THE MODULI STACK OF GIESEKER-SL_2-BUNDLES ON A NODAL CURVE

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1. Introduction

Let $\{Y_t\}$ be a family of smooth curves degenerating to a nodal curve X_0. It is an interesting problem to consider how the moduli spaces of vector bundles on Y_t degenerate. Since the moduli space of vector bundles on the nodal curve X_0 is not compact, we need to find a good compactification. One way to compactify it is to add torsion-free sheaves. Another way, which is originally due to Gieseker [G] and developed by Nagaraj-Seshadri [NS] and Kausz [K2], is to add those vector bundles on a certain semistalbe model of X_0, which let us call Gieseker vector bundles. In these works they consider moduli spaces of vector bundles with fixed degree. In this paper we'd like to consider moduli spaces of vector bundles with fixed determinant.(See [Sun] for related results.)

This paper is heavily based on the work of Kausz [K1] [K2]. So, let me here explain his results briefly. In [K1], Kausz introduced a concept of generalized isomorphisms and showed that a projective variety KGl_n that is a compactification of Gl_n is the fine moduli space of generalized isomorphisms. Then in [K2] he showed that the normalization of the moduli space of Gieseker vector bundles on X_0 is a KGl_n-bundle over the moduli space of vector bundles on the normalization $\widetilde{X}_0$ of X_0. The purpose of this paper is to show that with the techniques invented by Kausz we can also clarify the structure of the moduli space of Gieseker vector bundles of rank 2 with fixed determinant on an irreducible nodal curve X_0.

The contents of the sections are as follows. In section 2 we introduce basic definitions. In section 3 we define θ-determinant generalized isomorphisms (only for rank 2 case), and see that the equivalence classes of θ-determinant generalized isomorphisms form a projective variety KSL_2. In section 4 we define the moduli stack of Gieseker-SL_2-bundles. In section 5 we investigate the local structure of this stack. In section 6 first using the results in section 5 we see that the moduli stack of Gieseker-SL_2-bundles on X_0 is a union of two closed substacks. Then we describe the structure of each substack. Our main theorems (Theorem 6.4 and Theorem 6.5) say that one of the two closed substacks is a KSL_2-bundle over the moduli stack of vector bundles with fixed determinant on the normalization $\widetilde{X}_0$ of X_0, and that the other is non-reduced and its induced reduced substack is a $\overline{PGl_2}(\simeq \mathbb{P}^3)$-bundle over the moduli stack of vector bundles with fixed (but different from the former one) determinant on $\widetilde{X}_0$.

2. Preliminaries and Notations

In this section, we explain some notions and fix some notations that are used in this paper. Most of them are cited directly from [K2, §3].

Throughout this paper, $B := \mathrm{Spec}\mathbb{C}[[t]]$, $B_0 \hookrightarrow B$ is the closed point and B_η is the generic point. $\pi : \mathcal{X} \to B$ is a stable curve of genus $g \geq 2$ over B such that the generic fiber X_η is smooth, the special fiberX_0 is an irreducible curve with only one node Q. We assume that $\mathcal{X}$ is regular and fix a $\mathbb{C}[[t]]$-algebra isomorphism $\widehat{\mathcal{O}}_{\mathcal{X},Q} \simeq \mathbb{C}[[u,v,t]]/(uv-t)$. $\mathfrak{n} : \widetilde{X}_0 \to X_0$ denotes the normalization and put $\{P_1,\, P_2\} := \mathfrak{n}^{-1}(Q)$.

2.1. Let $R := R_1 \cup \cdots \cup R_l$ ($l \geq 1$) be a chain of rational curves, where $R_i \simeq \mathbb{P}^1$, and $R_i \cap R_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Let a, b be closed points of R_1, R_l respectively such that if $l = 1$ then $a \neq b$, and if $l > 1$ then $a \neq R_1 \cap R_2$ and $b \neq R_{l-1} \cap R_l$. Let X_l be the nodal curve that is obtained by identifying the pair of points (P_1, P_2) on $\widetilde{X}_0$ with (a, b) on R. We have the natural morphism $q : \widetilde{X}_0 \sqcup R \to X_l$. By abuse of notation, the points $q(P_1), q(P_2)$ on X_l are also denoted by P_1, P_2 respectively, and $R, R_i, \widetilde{X}_0$ also denote their isomorphic image in X_l by q. By collapsing R to the singular point Q on X_0, we have the morphism $k : X_l \to X_0$. Throughout this paper, we fix this X_l and the morphism k. By convention we let k also denote $\mathrm{id} : X_0 \to X_0$.

Definition 2.2. (i) Let T be a B-scheme and let $f : T \to B$ denote the structure morphism. A modification of $\mathcal{X}$ over T is a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad h \quad} & \mathcal{X} \times_B T \\ & {\scriptstyle pr_2 \circ h} \searrow \quad \swarrow {\scriptstyle pr_2} & \\ & T & \end{array}$$

such that $\mathcal{Y}$ is flat, projective and of finite type over T, and that for any field K and any morphism $\mathrm{Spec}K \to T$ if $f(\mathrm{Spec}K)$ is B_η then $h \times \mathrm{id}_{\mathrm{Spec}K} : \mathcal{Y} \times_T \mathrm{Spec}K \to \mathcal{X} \times_B \mathrm{Spec}K$ is an isomorphism, and if $f(\mathrm{Spec}K)$ is B_0 then for some $l \geq 0$ there is an isomorphism $g : X_l \times \mathrm{Spec}K \to \mathcal{Y} \times_T \mathrm{Spec}K$ satisfying $(h \times \mathrm{id}_{\mathrm{Spec}K}) \circ g = k \times \mathrm{id}_{\mathrm{Spec}K}$.

(ii) Let T be a B_0-scheme. A modification of X_0 over T is a modification of $\mathcal{X}$ over T, where T is regarded as a B-scheme by $B_0 \hookrightarrow B$.

(iii) If K is a field and $\mathrm{Spec}K \to B_0$ is a morphism and $Y \xrightarrow{h} X_0 \times \mathrm{Spec}K$ is a modification of X_0 over $\mathrm{Spec}K$, then l that appears in (i) is called the length of the modification.

Definition 2.3. Let K be a field over $\mathbb{C}$.

(i) $h := k \times \mathrm{id} : X_l \times \mathrm{Spec}K \to X_0 \times \mathrm{Spec}K$ is a modification of length $l \geq 0$ of X_0 over $\mathrm{Spec}K$. A vector bundle E on $X_l \times \mathrm{Spec}K$ is said to be admissible if either (a) or (b) below holds;
  (a) $l = 0$
  (b) $l \geq 1$ and $E|_{R_i}$ is isomorphic to $\mathcal{O}_{R_i}(1)^m \otimes \mathcal{O}_{R_i}^{\mathrm{rank}E-m}$ with $0 < m \leq \mathrm{rank}E$ for $1 \leq i \leq l$ and $\mathrm{H}^0(R, (E|_R)(-P_1 - P_2)) = 0$.

(ii) Let $h : Y \to X_0 \times \mathrm{Spec}K$ be a modification of X_0 over $\mathrm{Spec}K$ and let $g : X_l \times \mathrm{Spec}K \to Y$ be as in (i) of Definittion 2.2. A vector bundle E on Y is said to be admissible if g^*E is admissible.

(iii) Let $f : T \to B$ be a morphism and let $h : \mathcal{Y} \to \mathcal{X} \times_B T$ be a modification of $\mathcal{X}$ over T. A vector bundle $\mathcal{E}$ on $\mathcal{Y}$ is said to be admissible if for any $\mathrm{Spec}K \to T$, where K is a field, such that $f(\mathrm{Spec}K) = B_0$, the pullback of $\mathcal{E}$ to $\mathcal{Y} \times_T \mathrm{Spec}K$ is admissible.

3. KSL_2

Definition 3.1. Let S be a scheme. If we are given 2-bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ on S, and an isomorphism $\theta : \bigwedge^2 \mathcal{V}_1 \to \bigwedge^2 \mathcal{V}_2$, then a θ-determinant generalized isomorphism from $\mathcal{V}_1$ to $\mathcal{V}_2$ is the following data.

(i) 2-bundles $\mathcal{U}_i$ $(i = 1, 2)$ on S;
(ii) bf-morphisms of rank one (cf. [K1, Definition 5.1])
$$g_i := (\mathcal{M}_i, \mu_i, \mathcal{U}_i \xrightarrow{g_i^\sharp} \mathcal{V}_i, \mathcal{M}_i \otimes \mathcal{U}_i \xleftarrow{g_i^\flat} \mathcal{V}_i),$$
$(i = 1, 2)$ from $\mathcal{U}_i$ to $\mathcal{V}_i$;
(iii) an isomorphism $\upsilon : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\upsilon(\mu_1) = \mu_2$;
(iv) an isomorphism $\xi : \mathcal{U}_1 \to \mathcal{U}_2$,

where we require them to satisfy the conditions (a) and (b) below.

(a) For $\forall s \in S$, we have
$$\xi[s] \left(\mathrm{Ker} g_1^\sharp[s]\right) \cap \mathrm{Ker} g_2^\sharp[s] = \{o\},$$
where $?[s]$ means the restriction of $?$ to the fiber over s.
(b) The diagram

$$\begin{array}{ccc} \mathcal{M}_1 \otimes \bigwedge^2 \mathcal{U}_1 & \xrightarrow{\upsilon \otimes \wedge^2 \xi} & \mathcal{M}_2 \otimes \bigwedge^2 \mathcal{U}_2 \\ {\scriptstyle \wedge^{-2} g_1} \uparrow & & \uparrow {\scriptstyle \wedge^{-2} g_2} \\ \bigwedge^2 \mathcal{V}_1 & \xrightarrow[\theta]{} & \bigwedge^2 \mathcal{V}_2 \end{array} \tag{3.1}$$

commutes. (See [K1, Proposition 6.1] for the definition of $\wedge^{-2} g_i$.)

Definition 3.2. Keep the notation in Definition 3.1. Let $\Phi^{(l)} := \left(\mathcal{U}_i^{(l)}, g_i^{(l)}, \mathcal{M}_1^{(l)} \xrightarrow{v^{(l)}} \mathcal{M}_2^{(l)}, \mathcal{U}_1^{(l)} \xrightarrow{\xi^{(l)}} \mathcal{U}_2^{(l)}\right)$ $(l = 1, 2)$ be a θ-determinant generalized isomorphism from $\mathcal{V}_1$ to $\mathcal{V}_2$, where $g_i^{(l)} := (\mathcal{M}_i^{(l)}, \mu_i^{(l)}, \mathcal{U}_i^{(l)} \xrightarrow{g_i^{\sharp(l)}} \mathcal{V}_i^{(l)}, \mathcal{M}_i^{(l)} \otimes \mathcal{U}_i^{(l)} \xleftarrow{g_i^{\flat(l)}} \mathcal{V}_i^{(l)})$. An equivalence from $\Phi^{(1)}$ to $\Phi^{(2)}$ consists of isomorphisms $\mathcal{M}_j^{(1)} \simeq \mathcal{M}_j^{(2)}$ and $\mathcal{U}_j^{(1)} \simeq \mathcal{U}_j^{(2)}$ $(j = 1, 2)$ that are compatible with $v^{(l)}$, $\xi^{(l)}$ and $g_i^{(l)}$.

Definition 3.3. Keep the notation in Definition 3.1. $\mathcal{KSL}_2(\mathcal{V}_1, \mathcal{V}_2)$ is the functor from the category of S-schemes to the category of sets that associates to an S-scheme $T \xrightarrow{\phi} S$ the set of equivalence classes of $\phi^*(\theta)$-determinant generalized isomorphisms from $\phi^*\mathcal{V}_1$ to $\phi^*\mathcal{V}_2$.

Then, as in [K1], we have

Proposition 3.4. *The functor $\mathcal{KSL}_2(\mathcal{V}_1, \mathcal{V}_2)$ is representable by a projective S-scheme $KSL_2(\mathcal{V}_1, \mathcal{V}_2)$.*

4. Gieseker SL_2-bundles

In the rest of this paper, we fix a line bundle $\mathcal{P}$ on $\mathcal{X}$, of degree d on the fibers over B. Put $\mathcal{P}_0 := \mathcal{P}|_{X_0}$.

Definition 4.1. Let S be a B-scheme. A Gieseker-SL_2-bundle with determinat $\mathcal{P}$ on $\mathcal{X}$ over S, or a Gieseker-SL_2-bundle on $(\mathcal{X}; \mathcal{P})$ over S, is a triple $(h : \mathcal{Y} \to \mathcal{X} \times_B S, \mathcal{E}, \delta : \det \mathcal{E} \to (pr_1 \circ h)^*\mathcal{P})$, where $h : \mathcal{Y} \to \mathcal{X} \times_B S$ is a modification, $\mathcal{E}$ is an admissible 2-bundle on $\mathcal{Y}$ of degree d on the fibers over S, and δ is a morphism of $\mathcal{O}_{\mathcal{Y}}$-modules such that its restriction to every fiber of $\mathcal{Y}/S$ is nonzero.

$GSL_2B(\mathcal{X}/B; \mathcal{P})$ denotes the B-groupoid that associates to an affine B-scheme S the groupoid consisting of all the Gieseker-SL_2-bundles on $(\mathcal{X}; \mathcal{P})$ over S. $GSL_2B(X_0/B_0; \mathcal{P}_0)$, or simply $GSL_2B(X_0; \mathcal{P}_0)$, donotes the B_0-groupoid that is the restriction of $GSL_2B(\mathcal{X}/B; \mathcal{P})$ to the category of affine B_0-schemes.

Proposition 4.2. *$GSL_2B(\mathcal{X}/B; \mathcal{P})$ and $GSL_2B(X_0/B_0; \mathcal{P}_0)$ are algebraic stacks.*

Remark 4.3. Let $\{\varphi_{\lambda\mu} : T_\mu \to T_\lambda\}_{\lambda \prec \mu}$ be a projective system of affine B_0-schemes and let $T \xrightarrow{\varphi_\lambda} T_\lambda$ be a projective limit. By [EGAIV, §8 and (11.2.6)], we know that $\mathcal{G} := GSL_2B(X_0; \mathcal{P}_0)$ satisfies the conditions (i) and (ii) below;

(i) For any object $x \in \mathcal{G}(T)$, there exist λ and an object $x_\lambda \in \mathcal{G}(T_\lambda)$ such that $\varphi_\lambda^*(x_\lambda) \simeq x$;

(ii) Take λ_0 and $x_{\lambda_0}, y_{\lambda_0} \in \mathcal{G}(T_{\lambda_0})$. Then the map

$$\varinjlim \mathrm{Hom}_{\mathcal{G}(T_\mu)}(\varphi_{\lambda_0\mu}^* x_{\lambda_0}, \varphi_{\lambda_0\mu}^* y_{\lambda_0}) \to \mathrm{Hom}_{\mathcal{G}(T)}(\varphi_{\lambda_0}^* x_{\lambda_0}, \varphi_{\lambda_0}^* y_{\lambda_0})$$

is bijective.

By this fact, in many proofs we can assume that T is of finite type over B_0.

5. Local Structure

In this section, we investigate the local structure of the algebraic stack of Gieseker-SL_2-bundles.

Let K be a field extension of $\mathbb{C}$. Let $(Y \xrightarrow{h} X_0 \times_{B_0} \mathrm{Spec}K, \mathcal{E}, \det \mathcal{E} \xrightarrow{\delta} (pr_1 \circ h)^* \mathcal{P}_0)$ be a Gieseker-SL_2-bundle over $\mathrm{Spec}K$.

Lemma 5.1. *There are three possibilities:*

(Type 0) *Y is a modification of lengh 0, i.e. h is an isomorphism.*
(Type 1) *Y is a modification of lengh 1, moreover if R is $\mathbb{P}^1$ of Y colapsing to the singular point of $X_0 \times_{B_0} \mathrm{Spec}K$, then $\deg \mathcal{E}|_R = 2$.*
(Type 2) *Y is a modification of lengh 2, moreover if R_i ($i = 1, 2$) is $\mathbb{P}^1$ of Y colapsing to the singular point of $X_0 \times_{B_0} \mathrm{Spec}K$, then $\deg \mathcal{E}|_{R_i} = 1$ for $i = 1$and 2.*

Proof. We have only to exclude the possibility that Y is a modification of lengh 1 and $\deg \mathcal{E}|_R = 1$. Suppose that we had such a Gieseker-SL_2-bundle. Then $\delta|_R$ is zero since $\deg \mathcal{E}|_R = 1 > \deg(pr_1 \circ h)^*\mathcal{P}_0|_R = 0$. Hence $\delta|_{\tilde{X}_0 \times_{B_0} \mathrm{Spec}K}$ factors as $\det \mathcal{E}|_{\tilde{X}_0 \times_{B_0} \mathrm{Spec}K} \to (pr_1 \circ h)^*\mathcal{P}_0|_{\tilde{X}_0 \times_{B_0} \mathrm{Spec}K}(-P_1 - P_2) \hookrightarrow (pr_1 \circ h)^*\mathcal{P}_0|_{\tilde{X}_0 \times_{B_0} \mathrm{Spec}K}$. Since $\deg \mathcal{E}|_{\tilde{X}_0 \times_{B_0} \mathrm{Spec}K} = \deg \mathcal{E} - 1 > \deg(pr_1 \circ h)^*\mathcal{P}_0|_{\tilde{X}_0 \times_{B_0} \mathrm{Spec}K}(-P_1 - P_2)$, we have $\delta|_{\tilde{X}_0 \times_{B_0} \mathrm{Spec}K} = 0$, which implies $\delta = 0$. This contradicts the definition of a Gieseker-SL_2-bundle. □

Notation 5.2. Let $h : Y \to X_0 \times \mathrm{Spec}K$ be a modification of length $l \geq 0$ and let $g : X_l \times \mathrm{Spec}K \to Y$ be as in (i) of Definition 2.2. Recall from the paragraph 2.1 that if $l \geq 1$ then we have P_1, P_2 on X_l. From now on, for $l \geq 1$ the points $g(P_1), g(P_2)$ on Y are also denoted by P_1, P_2. If $l = 2$, then the point $g(R_1 \cap R_2)$ on Y is denoted by P_0. Moreover if $l = 0$, then the point $g(Q)$ on Y is denoted by P_0. The reason why we use this notation will be clear in Proposition 6.1.

In order to investigate the local structure of $GSL_2B(\mathcal{X}/B; \mathcal{P})$, we introduce several deformation functors. Let $\mathcal{A}$ be the category of artinian local $\mathbb{C}[[t]]$-algebra with residue field $\mathbb{C}$. Throughout this section, we fix an object $\mathbb{E}_0 := (Y \xrightarrow{h_0} X_0, E_0, \det E_0 \xrightarrow{\delta_0} h_0^*\mathcal{P}_0)$ of $GSL_2B(\mathcal{X}/B; \mathcal{P})(B_0)$. Put $L_0 := (\det E_0)^\vee \otimes h_0^*\mathcal{P}_0$, and let σ_0 be the global section of L_0 corresponding to δ_0. Let $\mathbb{L}_0$ denote the triple $(Y \xrightarrow{h_0} X_0, L_0, \sigma_0)$.

Definition 5.3. Three funtors $\mathcal{G}$,$\mathcal{F}$ and $\mathcal{M}$ from $\mathcal{A}$ to the category of sets are defined as follows. For $A \in \mathcal{A}$,

$$\mathcal{G}(A) := \left\{ \begin{array}{l} \mathbb{E} := (\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A, \mathcal{E}, \det \mathcal{E} \xrightarrow{\delta} (pr_1 \circ h)^*\mathcal{P}) \\ \in GSL_2B(\mathcal{X}/B; \mathcal{P})(\mathrm{Spec}A) \\ \text{with isomorphism } \mathbb{E} \times_{\mathrm{Spec}A} B_0 \xrightarrow{\alpha} \mathbb{E}_0. \end{array} \right\} \Big/ \sim_{\mathcal{G}},$$

$$\mathcal{F}(A) := \left\{ \begin{array}{l|l} & \mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A \text{ is} \\ \mathbb{L} := (\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A, \mathcal{L}, \sigma) & \text{a modification of } \mathcal{X}/B \\ \text{with isomorphism} & \text{over Spec}A. \\ \mathbb{L} \times_{\mathrm{Spec}A} B_0 \xrightarrow{\beta} \mathbb{L}_0 & \mathcal{L} \text{ is a line bundle on } \mathcal{Y}. \\ & \sigma \text{ is a global section of } \mathcal{L}. \end{array} \right\} \Big/ \sim_{\mathcal{F}},$$

$$\mathcal{M}(A) := \left\{ \begin{array}{l|l} \mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A & \\ \text{ with isomorphism} & \mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A \text{ is} \\ (\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A) \times_{\mathrm{Spec}A} B_0 & \text{a modification of } \mathcal{X}/B \\ \xrightarrow{\gamma} (Y \xrightarrow{h_0} X_0) & \text{over Spec}A. \end{array} \right\} \Big/ \sim_{\mathcal{M}},$$

where the equivalence relations $\sim_{\mathcal{G}}$, $\sim_{\mathcal{F}}$ and $\sim_{\mathcal{M}}$ are as below.

- $(\mathbb{E}, \alpha) \sim (\mathbb{E}', \alpha')$ if and only if there is an isomorphism $\mathbb{E} \xrightarrow{a} \mathbb{E}'$ such that $\alpha = \alpha' \circ (a \times_{\mathrm{Spec}A} B_0)$.
- $(\mathbb{L}, \sigma) \sim (\mathbb{L}', \sigma')$ if and only if there is an isomorphism $\mathbb{L} \xrightarrow{b} \mathbb{L}'$ such that $\beta = \beta' \circ (b \times_{\mathrm{Spec}A} B_0)$.
- $(\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A, \gamma) \sim (\mathcal{Y}' \xrightarrow{h'} \mathcal{X} \times_B \mathrm{Spec}A, \gamma')$ if and only if there is an isomorphism $(\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A) \xrightarrow{c} (\mathcal{Y}' \xrightarrow{h'} \mathcal{X} \times_B \mathrm{Spec}A)$ such that $\gamma = \gamma' \circ (c \times_{\mathrm{Spec}A} B_0)$.

Lemma 5.4. *$\mathcal{G}$, $\mathcal{F}$ and $\mathcal{M}$ satisfy the Schlessinger's condition (i.e. (H_1) (H_2) and (H_3) in Theorem2.11 of* [Sch]*). Therefore they have a hull.*

Proof. We omit the proof. □

We have the natural morphism $\Phi : \mathcal{F} \to \mathcal{M}$ of functors. Using the notation in Definition5.3, by associating $(\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A, (\det \mathcal{E})^\vee \otimes (pr_1 \circ h)^*\mathcal{P}, \sigma) \in \mathcal{F}(A)$ to $(\mathbb{E}, \alpha) \in \mathcal{G}(A)$ (where σ is the one determined by δ), we have the natural morphism $\Psi : \mathcal{G} \to \mathcal{F}$.

Lemma 5.5. *$\Psi : \mathcal{G} \to \mathcal{F}$ is smooth.*

Proof. Left to the reader. □

Let $\widehat{\mathcal{A}}$ be the category of complete noetherian local $\mathbb{C}[[t]]$-algebras A such that $A/\mathfrak{m}^n$ is in $\mathcal{A}$ for all $n \in \mathbb{N}$. For $R \in \widehat{\mathcal{A}}$, we set $h_R(A) := \mathrm{Hom}(R, A)$ to define a functor h_R on $\mathcal{A}$.

Theorem 5.6. *Let $h_R \to \mathcal{F}$ be a hull of $\mathcal{F}$.*

(0) *If $\mathbb{E}_0$ is of Type 0, then we have an isomorphism $R \simeq \mathbb{C}[[t]]$ of $\mathbb{C}[[t]]$-algebras.*

(1) *If* $\mathbb{E}_0$ *is of Type* 1*, then we have an isomorphism* $R \simeq \mathbb{C}[[t, t_1, u]]/(t - t_1^2)$ *of* $\mathbb{C}[[t]]$*-algebras.*
(2) *If* $\mathbb{E}_0$ *is of Type* 2*, then we have an isomorphism* $R \simeq \mathbb{C}[[t, t_0, t_1, u]]/(t - t_0 t_1^2)$ *of* $\mathbb{C}[[t]]$*-algebras.*

Corollary 5.7. *The algebraic* B*-stack* $GSL_2B(\mathcal{X}/B; \mathcal{P})$ *is regular.*

Proof. This follows from Lemma 5.5 and Theorem 5.6. □

The rest of this section is devoted to the proof of Theorem 5.6.

Proof of (0) of Theorem 5.6. It suffices to prove that for any $A \in \mathcal{A}$ $\mathcal{F}(A)$ is a set consisting of one element. Since $\mathbb{E}_0$ is of Type 0, we may assume that $\mathbb{L}_0 = (X_0 \xrightarrow{id} X_0, \mathcal{O}_{X_0}, 1)$. For $A \in \mathcal{A}$, $\mathbb{L} := (\mathcal{X} \times_B \mathrm{Spec}A \xrightarrow{id} \mathcal{X} \times_B \mathrm{Spec}A, \mathcal{O}_{\mathcal{X}\times_B \mathrm{Spec}A}, 1)$ with the canonical isomorphism $\mathbb{L} \times_{\mathrm{Spec}A} B_0 \xrightarrow{\beta} \mathbb{L}_0$ gives an element of $\mathcal{F}(A)$. Take an element $(\mathbb{L}', \beta')$ of $\mathcal{F}(A)$, where $\mathbb{L}' = (\mathcal{Y} \xrightarrow{h'} \mathcal{X} \times_B \mathrm{Spec}A, \mathcal{L}, \sigma)$ and $\beta' : \mathbb{L}' \times_{\mathrm{Spec}A} B_0 \xrightarrow{\sim} \mathbb{L}_0$. Let us prove that $(\mathbb{L}, \beta) \sim_{\mathcal{F}} (\mathbb{L}', \beta')$. Since h' is an isomorphism and σ is a nowhere-vanishing section of $\mathcal{L}$, we may assume that $\mathbb{L}' = (\mathcal{X} \times_B \mathrm{Spec}A \xrightarrow{id} \mathcal{X} \times_B \mathrm{Spec}A, \mathcal{O}_{\mathcal{X}\times_B \mathrm{Spec}A}, 1)$. Then β' must be the canonical isomorphism. Thus $(\mathbb{L}, \beta) \sim_{\mathcal{F}} (\mathbb{L}', \beta')$. □

We shall give a proof of only (2) of Theorem 5.6 because (1) of Theorem 5.6 is proved similarly. In the rest of the proof of Theorem 5.6, we assume that $\mathbb{E}_0$ is of Type 2. Put $W := \mathrm{Spec}\mathbb{C}[[t_0, t_1, t_2]]$ and let $f : W \to B$ be given by $f^*(t) = t_0 t_1 t_2$. By [G, §4], there exists a modification $\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B W$ of $\mathcal{X}/B$ over W that gives a hull of $\mathcal{M}$. Since $Y \xrightarrow{h_0} X_0$ is a modification of length 2, Y_0 is a union of $\widetilde{X}_0$ and a chain $R_1 \cup R_2$ of $\mathbb{P}^1$ with $\{P_i\} = \widetilde{X_0} \cap R_i$ and $\{P_0\} := R_1 \cap R_2$.(Recall the notation 5.2.) Moreover we can find an isomorphism

$$(\spadesuit) \qquad \widehat{\mathcal{O}}_{\mathcal{Y},P_i} \simeq \mathbb{C}[[t_0, t_1, t_2, x_i, y_i]]/(x_i y_i - t_i),$$

of $\mathbb{C}[[t_0, t_1, t_2]]$-algebra $(0 \le i \le 2)$. We fix $(\spadesuit)$ and injective morphisms (5.1)

$$\mathbb{C}[[t_0, t_1, t_2, x_i, y_i]]/(x_i y_i - t_i) \hookrightarrow \mathbb{C}[[t_0, t_1, t_2]]((x_i)) \oplus \mathbb{C}[[t_0, t_1, t_2]]((y_i)),$$

given by $x_i \mapsto (x_i, t_i/y_i)$ and $t_i \mapsto (t_i/x_i, y_i)$.

If A is an artinian local $\mathbb{C}[[t_0, t_1, t_2]]$-algebra with residue field $\mathbb{C}$, the pull-back of the versal deformation by $\mathrm{Spec}A \to W$ gives an infinitesimal deformation $\mathcal{Y}_A \xrightarrow{h_A} \mathcal{X} \times_B \mathrm{Spec}A$ of $Y \xrightarrow{h_0} X_0$. Let $j_A^{(i)}$ be the natural morphism $j_A^{(i)} : \mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}_A,P_i} \to \mathcal{Y}_A$ $(i = 1, 2)$. Put $U_A := \mathcal{Y}_A \setminus \{P_1, P_2\}$. The base change of (5.1) gives rise to the isomorphism

$$(\clubsuit_A) \qquad \mathrm{H}^0\left(\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}_A,P_i} \setminus \{P_i\}, \mathcal{O}\right) \simeq A((x_i)) \oplus A((y_i)).$$

Since by definition $L_0 = (\det E_0)^\vee \otimes h_0^*\mathcal{P}_0$, we have $\deg L_0|_{\widetilde{X}_0} = 2$ and $\deg L_0|_{R_i} = -1$ $(i = 1, 2)$. The nonzero section σ_0 vanishes on $R_1 \cup R_2$, and gives an isomorphism $\mathcal{O}_{\widetilde{X}_0} \xrightarrow{\sim} (L_0|_{\widetilde{X}_0})(-P_1 - P_2)$. Therefore L_0 is obtained by gluing at P_1 and P_2 the two line bundles $\mathcal{O}_{\widetilde{X}_0}(P_1+P_2)$ on $\widetilde{X}_0$ and $\mathcal{O}_{R_1\cup R_2}(-P_1 - P_2)$ on $R_1 \cup R_2$. By this, we have the trivializations $\varphi^{(i)}_{\mathbb{C}} : j^*_{\mathbb{C}} L_0 \xrightarrow{\sim} \widehat{\mathcal{O}}_{Y_0,P_i}$ $(i = 1, 2)$ and $\psi_{\mathbb{C}} : L_0|_U \xrightarrow{\sim} \mathcal{O}_{U_{\mathbb{C}}}$ such that on $\mathrm{Spec}\widehat{\mathcal{O}}_{Y_0,P_i} \setminus \{P_i\}$ $\psi_{\mathbb{C}} \circ \varphi^{(i)-1}_{\mathbb{C}}$ is given by $(a_i x_i, \frac{1}{y_i})$-multiplication for some nonzero complex number a_i, where $\mathbb{C}$ is considered as a $\mathbb{C}[[t_0, t_1, t_2]]$-algebra by $\mathbb{C} \simeq \mathbb{C}[[t_0, t_1, t_2]]/(t_0t_1t_2)$. By replacing the isomorphisms (♠) if necessary, we may assume that $a_1 = a_2 = 1$. Moreover replacing $\varphi^{(i)}_{\mathbb{C}}$ and $\psi_{\mathbb{C}}$ if necessary, we may assume that $\varphi^{(i)}_{\mathbb{C}}(j^*_{\mathbb{C}}\sigma_0) = y_i$ and

$$\psi_{\mathbb{C}}(\sigma_0|_{U_{\mathbb{C}}}) = \begin{cases} 1 & \text{on } \widetilde{X}_0 \setminus \{P_1, P_2\} \\ 0 & \text{on } R_1 \cup R_2 \setminus \{P_1, P_2\}. \end{cases} \tag{5.2}$$

Put $R := \mathbb{C}[[t_1, t_2, t_3, v]]/(t_1(1+v) - t_2)$ and let $\mathfrak{m}$ be its maximal ideal. For $\forall k > 0$, let $\mathcal{L}_{R/\mathfrak{m}^k}$ be a line bundle on $\mathcal{Y}_{R/\mathfrak{m}^k}$ (the pull-back by $\mathrm{Spec}R/\mathfrak{m}^k \to W$ of the versal deformation) that has the trivializations $\varphi^{(i)}_{R/\mathfrak{m}^k} : j^{(i)*}_{R/\mathfrak{m}^k}\mathcal{L}_{R/\mathfrak{m}^k} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{Y}_{R/\mathfrak{m}^k},P_i}$ and $\psi_{R/\mathfrak{m}^k} : \mathcal{L}_{R/\mathfrak{m}^k}|_{U_{R/\mathfrak{m}^k}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{U_{R/\mathfrak{m}^k}}$ such that $\psi_{R/\mathfrak{m}^k} \circ \varphi^{(1)-1}_{R/\mathfrak{m}^k}$ on $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}_{R/\mathfrak{m}^k}} \setminus \{P_1\}$ is given by $(x_1(1+v), \frac{1}{y_1})$-multiplication and $\psi_{R/\mathfrak{m}^k} \circ \varphi^{(2)-1}_{R/\mathfrak{m}^k}$ on $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}_{R/\mathfrak{m}^k}} \setminus \{P_2\}$ is given by $(x_2, \frac{1}{y_2})$-multiplication. Let $\sigma_{R/\mathfrak{m}^k}$ be the global section of $\mathcal{L}_{R/\mathfrak{m}^k}$ such that $\varphi^{(i)}_{R/\mathfrak{m}^k}(j^{(i)*}_{R/\mathfrak{m}^k}\sigma_{R/\mathfrak{m}^k}) = y_i$ and

$$\psi_{R/\mathfrak{m}^k}(\sigma_{R/\mathfrak{m}^k}|_{U_{R/\mathfrak{m}^k}}) = \begin{cases} 1 & \text{on } \widetilde{X}_0 \setminus \{P_1, P_2\} \\ t_1(1+v) = t_2 & \text{on } R_1 \cup R_2 \setminus \{P_1, P_2\}, \end{cases} \tag{5.3}$$

(note that, as a topological space, $U_{R/\mathfrak{m}^k}$ is a disjoint union of $\widetilde{X}_0 \setminus \{P_1, P_2\}$ and $R_1 \cup R_2 \setminus \{P_1, P_2\}$). These data give us the formal object $(\mathcal{L}_\infty, \sigma_\infty)$, where $\mathcal{L}_\infty$ is a line bundle on $\mathcal{Y} \times_W \mathrm{Spf} R$, and thus an element $\hat{\xi} = (\xi_k) \in \varprojlim \mathcal{F}(R/\mathfrak{m}^k)$, in other words, a morphism of functors $\Upsilon : h_R \to \mathcal{F}$.

The following proposition completes the proof of Theorem 5.6 (2).

Proposition 5.8. $\Upsilon : h_R \to \mathcal{F}$ *is a hull of* $\mathcal{F}$.

Proof. We will apply Propositon 7.1. Lemma 5.4 implies (a). Since $\mathcal{M}(\mathbb{C}[[t]]/(t^2)) = \phi$ and we have a morphism of functors $\mathcal{F} \xrightarrow{\Phi} \mathcal{M}$, (b) also holds. R satisfies (i). Let us see that (ii) holds. Let φ be the natural morphism

$$\mathrm{Hom}_{loc\,\mathbb{C}[[t]]-alg}(R, \mathbb{C}[\epsilon]) \to \mathrm{Hom}_{loc\,\mathbb{C}[[t]]-alg}(\mathbb{C}[[t_0, t_1, t_2]], \mathbb{C}[\epsilon]), \tag{5.4}$$

where $\epsilon^2 = 0$ and $\mathbb{C}[\epsilon]$ is considered as a $\mathbb{C}[[t]]$-algebra by $t \mapsto 0$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{loc\,\mathbb{C}[[t]]-alg}(R, \mathbb{C}[\epsilon]) & \longrightarrow & \mathcal{F}(\mathbb{C}[\epsilon]) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ \mathrm{Hom}_{loc\,\mathbb{C}[[t]]-alg}(\mathbb{C}[[t_0, t_1, t_2]], \mathbb{C}[\epsilon]) & \longrightarrow & \mathcal{M}(\mathbb{C}[\epsilon]). \end{array} \tag{5.5}$$

If $\mathbb{C}[[v]]$ is the quotient of R in which t_i's are mapped to zero, we have the natural isomorphism $\mathrm{Ker}\varphi \simeq \mathrm{Hom}_{loc\,\mathbb{C}[[t]]-alg}(\mathbb{C}[[v]], \mathbb{C}[\epsilon])$. Consider the category $\mathcal{K}$ whose objects are triples $(\mathcal{L}, \sigma, \beta : \mathcal{L}|_{X_2} \xrightarrow{\sim} L_0)$, where $\mathcal{L}$ is a line bundle on $X_2 \times_{\mathrm{Spec}\mathbb{C}} \mathrm{Spec}\mathbb{C}[\epsilon]$, σ is a global section of $\mathcal{L}$ and β is an isomorphism with $\beta(\sigma|_{X_2}) = \sigma_0$, and whose morphisms from $(\mathcal{L}, \sigma, \beta : \mathcal{L}|_{X_2} \xrightarrow{\sim} L_0)$ to $(\mathcal{L}', \sigma', \beta' : \mathcal{L}'|_{X_2} \xrightarrow{\sim} L_0)$, are pairs $(f : X_2 \times \mathrm{Spec}\mathbb{C}[\epsilon] \to X_2 \times \mathrm{Spec}\mathbb{C}[\epsilon], \mathcal{L} \xrightarrow{\tau} f^*\mathcal{L}')$, where f is $\mathbb{C}[\epsilon]$-isomorphism with $f|_{X_2} = \mathrm{id}_{X_2}$ and $(h_0 \times \mathrm{id}_{\mathrm{Spec}\mathbb{C}[\epsilon]}) \circ f = h_0 \times \mathrm{id}_{\mathrm{Spec}\mathbb{C}[\epsilon]}$, and τ is an isomorphism with $\tau(\sigma) = f^*\sigma'$. Then $\mathrm{Ker}\Phi$ is isomorphic to the set of isomorphism classes of the category $\mathcal{K}$.

Claim 5.8.1. $\mathrm{Ker}\varphi \to \mathrm{Ker}\Phi$ is bijective.

Proof of Claim 5.8.1. In the proof we will use Čech cohomologies involving formal neighborhoods. See Proposition 7.3 for the justification of this calculation.

Surjectivity: Take an object $(\mathcal{L}, \sigma, \beta : \mathcal{L}|_{X_2} \xrightarrow{\sim} L_0)$ of the categry $\mathcal{K}$. Since we have the trivial extension of L_0 over $X_2 \times \mathrm{Spec}\mathbb{C}[\epsilon]$, the equivalence classes of extensions of L_0 over $X_2 \times \mathrm{Spec}\mathbb{C}[\epsilon]$ are classified by $\mathrm{H}^1(X_2, \mathcal{O}_{X_2})$. Let $\mathrm{H}^1(X_2, \mathcal{O}_{X_2}) \xrightarrow{\mathrm{H}^1_\sigma} \mathrm{H}^1(X_2, L_0)$ be the morphism induced by the global section σ_0. For $a \in \mathrm{H}^1(X_2, \mathcal{O}_{X_2})$, let $\mathcal{L}^a$ be the corresponding extension of L_0. Then $\mathrm{H}^1_\sigma(a)$ is the obstruction for the existence of a lifting of σ_0 to $\mathcal{L}^a$. Hence $\mathcal{L}$ corresponds to a cohomology class in $\mathrm{Ker}\mathrm{H}^1_\sigma$. Note that H^1_σ factors as $\mathrm{H}^1(X_2, \mathcal{O}_{X_2}) \xrightarrow{\mathrm{H}^{1(1)}_\sigma} \mathrm{H}^1(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0}) \xrightarrow{\mathrm{H}^{1(2)}_\sigma} \mathrm{H}^1(X_2, L_0)$ and that $\mathrm{H}^{1(2)}_\sigma$ is injective because of the long exact sequence of cohomologies of the exact sequence $0 \to \mathcal{O}_{\widetilde{X}_0} \to L_0 \to \mathcal{O}_{R_1 \cup R_2}(-P_1 - P_2) \to 0$. Thus $\mathrm{Ker}\mathrm{H}^1_\sigma = \mathrm{Ker}\mathrm{H}^{1(1)}_\sigma$. The exact sequence $0 \to \mathcal{O}_{R_1 \cup R_2}(-P_1 - P_2) \to \mathcal{O}_{X_2} \to \mathcal{O}_{\widetilde{X}_0} \to 0$ gives rise to the exact sequence $0 \to \mathrm{H}^1(R_1 \cup R_2, \mathcal{O}_{R_1 \cup R_2}(-P_1 - P_2)) \to \mathrm{H}^1(\mathcal{O}_{X_2}) \to \mathrm{H}^1(\mathcal{O}_{\widetilde{X}_0}) \to 0$. Therefore

$$\mathrm{Ker}\mathrm{H}^1_\sigma \simeq \mathrm{H}^1(R_1 \cup R_2, \mathcal{O}_{R_1 \cup R_2}(-P_1 - P_2)). \tag{5.6}$$

For $a \in \mathbb{C}$, let $g_a : \mathrm{Spec}\mathbb{C}[\epsilon] \to \mathrm{Spec}\mathbb{C}[[v]] (\hookrightarrow \mathrm{Spec}R)$ be the morphism given by $g_a^\#(v) = a \cdot \epsilon$. Then, by the construction of $\mathcal{L}_\infty$, $(1_{X_2} \times g_a)^*\mathcal{L}_\infty$ corresponds to the cohomology class $[((a, 0), (0, 0))] \in \mathrm{H}^1(\mathcal{O}_{X_2})$, where $(a, 0) \in \mathbb{C}((x_1)) \oplus \mathbb{C}((y_1))$ and $(0, 0) \in \mathbb{C}((x_2)) \oplus \mathbb{C}((y_2))$. By the isomorphism (5.6), this class corresponds to $[(a, 0)] \in \mathrm{H}^1(R_1 \cup$

$R_2, \mathcal{O}_{R_1 \cup R_2}(-P_1 - P_2))$, where $a \in \mathbb{C}((x_1))$ and $0 \in \mathbb{C}((x_2))$. It is straightforward to chech that we have the isomorphism $\mathbb{C} \simeq \mathrm{H}^1(R_1 \cup R_2, \mathcal{O}_{R_1 \cup R_2}(-P_1 - P_2))$ by associating to $a \in \mathbb{C}$ the class $[(a, 0)]$. Hence for some $a \in \mathbb{C}$, we have an isomorphism $\tau : \mathcal{L} \xrightarrow{\sim} (1_{X_2} \times g_a)^* \mathcal{L}_\infty$ such that $\tau|_{X_2} = \beta$. We have $\tau(\sigma) = (1 + b \cdot \epsilon) \cdot (1_{X_2} \times g_a)^*(\sigma_\infty)$ for some $b \in \mathbb{C}$. Replacing τ by $(1 - b \cdot \epsilon) \cdot \tau$, we prove the surjectivity of $\mathrm{Ker}\varphi \to \mathrm{Ker}\Phi$.

Injectivity: First we recall the following general fact.

Fact. Let Z be a $\mathbb{C}$-scheme. Then $\mathrm{H}^0(T_Z)$ classifies $\mathbb{C}[\epsilon]$-automorphism of $Z \times \mathrm{Spec}\mathbb{C}[\epsilon]$ that is identity over $\mathrm{Spec}\mathbb{C}$. Moreover, let $\mathcal{U} = \{U_i\}$ be an affine open covering of Z and M_0 a line bundle on Z defined by a cocycle $\{\xi_{ij}\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{O}_Z^\times)$. Let f be a $\mathbb{C}[\epsilon]$-automorphism of $Z \times \mathrm{Spec}\mathbb{C}[\epsilon]$ determined by a derivation $\partial \in \mathrm{H}^0(T_Z)$. If a line bundle $\mathcal{M}$ on $Z \times \mathrm{Spec}\mathbb{C}[\epsilon]$ is an extension of M_0 determined by $\mu \in \mathrm{H}^1(Z, \mathcal{O}_Z)$ and if $\mu' \in \mathrm{H}^1(Z, \mathcal{O}_Z)$ is the cohomology class corresponding to $f^*\mathcal{M}$, then the cohomology class $\mu' - \mu \in \mathrm{H}^1(Z, \mathcal{O}_Z)$ is given by the cocycle $\{\partial \xi_{ij} / \xi_{ij}\}$.

Let us apply this fact to our situation. Let f be a $\mathbb{C}[\epsilon]$-automorphism of $X_2 \times \mathrm{Spec}\mathbb{C}[\epsilon]$ with $(h_0 \times \mathrm{id}_{\mathrm{Spec}\mathbb{C}[\epsilon]}) \circ f = h \times \mathrm{id}_{\mathrm{Spec}\mathbb{C}[\epsilon]}$. Around P_i, the derivation ∂ corresponding to f is written as $\partial\left(A(x_i, y_i)\right) = \eta_i x_i \frac{\partial}{\partial x_i} A(x_i, 0)$ for some $\eta_i \in \mathbb{C}[[x_i]]$ with $\eta_i(0) = 1$. Hence $\partial(x_i, 1/y_i)/(x_i, 1/y_i) = (\eta_i, 0) \in \mathbb{C}((x_i)) \oplus \mathbb{C}((y_i))$. Now the cohomology class of $\mathrm{H}^1(X_2, \mathcal{O}_{X_2})$ represented by the Čech cocycle $((\eta_1, 0), (\eta_2, 0))$ is zero because we have $[((\eta_1, 0), (\eta_2, 0))] = [((0, -\eta_1(0)), (0, -\eta_2(0)))]$ in $\mathrm{H}^1(X_2, \mathcal{O}_{X_2})$, and the latter is zero since $\eta_1(0) = \eta_2(0) = 1$. This implies $f^*(1_{X_2} \times g_a)^* \mathcal{L}_\infty \simeq (1_{X_2} \times g_a)^* \mathcal{L}_\infty$. Hence $\mathrm{Ker}\varphi \to \mathrm{Ker}\Phi$ is injective. This completes the proof of Claim 5.8.1. □

Claim 5.8.2. $\mathrm{Im}\varphi \to \mathrm{Im}\Phi$ is bijective.

proof of Claim 5.8.2. We have only to prove the surjectivity of $\mathrm{Im}\varphi \to \mathrm{Im}\Phi$. Let $g : \mathrm{Spec}\mathbb{C}[\epsilon] \to W = \mathrm{Spec}\mathbb{C}[[t_0, t_1, t_2]]$ be the morphism given by $g^\#(t_j) = a_j \cdot \epsilon$ ($a_j \in \mathbb{C}$). Assume that the pull-back $\mathcal{Y} \times_W \mathrm{Spec}\mathbb{C}[\epsilon]$ by g of the versal family $\mathcal{Y}/W$ is in $\mathrm{Im}\Phi$. This means that there is a line bundle $\mathcal{L}$ with a section σ on $\mathcal{Y} \times_W \mathrm{Spec}\mathbb{C}[\epsilon]$ that is an extension of the line bundle L_0 and its global section σ_0 on $\mathcal{Y} \times_W \mathrm{Spec}\mathbb{C} \simeq X_2$. At $P_i \in \mathcal{Y} \times_W \mathrm{Spec}\mathbb{C}[\epsilon]$, we have the isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{Y} \times_W \mathrm{Spec}\mathbb{C}[\epsilon], P_i} \simeq \mathbb{C}[\epsilon][[x_i, y_i]]/(x_i y_i - a_i \epsilon_i)$$

induced by (♠). Let $\mathcal{L}'$ be the extension of L_0 to $\mathcal{Y} \times_W \mathrm{Spec}\mathbb{C}[\epsilon]$ given by the Čech cocycle $\{(x_i, 1/y_i)\}_{i=1,2}$, where $(x_i, 1/y_i) \in \mathbb{C}[\epsilon]((x_i)) \oplus \mathbb{C}[\epsilon]((y_i)) \simeq \mathrm{H}^0(\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y} \times_W \mathrm{Spec}\mathbb{C}[\epsilon], P_i} - \{P_i\}, \mathcal{O})$. Let $o(\mathcal{L})$ and $o(\mathcal{L}') \in \mathrm{H}^1(X_2, L_0)$ be the obstructions for the existence of a lifting of σ_0 of L_0 to $\mathcal{L}$ and $\mathcal{L}'$ respectively. It is easy to see that $o(\mathcal{L}) - o(\mathcal{L}') = \sigma_0 \cdot \xi$, where ξ is an element of $\mathrm{H}^1(X_2, L_0)$ corresponding to the difference of

$\mathcal{L}$ and $\mathcal{L}'$. By assumption we have $o(\mathcal{L}) = 0$. As in the proof of Claim 5.8.1, we have the exact sequence

(5.7)
$$\mathrm{H}^1(X_2, \mathcal{O}_{X_2}) \xrightarrow{\mathrm{H}^1_\sigma} \mathrm{H}^1(X_2, L_0) \xrightarrow{\text{restr.}} \mathrm{H}^1(R_1 \cup R_2, \mathcal{O}(-P_1 - P_2)) \to 0.$$

By a concrete calculation, we find that $o(\mathcal{L}')|_{R_1 \cup R_2}$ is represented by the Čech cocycle (a_1, a_2), where $a_i \in \mathbb{C}$ is considered as an element of $\mathrm{H}^0(\mathrm{Spec}\widehat{\mathcal{O}}_{R_i,P_i} - \{P_i\}, \mathcal{O}(-P_i))$. It is easy to see an isomorphism $\mathrm{H}^1(R_1 \cup R_2, \mathcal{O}(-P_1 - P_2)|_{R_1 \cup R_2}) \simeq \mathbb{C}$ is given by associating $a_1 - a_2$ to the cohomology class $[(a_1, a_2)]$. $o(\mathcal{L}')|_{R_1 \cup R_2} = 0$ implies $a_1 = a_2$. This completes the proof of Claim 5.8.2. □

By the two claims above, we complete the proof of Proposition 5.8. □

Now for the fixed $\mathbb{E}_0 := (Y_0 \xrightarrow{h_0} X_0, E_0, \det E_0 \xrightarrow{\delta_0} h_0^*\mathcal{P}_0)$, assume that $Y_0 \xrightarrow{h_0} X_0$ is of type 1 or 2. For $A \in \mathcal{A}$ and $\mathbb{L} = (\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_B \mathrm{Spec}A, \mathcal{L}, \sigma) \in \mathcal{F}(A)$, let $\mathcal{Z}_i$ be the closed subscheme of $\mathcal{Y}$ whose support is $\{P_i\}$ and whose defining ideal is the first Fitting ideal of $\Omega_{\mathcal{Y}/\mathrm{Spec}A}$ at P_i. Then $(\mathrm{pr}_2 \circ h)|_{\mathcal{Z}_i} : \mathcal{Z}_i \to \mathrm{Spec}A$ is a closed immersion, and it is an isomorphism if and only if the infinitesimal deformation of the node P_i is a trivial deformation. Moreover, as a corollary of the proof of Theorem 5.6, we have

Corollary 5.9. $(\mathrm{pr}_2 \circ h)|_{\mathcal{Z}_i} : \mathcal{Z}_i \to \mathrm{Spec}A$ $(i = 1, 2)$ *define the same closed subscheme of* $\mathrm{Spec}A$.

We here prepare one proposition that is used in the next section.

Proposition 5.10. *For $A \in \mathcal{A}$, let*

(5.8)
$$\begin{array}{ccc} Y & \xrightarrow{g} & \mathcal{Y} \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h} \\ X_0 & \longrightarrow & \mathcal{X} \times_B \mathrm{Spec}A \\ \downarrow & & \downarrow \\ B_0 = \mathrm{Spec}\mathbb{C} & \longrightarrow & \mathrm{Spec}A \end{array}$$

be an object of $\mathcal{M}(A)$. Let $\iota_0 : \widetilde{X}_0 \to Y$ denote the unique morphism satisfying $h_0 \circ \iota_0 = \mathfrak{n}$. Assume that $(\mathrm{pr}_2 \circ h)|_{\mathcal{Z}_i} : \mathcal{Z}_i \to \mathrm{Spec}A$ $(i = 1, 2)$ are isomorphisms, or equivalently that the infinitesimal deformations of the nodes P_i are trivial. Then there exists a unique closed immersion $\iota : \widetilde{X}_0 \times_{\mathrm{Spec}\mathbb{C}} \mathrm{Spec}A \to \mathcal{Y}$ such that $\iota|_{\widetilde{X}_0} = g \circ \iota_0$ and that the closed subscheme of $\mathcal{Y}$ determined by $\iota|_{\{P_i\} \times \mathrm{Spec}A} : \{P_i\} \times \mathrm{Spec}A \to \mathcal{Y}$ is $\mathcal{Z}_i$ for $i = 1, 2$.

Proof. Fix two distinct points a_1, a_2 on $\mathbb{P}^1$. The section $(a_1, \mathrm{id}_{\mathrm{Spec}\mathbb{C}[[u]]})$: $\mathrm{Spec}\mathbb{C}[[u]] \to \mathbb{P}^1 \times \mathrm{Spec}\mathbb{C}[[u]]$ of $\mathrm{pr}_2 : \mathbb{P}^1 \times \mathrm{Spec}\mathbb{C}[[u]] \to \mathrm{Spec}\mathbb{C}[[u]]$ is denoted by α_i. Let $\beta : G \to \mathbb{P}^1 \times \mathrm{Spec}\mathbb{C}[[u]]$ be the blowing-up at $(a_1, 0)$. Let $s_i : \mathrm{Spec}\mathbb{C}[[u]] \to G$ be the section of $G \xrightarrow{\mathrm{pr}_2 \circ \beta} \mathrm{Spec}\mathbb{C}[[u]]$ with $\beta \circ s_i = \alpha_i$. The section $(P_i, \mathrm{id}) : \mathrm{Spec}\mathbb{C}[[u]] \to \widetilde{X}_0 \times \mathrm{Spec}\mathbb{C}[[u]]$ is denoted by s'_i $(i = 1, 2)$. Let $q : \mathbb{P}^1 \to X_0$ be the composite $\mathbb{P}^1 \xrightarrow{pr} \mathrm{Spec}\mathbb{C} \to \{Q\} \subset X_0$. (Recall that Q is the unique node of X_0.) Put $m := (q \times \mathrm{id}_{\mathrm{Spec}\mathbb{C}[[u]]}) \circ \beta$. If $\mathcal{Y}^*$ denotes the flat family over $\mathrm{Spec}\mathbb{C}[[u]]$ that is constructed from $\widetilde{X}_0 \times \mathrm{Spec}\mathbb{C}[[u]]$ and G by gluing the sections s_i and s'_i $(i = 1, 2)$, we have a morphism $h : \mathcal{Y}^* \to X_0 \times \mathrm{Spec}\mathbb{C}[[u]]$ because $m \circ s_i = (\mathfrak{n} \times 1) \circ s'_i$. Regard $\mathbb{C}[[u]]$ as a $\mathbb{C}[[t]]$-algebra by $\mathbb{C}[[t]]/(t) \hookrightarrow \mathbb{C}[[u]]$. Applying Proposition 7.1, it is easily seen that $h : \mathcal{Y}^* \to X_0 \times \mathrm{Spec}\mathbb{C}[[u]]$ is a versal family of the deformation of the modification $h_0 : Y \to X_0$ with the singularities P_1, P_2 non-deformed. This implies the existence of $\iota : \widetilde{X}_0 \times \mathrm{Spec}A \to \mathcal{Y}$ in the proposition. The uniqueness follows since infinitesimal automorphism of $\widetilde{X}_0$ with $\{P_i\}$ fixed are trivial because the 2-pointed curve $(\widetilde{X}_0; P_1, P_2)$ is stable. □

6. Global Structure

Proposition 6.1. *Let T be a B_0-scheme. Let $(h : \mathcal{Y} \to X_0 \times_{B_0} T, \mathcal{E}, \delta : \wedge^2\mathcal{E} \to (pr_1 \circ h)^*\mathcal{P}_0)$ be a Gieseker-SL_2-bundle on $(X_0, \mathcal{P}_0)$ over T. Then there are closed subsets Π_i $(i = 0, 1, 2)$ of $\mathcal{Y}$ such that $\Pi_i \times_T \mathrm{Spec}\mathbb{C}(t) = \{P_i\}$ for every $t \in T$. (Here recall the notation 5.2.)*

Proof. We may assume that T is reduced irreducible and of finite type over B_0. We may also assume that T is normal. For $0 \leq l \leq 2$, T_l is defined to be the subset of T that consists of all points $t \in T$ such that $h \times_T \mathrm{id} : \mathcal{Y} \times_T \mathrm{Spec}\mathbb{C}(t) \to X_0 \times_{B_0} \mathrm{Spec}\mathbb{C}(t)$ is of length l. It is easy to see that $\bigcup_{l \leq m} T_l$ is open in T. Let η be the generic point of T. We have an isomorphism

$$\mathcal{Y} \times_T \mathrm{Spec}\mathbb{C}(\eta) \simeq X_l \times_{B_0} \mathrm{Spec}\mathbb{C}(\eta) \tag{6.1}$$

over $X_0 \times_{B_0} \mathrm{Spec}\mathbb{C}(\eta)$ for some $0 \leq l \leq 2$. Let $\sigma_i : \mathrm{Spec}\mathbb{C}(\eta) \to \mathcal{Y} \times_T \mathrm{Spec}\mathbb{C}(\eta)$ be the morphism that maps η to P_i, and put $\Pi_i := \overline{\sigma_i(\eta)} \subset \mathcal{Y}$, which is given the reduced scheme structure, where $i = 0$ if $l = 0$, $i \in \{1, 2\}$ if $l = 1$, and $i \in \{0, 1, 2\}$ if $l = 2$. Since the isomorphism (6.1) extends over a nonempty open subset $U \subset T$, we have $\Pi_i \times_T \mathrm{Spec}\mathbb{C}(t) = \{P_i\}$ for $\forall t \in U$. If $\mathcal{Z}$ is the closed subscheme of $\mathcal{Y}$ defined by the first Fitting ideal of $\Omega_{\mathcal{Y}/T}$, then $\Pi_i \subset \mathcal{Z}$. By this, we know that $\Pi_i \to T$ is a finite birational morphism, hence an isomorphism because of the assumption that T is normal.

Claim 6.1.1. $\Pi_i \cap \Pi_j = \emptyset$ for $i \neq j$.

Proof of Claim 6.1.1. Suppose that $\Pi_i \cap \Pi_j \neq \emptyset$ for $i \neq j$. Take a $\mathbb{C}$-valued point $t_0 \in T$ such that $\Pi_i \times_T \mathrm{Spec}\mathbb{C}(t_0) \cap \Pi_j \times_T \mathrm{Spec}\mathbb{C}(t_0) \neq$

$\emptyset$. Then we can find a morphism $V := \mathrm{Spec}\mathbb{C}[[v]] \xrightarrow{\alpha} T$ such that α(the closed point of V) $= t_0$ and α(the generic point of V) $\in U$. The base-changes of Π_i and Π_j give us two sections $\sigma_i, \sigma_j : V \to \mathcal{Y} \times_T V$ such that σ_i(the closed point of V) $=$ σ_j(the closed point of V) and σ_i(the generic point of V) $=$ $\{P_i\}$ and σ_j(the generic point of V) $=$ $\{P_j\}$. Taking into account the fact that both σ_i and σ_j factor through $\mathcal{Z} \times_T V \hookrightarrow \mathcal{Y} \times_T V$, we know σ_i and σ_j coincide on the closed subscheme $\mathrm{Spec}\mathbb{C}[[v]]/(v^N) \subset V$ for $\forall N > 0$. Then we have $\sigma_i(V) = \sigma_j(V)$. This is a contradiction. □

Claim 6.1.2. If $\Pi_i \times_T \mathrm{Spec}\mathbb{C}(t) = \{P_m\}$ for $m = 1$ or 2 and $t \in T$, then $i = m$.

Proof of Claim 6.1.2. If this holds, it holds for some $\mathbb{C}$-valued point $t_0 \in T$. Take $V := \mathrm{Spec}\mathbb{C}[[v]] \xrightarrow{\alpha} T$ as in the proof of Claim 6.1.1. Let $(\mathcal{Y}_V \xrightarrow{h_V} X_0 \times V, \mathcal{E}_V, \wedge^2\mathcal{E}_V \xrightarrow{\delta_V} (pr_1 \circ h_V)^*\mathcal{P}_0)$ be the pull-back by α of the given Gieseker-SL_2-bundle over T. Put $\mathcal{Z}_V := \mathcal{Z} \times_T V$ and $\Pi_{iV} := \Pi_i \times_T V$. Put $V_N := \mathrm{Spec}\mathbb{C}[[v]]/(v^{N+1})(\hookrightarrow V)$. $\mathcal{Z}_V$ is a disjoint union of the closed subschemes $\mathcal{Z}_V^{(i)}$ such that for $\forall N > 0$ the support of $\mathcal{Z}_{V_N}^{(k)}(:= \mathcal{Z}_V \times_V V_N)$ is $\{P_k\}$, where $k \in \{1, 2\}$ if the length of the modification $\mathcal{Y}_V \times_V V_0$ is 1, and $k \in \{0, 1, 2\}$ if the length is 2. Since $\Pi_{iV} \to V$ is an isomorphism, we have $\Pi_{iV_N} \xrightarrow{\sim} \mathcal{Z}_{iV_N}^{(m)} \xrightarrow{\sim} V_N$ for $\forall N > 0$, hence $\Pi_{iV} = \mathcal{Z}_V^{(m)}$. By this, we know that the deformation of the singularity of $\mathcal{Y}_V \times_V V_N$ at P_m is trivial. By Corollary 5.9, the deformation of the singularity of $\mathcal{Y}_V \times_V V_N$ at P_{3-m} is also trivial. Then by Proposition 5.10 and algebraization (cf. (5.1.8) of [EGAIII]), we have the closed the closed immersion $g : \widetilde{X}_0 \times V \hookrightarrow \mathcal{Y}_V$ with $h_V \circ g = \mathfrak{n} \times \mathrm{id}_V$ such that $g(\{P_j\} \times V) = \mathcal{Z}_V^{(j)}$ $(j = 1, 2)$. Therefore $\mathcal{Z}_V^{(m)} \times_V \mathrm{Spec}\mathbb{C}((v)) = \{P_m\}$. Since $\Pi_{iV} = \mathcal{Z}_V^{(m)}$, $\Pi_{iV} \times_V \mathrm{Spec}\mathbb{C}((v)) = \{P_m\}$. This implies $i = m$ since α(the generic point of V) $\in U$. □

With these claims prepared, we will prove the proposition.

Case (i). $\underline{T = T_2}$: In this case, the above claims imply that Π_0, Π_1, Π_2 have the desired property.

Case (ii). $\underline{T_0 = \emptyset \text{ and } T_1 \neq \emptyset}$: In this case we have Π_1, Π_2. Using the above claims plus a similar argument as in the proof of Claim 6.1.2, one can check that $\Pi_j \times_T \mathrm{Spec}\mathbb{C}(t) = \{P_j\}$ for $\forall t \in T$, $j = 1, 2$. On $\mathcal{Y} \times_T T_2$, by Case(i) we have the desired $\Pi_0 \subset \mathcal{Y} \times_T T_2$. These Π_0, Π_1, Π_2 are what we want.

Case (iii). $\underline{T_0 \neq \emptyset}$: We have Π_0. By Claim 6.1.2, we have $\Pi_0 \times_T \mathrm{Spec}\mathbb{C}(t) = \{P_0\}$ for $\forall t \in T$. Therefore $T_1 = \emptyset$. On $\mathcal{Y} \times_T T_2$, by Case(i) we have $\Pi_1, \Pi_2 \subset \mathcal{Y} \times_T T_2$ having the desired property. These Π_0, Π_1, Π_2 are what we want.

This is the end of the proof of Proposition 6.1. □

6.2. By this proposition, given a Gieseker SL_2-bundle $(h : \mathcal{Y} \to X_0 \times_{B_0} T, \mathcal{E}, \wedge^2\mathcal{E} \xrightarrow{\delta} (pr_1 \circ h)^*\mathcal{P}_0)$ on $(X_0, \mathcal{P}_0)$ over T, the locus of $\mathcal{Y}$ where the morphism $pr_2 \circ h$ is not smooth is the disjoint union of three closed subsets Π_0, Π_1, Π_2. Instead of reduced scheme structure, let us now endow each Π_i with the scheme structure defined by the first Fitting ideal of $\Omega_{\mathcal{Y}/T}$.

Then $(pr_2 \circ h)|_{\Pi_i} : \Pi_i \to T$ is a closed immersion. Let $\mathcal{I}_i (\subset \mathcal{O}_T)$ be its defining ideal. By Corollary 5.9, we have $\mathcal{I}_1 = \mathcal{I}_2$. Moreover, by the description of the versal family, we have $\mathcal{I}_0\mathcal{I}_1\mathcal{I}_2 (= \mathcal{I}_0\mathcal{I}_1^2 = \mathcal{I}_0\mathcal{I}_2^2) = 0$. Using these ideals, we shall define closed substacks of $GSL_2B(X_0; \mathcal{P}_0)$.

Definition 6.3. We define closed substacks $GSL_2B(X_0; \mathcal{P}_0)^{(0)}$, $GSL_2B(X_0; \mathcal{P}_0)^{(1)}$ and $GSL_2B(X_0; \mathcal{P}_0)^{(1)}_{red}$ of $GSL_2B(X_0; \mathcal{P}_0)$ as follows. For an affine B_0-scheme T, an object $(h : \mathcal{Y} \to X_0 \times_{B_0} T, \mathcal{E}, \wedge^2\mathcal{E} \xrightarrow{\delta} (pr_1 \circ h)^*\mathcal{P}_0)$ of $GSL_2B(X_0; \mathcal{P}_0)(T)$ is in $GSL_2B(X_0; \mathcal{P}_0)^{(0)}(T)$ [resp. $GSL_2B(X_0; \mathcal{P}_0)^{(1)}(T)$ or $GSL_2B(X_0; \mathcal{P}_0)^{(1)}_{red}(T)$] if and only if $\mathcal{I}_0 = 0$ [resp. $\mathcal{I}_1^2 = 0$ or $\mathcal{I}_1 = 0$].

Put $\widetilde{\mathcal{P}}_0 := \mathfrak{n}^*\mathcal{P}_0$. Let $\mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0)$ be the moduli stack of 2-bundles on $\widetilde{X}_0$ with determinant $\widetilde{\mathcal{P}}_0$. More precisely, for an affine B_0-scheme T, objects of the groupoid $\mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0)(T)$ are 2-bundles $\mathcal{F}$ on $\widetilde{X}_0 \times_{B_0} T$ together with an isomorphism $\wedge^2\mathcal{F} \to pr_1^*\widetilde{\mathcal{P}}_0$. Put $\sigma_i := (P_i, \mathrm{id}) : \mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0) \to \widetilde{X}_0 \times \mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0)$, $i = 1, 2$. On $\widetilde{X}_0 \times \mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0)$, we have the universal 2-bundle $\mathcal{F}_{univ}$ together with the isomorphism $\wedge^2\mathcal{F}_{univ} \to pr_1^*\widetilde{\mathcal{P}}_0$. Note that we have the canonical isomorphism $\sigma_1^*pr_1^*\widetilde{\mathcal{P}}_0 \simeq \sigma_2^*pr_1^*\widetilde{\mathcal{P}}_0$ and thus the canonical isomorphism $\theta : \wedge^2\sigma_1^*\mathcal{F}_{univ} \xrightarrow{\sim} \wedge^2\sigma_2^*\mathcal{F}_{univ}$. This allows us to consider the stack $KSL_2(\sigma_1^*\mathcal{F}_{univ}, \sigma_2^*\mathcal{F}_{univ})$.

Theorem 6.4. *We have an isomorphism of B_0-stacks*

$$GSL_2B(X_0; \mathcal{P}_0)^{(0)} \simeq KSL_2(\sigma_1^*\mathcal{F}_{univ}, \sigma_2^*\mathcal{F}_{univ}). \tag{6.2}$$

Next consider the moduli stack $\mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0 \otimes \mathcal{O}_{\widetilde{X}_0}(-P_1 + P_2))$. On $\widetilde{X}_0 \times \mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0 \otimes \mathcal{O}_{\widetilde{X}_0}(-P_1 + P_2))$, we have the universal 2-bundle $\mathcal{W}_{univ}$ together with the isomorphism $\wedge^2\mathcal{W}_{univ} \simeq pr_1^*(\widetilde{\mathcal{P}}_0 \otimes \mathcal{O}_{wtXz}(-P_1 + P_2))$. Put $\tau_i := (P_i, \mathrm{id}) : \mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0 \otimes \mathcal{O}_{\widetilde{X}_0}(-P_1 + P_2)) \to \widetilde{X}_0 \times \mathcal{SU}_2(\widetilde{X}_0, \widetilde{\mathcal{P}}_0 \otimes \mathcal{O}_{\widetilde{X}_0}(-P_1 + P_2))$.

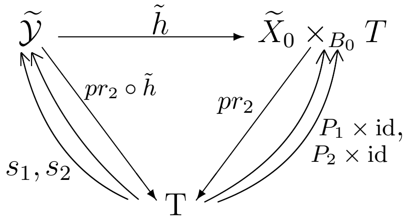
Theorem 6.5. *We have an isomorphism of B_0-stacks*

$$GSL_2B(X_0; \mathcal{P}_0)^{(1)}_{red} \simeq \overline{PGl}(\tau_1^*\mathcal{W}_{univ}, \tau_2^*\mathcal{W}_{univ}). \tag{6.3}$$

(See §8 and §9 of [K1] *for the definition of $\overline{PGl}$.)*

The rest of this section is devoted to the proof of the above two theorems.

Definition 6.6. Let



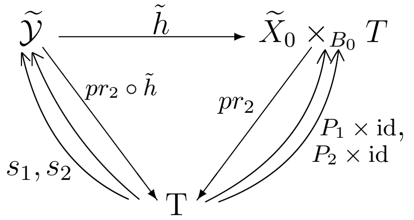
be a modification of the two pointed curve $(\widetilde{X}_0; P_1, P_2)$ over T (cf. Definition 4.4 of [K2]). It is said to be bi-simple if and only if for any $t : \mathrm{Spec}\mathbb{C}(t) \to T$ either (i) or (ii) below holds.

(i) $\widetilde{\mathcal{Y}} \times_T \mathrm{Spec}\mathbb{C}(t) \xrightarrow{\tilde{h}\times t} \widetilde{X}_0 \times_{B_0} \mathrm{Spec}\mathbb{C}(t)$ is an isomorphism.
(ii) Both $(\tilde{h} \times t)^{-1}(P_1)$ and $(\tilde{h} \times t)^{-1}(P_2)$ are isomorphic to $\mathbb{P}^1_{\mathbb{C}(t)}$.

Definition 6.7. The B_0-groupoid $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}$ of Gieseker-SL_2-bundle data is defined as follows.

For an affine B_0-scheme T, an object of $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}(T)$ is the following collection of data.

(i) A bi-simple modification $(\widetilde{\mathcal{Y}}, s_1, s_2, \tilde{h})$ of $(\widetilde{X}_0, P_1, P_2)$ over T,



(ii) A 2-bundle $\widetilde{\mathcal{E}}$ on $\widetilde{\mathcal{Y}}$,
(iii) An isomorphism $\xi : s_1^*\widetilde{\mathcal{E}} \xrightarrow{\sim} s_2^*\widetilde{\mathcal{E}}$,
(iv) An isomorphism $\eta : \mathcal{O}(-s_1 - s_2) \otimes (pr_1 \circ \tilde{h})^*\mathcal{O}(P_1 + P_2) \xrightarrow{\sim} (\wedge^2\widetilde{\mathcal{E}})^\vee \otimes (pr_1 \circ \tilde{h})^*\widetilde{\mathcal{P}}_0$.

Furthermore, we require that they satisfy the following condition.

(a) The pair $(\widetilde{\mathcal{E}}, \xi : s_1^*\widetilde{\mathcal{E}} \xrightarrow{\sim} s_2^*\widetilde{\mathcal{E}})$ is admissible for $(\widetilde{\mathcal{Y}}, s_1 s_2, \tilde{h})$ in the sense of Definition 4.5 of [K2].
(b) The diagram (♡)

$$\begin{array}{ccc} s_1^*\Big(\mathcal{O}(-s_1-s_2)\otimes(pr_1\circ\tilde{h})^*\mathcal{O}(P_1+P_2)\Big) & \xrightarrow{s_1^*(\eta)} & s_1^*(\wedge^2\widetilde{\mathcal{E}})^\vee\otimes s_1^*(pr_1\circ\tilde{h})^*\widetilde{\mathcal{P}}_0 \\ \uparrow{\scriptstyle s_1^*(\mu)} & & \\ \mathcal{O} & & \downarrow{\scriptstyle (\wedge\xi)^\vee\otimes(\text{canonical})} \\ \downarrow{\scriptstyle s_2^*(\mu)} & & \\ s_2^*\Big(\mathcal{O}(-s_1-s_2)\otimes(pr_1\circ\tilde{h})^*\mathcal{O}(P_1+P_2)\Big) & \xrightarrow{s_2^*(\eta)} & s_2^*(\wedge^2\widetilde{\mathcal{E}})^\vee\otimes s_2^*(pr_1\circ\tilde{h})^*\widetilde{\mathcal{P}}_0 \end{array}$$

commutes, where μ is the section of $\mathcal{O}(-s_1-s_2)\otimes(pr_1\circ\tilde{h})^*\mathcal{O}(P_1+P_2)$ such that its image by the canonical injection $\mathcal{O}(-s_1-s_2)\otimes(pr_1\circ\tilde{h})^*\mathcal{O}(P_1+P_2) \hookrightarrow (pr_1\circ\tilde{h})^*\mathcal{O}(P_1+P_2)$ is the pull-back by $pr_1 \circ \tilde{h}$ of the canonical section of $\mathcal{O}_{\widetilde{X}_0}(P_1 + P_2)$.

Morphisms of the groupoid $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}(T)$ are defined obviously.

Lemma 6.8. *Let $(\widetilde{\mathcal{Y}}, s_1, s_2, \tilde{h})$ be a bi-simple modification of the 2-pointed curve $(\widetilde{X}_0, P_1, P_2)$ over an affine B_0-scheme T. Let $\mathcal{L}$ be a line bundle on $\widetilde{\mathcal{Y}}$ and λ a section of $\mathcal{L}$ such that $\lambda|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}\mathbb{C}(t)} \neq 0$ for $\forall t \in T$. Put $\mathcal{M} := \mathcal{O}(-s_1-s_2)\otimes(pr_1\circ\tilde{h})^*\mathcal{O}(P_1+P_2)$ and μ the canonical section defined in Definition 6.3(b). Assume that $\mathcal{L}|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}\mathbb{C}(t)} \simeq \mathcal{M}|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}\mathbb{C}(t)}$ for $\forall t \in T$. Then there is a unique isomorphism $\mathcal{L} \simeq \mathcal{M}$ in which λ and μ corrspond.*

Proof. We may assume that T is of finite type over B_0. We have that $\mathcal{O}_T \xrightarrow{\sim} (pr_2\circ\tilde{h})_*\mathcal{O}_{\widetilde{\mathcal{Y}}}$ (cf. Lemma3.9 of [K2]). If we have two isomorphism $\alpha_i : \mathcal{L} \to \mathcal{M}$ $(i = 1, 2)$ with $\alpha_i(\lambda) = \mu$, then $\alpha_2 \circ \alpha_1^{-1}$ is $(pr_2 \circ \tilde{h})^*(a)$-multiplication for some $a \in \mathcal{O}_T$. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_T = (pr_2 \circ \tilde{h})_*\mathcal{O}_{\widetilde{\mathcal{Y}}} & \xrightarrow{(pr_2\circ\tilde{h})_*(\mu)} & (pr_2 \circ \tilde{h})_*\mathcal{M} \\ \Vert & & \downarrow \times a \\ (pr_2 \circ \tilde{h})_*\mathcal{O}_{\widetilde{\mathcal{Y}}} & \xrightarrow{(pr_2\circ\tilde{h})_*(\mu)} & (pr_2 \circ \tilde{h})_*\mathcal{M}. \end{array} \tag{6.4}$$

Since $\dim \mathrm{H}^0(\mathcal{M}|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}\mathbb{C}(t)}) = 1$ and $\mu|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}\mathbb{C}(t)}) \neq 0$ for $\forall t \in T$, $(pr_2 \circ \tilde{h})_*(\mu)$ is an isomorphism by base-change theorem. So $a = 1$, which proves the uniqueness. By the uniqueness it suffices to prove the lemma locally on T. Moreover it suffices to prove that, for any closed point $t \in T$, there is a Zariski open neighborhood $U \subset T$ such that $\mathcal{L}|_{(pr_2\circ\tilde{h})^{-1}(U)} \simeq \mathcal{M}|_{(pr_2\circ\tilde{h})^{-1}(U)}$. In fact, if so, we can adjust the isomorphism so that λ and μ correspond because $(pr_2 \circ \tilde{h})_*(\lambda) : \mathcal{O}_T \to (pr_2 \circ \tilde{h})_*\mathcal{L}$ and $(pr_2 \circ \tilde{h})_*(\mu) : \mathcal{O}_T \to (pr_2 \circ \tilde{h})_*\mathcal{M}$ are isomorphisms.

Claim 6.8.1. If $T = \mathrm{Spec}A$, where $(A, \mathfrak{m})$ is an artinian local $\mathbb{C}$-algebra with $\mathbb{C} \xrightarrow{\sim} A/\mathfrak{m}$, then the lemma holds.

Proof of Claim 6.8.1. We prove the claim by induction on $l = \dim_{\mathbb{C}} A$. If $l = 1$, by assumption the calim is true. If $l > 1$, let $I \subset A$ be an ideal of length one and put $T' := \mathrm{Spec}A/I$ and $\widetilde{\mathcal{Y}}' := \widetilde{\mathcal{Y}} \times_T T'$. By induction we have $\mathcal{L}|_{\widetilde{\mathcal{Y}}'} \simeq \mathcal{M}|_{\widetilde{\mathcal{Y}}'}$ in which $\lambda|_{\widetilde{\mathcal{Y}}'}$ and $\mu|_{\widetilde{\mathcal{Y}}'}$ correpond. $\mathcal{L}$ and $\mathcal{M}$ are two extensions over $\widetilde{\mathcal{Y}}$ of the line bunle $\mathcal{L}|_{\widetilde{\mathcal{Y}}'}(\simeq \mathcal{M}|_{\widetilde{\mathcal{Y}}'})$ on $\widetilde{\mathcal{Y}}'$. Their difference is expressed by an element $e \in \mathrm{H}^1(\widetilde{\mathcal{Y}} \times_T \mathrm{Spec}A/\mathfrak{m}, \mathcal{O})$. Since the sections $\lambda|_{\widetilde{\mathcal{Y}}'}$ and $\mu|_{\widetilde{\mathcal{Y}}'}$ extend over $\widetilde{\mathcal{Y}}'$, we have $e \cdot (\mu|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}A/\mathfrak{m}} = 0$ in $\mathrm{H}^1(\widetilde{\mathcal{Y}} \times_T \mathrm{Spec}A/\mathfrak{m}, \mathcal{M}|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}A/\mathfrak{m}})$. Since $\mathrm{H}^1(\widetilde{\mathcal{Y}} \times_T \mathrm{Spec}A/\mathfrak{m}, \mathcal{O}) \xrightarrow{\mu} \mathrm{H}^1(\widetilde{\mathcal{Y}} \times_T \mathrm{Spec}A/\mathfrak{m}, \mathcal{M}|_{\widetilde{\mathcal{Y}}\times_T \mathrm{Spec}A/\mathfrak{m}})$ is bijective, we have $e = 0$, by which we have $\mathcal{L} \simeq \mathcal{M}$. Adjusting this so that λ and μ correspond, we prove the claim. $\square$

Take a closed point $t \in T$. Since $(pr_2 \circ \tilde{h})_* \mathcal{H}om_{\mathcal{O}_{\widetilde{\mathcal{Y}}}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}_T} \widehat{\mathcal{O}}_{T,t} \xrightarrow{\sim} \varprojlim \mathrm{Hom}_{\mathcal{O}_{\widetilde{\mathcal{Y}} \times_T \mathrm{Spec} \mathcal{O}/\mathfrak{m}^n}}(\mathcal{L} \otimes_{\mathcal{O}_T} \mathcal{O}_T/\mathfrak{m}^n, \mathcal{M} \otimes_{\mathcal{O}_T} \mathcal{O}_T/\mathfrak{m}^n)$, by the above claim we can find $\varphi \in (pr_2 \circ \tilde{h})_* \mathcal{H}om_{\mathcal{O}_{\widetilde{\mathcal{Y}}}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}_T} \widehat{\mathcal{O}}_{T,t}$ such that $\varphi \otimes_{\mathcal{O}_T} \mathcal{O}_{T,t}/\mathfrak{m} : \mathcal{L}|_{\widetilde{\mathcal{Y}} \times_T \mathrm{Spec} \mathcal{O}_{T,t}/\mathfrak{m}} \to \mathcal{M}|_{\widetilde{\mathcal{Y}} \times_T \mathrm{Spec} \mathcal{O}_{T,t}/\mathfrak{m}}$ is an isomorphism. Extending φ over some Zariski open neighborhood of t, we complete the proof of the lemma. □

Proposition 6.9. *We have an isomorphism of B_0-groupoids*

$$GSL_2B(X_0; \mathcal{P}_0)^{(0)} \xrightarrow{\sim} GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}.$$

Proof. It suffices to establish the isomorphism over the full subcategory of affine schemes of finite type over B_0.

<u>Construction of $\Phi : GSL_2B(X_0; \mathcal{P}_0)^{(0)} \to GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}$.</u>

Let T be an affine scheme of finite over B_0 and let $(\mathcal{Y} \xrightarrow{h} X_0 \times T, \mathcal{E}, \wedge^2 \mathcal{E} \xrightarrow{\delta} (pr_1 \circ h)^* \mathcal{P}_0)$ be an object of $GSL_2B(X_0; \mathcal{P}_0)^{(0)}(T)$. Let $\Pi_0 \subset \mathcal{Y}$ be as in the paragraph 6.2. In our situation Π_0 is a section over T. Let $\widetilde{\mathcal{Y}} \xrightarrow{g} \mathcal{Y}$ be the blowing-up along Π_0. It is easily checked that $\widetilde{\mathcal{Y}}$ is flat over T and that there is a unique morphism $\tilde{h} : \widetilde{\mathcal{Y}} \to \widetilde{X}_0 \times T$ satisfying $(\mathfrak{n} \times \mathrm{id}_T) \circ \tilde{h} = h \circ g$.

$$\begin{array}{ccc} \widetilde{\mathcal{Y}} & \xrightarrow{\tilde{h}} & \widetilde{X}_0 \times T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathfrak{n} \times \mathrm{id}} \\ \mathcal{Y} & \xrightarrow{h} & X_0 \times T \end{array} \tag{6.5}$$

$g^{-1}(\Pi_0)$ consists of the disjoint two sections s_1 and s_2 over T such that $\tilde{h} \circ s_i = (P_i, \mathrm{id}_T)$. Then $(\tilde{h} : \widetilde{\mathcal{Y}} \to \widetilde{X}_0 \times T, s_1, s_2)$ is a bi-simple modification of $(\widetilde{X}_0, P_1, P_2)$ over T. Put $\widetilde{\mathcal{E}} := g^* \mathcal{E}$. Let ξ be the composite of natural isomorphisms $s_1^* \widetilde{\mathcal{E}} \simeq s_1^* g^* \mathcal{E} \simeq \Pi_0^* \mathcal{E} \simeq s_2^* g^* \mathcal{E} \simeq s_2^* \widetilde{\mathcal{E}}$. We have $g^*(\delta) : \wedge^2 \widetilde{\mathcal{E}} \simeq g^* \wedge^2 \mathcal{E} \to g^*(pr_1 \circ h)^* \mathcal{P}_0 \simeq (pr_1 \circ \tilde{h})^* \widetilde{\mathcal{P}}_0$, which induces a morphism $\lambda : \mathcal{O}_{\widetilde{\mathcal{Y}}} \to (\wedge^2 \widetilde{\mathcal{E}})^\vee \otimes (pr_1 \circ \tilde{h})^* \widetilde{\mathcal{P}}_0$. By Lemma 6.8, there is a unique isomorphism $\eta : \mathcal{M} := \mathcal{O}(-s_1 - s_2) \otimes (pr_1 \circ \tilde{h})^* \mathcal{O}(P_1 + P_2) \xrightarrow{\sim} (\wedge^2 \widetilde{\mathcal{E}})^\vee \otimes (pr_1 \circ \tilde{h})^* \widetilde{\mathcal{P}}_0$ such that λ corresponds to the canonical section μ of $\mathcal{M}$. Since λ is a pull-back of the morphism on $\mathcal{Y}$, the diagram (♡) in Definition 6.7 commutes. These data define an object of $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}(T)$.

<u>Construction of $\Psi : GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)} \to GSL_2B(X_0; \mathcal{P}_0)^{(0)}$.</u>

Given an object $((\widetilde{\mathcal{Y}} \xrightarrow{\tilde{h}} \widetilde{X}_0 \times T, s_1, s_2), \widetilde{\mathcal{E}}, s_1^* \widetilde{\mathcal{E}} \xrightarrow{\xi} s_2^* \widetilde{\mathcal{E}}, \eta : \mathcal{O}(-s_1 - s_2) \otimes (pr_1 \circ \tilde{h})^* \mathcal{O}(P_1 + P_2) \xrightarrow{\sim} (\wedge^2 \widetilde{\mathcal{E}})^\vee \otimes (pr_1 \circ \tilde{h})^* \widetilde{\mathcal{P}}_0)$ of $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}(T)$, let $g : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be a cokernel of $T \overset{s_1}{\underset{s_2}{\rightrightarrows}} \widetilde{\mathcal{Y}}$. There is a unique T-morphism $h : \mathcal{Y} \to X_0 \times T$ with $h \circ g = (\mathfrak{n} \times \mathrm{id}_T) \circ \tilde{h}$. Put $\Pi_0 := g \circ s_i$ and

$\mathcal{E} := \mathrm{Ker}(g_*\widetilde{\mathcal{E}} \xrightarrow{s_2^\# - \xi\circ s_1^\#} s_2^*\widetilde{\mathcal{E}})$. The commutativity of the diagram ($\heartsuit$) in Definition 6.7 induces a morphism $\mathcal{O} \to (\wedge^2\mathcal{E})^\vee \otimes (pr_1 \circ h)^*\mathcal{P}_0$, which gives $\delta : \wedge^2\mathcal{E} \to (pr_1 \circ h)^*\mathcal{P}_0$.

We can see that the construction Φ and Ψ commute with isomorphisms and base changes and that they are inverses to each other. $\square$

Proposition 6.10. *We have an isomorphism of B_0-groupoids*

$$GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)} \simeq KSL(\sigma_1^*\mathcal{F}_{univ}, \sigma_2^*\mathcal{F}_{univ}).$$

Lemma 6.11. *Let S be a locally noetherian scheme, and let $\pi : \mathcal{C} \to S$ be a proper, flat morphism with connected geometric fibers of dimension one, and let $s : S \to \mathcal{C}$ be a section over S such that $s(S)$ is in the smooth locus of π. Let $(\mathcal{C}', f, \pi', s')$ be a simple modification of $(\mathcal{C}, \pi, s)$ (cf. Definition 5.1 of* [K2]*). Let $n \geq 1$ and $1 \leq d \leq n$, and let $\mathcal{E}'$ be a rank n vector bundle on $\mathcal{C}'$ that is admissible of degree d for $(\mathcal{C}', f, \pi, s')$ (cf. Definition 7.1 of* [K2]*). Put $\mathcal{N} := (f^*\mathcal{O}_\mathcal{C}(s))(-s')$, $N := s'^*\mathcal{N}$ and $\mathcal{F} := (f_*\mathcal{E}'(-s'))(s)$. Let $g : s'^*\mathcal{E}' \overset{\otimes N}{\curvearrowright\!\!\!\longrightarrow} s^*\mathcal{F}$ be the bf-morphism of rank $n-d$ constructed in* §7 *of* [K2]*. Then there exists a unique isomorphism $\rho : \wedge^n\mathcal{E}' \xrightarrow{\sim} f^*(\wedge^n\mathcal{F}) \otimes \mathcal{N}^{-d}$ such that $s'^*(\rho) = \wedge^n g$ (See* §6 *of* [K2] *for the definition of $\wedge^n g$).*

Proof. By Lemma 7.6 of [K2], for $\forall x \in \mathcal{C}$ there exists an open neighborhood U of x and an isomorphims $\alpha : \mathcal{E}'|_{f^{-1}(U)} \to (\mathcal{N}^{-1})^{\oplus d} \oplus \mathcal{O}_{f^{-1}(U)}^{\oplus n-d}$. Let β be the composite of isomorphims

$$\mathcal{F}|_U = f_*(\mathcal{E}'\otimes\mathcal{N})|_U \to \mathcal{O}^{\oplus d}\oplus(f_*(\mathcal{N})|_U)^{\oplus n-d} \xleftarrow{(1^{\oplus d}, f_*(\nu)^{\oplus n-d})} \mathcal{O}_U^{\oplus d}\oplus\mathcal{O}_U^{\oplus n-d},$$

where ν is the canonical section of $\mathcal{N}$. We define $\rho|_{f^{-1}(U)} : \wedge^n\mathcal{E}'|_{f^{-1}(U)} \to f^*(\wedge^n\mathcal{F}) \otimes \mathcal{N}^{-d}|_{f^{-1}(U)}$ by $(\wedge^n\alpha) \circ (f^*(\wedge^n\beta) \otimes \mathrm{id}_{\mathcal{N}^{-d}})^{-1}$. One can check that $\rho|_{f^{-1}(U)}$ is independent of the choice of α. Therefore we have globally an isomorphism $\rho : \wedge^n\mathcal{E}' \xrightarrow{\sim} f^*(\wedge^n\mathcal{F}) \otimes \mathcal{N}^{-d}$. $s'^*(\rho) = \wedge^n g$ follows from Lemma 7.5 of [K2]. The uniqueness follows from the isomorphism $\pi'_*(\mathcal{O}_{\mathcal{C}'}) \xrightarrow{\sim} s'^*\mathcal{O}_{\mathcal{C}'}$. $\square$

Proof of Proposition 6.10. Construction of $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)} \to KSL_2(\sigma_1^*\mathcal{F}_{univ}, \sigma_2^*\mathcal{F}_{univ})$.

Let T be an affine B_0-scheme. As in the proof of Proposition 6.9, we may assume that T is of finite type over B_0. Let $((\widetilde{\mathcal{Y}} \xrightarrow{\tilde{h}} \widetilde{X}_0 \times T, s_1, s_2), \widetilde{\mathcal{E}}, s_1^*\widetilde{\mathcal{E}} \xrightarrow{\xi} s_2^*\widetilde{\mathcal{E}}, \eta : \mathcal{M} := \mathcal{O}(-s_1 - s_2) \otimes (pr_1 \circ \tilde{h})^*\mathcal{O}(P_1 + P_2) \xrightarrow{\sim} (\wedge^2\widetilde{\mathcal{E}})^\vee \otimes (pr_1 \circ \tilde{h})^*\widetilde{\mathcal{P}}_0)$ be an object of $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}(T)$. Let μ be the canonical section of $\mathcal{M}$. Put $\mathcal{F} := \tilde{h}_*(\widetilde{\mathcal{E}} \otimes \mathcal{M})$, $M_i := s_i^*\mathcal{M}$ and $\mu_i := s_i^*(\mu)$. Then by §7 of [K2], we have bf-morphisms of rank one $\alpha_i : s_i^*\widetilde{\mathcal{E}} \overset{\otimes \mathcal{M}_i}{\curvearrowright\!\!\!\longrightarrow} (P_i, \mathrm{id}_T)^*\mathcal{F}$. Taking ξ into account, we have diagram ($\diamondsuit$),

$$\begin{array}{ccc} s_1^*\widetilde{\mathcal{E}} & \xrightarrow{\quad\xi\quad} & s_2^*\widetilde{\mathcal{E}} \\ {\scriptstyle \otimes M_1}\big\uparrow\big\downarrow{\scriptstyle \alpha_1} & & {\scriptstyle \alpha_2}\big\downarrow\big\uparrow{\scriptstyle \otimes M_2} \\ (P_1, \mathrm{id}_T)^*\mathcal{F} & & (P_2, \mathrm{id}_T)^*\mathcal{F}. \end{array}$$

By Lemma 6.11, we have the natural isomorphism $\wedge^2\mathcal{F} \simeq \tilde{h}_*(\wedge^2\widetilde{\mathcal{E}}\otimes\mathcal{M})$. Combining this with $\tilde{h}_*(\eta)$, we have the isomorphism $\zeta : \wedge^2\mathcal{F} \xrightarrow{\sim} pr_1^*\widetilde{\mathcal{P}}_0$ such that $(P_i, \mathrm{id}_T)^*(\zeta)$ is the composite

$$\wedge^2(P_i, \mathrm{id})^*\mathcal{F} \xrightarrow{\wedge^{-2}\alpha_i} (\wedge^2 s_i^*\widetilde{\mathcal{E}}) \otimes M_1$$
$$\xrightarrow{s_i^*(\eta\otimes\mathrm{id}_{\wedge^2\widetilde{\mathcal{E}}})} s_i^*(pr_1 \circ \tilde{h})^*\widetilde{\mathcal{P}}_0 = (P_i, \mathrm{id}_T)^* pr_1^*\widetilde{\mathcal{P}}_0.$$

There is a unique isomorphism $v : M_1 \to M_2$ such that the composite

$$(P_1, \mathrm{id}_T)^* pr_1^*\widetilde{\mathcal{P}}_0 \xrightarrow{(P_1,\mathrm{id}_T)^*(\zeta)^{-1}} (P_1, \mathrm{id}_T)^* \wedge^2 \mathcal{F}$$
$$\xrightarrow{\wedge^{-2}\alpha_1} \wedge^2 s_1^*\widetilde{\mathcal{E}} \otimes M_1$$
$$\xrightarrow{\wedge^2\xi\otimes v} \wedge^2 s_2^*\widetilde{\mathcal{E}} \otimes M_2$$
$$\xrightarrow{(\wedge^{-2}\alpha_2)^{-1}} \wedge^2(P_2, \mathrm{id}_T)^*\mathcal{F}$$
$$\xrightarrow{(P_2,\mathrm{id}_T)^*(\zeta)} (P_2, \mathrm{id}_T)^* pr_1^*\widetilde{\mathcal{P}}_0$$

is the canonical morphism. Moreover by the diagram ($\heartsuit$), we have $v(\mu_1) = \mu_2$. The admissibility of the pair $(\widetilde{\mathcal{E}}, \xi)$ implies that $\cap_{i=1}^2 \mathrm{Ker}(s_i^*\mathcal{E}_i \to (P_i, \mathrm{id}_T)^*\mathcal{F}) = \{o\}$. Therefore these data give an object of $KSL_2(\sigma_1^*\mathcal{F}_{univ}, \sigma_2^*\mathcal{F}_{univ})(T)$.

Construction of $KSL_2(\sigma_1^*\mathcal{F}_{univ}, \sigma_2^*\mathcal{F}_{univ}) \to GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}$

Take an object of $KSL_2(\sigma_1^*\mathcal{F}_{univ}, \sigma_2^*\mathcal{F}_{univ})(T)$, that is, the following data.

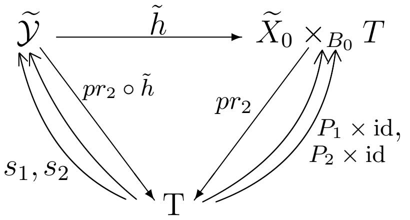
- a 2-bundle $\mathcal{F}$ on $\widetilde{X}_0 \times T$;
- an isomorphism $\zeta : \wedge^2\mathcal{F} \xrightarrow{\sim} pr_1^*\widetilde{\mathcal{P}}_0$;
- two line bundles M_1, M_2 on T and their sections μ_1, μ_2;
- an isomorphism $v : M_1 \xrightarrow{\sim} M_2$ satisfying $v(\mu_1) = \mu_2$;
- two line bundles E_1, E_2 on T;
- an isomorphism $\xi : E_1 \xrightarrow{\sim} E_2$;
- two bf-morphisms $\alpha_i : E_i \overset{\otimes M_1}{\curvearrowleft\!\!\longrightarrow} (P_i, \mathrm{id}_T)^*\mathcal{F}$, $i = 1, 2$,

such that
(6.6)
$$\xi\left(\mathrm{Ker}(E_1[t] \to (P_1, \mathrm{id}_T)^*\mathcal{F}[t])\right) \cap \mathrm{Ker}(E_2[t] \to (P_2, \mathrm{id}_T)^*\mathcal{F}[t]) = \{o\},$$

and the diagram

$$
\begin{array}{ccc}
\wedge^2 E_1 \otimes M_1 & \xrightarrow{\wedge^2\xi\otimes v} & \wedge^2 E_2 \otimes M_2 \\
{\scriptstyle \wedge^2\alpha_1}\big\uparrow & & \big\uparrow{\scriptstyle \wedge^2\alpha_2} \\
\wedge^2 (P_1, \mathrm{id}_T)^*\mathcal{F} & & \wedge^2 (P_2, \mathrm{id}_T)^*\mathcal{F} \\
{\scriptstyle (P_1,\mathrm{id}_T)^*(\zeta)}\big\downarrow & & \big\downarrow{\scriptstyle (P_2,\mathrm{id}_T)^*(\zeta)} \\
(P_1, \mathrm{id}_T)^* pr_1^*\widetilde{\mathcal{P}}_0 & \xrightarrow{\text{canonical isom.}} & (P_2, \mathrm{id}_T)^* pr_1^*\widetilde{\mathcal{P}}_0
\end{array}
\tag{6.7}
$$

commutes. Making use of the bf-morphisms α_1 and α_2, we obtain a bi-simple modification $(\tilde{h} : \widetilde{\mathcal{Y}} \to \widetilde{X}_0 \times T, s_1, s_2)$



plus isomorphisms $\epsilon_i : s_i^*\mathcal{M} \xrightarrow{\sim} M_i$ with $\epsilon_i(\mu) = \mu_i$ by §5 of [K2], where $\mathcal{M} := \mathcal{O}(-s_1 - s_2) \otimes (pr_1 \circ \tilde{h})^*\mathcal{O}(P_1 + P_2)$ and μ is the canonical section of $\mathcal{M}$. By Construction 7.1 in [K2], there exists a 2-bundle $\widetilde{\mathcal{E}}$ on $\widetilde{\mathcal{Y}}$ together with isomorphisms $\tilde{h}_*(\widetilde{\mathcal{E}} \otimes \mathcal{M}) \simeq \mathcal{F}$ and $s_i^*\widetilde{\mathcal{E}} \simeq E_i$ such that they give rise to the bf-morphisms α_i. $\widetilde{\mathcal{E}}$ and the isomorphisms are unique up to unique isomorphism by Lemma 7.7 of [K2]. By Lemma 6.11, we have a unique isomorphism $\beta : \mathcal{M} \xrightarrow{\sim} (\wedge^2\widetilde{\mathcal{E}})^\vee \otimes \wedge^2\tilde{h}^*\mathcal{F}$ such that $s_i^*(\beta) \otimes 1_{\wedge^2\widetilde{\mathcal{E}}} = 1_{\mathcal{M}} \otimes (\wedge^2\alpha_i)$. Put $\eta := (1_{(\wedge^2\widetilde{\mathcal{E}})^\vee} \otimes \tilde{h}^*(\zeta)) \circ \beta : \mathcal{M} \xrightarrow{\sim} (\wedge^2\widetilde{\mathcal{E}})^\vee \otimes (pr_1 \circ \tilde{h})^*\widetilde{\mathcal{P}}_0$. $\xi : E_1 \xrightarrow{\sim} E_2$ induces the isomorphism $s_1^*\widetilde{\mathcal{E}} \xrightarrow{\sim} s_2^*\widetilde{\mathcal{E}}$, which, by abuse of notation, we denote also by ξ. Then by (6.6), $(\widetilde{\mathcal{E}}, \xi)$ is admissible for the above bi-simple modification. The diagram (6.7) implies the commutativity of the diagram (♡) in Definition 6.7. Thus these data give an object of $GSL_2BD(\widetilde{X}_0, P_1, P_2; \widetilde{\mathcal{P}}_0)^{(0)}(T)$.

One can check that the above constructions commute with isomorphisms and base-changes and that they are inverses to each other. □

Sketch of proof of Theorem 6.5. The proof is analogous to that of Theorem 6.4. Here we shall just give $\Phi : GSL_2B(X_0; \mathcal{P}_0)^{(1)}_{red} \to \overline{PGl}(\tau_1^*\mathcal{W}_{univ}, \tau_2^*\mathcal{W}_{univ})$ and leave the construction of its inverse to the reader.

Let T be an affine scheme of finite type over B_0. Take an object $(h : \mathcal{Y} \to X_0 \times T, \mathcal{E}, \delta : \wedge^2\mathcal{E} \to h^*\mathcal{P}_0)$ of $GSL_2B(X_0; \mathcal{P}_0)^{(1)}_{red}(T)$. By the definition of $GSL_2B(X_0; \mathcal{P}_0)^{(1)}_{red}$, $(pr_2 \circ h)|_{\Pi_1} : \Pi_1 \to T$ is an isomorphism, where Π_i is the one described in the paragraph 6.2. Let $g : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be the blowing-up along Π_1. Let $\tilde{h} : \widetilde{\mathcal{Y}} \to \widetilde{X}_0 \times T$ be such that $(\mathfrak{n} \times \mathrm{id}_T) \circ \tilde{h} = h \circ g$. $g^{-1}(\Pi_1)$ consists of two sections of $\widetilde{\mathcal{Y}} \to T$, and let $\tilde{s}$ be one of them such that $\tilde{h} \circ \tilde{s} = (P_2, \mathrm{id}_T)$. Put $\widetilde{\mathcal{E}} := g^*\mathcal{E}$.

Then by Proposition 8.6 of [K2], we have a family of nodal curves $\mathcal{Z} \xrightarrow{\pi'} T$ over T with a section s' and a T-morphism $f_2 : \widetilde{\mathcal{Y}} \to \mathcal{Z}$ and $f_1 : \mathcal{Z} \to \widetilde{X}_0 \times T$ such that f_2 is a simple modification of $(\mathcal{Z}, \pi', s')$ and that f_1 is a simple modification of $(\widetilde{X}_0 \times T, pr_2, (P_2, \mathrm{id}_T))$ and that $\widetilde{\mathcal{E}}$ is admissible of degree one for $(\mathcal{Z}, \pi', s')$ (See Definition 7.1 of [K2]). Then Constructions 9.1 and 9.2 of [K2] give a 2-bundle $\mathcal{F}$ on $\widetilde{X}_0 \times T$ and bf-morphisms $g_1 : (P_1, \mathrm{id}_T)^*\mathcal{F} \overset{\otimes(L_1 \ni \lambda_1)}{\curvearrowright\!\!\!\longrightarrow} F_1$ and $g_0 : F_1 \overset{\otimes(L_0 \ni \lambda_0)}{\curvearrowright\!\!\!\longrightarrow} (P_2, \mathrm{id}_T)^*\mathcal{F}$. Since $(pr_2 \circ h)|_{\Pi_2} : \Pi_2 \to T$ is also an isomorphism by Corollary 5.9, we have $\lambda_0 = 0$. Moreover, if we let $\mathcal{M}$ be the line bundle $(pr_1 \circ f_1 \circ f_2)^*\mathcal{O}_{\tilde{X}_0}(P_1 + P_2) \otimes f_2^*\mathcal{O}_{\mathcal{Z}}(-s') \otimes \mathcal{O}_{\tilde{\mathcal{Y}}}((P_1, \mathrm{id}_T)(T))$ on $\widetilde{\mathcal{Y}}$ and μ be its canonical global section, and λ be the global section of $(\wedge^2\mathcal{E})^\vee \otimes h^*\mathcal{P}_0$ induced by δ, then just as Lemma 6.8 we have the unique isomorphism

$$(\wedge^2\widetilde{\mathcal{E}})^\vee \otimes \tilde{h}^*\widetilde{\mathcal{P}}_0 \simeq \mathcal{M} \tag{6.8}$$

in which $g^\#(\lambda)$ and μ correspond. The isomorphism (6.8) and Lemma 6.11 give the isomorphism $\wedge^2\mathcal{F} \simeq pr_1^*(\widetilde{\mathcal{P}}_0 \otimes \mathcal{O}(-P_1 + P_2))$. These data determine an object of $\overline{PGl}(\tau_1^*\mathcal{W}_{univ}, \tau_2^*\mathcal{W}_{univ})(T)$. □

7. Appendix

In this appendix, we gather several propositions that are used in this paper.

Proposition 7.1. *Let k be a field. Let Λ be a complete noetherian local k-algebra with maximal ideal μ such that $k \xrightarrow{\sim} \Lambda/\mu$. Let $\mathcal{A}$ [resp. $\widehat{\mathcal{A}}$] be the category of artinian local Λ-algebra [resp. complete noetherian local Λ-algebra] having residue field k. Let F be a functor from $\mathcal{A}$ to the category of sets. Assume that*

(a) *F has a hull,*
(b) *for any ideal J of Λ with $\mu \supset J \supset \mu^2$ and $\dim_k \mu/J = 1$, we have $F(\Lambda/J) = \phi$.*

Assume that we are given $S \in \widehat{\mathcal{A}}$ and $h_S(:= \mathrm{Hom}(S, -)) \xrightarrow{u} F$ such that

(i) *as a k-algebra, S is a ring of formal power series over k,*
(ii) *$h_S((\Lambda/\mu)[\epsilon]) \simeq F((\Lambda/\mu)[\epsilon])$, where $(\Lambda/\mu)[\epsilon]$ is the Λ/μ-algebra with $\epsilon^2 = 0$.*

Then $h_S \xrightarrow{u} F$ is a hull.

Proof. By (a) we can find a hull $h_R \xrightarrow{v} F$, where $R \in \widehat{\mathcal{A}}$. Then we have a morphism $\varphi : R \to A$ in $\widehat{\mathcal{A}}$ such that $v \circ h_\varphi = u$, where $h_\varphi : h_S \to h_R$ is the morphism induced by φ. Let $\mathfrak{m}$, $\mathfrak{n}$ be the maximal ideals of R and S, respectively. By (ii), the morphism $\mathfrak{m}/\mu R + \mathfrak{m}^2 \xrightarrow{\bar{\varphi}} \mathfrak{n}/\mu S + \mathfrak{n}^2$ induced by φ is an isomorphism. By (b) $\mathrm{Hom}_{loc\,\Lambda-alg}(R, \Lambda/J) =$

$\mathrm{Hom}_{loc\,\Lambda-alg}(S,\Lambda/J)=\phi$, for any ideal J of Λ as in (b). This implies that $\mu R\subset\mathfrak{m}^2$ and $\mu S\subset\mathfrak{n}^2$. Therefore φ induces the isomorphism $\bar\varphi:\mathfrak{m}/\mathfrak{m}^2\to\mathfrak{n}/\mathfrak{n}^2$. By (i), there exists a local k-algebra homomorphism $\theta:S\to R$ such that the induced morphism $\bar\theta:\mathfrak{n}/\mathfrak{n}^2\to\mathfrak{m}/\mathfrak{m}^2$ is $\bar\varphi^{-1}$. Then θ is surjective and $\varphi\circ\theta$ is bijective. Hence φ is an isomorphism. □

Lemma 7.2. *Let R be a commutative ring with identity. Let $f\in R$ and let $\widehat{R}$ be the (f)-adic completion of R. Then for any R-module M, the complex*

$$M\xrightarrow{\alpha}(M\otimes_R\widehat{R})\oplus M_f\xrightarrow{\beta}M\otimes_R\widehat{R}_f \tag{7.1}$$

is exact, where $\alpha(m):=(m\otimes 1,m)$ and $\beta((m\otimes a,\frac{m'}{f^N})):=m\otimes a-m'\otimes\frac{1}{f^N}$. Moreover either if M is f-regular, or if $\widehat{R}$ is flat over R, then α is injective.

Proof. Put $T:=\mathrm{Ker}(M\to M_f)$, $M_f/M:=\mathrm{Coker}(M\to M_f)$ and $\overline{M}:=M/T$. Then we have the natural morphism $M_f/M\simeq\overline{M}_f/\overline{M}$. Since $\overline{M}$ is f-regular, by Lemme 3(a) in [BL], we have $\mathrm{Tor}_1^R(\widehat{R},\overline{M}_f/\overline{M})=0$. Hence we have the exact sequence

$$0\to(M/T)\otimes_R\widehat{R}\to M_f\otimes_R\widehat{R}\to(M_f/M)\otimes_R\widehat{R}\to 0. \tag{7.2}$$

Consider the commutative diagram

(7.3)

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & M_f & \longrightarrow & M_f/M & \longrightarrow & 0\\
 & & \varphi\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow\psi & & \\
 & & T\otimes_R\widehat{R} & \xrightarrow{\theta} & M\otimes_R\widehat{R} & \longrightarrow & M_f\otimes_R\widehat{R} & \longrightarrow & (M_f/M)\otimes_R\widehat{R} & \longrightarrow & 0,
\end{array}$$

in which both top and bottom rows are exact. By Lemme 1 in [BL], φ and ψ are bijective. This and diagram chasing prove the first part of the lemma. If M is f-regular or if $\widehat{R}$ is R-flat, then θ in the above diagram is injective. This implies the latter part of the lemma. □

Proposition 7.3. *Let k be a field. Let X be a scheme of finite type over k and of pure dimension one. Let P_i $(1\le i\le N)$ be closed points of X such that $U:=X-\{P_1,\dots,P_N\}$ is affine. Let $\mathfrak{m}_{P_i}$ and $\widehat{\mathfrak{m}}_{P_i}$ denote the maximal ideal of $\mathcal{O}_{X,P_i}$ and $\widehat{\mathcal{O}}_{X,P_i}$ respectively. Put $Q_i:=\mathrm{H}^0(\mathrm{Spec}\mathcal{O}_{X,P_i}-\{\mathfrak{m}_{P_i}\},\mathcal{O})$ and $\widehat{Q}_i:=\mathrm{H}^0(\mathrm{Spec}\widehat{\mathcal{O}}_{X,P_i}-\{\widehat{\mathfrak{m}}_{P_i}\},\mathcal{O})$. For a quasi-coherent sheaf $\mathcal{F}$ on X, we have natural morphisms $\rho_i:\mathrm{H}^0(U,\mathcal{F})\to\mathcal{F}_{P_i}\otimes_{\mathcal{O}_{P_i}}\widehat{Q}_i$ and $\gamma_i:\mathcal{F}_{P_i}\otimes_{\mathcal{O}_{P_i}}\widehat{\mathcal{O}}_{P_i}\to\mathcal{F}_{P_i}\otimes_{\mathcal{O}_{P_i}}\widehat{Q}_{P_i}$. We define the complex $C^\bullet(\mathcal{F})$ of k-vector spaces as follows. $C^0(\mathcal{F}):=\mathrm{H}^0(U,\mathcal{F})\oplus\oplus_{i=1}^N(\mathcal{F}_{P_i}\otimes_{\mathcal{O}_{P_i}}\widehat{\mathcal{O}}_{P_i})$ and $C^1(\mathcal{F}):=\oplus_{i=1}^N(\mathcal{F}_{P_i}\otimes_{\mathcal{O}_{P_i}}\widehat{Q}_i)$ and $C^m(\mathcal{F})=0$ for $m\ne 0,1$. $d^0:C^0(\mathcal{F})\to C^1(\mathcal{F})$ maps $(s_U,(s_i)_{i=1}^N)\in C^0(\mathcal{F})$ to $(\rho_i(s_U)-\gamma_i(s_i))_{i=1}^N$ and $d^m=0$ for $m\ne 0$. Then*

(1) *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence of quasi-coherent sheaves, we have a long exact sequence of cohomologies* $\mathrm{H}^m(C^\bullet(-))$.

(2) *For a quasi-coherent sheaf* $\mathcal{F}$*, we have an isomorphism* $\mathrm{H}^m(X, \mathcal{F}) \to \mathrm{H}^m(C^\bullet(\mathcal{F}))$ *that is functorial in* $\mathcal{F}$ *and compatible with long exact sequences.*

Proof. (1) follows easily from the flatness of $\widehat{\mathcal{O}}_{P_i}$ and Q_i over $\mathcal{O}_{P_i}$. We have the natural morphism $\mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(C^\bullet(\mathcal{F}))$. Let $f_i \in \mathfrak{m}_{P_i}$ be such that $\mathcal{O}_{X,P_i}/(f_i)$ is artinian. Then we have the isomorphisms $Q_i \simeq (\mathcal{O}_{X,P_i})_{f_i}$ and $\widehat{Q}_i \simeq (\widehat{\mathcal{O}}_{X,P_i})_{f_i}$. Then applying Lemma 7.2, we know that $\mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(C^\bullet(\mathcal{F}))$ is bijective. To establish an isomorphism $\mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^1(C^\bullet(\mathcal{F}))$, it suffices to prove $H^1(C^\bullet(\mathcal{F})) = 0$ if $\mathcal{F}$ is a flasque quasi-coherent sheaf. Suppose $\mathcal{F}$ is flasque. Then $\mathcal{F}_{P_i} \to \mathcal{F} \otimes_{\mathcal{O}_{P_i}} Q_i$ are surjective, hence so are γ_i $(1 \leq i \leq N)$. Hence $\mathrm{H}^1(C^\bullet(\mathcal{F})) = 0$. □

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References

[BL] A. Beauville and Y. Laszlo: *Un lemme de descente*, C. R. Acad. Sci. Paris, t.320, Série I, 335-340 (1995)

[G] D. Gieseker: *A degeneration of the moduli space of stable bundles*, J. Differential Geom., 19, 173-206 (1984)

[EGAIII] A. Grothendieck and J. Dieudonné: *Éléments de géometrie algébrique III* Publications Mathémathiques IHES, N. 11, 17 (1961, 1963)

[EGAIV] A. Grothendieck and J. Dieudonné: *Éléments de géometrie algébrique IV* Publications Mathémathiques IHES, N. 20, 24, 28, 32 (1964, 1965, 1966, 1967)

[K1] I. Kausz: *A modular compactification of the general linear group*, Doc. Math. Vol. 5, 553-594 (2000)

[K2] I. Kausz: *A Gieseker type degeneration of moduli stacks of vector bundles on curves*, arXiv:math.AG/0201197

[NS] D. S. Nagaraj and C. S. Seshadri: *Degenerations of the moduli spaces of vector bundles on curves II (Generalized Gieseker moduli spaces)*, Proc. Indian Acad. Sci. Vol. 109, 165-201 (1999)

[Sch] M. Schlessinger: *Functors of artin rings*, Trans. Amer. Math. Soc., 130, 208–222 (1968).

[Sun] X. Sun: *Moduli spaces of* $\mathrm{SL}(r)$*-bundles on singular irreducible curves*, arXiv:math.AG/0303198

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