_n crystal might be embedded into $\mathcal{A}(n,m)$ for an appropriate m. In fact, let B be a normal crystal and let $\lambda(1), \lambda(2), \dots, \lambda(k)$ be a tuple of the highest weights of the irreducible components of B, that is a tuple of partitions. Let us pick a tuple without repetitions of semi-standard Young tableaux of shapes $\lambda(1), \lambda(2), \dots, \lambda(k)$, obviously we can always do that with an appropriate alphabet $1 < 2 < \dots < m$. Now we consider the **L**-tight arrays corresponding to these tableaux, than we consider the orbits of these arrays under the action of operations R_i, $i = 1, \dots, n-1$. The resulting set of arrays provide an embedding of B and this embedding is a crystal isomorphism. Of course, such an embedding is not unique.

Now, we specify a category $\mathcal{A}_n$ which corresponds to crystals of $\mathcal{A}(n,\infty)$.

The set $Ob(\mathcal{A}_n)$ of objects of the category is constituted of finite subsets of $\mathcal{A}(n,\infty)$ stable under actions of operations L_i, R_i, $i = 1, \dots, n-1$. We will consider such sets modulo the following equivalence: two objects B and $B' \in Ob(\mathcal{A}_n)$ are equivalent if there exists $B'' \in \mathcal{A}$, such that B and B' might be obtained by inserting some zero rows to B''. The set of morphisms $Mor(B', B'')$, B', $B'' \in Ob(\mathcal{A}_n)$ consists of all crystal morphisms from B' to B''. One can check that we obtain a category indeed. (Note, that, due to Proposition 1, the set of objects of the category $\mathcal{A}_n$, is in one-to-one correspondence to the set of finite tuples without repetitions of semi-standard Young tableaux whose diagrams have at most n rows. A morphism of two such tuples is a mapping $h : \{\Lambda_1, \dots\} \to \{\Lambda'_1, \dots\}$ such that the shape of Λ_i coincides with the shape of $h(\Lambda_i)$.)

The zero element of this category is the zero array.

The category $\mathcal{A}_n$ is a tensor category. Namely, let B, $B' \in Ob(\mathcal{A}_n)$, then $B \otimes B'$ is a subset of $\mathcal{A}(n,\infty)$ obtained by putting B' on the top of B, that is let $B \subset \mathcal{A}(n,m)$, for an appropriate m, and $B' \subset \mathcal{A}(n,m')$, for some m', then $B \otimes B' \subset A(n, m+m')$ and the elements of the tensor product are $n \times (m+m')$ arrays of the form of concatenation of $n \times m$ arrays of B and $n \times m'$ arrays of B'.

Now we define an involution on the category $* : \mathcal{A}_n \to \mathcal{A}_n$. Namely, let $B \in Ob(\mathcal{A}_n)$ be an object of the category, that is an invariant (under L_i, R_i, $i = 1,\dots,n-1$) finite subset of $\mathcal{A}(n,\infty)$, and let m be minimal integer such that $B \subset \mathcal{A}(n,m)$. Then we define $*B \subset \mathcal{A}(n,m)$ to be a set constituted of the arrays centrally symmetric to arrays of B, that is, to $a \in B$ is corresponded the centrally symmetric array $a^* \in *B$, such that $a^*(i,j) = a(n-i+1, m-j+1)$, $i = 1,\dots,n$, $j = 1,\dots,m$.

Lemma. Let B be an invariant finite subset of $\mathcal{A}(n,\infty)$. Then $*B$ is an invariant subset of $\mathcal{A}(n,\infty)$.

Proof. One can check that there hold $L_i(a^*) = (R_{n-i}(a))^*$ and $R_i(a^*) = (L_{n-i}(a))^*$. Q.E.D.

Note, that the mapping $B \to *B$, $a \to a^*$, $a \in B \subset \mathcal{A}(n,m)$, is not a crystal morphism. For example, let $a \in B$ be a highest weight vector (i.e. $L_i a = a$, $i = 1,\dots,n-1$), then $R_i(a^*) = a^*$, $i = 1,\dots,n-1$.

Let us define a crystal isomorphism $S : B \to *B$. This isomorphism might be seen as a generalization of the Schützenberger involution. Namely, let $a \in B$ and let w be an *effective word* for a, that is a word $L_{i_s} \dots L_{i_1}$ in the alphabet $L_1,\dots,L_{n-1}$ such that

1) $wa = \mathbf{L}(a)$;

2) for any $t = 1,\dots,s-1$, $L_{i_t} \dots L_{i_1}\, a \neq L_{i_{t+1}} L_{i_t} \dots L_{i_1}\, a$.

Then the reading w from left to right and simultaneous replacing L_i by R_i, $i = 1,\dots,n-1$, produce the word $w' = R_{i_1} \dots R_{i_s}$ in the alphabet $R_1,\dots,R_{n-1}$, such that $a = w'(\mathbf{L}(a))$. Then the mapping $a \to w'(\mathbf{L}(a^*))$ is a crystal isomorphism, which we will denote $S : B \to *B$. (Note that $S(a)$ does not depend on the effective word for a.)

Example. Let

$$a = \begin{pmatrix} 5 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & 3 & 0 \\ 2 & 1 & 3 \end{pmatrix}, \text{ then } L_1^3 L_2^5 L_1^3 \text{ is a reading word for } a \text{ and}$$

$$S(a) = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 1 \\ 8 & 0 & 1 \\ 0 & 3 & 3 \end{pmatrix}.$$

Note that this crystal isomorphism $S : \mathcal{A}(n,m) \to \mathcal{A}(n,m)$ sends an **L**-tight array a to the **L**-tight array $\mathbf{L}(a^*)$. Using the bijection between **L**-tight arrays of $\mathcal{A}(n,m)$ and semi-standard Young tableaux in the alphabet $\{1,\ldots,m\}$ with at most n rows (Proposition 1), we established ([2]) that the above defined crystal isomorphism coincides with the well-known Schützenberger involution on tableaux (for the original definition of the Schützenberger involution see, for example, [4]).

This crystal isomorphism gives rise to an interesting isomorphism $B \otimes B' \cong B' \otimes B$.

Namely, we set $R(B \otimes B') = S(S(B) \otimes S(B'))$. Obviously the set $S(B \otimes B')$ coincides with the set $*B' \times *B$, and the latter set equals $S(B') \times S(B)$. Since S is an involution, we get the isomorphism

$$R : B \otimes B' \cong B' \otimes B.$$

Conjecture. The isomorphism R is an involution and there holds the Yang-Baxter equation

$$R_{12}R_{23}R_{12}(B \otimes B' \otimes B'') = R_{23}R_{12}R_{23}(B \otimes B' \otimes B'').$$

Here some evidences supporting this conjecture.

Let $B = B_V(k)$, $B' = B_V(l)$, $B'' = B_V(m)$ be crystals corresponding to the irreducible representations $S^k(V)$, $S^l(V)$ and $S^m(V)$ of $GL(V)$, respectively. Then, since there exists unique isomorphism $B_V(k) \otimes B_V(l) \cong B_V(l) \otimes B_V(k)$, R coincides with this isomorphism. Moreover, this isomorphism is exactly the symmetric group involution $\sigma : \mathcal{A}(n,2) \to \mathcal{A}(n,2)$, $\sigma(a) = D^{\epsilon'_1(a)-\phi'_1(a)}(a)$ if $\epsilon'_1(a) \geq \phi'_1(a)$, and $\sigma(a) = U^{\phi'_1(a)-\epsilon'(a)}(a)$ overwise. Since the generators of the symmetric group satisfy the Coxeter-Moore relation, the Yang-Baxter equation holds true in the case $B = B_V(k)$, $B' = B_V(l)$, $B'' = B_V(m)$.

Lemma. Let B, $B' \in Ob(\mathcal{A}_3)$. Then the isomorphism $R : B \otimes B' \cong B' \otimes B$ is an involution.

Proof. We have to check the following diagram is commutative.

$$\begin{array}{ccc} B \otimes B' & \to_{S\otimes S} & S(B) \otimes S(B') \\ \downarrow & & \downarrow \\ S(B \otimes B') & \to_{S\otimes S} & B' \otimes B \end{array}$$

In the case $n \leq 3$, this might be done by routine verification for irreducible orbits. Let B and B' be the orbit of the weights $(\lambda_1, \lambda_2, \lambda_3)$ and (ν_1, ν_2, ν_3), respectively. To check the commutativity, it suffices to do that for highest weight vectors of $B \otimes B'$, i.e. **L**-tight arrays of the form $\substack{b \\ b'}$, with some $b \in B$, $b' \in B'$ (obviously, b is **L**-tight). These arrays take the form

$$\begin{matrix} 0 & 0 & \nu_3 \\ 0 & \nu_2 - c & c \\ \nu_1 - a - b & a & b \\ 0 & 0 & \lambda_3 \\ 0 & \lambda_2 & 0 \\ \lambda_1 & 0 & 0 \end{matrix}$$

where $a \leq \lambda_1 - \lambda_2$ and $\max\{b, b + c - a\} \leq \lambda_2 - \lambda_3$.

There are several cases for checking. For $a \leq c$, we have $S(S(b) \otimes S(b')) = S \otimes S(S(b \otimes b')) =$

$$\begin{matrix} 0 & 0 & \lambda_3 \\ 0 & \lambda_2 - (b + c - a) & b + c - a \\ \lambda_1 - a - b & b & a \\ 0 & 0 & \nu_3 \\ 0 & \nu_2 & 0 \\ \nu_1 & 0 & 0 \end{matrix}$$

Other cases are left to the reader. Q.E.D.

Remark. There is the isomorphism $R' : B_V(\lambda) \otimes B_V(\nu) \cong B_V(\nu) \otimes B_V(\lambda)$ which obtains via degenerations of quantum deformations of the tensor product of the irreducible representations of $GL(V)$, $V_\mu \otimes V_\nu$, at $q = 0$ and $q = \infty$ (see [3]). It is interesting to compare R and R'.

6 Bi-crystal structure of $\Lambda^\bullet(V \otimes W)$

The subset $\mathcal{B}(n, m)$ of Boolean arrays of $\mathcal{A}(n, m)$, that is the set of arrays with {0, 1} entries might be seen as the ground set for the $A_{n \times m}$-type crystal corresponding to $\Lambda^\bullet(V \otimes W)$. We introduce two commuting crystal structures on $\mathcal{B}(n, m)$. These structures will be different of what structures in $\mathcal{A}(n, m)$. (Obviously, $B(n, m)$ is not stable under the crystal operations L_i, R_i and D_j, U_j, $i = 1, \ldots, n - 1$, $j = 1, \ldots, m - 1$.)

Let us define the operations $\hat{L}_1$ and $\hat{R}_1$ in two column's arrays $\mathcal{B}(2, m)$.

Consider such an array a

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,2) & a(2,2) \\ a(1,1) & a(2,1) \end{pmatrix}$$

In order to define the action of $\hat{L}_1$, consider the following function $\hat{l}_a : [m] \to \mathbb{Z}$,

$$\hat{l}_a(j) = \sum_{j'=j}^{m} (a(2,j') - a(1,j')).$$

Denote j^* the greatest element of the set $\operatorname{Argmax} \hat{l}_a(\cdot)$. If $\hat{l}_a(j^*) \leq 0$, then $\hat{L}_1(a) = a$, otherwise $\hat{L}_1(a)$ takes the form

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,j^*+1) & a(2,j^*+1) \\ a(1,j^*)+1 & a(2,j^*)-1 \\ a(1,j^*-1) & a(2,j^*-1) \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

Note, that in this case, since we consider Boolean arrays, we get $a(1,j^*) = 0$ and $a(2,j^*) = 1$ from the definition of j^*. In order to define the action of $\hat{R}_1$, consider a function $\hat{r}_a : [m] \to \mathbb{Z}$,

$$\hat{r}_a(j) = \sum_{j'=1}^{j} (a(1,j') - a(2,j')).$$

Denote $\hat{j}$ the least element of the set $\operatorname{Argmax} \hat{r}_a(\cdot)$. If $\hat{r}_a(\hat{j}) \leq 0$, then $\hat{R}_1(a) = a$, otherwise $\hat{R}_1(a)$ takes the form

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,\hat{j}+1) & a(2,\hat{j}+1) \\ a(1,\hat{j})-1 & a(2,\hat{j})+1 \\ a(1,\hat{j}-1) & a(2,\hat{j}-1) \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

It is easy to check that these operations set A_2-type crystal structure on the set of two columns arrays $\mathcal{B}(2,m)$ with the weight function being column's sum.

The pair of commuting operations $\hat{D}_j$ and $\hat{U}_j$ do not come of the form of the transposition as it was in the case of arrays of $\mathcal{A}(n,m)$. Specifically, we set U_1 on the two-row's array as follows: firstly we transform a two-row's array $a \in \mathcal{B}(n,2)$

$$a = \begin{pmatrix} a(1,2) & a(2,2) & \dots & a(n,2) \\ a(1,1) & a(2,1) & \dots & a(n,1) \end{pmatrix}$$

into two-column's array $a^\star \in \mathcal{B}(2,n)$ with entries

$$a^\star = \begin{pmatrix} a(1,1) & a(1,2) \\ a(2,1) & a(2,2) \\ \vdots & \vdots \\ a(n,1) & a(n,2) \end{pmatrix}$$

Then, we consider the inverse array in $Z(n,2)$ for the array $L_1(a^\star)$, that array we set as $D_1(a)$, i.e.

$$\hat{D}_1(a) = (\hat{L}_1(a^\star))^{-\star}. \tag{13}$$

Of course, $U_1(a) = (\hat{R}_1(a^\star))^{-\star}$. Thus, we get two crystal structures on the set $\mathcal{B}(n,m)$. We claim that these A-type crystal structures commute. Namely, there holds

Theorem 2. The operation $\hat{L}_i$ commutes with the operation $\hat{D}_j$.

Proof. We have to check the proposition for two cases. The first one, the operation $\hat{L}_i$ acts identically and $\hat{D}_j$ act either in the column i, or $i+1$. The second case, when $\hat{L}_i$ acts either in the row $j+1$, or j.

We will imagine 1 at the (l,k)-th place as a ball in the corresponding box.

In the first case, all balls in the $i+1$-th column could be matched with the balls in the i-th column such that each ball of the $i+1$-th column has a paired partner located west or west-north of it. Assume that $\hat{D}_j$ transforms a west-located partner down. Then the corresponding function $\hat{r}_j$ attains first maximum at $\hat{j} = i$, that forces the identity $a(i+1,j) = a(i+1,j+1) = 1$. Now $\hat{L}_i\hat{D}_j = \hat{D}_j\hat{L}_i$ follows since, $\hat{L}_i$ also does not acts on $\hat{D}_j(a)$, since $a(i+1,j)$ and $a(i+1,j+1)$ will exchange their partners in $\hat{D}_j(a)$, comparing the partnerships in a.

In the second case, either L_i acts in the j-th or $(j+1)$-th rows (otherwise commuting is obvious). In the first case, we have $a(i,j+1) = a(i+1,j+1)$, and one can check that $\hat{L}_i\hat{D}_j = \hat{D}_j\hat{L}_i$. In the second case, $a(i,j) \geq a(i+1,j)$, and again it is easy to check $\hat{L}_i\hat{D}_j = \hat{D}_j\hat{L}_i$. Q.E.D.

Because of this theorem, we can consider bi-invariant subsets of $\mathcal{B}(n,m)$, and such subsets have bi-crystal structure. Let us consider $\hat{\mathbf{D}}$-tight, $\hat{\mathbf{L}}$-tight and $\hat{\mathbf{L}}\hat{\mathbf{D}}$-tight arrays of $\mathcal{B}(n,m)$.

Proposition 2. 1) A $\hat{\mathbf{L}}\hat{\mathbf{D}}$-tight array of $\mathcal{B}(n,m)$ takes the form of an array which has 1's located at nodes of a Young diagram and 0's outside.

2) There is a canonical bijection between $\hat{\mathbf{D}}$-tight (and $\hat{\mathbf{L}}$-tight) and semi-standard Young tableaux.

3) If the tableau for $\hat{\mathbf{L}}(a)$ has the shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$, then the tableau for $\hat{\mathbf{D}}(a)$ has the transposed shape λ'. Moreover, the shape of $\hat{\mathbf{L}}(a)$ coincides with the shape of the Young diagram constituted of the 1's of the array $\hat{\mathbf{L}}(\hat{\mathbf{D}}(a))$.

Proof. We start from proving the item 2. The bijection is set by the following rule. Given a $\hat{\mathbf{D}}$-tight array a, we associate to it a tableau such that the j-th column of this tableau is obtained by reading from left to right the j-th row of a in the alphabet $1 < 2 < \ldots < n$. Namely if $a(j,i) = 1$ then we insert the letter i in the column, otherwise we go to $a(j,i+1)$. For example, the row 110001010100 reads as the column $(1\,2\,6\,8\,10)^t$. Thus, we have to verify that the result of such a transforming rows of arrays into columns of a tableau will get a semi-standard Young tableau. Because of the reading rule and since the arrays are Boolean, we get strict increasing along the columns. So, we have to check that each row of such a tableau is weakly increasing. It suffices to check this for a pair of adjoint rows, or, equivalently, for two row's arrays. Due to the definition of $\hat{\mathbf{D}}$-tightness, there exists a matching such that each ball in the second row has a partner in the south-west or south direction in the first row. This implies weak increasing along the row of the corresponding column. Obviously, this construction reverses, and, thus, the claimed bijection is established.

We associate a semi-standard tableau to an $\hat{\mathbf{L}}$-tight array of $\mathcal{B}(n,m)$ as follows. We fill up the i-th column of the tableau by reading i-th column of a from top to bottom in the alphabet $1 < 2 < \ldots < m$. Namely, if $a(i,j) = 1$, then we fill the letter $m - j + 1$, overwise we go to $a(i,j-1)$. For example, the $\hat{\mathbf{L}}$-tight array

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$

produces the tableaux

$$\begin{matrix} 5 & & & \\ 3 & 4 & 4 & 4 \\ 1 & 1 & 2 & 3 \end{matrix}$$

Thus we established the item 2.

The item 3 follows from the above construction and the item 1. To establish the item 1, we show that $\hat{\mathbf{L}}\hat{\mathbf{D}}$-tight arrays can not have "holes", i.e. in such an array it can not happen that $a(i,j) = 0$ and either $a(i, j+1) = 1$ or $a(i+1, j) = 1$. In fact, assume that such a hole exists, say $a(i,j) = 0$ and $a(i, j+1) = 1$ for some (i,j). We call such a hole *vertical*. Then, for some $i' < i$, there holds $a(i', j+1) = 0$ and $a(i'+1, j+1) = 1$. We call such a hole *horizontal*. In fact, if $a(i', j+1) = 1$ with any $i' < i$, then $\hat{U}_j(a) \neq a$. Furthermore, there holds $a(i', j') = 1$ with some $j' > j+1$, otherwise $\hat{L}_{i'}(a) \neq a$. That implies existence of a vertical hole being located strictly north-west to (i,j), that is $a(i', \tilde{j}) = 0$ and $a(i', \tilde{j}+1) = 1$ with $i' < i$ and $\tilde{j} > j$. Hence, we can get an infinite sequence of vertical holes, that is not the case. Q.E.D.

Let us illustrate this proposition by the following example. Consider the array $a = \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$. Then $\hat{\mathbf{D}}(a) = \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$ and the corresponding Young tableau is $\begin{matrix} 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & 3 \end{matrix}$, $\hat{\mathbf{L}}(a) = \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$ and the corresponding tableau is $\begin{matrix} 3 & & \\ 2 & 4 & 4 \\ 1 & 2 & 3 \end{matrix}$, and, finally,

$$\hat{\mathbf{D}}\hat{\mathbf{L}}(a) = \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}.$$

Remark. There is a bijection between the set of $\hat{\mathbf{D}}$-tight arrays of $\mathcal{B}(n,m)$ and functions $f : [m] \to 2^{[n]}$ which are monotone with respect to the following partial order[4] on $2^{[n]}$: for subsets $K = \{i_1, \dots, i_k\}$ and $L = \{j_1, \dots, j_l\}$ of $[n]$, we set

$$K \succ\succ L \text{ if } k \leq l \text{ and } i_t \geq j_t \text{ for } t = 1, \dots, k.$$

In fact, let a be a $\hat{\mathbf{D}}$-tight array. Then $\hat{D}_j(a) = a$ is equivalent to that we can provide to each 1 in the $j+1$-th row of a its own partner 1 located south or south-west in the j-th row. Thus, the collection $f_a(j) := \{i \in [n] : a(i,j) = 1\}$, $j = 1, \dots, m$, of subsets of $[n]$ is monotone, $f_a(j+1) \succ\succ f_a(j)$, $j = 1, \dots, m-1$.

Similarly $\hat{\mathbf{L}}$-tight arrays of $\mathcal{B}(n,m)$ are in bijection with monotone functions $g : [n] \to 2^{[m]}$. Specifically, $a \to g_a$, where $g_a(i) := \{m - j + 1 : a(i,j) = 1\}$, $i = 1, \dots, n$.

There is a bijection $*_c : \mathcal{B}(n,m) \to \mathcal{B}(n,m)$, which is the composition of two mappings: the first one sends a Boolean array a to its complement a^c, $a^c(i,j) = 1$ iff $a(i,j) = 0$ and $a^c(i,j) = 0$ iff $a(i,j) = 1$, and the second sends an array a to its middle-axe symmetry $a^s(i,j) = a(n-i+1, j)$. Thus

$$(*_c a)(i,j) = a^c(n-i+1, j),\ i = 1, \dots, n,\ j = 1, \dots, m.$$

Proposition 3. The mapping $*_c$ sends the set of $\hat{\mathbf{L}}$-tight arrays to itself.
Proof. The proposition follows from the above remark and the following property of the ordering $\succ\succ$:
Let K, L be subsets of $[n]$. Then $K \succ\succ L$ if and only if $L^c \succ\succ K^c$, where $K^c = [n] \setminus K$ denotes the complement. Q.E.D.

As a consequence of this proposition and Proposition 3, we get the following bijection (noted in [6])

The mapping $*_c$ provides is a bijection between the set of semi-standard Young tableaux of shape λ, $n \geq \lambda_1 \geq \ldots \geq \lambda_m \geq 0$, filled out from the alphabet $1, \dots, m$, of weight $(\mu_1, \dots, \mu_m)$ and the set of semi-standard Young tableaux of the dual shape $\lambda^d := (n - \lambda_m, \dots, n - \lambda_1)$ and the dual weight $(n - \mu_1, \dots, n - \mu_m)$.

Now we are going to define less trivial bijection. The mapping $a \to (\hat{\mathbf{L}}(a), \hat{\mathbf{D}}(a))$ is a bijection. We have to show how to invert this mapping. The inversion procedure is parallel to that for the case of $\mathcal{A}(n,m)$ ([2]). Namely, a word $w = \hat{L}_{i_1} \dots \hat{L}_{i_s}$ in the alphabet $\hat{L}_1, \dots, \hat{L}_{n-1}$ is said to be

[4] This order was considered in [7].

effective for an array $a \in \mathcal{B}(n, m)$ if
1) $wa = \hat{\mathbf{L}}(a)$;
2) for any $t = 1, \ldots, s-1$, $\hat{L}_{i_t} \ldots \hat{L}_{i_s}(a) \neq \hat{L}_{i_{t+1}} \ldots L_{i_s}(a)$.

Let l be an $\hat{\mathbf{L}}$-tight array, let d be an $\hat{\mathbf{D}}$-tight array and $\hat{\mathbf{D}}(l) = \hat{\mathbf{L}}(d)$ (that is we take a pair of semi-standard Young tableaux of conjugate shapes). Let $w = \hat{L}_{i_1} \ldots \hat{L}_{i_s}$ be an effective word for d. Then we set

$$a = \hat{R}_{i_s} \ldots \hat{R}_{i_1}(l).$$

We claim that $l = \hat{\mathbf{L}}(a)$ and $d = \hat{\mathbf{D}}(a)$. The first equality is obvious, the second holds due to commutativity of the operations $\hat{L}_i$ and $\hat{D}_j$, and due to $\hat{\mathbf{D}}(l) = \hat{\mathbf{L}}(d)$.

Any bi-invariant subset of $\mathcal{B}(n, m)$ contains a unique $\hat{\mathbf{L}}\hat{\mathbf{D}}$-tight array, and its shape completely characterizes that subset. Moreover, the bi-invariant subset of $\mathcal{B}(n, m)$ of the shape λ takes the form of the external tensor product $B_V(\lambda)\hat{\otimes}B_W(\lambda')$, and so, we get the decomposition

$$\mathcal{B}(n, m) \cong \sum_{\lambda} B_V(\lambda)\hat{\otimes}B_W(\lambda'),$$

where summations is take over partitions with at most n columns and m rows.

Remarks. 1) The bijection $a \to (\hat{\mathbf{L}}(a), \hat{\mathbf{D}}(a))$ resembles the Knuth bijection[5] between the set of 0, 1-matrices and pairs of semi-standard tableaux of conjugate shapes. Alike, the RSK bijection and the bi-crystal decomposition $\mathcal{A}$ differ by the Schützenberger involution one of tableaux, this bijection also has the same difference from the Knuth bijection.

2) Similarly to the case of integer arrays $\mathcal{A}(n, m)$ ([2]), there exists two pairs of commuting crystal structures on $\mathcal{B}(n, m)$, crystal structures in each pair are related via the crystal isomorphism $S : \mathcal{A}(n, m) \to \mathcal{A}(n, m)$ (see Section 5), while there is no even relations between shapes of an array a with respect to crystal structures from different pairs.

3) Let us note, that due to the definitions of the crystal structures (and we have yet seen that in the proof of the above proposition), tensor product of A_m-crystals $\mathcal{B}(n, m)\otimes\mathcal{B}(n', m)$ is obtained by putting arrays of $\mathcal{B}(n', m)$ to the right $\mathcal{B}(n, m)$, while the tensor product of A_n-crystals $\mathcal{B}(n, m)\otimes\mathcal{B}(n, m')$ is obtained by putting arrays of $\mathcal{B}(n, m')$ to the bottom $\mathcal{B}(n, m)$, i.e. $b \otimes b'$

[5]The Knuth bijection is also known under the name dual RSK correspondence. Our constructions do not revealed any duality.

reads as the array $\begin{smallmatrix} b \\ b' \end{smallmatrix}$. Because of this and the above established isomorphism between $\hat{\mathbf{L}}$-tight 0, 1-arrays and semistandard tableaux, $\mathcal{B}(\infty, \infty)$ has to be the set of 0, 1-arrays (with finite support) located in the ortant $x \geq 0$, $y \leq 0$.

4) Similarly to the case of $\mathcal{A}(n, \infty)$, any normal A_n-type crystal might be embedded into $\mathcal{B}(n, \infty)$. This gives rise to the category $\mathcal{B}_n$ constituted of $\hat{L}_i$, $\hat{R}_i$-invariant (finite) subsets of $\mathcal{B}(n, \infty)$. The set of objects of $\mathcal{B}_n$ might be parametrized by finite tuples without repetitions of semistandard Young tableaux with at most n columns. The same reasoning as for the category $\mathcal{A}_n$ take place in the category $\mathcal{B}_n$. Thus, this category has the involution $* : \mathcal{B}_n \to \mathcal{B}_n$ and hypothetical R-matrix. So, we get another tensor category equivalent to the category of normal crystals.

7 Continuous crystals

Here we introduce the notion of continuous A_n-type crystals, and show that the set of arrays $\mathcal{A}^{\mathbb{R}}(n, m)$ with non-negative real entries has two commuting A_n and A_m-continuous crystal structures. Thus, $\mathcal{A}^{\mathbb{R}}(n, m)$ is a continuous bi-crystal. It is interesting to note that there are no relations among usual crystal operations E_i and E_{i+1}, while, for continuous normal crystal, the Verma relations hold true infinitesimally. This could provide a link between the continuous crystals and geometric crystals due to Berenstein and Kazhdan [1].

Definition. A set B is said to be a *continuous* A_n-type crystal if, for any $\alpha \geq 0$, there are operations $E_i^\alpha : B \mapsto B$, $F_i^\alpha : B \mapsto B$, $i = 1, \ldots, n-1$, and functions $\epsilon_i : B \to \mathbb{R}$, $\phi_i : B \to \mathbb{R}$, $i = 1, \ldots, n-1$, $wt : B \to \mathbb{R}^n$, such that there holds

$$E_i^\alpha(b) = b' \Leftrightarrow F_i^\alpha(b') = b \text{ if, for any } \delta > 0,\ E_i^{\alpha-\delta} b \neq b'; \tag{14}$$

$$wt(E_i^\alpha(b)) = wt(b) + \alpha(e_i - e_{i+1}) \text{ if , for any } \delta > 0,\ E_i^{\alpha-\delta} b \neq E_i^\alpha(b); \tag{15}$$

$$wt(F_i^\alpha(b)) = wt(b) - \alpha(e_i - e_{i+1}) \text{ if , for any } \delta > 0,\ F_i^{\alpha-\delta} b \neq F_i^\alpha(b); \tag{16}$$

$$wt(b)_i - wt(b)_{i+1} = \phi_i(b) - \epsilon_i(b), \tag{17}$$

where $\epsilon_i(b) = \max\{\alpha :$ for any $\delta > 0,\ E_i^{\alpha-\delta} b \neq E_i^\alpha(b)\}$, and $\phi_i(b) = \max\{\beta :$ for any $\delta > 0,\ F_i^{\beta-\delta} b \neq E_i^\beta(b)\}$.

A continuous crystal is *irreducible* if it is a connected set with respect to the action of operators E_i^α, F_i^α, $\alpha \geq 0$, $i = 1, \ldots, n-1$.

Any irreducible crystal B is characterized by the weight $wt(u_B)$ of its "highest weight vector", i.e. $E_i^\alpha(u_B) = u_B$ with any $i = 1, \ldots, n-1$, $\alpha \geq 0$.

Due to (17) and definition of the functions ϵ_i, $i = 1, \ldots, n-1$, we get that $wt(u_B)$ belongs to the Weil chamber $W(\mathbb{R}^n) := \{x_1 \geq \ldots \geq x_n\}$.

Alike the set of semi-standard Young tableaux of shape $\lambda \in W(\mathbb{Z}^n)$ might be endowed with crystal operations of A_n-type, the set of "continuous" Young tableaux, or, equivalently, the set of all points of the Gelfand-Ceitlin polytope might be endowed with continuous crystal operations of A_n-type. Namely, according to Proposition 1, we may identify the set of semi-standard Young tableaux of shape λ with the orbit of **LD**-tight array with λ on the diagonal under the action of operations R_i, $i = 1, \ldots, n-1$. Similarly, the orbit of **LD**-tight array with $\lambda \in W(\mathbb{R}^n)$ on the diagonal under the action of operations R_i^a, $a \geq 0$, $i = 1, \ldots, n-1$, is in bijection with the points of the Gelfand-Ceitlin polytope $GC(\lambda)$. Denote by $B_V^c(\lambda)$ this crystal.

Remark. Note that $B_V^c(\lambda)$ extends the crystal $B_{\mathbb{Q}}(\lambda)$ due to Definition 8.1.7 [8], that is, due to the Kashiwara definition $\lambda \in \mathbb{Q}^n$ and crystal operations are taken in rational powers, while due to ours $\lambda \in \mathbb{R}^n$ and crystal operations are taken in real powers.

Now we introduce several notions. A continuous crystal is *simple* if it contains finitely many connected components (with respect to the action of operators E_i^α, F_i^α, $\alpha \geq 0$, $i = 1, \ldots, n-1$).

A *measurable* G-C crystal takes the form of the direct sum

$$\coprod_{\lambda \in W(\mathbb{R}^n)} C(\lambda) \times B_V^c(\lambda)$$

with a measurable set-valued function $C : \mathbb{R}^n \to \mathbb{R}_+$.

A mapping $f : B_1 \to B_2$ of continuous crystals B_1 and B_2 is a *morphism* if f commutes with the crystal operations.

A crystal is measurable if it is isomorphic to a measurable G-C crystal.

Tensor product $B_1 \otimes B_2$ of continuous crystals is the set $B_1 \times B_2$ with operations $wt(b_1, b_2) = wt(b_1) + wt(b_2)$ and

$$E_i^a(b \otimes b') = E_i^{a'} \otimes E_i^{a''}(b'), \tag{18}$$

where $a'' = \min\{a, \phi_i(b) - \epsilon_i(b')\}$, $a' = a - a''$, and

$$F_i^a(b \otimes b') = F_i^{\tilde{a}} \otimes F_i^{\tilde{\tilde{a}}}(b'), \tag{19}$$

where $\tilde{a} = \min\{a, \phi_i(b) - \epsilon_i(b')\}$, $\tilde{\tilde{a}} = a - \tilde{a}$.

The same formulae as in the case of usual crystals define the action of Weil group and Hecke algebra $H_n(0)$ for continuous crystals.

We are going to endow the set of real-valued arrays $\mathcal{A}^{\mathbb{R}}(n,m)$ with the structure of continuous A_n-crystal (A_m-crystal and bi-crystal).

Similarly to the case of $\mathcal{A}(n,m)$, to define the operators E_i^α, F_i^α, $\alpha \geq 0$, $i = 1,\ldots,n-1$. we have to define operations $L^\alpha := E_1^\alpha$ and $R^\alpha := F_1^\alpha$ for two columns' arrays $\mathcal{A}^{\mathbb{R}}(2,m)$.

Consider an array a

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,2) & a(2,2) \\ a(1,1) & a(2,1) \end{pmatrix}$$

In order to define the action of L_1^α, consider the function $l_a : [m] \to \mathbb{R}$,

$$l_a(j) = a(2,1) + \sum_{j'=2}^{j} (a(2,j') - a(1,j'-1)).$$

Define $\epsilon_1(a) = \max_j l_a(j)$. Since for $\alpha \geq \epsilon_1(a)$, $L_1^\alpha = L_1^{\epsilon_1(a)}(a)$, we consider the case $\alpha \leq \epsilon_1(a)$. Obviously, if $\epsilon_1(a) = 0$, then $L_1^\alpha(a) = a$ for any α.

Denote j_1^* the smallest element of the set $\operatorname{Argmax} l_a(j)$, denote j_2^* the smallest element of the set $\operatorname{Argmax}_{j<j_1^*} l_a(j)$ and so on, denote j_t^* the smallest element of the set $\operatorname{Argmax}_{j<j_{t-1}^*} l_a(j)$. A resulting sequence is $j_1^* > \ldots > j_k^* = 1$. Denote $\delta_{j_t^*} = l_a(j_t^*) - l_a(j_{t+1}^*)$, $t = 1,\ldots,k-1$. Let t^* be the first occurrence of $\delta_{j_1^*} + \ldots \delta_{j_t^*} \geq \alpha$. Then $L_1(a)$ takes the form

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,j_1^*) + \delta_{j_1^*} & a(2,j_1^*) - \delta_{j_1^*} \\ a(1,j^*-1) & a(2,j^*-1) \\ \vdots & \vdots \\ a(1,j_2^*) + \delta_{j_2^*} & a(2,j_2^*) - \delta_{j_1^*} \\ \vdots & \vdots \\ a(1,j_{t^*}^*) + \alpha - (\delta_{j_1^*} + \ldots + \delta_{j_{t^*-1}^*}) & a(2,j_{t^*}^*) - \alpha + (\delta_{j_1^*} + \ldots + \delta_{j_{t^*-1}^*}) \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

For example, if $t^* = 1$, then $L_1(a)$ takes the form

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,j_1^*)+\alpha & a(2,j_1^*)-\alpha \\ a(1,j^*-1) & a(2,j^*-1) \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

In order to define the action of R_1, consider a function $r_a : [m] \to \mathbb{R}$,

$$r_a(j) = a(1,m) + \sum_{j'=j}^{m-1} (a(1,j') - a(2,j'+1)).$$

Since for $\alpha \geq \phi_1(a)$, $R_1^\alpha = R_1^{\phi_1(a)}(a)$, we consider the case $\alpha \leq \phi_1(a)$. We set $\phi_1(a) = \max_j r_a(j)$.

Denote j_1^* the smallest element of the set $\mathrm{Argmax} r_a(j)$, denote j_2^* the smallest element of the set $\mathrm{Argmax}_{j>j_1^*} l_a(j)$ and so on, denote j_t^* the smallest element of the set $\mathrm{Argmax}_{j>j_{t-1}^*} l_a(j)$. A resulting sequence is $j_1^* < \ldots < j_k^* = m$. Denote $\kappa_{j_t^*} = l_a(j_t^*) - l_a(j_{t+1}^*)$, $t = 1, \ldots, k-1$. Let s^* be the first occurrence of $\kappa_{j_1^*} + \ldots \kappa_{j_s^*} \geq \alpha$. Then $R_1(a)$ takes the form

$$\begin{pmatrix} a(1,m) & a(2,m) \\ \vdots & \vdots \\ a(1,j_{m^*}^*)+\alpha-(\delta_{j_1^*}+\ldots+\delta_{j_{m^*-1}^*}) & a(2,j_{m^*}^*)-\alpha+(\delta_{j_1^*}+\ldots+\delta_{j_{m^*-1}^*}) \\ a(1,j_2^*)+\delta_{j_2^*} & a(2,j_2^*)-\delta_{j_1^*} \\ \vdots & \vdots \\ a(1,j^*+1) & a(2,j^*+1) \\ \vdots & \vdots \\ a(1,j_1^*)+\kappa_{j_1^*} & a(2,j_1^*)-\kappa_{j_1^*} \\ \vdots & \vdots \\ a(1,1) & a(2,1) \end{pmatrix}$$

It is not difficult to check that these operations and the functions ϵ_1, ϕ_1 and $wt(a) = (\sum_j a(1,j), \sum_j a(2,j)$ endow the set of two-columns arrays with the structure of a continuous A_2-type crystal. Thus we get the continuous A_n-type crystal structure on $\mathcal{A}^{\mathbb{R}}(n,m)$.

Using transposition $a(i,j) \to a(j,i)$, we get the continuous A_m-type crystal structure on $\mathcal{A}^{\mathbb{R}}(n,m)$.

The proof of Theorem 1 ([2]) provide the following results

Theorem 3. The above defined continuous A_n- and A_m-crystal structures on $\mathcal{A}^{\mathbb{R}}(n,m)$ commute.

The Weil group S_n acts on columns of $\mathcal{A}^{\mathbb{R}}(n,m)$ as follows. The transpositions are set by

$$s_i(a) = \begin{cases} L_i^{\epsilon_i(a)-\phi_i(a)}(a) & \text{if } \epsilon_i(a)-\phi_i(a) \geq 0 \\ R_i^{\phi_i(a)-\epsilon_i(a)}(a) & \text{if } \epsilon_i(a)-\phi_i(a) \leq 0 \end{cases}$$

Correspondingly, elementary transpositions of the Weil group S_m action on rows of $\mathcal{A}^{\mathbb{R}}(n,m)$ takes the form

$$s_j(a) = \begin{cases} D_i^{\epsilon_j(a^*)-\phi_j(a^*)}(a) & \text{if } \epsilon_j(a^*)-\phi_j(a^*) \geq 0 \\ U_j^{\phi_j(a^*)-\epsilon_j(a^*)}(a) & \text{else} \end{cases}$$

Two Hecke algebras' $H_n(0)$ acts on on columns of $\mathcal{A}^{\mathbb{R}}(n,m)$ as $t_i(a) = L_i^{\epsilon_i(a)}(a)$, $i = 1, \dots, n-1$, and $t'_i(a) = R_i^{\phi_i(a)}(a)$, $i = 1, \dots, n-1$, correspondingly.

Two Hecke algebras' $H_m(0)$ acts on on rows of $\mathcal{A}^{\mathbb{R}}(n,m)$ as $t_j(a) = D_i^{\epsilon_j(a^*)}(a)$, $i = 1, \dots, m-1$, and $t'_j(a) = U_i^{\phi_i(a^*)}(a)$, $i = 1, \dots, m-1$.

Corollary. a) The Weil groups' S_n and S_m actions on $\mathcal{A}^{\mathbb{R}}(n,m)$ commute[6];

b) The Hecke algebras' $H_n(0)$ and $H_m(0)$ actions on $\mathcal{A}^{\mathbb{R}}(n,m)$ commute.

Denote by $\mathbf{L}\mathcal{A}^{\mathbb{R}}$, $\mathbf{D}\mathcal{A}^{\mathbb{R}}$ and $\mathbf{P}\mathcal{A}^{\mathbb{R}}$ the sets of **L**-tight arrays, **D**-tight arrays and **LD**-tight (or perfect arrays), respectively. Perfect arrays might have non-zero entries at the diagonal only (they correspond to the continuous Yamanouchi tableaux). Thus, we get a mapping

$$(\mathbf{L}, \mathbf{D}) : \mathcal{A}^{\mathbb{R}} \to \mathbf{L}\mathcal{A}^{\mathbb{R}} \times_{\mathbf{LD}\mathcal{A}^{\mathbb{R}}} \mathbf{D}\mathcal{A}^{\mathbb{R}} \tag{20}$$

The following proposition is the continuous (modified) RSK bijection.

Proposition 4. This mapping is a bijection.

Proposition 5. There is a bijection between the set of **L**-tight arrays of shape λ and the set of points of the Gelfand-Cetlin polytope $GC(\lambda)$.

The bi-crystal decomposition of $\mathcal{A}^{\mathbb{R}}(n,m)$ takes the form

$$\mathcal{A}^{\mathbb{R}}(n,m) \cong \coprod_{\lambda} B_V^c(\lambda) \hat{\otimes} B_W^c(\lambda). \tag{21}$$

[6]In [9] is constructed another pair of commuting actions of S_n and S_m on $\mathcal{A}^{\mathbb{R}}(n,m)$.

We say that a (continuous) crystal B is *normal* if, for some W, or $m = \dim W$, it is isomorphic (crystally) to sub-crystal of $\mathcal{A}^{\mathbb{R}}(n,m)$ of the form

$$\coprod_{\lambda\in\Lambda} D(\lambda) \times B_V^c(\lambda),$$

where Λ is a polytope in $W(\mathbb{R}^n)$ and $D(\lambda)$ is a polytope (or a finite disjoint union of polytopes) in $B_W^c(\lambda)$.

Here is a continuous variant of the generalized Schützenberger involution (see Section 5).

Lemma. Let B be an invariant subset of $\mathcal{A}^{\mathbb{R}}(n,m)$. Then $*B$ is an invariant subset of $\mathcal{A}^{\mathbb{R}}(n,m)$.

Then $S(a) = w'(\mathbf{L}(a^*))$ is the continuous generalized Schützenberger involution, where w is an effective word for a in the alphabet L_i^t, $0 < t \le 1$, $i = 1,\dots,n-1$, and w' its reverse reading in the alphabet R_i^t, $0 < t \le 1$, $i = 1,\dots,n-1$. On this way, we, hypothetically, get an R-matrix

$$R^c : B_V^c(\lambda)\otimes B_V^c(\nu) \cong B_V^c(\nu)\otimes B_V^c(\lambda), \quad R^c(b\otimes b') = S(S(b)\otimes S(b')).$$

7.1 Infinitesimal Verma relations

The following proposition shows that, for sub-crystals of $\mathcal{A}^{\mathbb{R}}(n,m)$ $(m \ge n)$, the Verma relations hold true infinitesimally.

Proposition 6. Let B be a sub-crystal of $\mathcal{A}^{\mathbb{R}}(n,m)$ and let $b \in B$. Then, there exists $\kappa(b) > 0$ such that, for any $i = 1,\dots,n-2$,

$$E_i^{\alpha} E_{i+1}^{\alpha+\beta} E_i^{\beta} = E_{i+1}^{\beta} E_i^{\alpha+\beta} E_{i+1}^{\alpha} \tag{22}$$

holds with any α, $\beta \le \kappa(b)$.

Proof. One has directly to verify several possible configurations. We will do this for one of them, leaving other cases to the reader. Namely, let $\epsilon_i(b) > 0$ and $\epsilon_{i+1}(b) > 0$. Let j_{i+1}^* be such that $(E_{i+1}^{\alpha}(b))(i+1, j_{i+1}^*) = b(i+1, j_{i+1}^*) - \alpha$, let j_i^* be such that $(E_i^{\alpha+\beta}(E_{i+1}^{\alpha}(b)))(i, j_i^*) = b(i, j_i^*) - (\alpha+\beta)$, and let j_{i+1}^{**} be such that $(E_{i+1}^{\beta}((E_i^{\alpha+\beta}(E_{i+1}^{\alpha}(b)))))(i+1, j_{i+1}^{**}) = b(i+1, j_{i+1}^{**}) - \beta$. Such row numbers exist for appropriate choice of α and β. Consider a case $j_{i+1}^{**} > j_i^* \ge j_{i+1}^*$. Then, j_i^* is that row number that $(E_i^{\beta}(b))(i, j_i^*) = b(i, j_i^*) - \beta$; j_{i+1}^{**} is that row number that $(E_{i+1}^{\beta}(E_i^{\beta}(b)))(i+1, j_{i+1}^{**}) = b(i+1, j_{i+1}^{**}) - \beta$ and j_{i+1}^* is that row number that $(E_{i+1}^{\alpha}(E_{i+1}^{\beta}(E_i^{\beta}(b))))(i+1, j_{i+1}^*) = b(i+1, j_{i+1}^*) - \alpha$; and, finally, j_i^* is that row number that $(E_i^{\alpha}(E_{i+1}^{\alpha+\beta}(E_i^{\beta}(b))))(i, j_i^*) =$

$(E_{i+1}^{\alpha+\beta}(E_i^{\beta}(b(i,j_i^*)) - \alpha$. All this together yields the validity of (22). Q.E.D.

Remark. The Verma relations do not hold globally. For example $E_1E_2^2E_1(b) \neq E_2E_1^2E_2(b)$ for the array

$$b = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}.$$

In fact,

$$E_iE_{i+1}^2E_i(b) = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \quad E_{i+1}E_i^2E_{i+1}(b) = \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}.$$

8 Littlewood-Richardson rule

One of important problems is the decomposition into irreducibles the tensor product of irreducible representations. The Weil formula provides an answer, however, because the formula is of an alternating form and many simplifying cancellations might be done, it is not easy to use this formula to analyze these coefficients.

For the case of $GL(n)$, combinatorial rule for computing these coefficients is given by the Littlewood-Richardson rule [11, 4]. Using the crystals $\mathcal{A}(n,\infty)$ and $\mathcal{B}(n,\infty)$, we provide several reformulations of this rule.

Recall, that the tensor product of an A_n-type crystal $B_1 \subset \mathcal{A}(n,m_1)$ and a crystal $B_2 \subset \mathcal{A}(n,m_2)$ is isomorphic to the crystal in $\mathcal{A}(n,m_1+m_2)$ obtained by putting arrays of B_2 on the top of arrays of B_1, i.e. $b_1 \otimes b_2 = \begin{matrix} b_2 \\ b_1 \end{matrix}$. And the tensor product of an A_n-type crystal $B_1 \subset \mathcal{B}(n,m_1)$ and a crystal $B_2 \subset \mathcal{B}(n,m_2)$ is isomorphic to the crystal in $\mathcal{B}(n,m_1+m_2)$ obtained by putting arrays of B_1 on the top of arrays of B_2, i.e. $b_1 \otimes b_2 = \begin{matrix} b_1 \\ b_2 \end{matrix}$.

Now, in order to decompose $B(\mu) \otimes B(\nu) \subset \mathcal{A}(n,2n)$ into irreducibles, we have to find $\mathbf{L}$-tight arrays in $B(\mu) \otimes B(\nu)$. Denote by $\mathbf{L}(B(\lambda) \otimes B(\mu))$ this set of arrays. Then the decomposition formula takes the form

$$B(\mu) \otimes B(\nu) = \sum_{a \in \mathbf{L}(B(\mu)\otimes B(\nu))} B(wt(a)).$$

Thus, the Littlewood-Richardson coefficient $c_{\mu,\nu}^{\lambda}$ equals the cardinality of the set of $\mathbf{L}$-tight arrays in $B(\mu) \otimes B(\nu)$ of the weight λ. We will describe this set in several languages.

On the crystal language this might be rewritten as follows

$$B(\mu) \otimes B(\nu) = \sum_{b \in B(\nu) \,:\, \mu_i \geq \epsilon_i(b) \,\forall\, i} B(\mu + wt(b)).$$

This formula coincides with that asserted in Corollary 4.1.7 [8]. This immediately gives the well-known PKV-formula (see, for example, [13])

$$c^{\lambda}_{\mu,\nu} = \dim\{v \in V_{\nu}^{\lambda-\mu} \;:\; e_i^{\mu_i+1} v = 0\},$$

where V_ν^κ denotes the subspace of vectors of weight κ, and e_i, $i = 1, \ldots, n-1$, denote the e's part of the Chevalley generators for $gl(n)$.

On the language of discretely concave functions the set **L**-tight arrays in $B(\mu) \otimes B(\nu)$ reads as the set $DC(\lambda; \mu, \nu)$ of integer-valued discretely concave functions on the grid $\Delta(n)$ with the boundary increments μ, ν and λ correspondingly. Specifically, by the (two-dimensional triangular) *grid* of size n, we mean the following subset $\Delta(n)$ in $\mathbb{Z}^2$ (see Fig. 1):

$$\Delta(n) = \{(i,j) \in \mathbb{Z}^2,\ 0 \leq i, j \leq n, j \leq i\}.$$

In other words, this is integer points in the triangle with vertices $(0,0)$, $(n,0)$ and (n,n) in the plane $\mathbb{R}^2$ (we call it the *triangle* of $\Delta(n)$). We are interested in functions defined at the points of $\Delta(n)$. (If desired, we can assume that they are defined at all points of $\mathbb{Z}^2$ but take the value $-\infty$ outside $\Delta(n)$.)

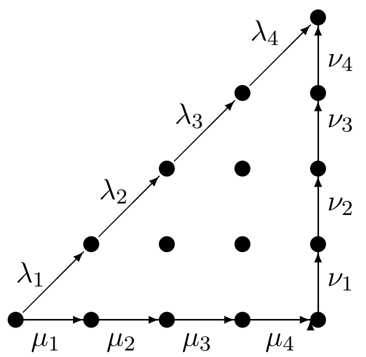


Figure 1.

A function f on the grid $\Delta(n)$ is called *discretely concave (or a DC function)* if, for any integers i and j, the following inequalities hold:

(i) $f(i,j) + f(i+1, j+1) \geq f(i+1, j) + f(i, j+1)$;

(ii) $f(i+1,j)+f(i+1,j+1) \geq f(i,j)+f(i+2,j+1)$;

(iii) $f(i,j+1)+f(i+1,j+1) \geq f(i,j)+f(i+1,j+2)$.

The terminology can be partially justified as follows. Let us cut the plane $\mathbb{R}^2$ into parts by lines of three sorts: $x = a$, where a are integers; $y = b$, where b are integers; and $x - -y = c$, where c are integers. We obtain a standard triangulation of the plane (see Fig. 2). Let us extend our function over each triangle by linearity; as a result of such a linear interpolation, we obtain a function $\tilde{f}$. The function f is discretely concave if and only if $\tilde{f}$ is concave.

For example, any affine function is discretely concave. For such a function, all inequalities (i)–(iii) are equalities.

There exist more interesting quasi-separable functions. Suppose given three discretely concave functions f, g, and h of one (integer) variable. Then, the function

$$F(x,y) = f(x) + g(y) + h(x-y),$$

of two (integer) variables is, obviously, a DC function.

Integer-valued discretely concave functions in the language of skew tableaux are nothing but L-R skew tableaux precisely. Namely, to a function $f \in DC(\lambda;\mu,\nu)$ we associate the array

$$a_f \in \mathcal{A}(n,n), \text{ such that } a_f(i,j) = f(i,j) - f(i,j-1), \quad 1 \leq i \leq j \leq n.$$

To this array a_f we associate a skew tableau T_f of shape $\lambda \setminus \mu$ such that the word $1^{a_f(i,1)}2^{a_f(i,2)} \ldots i^{a(i,i)}$ comes as the reading of the i-th row of T_f. Then the conditions (i) and (ii) imply that the corresponding skew tableau is semistandard and the condition (iii) implies that this tableau is an L-R tableau. Recall, that a semistandard tableau is a *L-R tableau* if the reading of this tableau from right to left and from bottom to top produces a lattice word, that is a word with the property that, having read this word until any place the number of occurrences of i is inferior than that of $i+1$ for any letter i of the alphabet. (For details, see [2].)

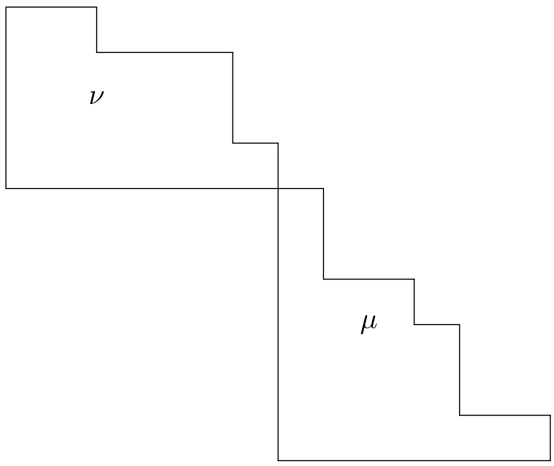
Vice versa, L-R tableau T of shape $\lambda \setminus \mu$ and of the weight ν gives rise to a function $f_T \in DC(\lambda;\mu,\nu)$ by the rule

$$f_T(i,j) = \sum_{i' \leq i} \mu_{i'} + \sum_{i' \leq i, j' \leq j} wt_T^{i'}(j'),$$

where $wt_T^{i'}(j')$ equals the multiplicity of the letter j' in the reading word of the i'-th row of T.

Thus we obtain the famous L-R rule: *the set of integer-valued discretely concave function on* $\Delta(n)$ *with boundary increments* μ, ν *and* λ *is isomorphic to the set of L-R skew tableaux of shape* $\lambda \setminus \mu$ *and of weight* ν.

Another interpretation of the set $DC(\lambda; \mu, \nu)$ of discretely concave functions on the grid $\Delta(n)$ with given boundary increments take the form of the set of *semi-standard skew tableaux of the shape* $\mu * \nu$, *that is a skew diagram obtained by concatenation of the diagram of shape* μ *and the diagram of shape* ν *(see Picture 2), such that the restriction of the tableau to the diagram* μ *yields the Yamanuchi tableaux and the reading of* $\mu * \nu$ *(as above) produces a lattice word.*



$\mu * \nu$

Figure 2.

Now we come to the interpretation of the set of $\hat{\mathbf{L}}$-tight arrays in $B(\mu) \otimes B(\nu) \subset \mathcal{B}(n, \mu_1^* + \nu_1^*)$. (Recall that the $\hat{\mathbf{L}}\hat{\mathbf{D}}$-tight array of $B(n, \infty)$ of shape κ produce the crystal $B(\kappa) \subset \mathcal{B}(n, (\kappa)'_1)$.) Denote by $\mathbf{L}(B(\mu) \otimes B(\nu))$ this set of arrays.

Now we use imbedding of irreducible A_n-type crystals into $\mathcal{B}(n, \infty)$ for another combinatorial interpretation of L-R coefficients.

Recall, that a $\hat{\mathbf{D}}$-tight array of $\mathcal{B}(n, m)$ might be set as a $\prec\prec$-monotone functions $f : [m] \to 2^{[n]}$. In view of this, we can identify arrays of $\mathbf{L}(B(\mu) \otimes B(\nu))$ of shape λ' with the set of *skew semi-standard tableaux* T *of shape* $\lambda' \setminus \mu'$ *and of the weight* $w_0(\nu') := (\nu'_{\nu_1}, \ldots, \nu'_1)$, *where* w_0 *is the longest* *permutation, such that the set constituted of the columns of T in which j is placed dominates (with respect the order $\succ\succ$) that set for $j+1$, $j = 1, \dots, \nu'_1 - 1$.* This combinatorial rule for counting the coefficients $c^{\lambda}_{\mu,\nu}$ is a new one and has a flavour of the Schützenbereger involution to the classical combinatorial rule of L-R rule.

8.1 L-R rule for continuous crystals

For describing the decomposition of the tensor product of continuous crystals into irreducibles, the necessity to count crystals by means of sets and not numbers, the cardinality of the corresponding sets, comes naturally. Thus, we will describe the sets $C^{\lambda}_{\mu,\nu}$ of a decomposition

$$B^c(\mu) \otimes B^c(\nu) \cong \coprod_{\lambda} C^{\lambda}_{\mu,\nu} \times B^c(\lambda) \tag{23}$$

For the decomposition $B^c(\mu) \otimes B^c(\nu)$ of the tensor product of irreducible continuous crystals, the same recipe as for the case of $\mathcal{A}(n,m)$ has sense. Namely, $\mathbf{L}$-tight arrays of $B^c(\mu) \otimes B^c(\nu) \subset \mathcal{A}^{\mathbb{R}}(n,2n)$ of the weight λ is isomorphic to the set $DC(\lambda;\mu,\nu)$ of discretely concave functions on the triangle greed $\Delta(n)$ with the boundary increments μ, ν, and λ Moreover, the set $DC(\lambda;\mu,\nu)$ is naturally isomorphic to a convex polytope. Since $B^c(\lambda)$ is isomorphic to the G-C polytope $GC(\lambda)$, we obtain the following polytopal decomposition

$$GC(\mu) \otimes GC(\nu) \cong \coprod_{\lambda} DC(\lambda;\mu,\nu) \times GC(\lambda). \tag{24}$$

Note, that this isomorphism together with the isomorphism $R^c : GC(\mu) \otimes GC(\nu) \cong GC(\nu) \otimes GC(\mu)$ provide an isomorphism $DC(\lambda;\mu,\nu) \cong DC(\lambda;\nu,\mu)$. The latter isomorphism is a refinement of the symmetry $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$ of LR coefficients.

For integer λ, μ, ν, we pointed out above that the set of $\mathbf{L}$-tight arrays of $B(\mu) \otimes B(\nu) \subset \mathcal{A}(n,2n)$ of the weight λ is isomorphic to the set of integer-valued discretely concave function $DC^{\mathbf{Z}}(\lambda;\mu,\nu)$. Since, for integer λ, μ and ν, $DC^{\mathbf{Z}}(\lambda;\mu,\nu)$ is constituted of the integer points of the polytope $DC(\lambda;\mu,\nu)$, we get

$$GC(\mu)(\mathbb{Z}) \otimes GC(\nu)(\mathbb{Z}) \cong \coprod_{\lambda} DC^{\mathbb{Z}}(\lambda;\mu,\nu) \times GC(\lambda)(\mathbb{Z}), \tag{25}$$

where, for a polytope $P \subset \mathbb{R}^n$, $P(\mathbb{Z}) := P \cap \mathbb{Z}^n$. Since $B(\lambda) \cong GC(\lambda)(\mathbb{Z})$, we can rewrite (25) as

$$B(\mu) \otimes B(\nu) \cong \coprod_{\lambda} DC^{\mathbb{Z}}(\lambda; \mu, \nu) \times B(\lambda). \tag{26}$$

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