Extensions of representations of the CAR algebra to the Cuntz algebra $\mathcal{O}_2$ —the Fock and the infinite wedge—

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Fermions are expressed by polynomials of canonical generators of the Cuntz algebra $\mathcal{O}_2$ and they generate the $U(1)$-fixed point subalgebra $\mathcal{A} \equiv \mathcal{O}_2^{U(1)}$ of $\mathcal{O}_2$ by the canonical gauge action. We extend the Fock and the infinite wedge representations of $\mathcal{A}$ to permutative representations of $\mathcal{O}_2$. By these extensions, the boson-fermion correspondence is rewritten by canonical generators of $\mathcal{O}_2$.

1. Introduction

Let $\mathcal{A}_0$ be the Clifford algebra generated by fermions $a_n, a_n^*, n \in \mathbf{N} \equiv \{1,2,3,\ldots\}$ which satisfy the canonical anticommutation relations(=CAR):

$$a_n a_m^* + a_m^* a_n = \delta_{n,m} I, \quad a_n^* a_m^* + a_m^* a_n^* = a_n a_m + a_m a_n = 0 \tag{1.1}$$

for $n, m \in \mathbf{N}$. $\mathcal{A}_0$ always has unique C*-norm $\|\cdot\|$ and the completion $\mathcal{A} \equiv \overline{\mathcal{A}_0}$ with respect to $\|\cdot\|$ is called the *CAR algebra* in theory of operator algebras([**6**]). In [**1**, **2**, **3**, **4**], we construct several polynomial embeddings of $\mathcal{A}$ into the Cuntz algebras $\mathcal{O}_N$. For example, if s_1, s_2 are canonical generators of $\mathcal{O}_2$, that is, they satisfy

$$s_i^* s_j = \delta_{ij} I \quad (i,j=1,2), \quad s_1 s_1^* + s_2 s_2^* = I, \tag{1.2}$$

then

$$a_1 \equiv s_1 s_2^*, \quad a_n \equiv \sum_{J \in \{1,2\}^{n-1}} (-1)^{n_2(J)} s_J s_1 s_2^* s_J^* \quad (n \geq 2) \tag{1.3}$$

satisfy (1.1) where $n_2(J) \equiv \sum_{l=1}^k (j_l - 1)$ and $s_J = s_{j_1} \cdots s_{j_k}$, $s_{J^*} = s_{j_k}^* \cdots s_{j_1}^*$ for $J = (j_1, \ldots, j_k)$, and $C^* < \{a_n \in \mathcal{O}_2 : n \in \mathbf{N}\} >$ coincides with a fixed-point subalgebra $\mathcal{O}_2^{U(1)}$ of $\mathcal{O}_2$ by the canonical gauge action. Put a linear map ζ on $\mathcal{O}_2$ by

$$\zeta(x) \equiv s_1 x s_1^* - s_2 x s_2^* \quad (x \in \mathcal{O}_2). \tag{1.4}$$

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Then $a_n = \zeta(a_{n-1})$ for each $n \geq 2$. In this sense, $\{a_n\}_{n\in\mathbf{N}}$ in (1.3) is called the *recursive fermion system(=RFS)* in $\mathcal{O}_2$.

In this paper, we extend the Fock and the infinite wedge representations([**12**, **13**]) of $\mathcal{A}$ to permutative representations of $\mathcal{O}_2$ under identification of $\mathcal{A}$ as $\mathcal{O}_2^{U(1)} \subset \mathcal{O}_2$ by (1.3). At first, we give our main theorem for abstract formulations of representations of $\mathcal{A}$.

Theorem 1.1. (i) *Let $(\mathcal{H}_F, \pi_F)$ be the Fock representation of $\mathcal{A}$, that is, $(\mathcal{H}_F, \pi_F)$ is a cyclic representation with a cyclic vector $\Omega \in \mathcal{H}_F$ such that*

$$\pi_F(a_n)\Omega = 0 \quad ({}^\forall n \in \mathbf{N}).$$

Then $(\mathcal{H}_F, \pi_F)$ is extended to an irreducible representation $(\mathcal{H}_F, \tilde{\pi}_F)$ of $\mathcal{O}_2$ defined by

$$\tilde{\pi}_F(s_1) \equiv L, \quad \tilde{\pi}_F(s_2) \equiv \pi(a_1^*) \cdot L$$

where L is the one-sided shift operator on $\mathcal{H}_F$ defined by

$$L\Omega \equiv \Omega, \quad L\pi_F(a_{n_1}^* \cdots a_{n_k}^*)\Omega \equiv \pi_F(a_{n_1+1}^* \cdots a_{n_k+1}^*)\Omega$$

for each $n_1, \ldots, n_k \in \mathbf{N}$ and $k \in \mathbf{N}$.

(ii) *Let $(\Lambda^{\frac{\infty}{2}}V, \pi_{\infty,+})$ be the infinite wedge representation of $\mathcal{A}$, that is, $(\Lambda^{\frac{\infty}{2}}V, \pi_{\infty,+})$ is a cyclic representation with a cyclic vector $|\mathrm{vac}{>}_+ \in \Lambda^{\frac{\infty}{2}}V$ such that*

$$\psi_{-k}|\mathrm{vac}{>}_+ = \psi_k^*|\mathrm{vac}{>}_+ = 0 \quad ({}^\forall k \in \mathbf{Z} + \tfrac{1}{2},\, k > 0)$$

where

$$\psi_k \equiv \pi_{\infty,+}(a_{2k+1}), \quad \psi_{-k} \equiv \pi_{\infty,+}(a_{2k}) \quad (k \in \mathbf{Z} + \tfrac{1}{2},\, k > 0) \tag{1.5}$$

and $\mathbf{Z} + \frac{1}{2} \equiv \{n + 1/2 : n \in \mathbf{Z}\}$. Then $(\Lambda^{\frac{\infty}{2}}V, \pi_{\infty,+})$ is extended to an irreducible representation $(\Lambda^{\frac{\infty}{2}}V \oplus \Lambda^{\frac{\infty}{2}}V^*, \Pi)$ of $\mathcal{O}_2$ which satisfies

$$\Pi(s_1 s_2)|\mathrm{vac}{>}_+ = |\mathrm{vac}{>}_+.$$

Both representations $(\mathcal{H}, \tilde{\pi}_F)$ and $(\Lambda^{\frac{\infty}{2}}V \oplus \Lambda^{\frac{\infty}{2}}V^*, \Pi)$ of $\mathcal{O}_2$ in Theorem 1.1 are permutative representations([**5**, **8**, **9**]) and they are not equivalent each other. Well-known Fock and infinite wedge representations are just realizations of those in Theorem 1.1. The extension for a concrete infinite wedge is given in § 4.

On the other hand, the boson-fermion correspondence on the infinite wedge representation is given by

$$\alpha_n = \sum_{k\in\mathbf{Z}+\frac{1}{2}} \psi_{k-n}\psi_k^* \quad (n \in \mathbf{Z} \setminus \{0\}). \tag{1.6}$$

$\{a_n\}_{n\in\mathbf{Z}}$ satisfies $\alpha_{-n}=\alpha_n^*$, $\alpha_n\alpha_m-\alpha_m\alpha_n=n\cdot\delta_{n,-m}I$. By identifying s_i and $\Pi(s_i)$ in Theorem 1.1 (ii) and combining (1.3) and (1.5), we have

$$\begin{cases} \psi_k = \displaystyle\sum_{J\in\{1,2\}^{2k}} (-1)^{n_2(J)} s_J s_1 s_2^* s_J^* \\ \\ \psi_{-k} = \displaystyle\sum_{J\in\{1,2\}^{2k-1}} (-1)^{n_2(J)} s_J s_1 s_2^* s_J^* \end{cases} \quad (k\in\mathbf{Z}+\tfrac{1}{2},\ k>0).$$

From these and (1.6), we have a direct expression of bosons by canonical generators of $\mathcal{O}_2$ as follows:

$$\alpha_n=\sum_{l\in\mathbf{N}}\rho^{2l-2}(X_n)+B_n \quad (n\geq 1) \tag{1.7}$$

where

$$X_n\equiv\rho(s_1s_2^*\zeta^{2n}(s_2s_1^*))+\zeta^{2n}(s_1s_2^*)s_2s_1^* \quad (n\geq 1), \tag{1.8}$$

$$B_1\equiv -s_1s_2s_1^*s_2^*,\quad B_n\equiv\rho(B_{n-1}^*)-s_1\zeta^{2n-2}(s_2s_1^*)s_2^* \quad (n\geq 2), \tag{1.9}$$

ζ is in (1.4) and ρ is the canonical endomorphism of $\mathcal{O}_2$, that is, $\rho(x)\equiv s_1xs_1^*+s_2xs_2^*$ for $x\in\mathcal{O}_2$. Furthermore $\alpha_n^*|\mathrm{vac}>_+ = B_n^*|\mathrm{vac}>_+$ for each $n\geq 1$.

In § 2, we review representations of $\mathcal{A}$ and $\mathcal{O}_2$ and the RFS. In § 3, we introduce a branching function system on the space of Maya diagrams and review the infinite wedge space and its dual space. In § 4, we show extensions of the Fock and the infinite wedge representations to $\mathcal{O}_2$. A relation between a branching law of a permutative representation of $\mathcal{O}_2$ and the extension of the infinite wedge is concretely illustrated.

2. The recursive fermion system and permutative representations of $\mathcal{O}_2$

Both $\mathcal{O}_2$ and the CAR algebra

$$CAR\equiv C^*\!<\{a_n : n\in\mathbf{N}\}>$$

($=\mathcal{A}$ in § 1) are simple, infinite dimensional, noncommutative C*-algebras([**6**, **7**]). Remark that $a_n^*\in CAR$ for each $n\in\mathbf{N}$ by definition of C*-algebra. Unital $*$-homomorphisms (specially, unital $*$-representations) from these algebras to other algebras are always faithful. Algebras which are generated by generators s_1,s_2 in (1.2) and a_n, $n\in\mathbf{N}$ in (1.1) are unique up to $*$-isomorphisms, respectively. Therefore their representations are determined by only operators on a Hilbert space, which satisfy relations of their generators without ambiguity. In this paper, a representation and an embedding always mean a unital $*$-representation and a unital $*$-embedding, respectively.

2.1. Representations of CAR and the RFS. We review representations of CAR in theory of operator algebras in [**6**].

Definition 2.1. *Let $(\mathcal{H},\pi)$ be a representation of CAR.*

(i) *$(\mathcal{H},\pi)$ is the (abstract)Fock representation of CAR if there is a cyclic unit vector $\Omega\in\mathcal{H}$ such that $\pi(a_n)\Omega=0$ for each $n\in\mathbf{N}$. Ω is called the vacuum of $(\mathcal{H},\pi)$. We denote $(\mathcal{H},\pi)$ by $\mathcal{H}_{Fock}$ simply.*

(ii) *$(\mathcal{H},\pi)$ is $P[12]$ if there is a cyclic unit vector $\Omega\in\mathcal{H}$ such that $\pi(a_{2n-1})\Omega=\pi(a_{2n}^*)\Omega=0 \quad ({}^\forall n\in\mathbf{N})$.*

(iii) *$(\mathcal{H},\pi)$ is $P[21]$ if there is a cyclic unit vector $\Omega\in\mathcal{H}$ such that $\pi(a_{2n-1}^*)\Omega=\pi(a_{2n})\Omega=0 \quad ({}^\forall n\in\mathbf{N})$.*

For consistency with after statements, any Ω in the above is normalized. $\mathcal{H}_{Fock}$, $P[12]$, $P[21]$ appear in [**5**] as components of irreducible decomposition of permutative representation of $\mathcal{O}_2$, which are called “atom”. This fact is explained in Proposition 2.6.

Proposition 2.2. *All of $\mathcal{H}_{Fock},P[12],P[21]$ are unique up to unitary equivalences and irreducible. Any two of $\mathcal{H}_{Fock},P[12],P[21]$ are not unitarily equivalent.*

Proof. See (5.18) in [**2**], and [**5**]. In Appendix A, their inequivalences are shown. □

By Proposition 2.2, symbols $\mathcal{H}_{Fock}$, $P[12]$ and $P[21]$ make sense as equivalence classes of representations. Since fermions are often treated as operators on a concrete Hilbert space, any representation which is different with the Fock representation in only permutation of creations and annihilations and their phase factors, are called the Fock representation, too in such situation. In this paper, we do not call such representation by the Fock representation.

We review a concrete example: Put $H\equiv l_2(\mathbf{N})$ and the completely antisymmetric Fock space $F_-(H)\equiv\mathbf{C}\Omega\oplus\bigoplus_{k=1}^{\infty}H^{\wedge k},\quad H^{\wedge k}\equiv P_-^{(k)}H^{\otimes k}$ where $P_-^{(k)}$ is the antisymmetrizer on $H^{\otimes k}$ defined by $P_-^{(k)}(v_1\otimes\cdots\otimes v_k)\equiv(\sqrt{k}!)^{-1/2}\sum_{\sigma\in\mathfrak{S}_k}\mathrm{sgn}(\sigma)\cdot v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}$ for $k\geq1$. We denote $v_1\wedge\cdots\wedge v_k=P_-^{(k)}(v_1\otimes\cdots\otimes v_k)$. We see $v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(k)}=\mathrm{sign}(\sigma)(v_1\otimes\cdots\otimes v_k)$ for each $\sigma\in\mathfrak{S}_k$. For $f\in H$, define $A^*(f)\Omega\equiv f,\quad A^*(f)v\equiv f\wedge v$ for $f\in H$, $v\in H^{\wedge n}$, $n\geq1$. $A(f)$ is defined by the adjoint operator of $A^*(f)$ on $F_-(H)$. We see that $A(f)\Omega=0$ for each $f\in H$. Then $A(f)A^*(g)+A^*(g)A(f)=<f|g>I$ for each $f,g\in H$. For the canonical basis $\{e_n\}_{n\in\mathbf{N}}$ of $H=l_2(\mathbf{N})$, put $\pi_F(a_n)\equiv A(e_n)$ for $n\in\mathbf{N}$. Then $(F_-(H),\pi_F)$ is a representation of CAR. $(F_-(H),\pi_F)$ is the (concrete)Fock representation.

Let s_1,s_2 be canonical generators of $\mathcal{O}_2$. Define an embedding

$$\varphi_S: CAR\hookrightarrow\mathcal{O}_2;\quad \varphi_S(a_n)\equiv\zeta^{n-1}(s_1s_2^*)\quad(n\geq1) \tag{2.1}$$

where ζ is in (1.4). For example, $\varphi_S(a_1) = s_1 s_2^*$, $\varphi_S(a_2) = s_1 s_1 s_2^* s_1^* - s_2 s_1 s_2^* s_2^*$. We call φ_S by the *standard embedding* of CAR into $\mathcal{O}_2$. $C^* < \{\varphi_S(a_n)\}_{n\in\mathbf{N}} >= \mathcal{O}_2^{U(1)} = \{x \in \mathcal{O}_2 : {}^\forall z \in U(1),\, \gamma_z(x) = x\} \cong UHF_2$ where γ is the canonical $U(1)$-action of $\mathcal{O}_2$, $\gamma_z(s_i) \equiv z s_i$ for $z \in U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$ and $i = 1, 2$. By identifying a_n and $\varphi_S(a_n)$, a_n's coincide with those in (1.3) and we have the following intertwining relations:

Lemma 2.3. *For $n \geq 1$,*

$$s_1 a_n = a_{n+1} s_1, \quad s_1 a_n^* = a_{n+1}^* s_1, \quad s_2 a_n = -a_{n+1} s_2, \quad s_2 a_n^* = -a_{n+1}^* s_2,$$

$$s_1^* a_{n+1} = a_n s_1^*, \quad s_1^* a_{n+1}^* = a_n^* s_1^*, \quad s_2^* a_{n+1} = -a_n s_2^*, \quad s_2^* a_{n+1}^* = -a_n^* s_2^*.$$

2.2. Permutative representations of $\mathcal{O}_2$ and their branching laws. Permutative representations of the Cuntz algebras are well-studied([**5**, **8**, **9**]). We introduce two permutative representations of $\mathcal{O}_2$ according to [**10**].

Definition 2.4. *A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_2$ is $P(1)$(resp. $P(12)$) if there is a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(s_1)\Omega = \Omega$(resp. $\pi(s_1 s_2)\Omega = \Omega$). We call Ω by the GP vector of $(\mathcal{H}, \pi)$.*

Both $P(1)$ and $P(12)$ exist uniquely up to unitary equivalences, and they are irreducible and not unitarily equivalent each other.

Assume that $(\mathcal{H}, \pi)$ is $P(12)$ of $\mathcal{O}_2$ with the GP vector Ω and α is an automorphism of $\mathcal{O}_2$ defined by $\alpha(s_1) \equiv s_2$, $\alpha(s_2) \equiv s_1$. Define an operator U on $\mathcal{H}$ by

$$U\Omega \equiv \pi(s_2)\Omega, \quad U\pi(s_J)\Omega \equiv \pi(\alpha(s_J)s_2)\Omega \quad (J \in \{1,2\}^k,\ k \geq 1). \tag{2.2}$$

Then U is a unitary which satisfies $U^2 = I$ and $\mathrm{Ad}U \circ \pi = \pi \circ \alpha$. In consequence, $(\mathcal{H}, \pi, U)$ is a covariant representation of a C*-dynamical system $(\mathcal{O}_2, \alpha, \mathbf{Z}_2)$.

Example 2.5. (i) Define a representation $(l_2(\mathbf{N}), \pi_S)$ of $\mathcal{O}_2$ by

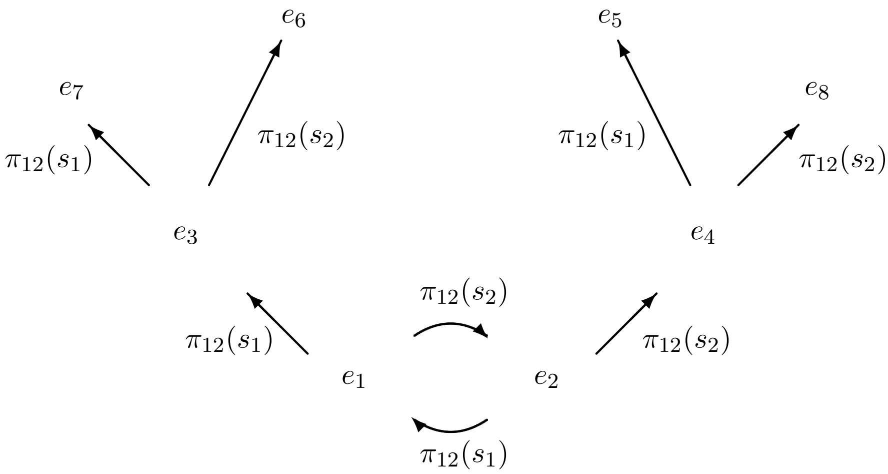
$$\pi_S(s_1)e_n \equiv e_{2n-1}, \quad \pi_S(s_2)e_n \equiv e_{2n} \quad (n \in \mathbf{N}).$$

Then $(l_2(\mathbf{N}), \pi_S)$ is $P(1)$. We call $(l_2(\mathbf{N}), \pi_S)$ by the *standard representation* of $\mathcal{O}_2$.

(ii) Define a representation $(l_2(\mathbf{N}), \pi_{12})$ of $\mathcal{O}_2$ by

$$\pi_{12}(s_1)e_{2n-1} \equiv e_{4n-1}, \quad \pi_{12}(s_1)e_{2n} \equiv e_{4n-3}, \quad \pi_{12}(s_2)e_n \equiv e_{2n} \quad (n \in \mathbf{N}).$$

The action of s_1, s_2 on the canonical basis of $l_2(\mathbf{N})$ is illustrated as follows:



This system looks like the Fock representation with two vacuums e_1 and e_2. This diagram appears in § 3 and § 4 again. $(l_2(\mathbf{N}), \pi_{12})$ is $P(12)$. Remark that $\pi_{12}(s_1 s_2)e_1 = e_1$ is expressed as a cycle in the above. For this type example, see [**11**].

By φ_S in (2.1), we identify CAR and a subalgebra $\varphi_S(CAR) = \mathcal{O}_2^{U(1)} \subset \mathcal{O}_2$. For a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_2$, we have the restriction $(\mathcal{H}, \pi|_{CAR})$ of $(\mathcal{H}, \pi)$ on CAR.

Proposition 2.6. *([**2**]) The following branching laws hold:*

$$P(1)|_{CAR} = \mathcal{H}_{Fock}, \quad P(12)|_{CAR} = P[12] \oplus P[21].$$

Specially, all of these are irreducible decompositions.

We consider the branching of $P(12)$ on CAR more.

Lemma 2.7. *Let $(\mathcal{H}, \pi)$ be $P(12)$ of $\mathcal{O}_2$ with the GP vector $\Omega_1 \equiv \Omega$ and put $\Omega_2 \equiv \pi(s_2)\Omega_1$. Then we have the followings:*

$$\pi(a_{2k-1})\Omega_1 = 0, \quad \pi(a_{2k})\Omega_1 = (-1)^{k-1}\pi(s_{(12)^{k-1}} s_1 s_1)\Omega_1,$$

$$\pi(a_{2k-1}^*)\Omega_1 = (-1)^{k-1}\pi(s_{(12)^{k-1}} s_2 s_2)\Omega_1, \quad \pi(a_{2k}^*)\Omega_1 = 0,$$

$$\pi(a_{2k-1})\Omega_2 = (-1)^{k-1}\pi(s_{(21)^{k-1}} s_1)\Omega_1, \quad \pi(a_{2k})\Omega_2 = 0,$$

$$\pi(a_{2k-1}^*)\Omega_2 = 0, \quad \pi(a_{2k}^*)\Omega_2 = (-1)^k \pi(s_2 s_{(12)^{k-1}} s_2 s_2)\Omega_1$$

for each $k \in \mathbf{N}$.

Proof. By $\pi(\zeta(x)s_2)\Omega_1 = -\pi(s_2 x)\Omega_1$ for any $x \in \mathcal{O}_2$, statements hold. $\square$

Let $V_{12} \equiv \pi(CAR)\Omega_1$, $V_{21} \equiv \pi(CAR)\Omega_2$. Then $\mathcal{H} = V_{12} \oplus V_{21}$ and we see that

	V_{12}	V_{21}
vacuum	Ω_1	Ω_2
creation	a^*_{2k-1}, a_{2k}	a_{2k-1}, a^*_{2k}
annihilation	a_{2k-1}, a^*_{2k}	a^*_{2k-1}, a_{2k}

where $k \in \mathbf{N}$. Specially,

$$(2.3)\qquad \begin{cases} \pi(a_1)\Omega_2 = \pi(s_1)\Omega_1, \quad \pi(a_2)\Omega_1 = \pi(s_1s_1)\Omega_1, \\ \\ \pi(a_1^*)\Omega_1 = \pi(s_2)\Omega_2, \quad \pi(a_2^*)\Omega_2 = -\pi(s_2s_2)\Omega_2. \end{cases}$$

If α is the $\mathbf{Z}_2$-action on $\mathcal{O}_2$ and a_n is the RFS in $\mathcal{O}_2$, then $\alpha(a_n) = (-1)^{n-1}a_n^*$. Hence U in (2.2) satisfies

$$U\Omega_1 = \ \Omega_2, \quad U\Omega_2 = \Omega_1, \quad U\pi(a_K a_L^*)\Omega_1 = (-1)^{|K|_1+|L|_1}\pi(a_K^* a_L)\Omega_2$$

where $a_K \equiv a_{k_1}\cdots a_{k_n}$, $|K|_1 \equiv \sum_{i=1}^n (k_i-1)$ for $K = \{k_1,\ldots,k_n\} \subset \mathbf{N}$. Hence $UV_{12} = V_{21}$.

Example 2.8. (i) In Example 2.5 (i), $\pi_S \circ \varphi_S$ is $\mathcal{H}_{Fock}$ with the vacuum e_1. See [**1**] for more detail.

(ii) In Example 2.5 (ii), we consider $\pi_{12} \circ \varphi_S$. Then $(l_2(2\mathbf{N}-1), \pi_{12} \circ \varphi_S)$ is $P[12]$ and $(l_2(2\mathbf{N}), \pi_{12} \circ \varphi_S)$ is $P[21]$. If we identify a_n and $(\pi_{12} \circ \varphi_S)(a_n)$, then

$$a_{2n-1}e_1 = a^*_{2n}e_1 = a_{2n}e_2 = a^*_{2n-1}e_2 = 0,$$
$$a_{2n}e_1 = (-1)^{n-1}e_{4^{n-1}\cdot 6+1}, \quad a^*_{2n-1}e_1 = (-1)^{n-1}e_{4^{n-1}\cdot 3+1},$$
$$a_{2n-1}e_2 = (-1)^{n-1}e_{4^{n-1}\cdot 3+2}, \quad a^*_{2n}e_2 = (-1)^{n}e_{4^{n-1}\cdot 6+2}$$

for each $n \in \mathbf{N}$. These statements are shown by using $(\pi_{12}(s_1s_2))^m e_n = e_{4^m(n-1)+1}$ for each $m, n \in \mathbf{N}$. Specially, when $n_2 > n_1$,

$$a_{2n_1}a_{2n_2}e_1 = (-1)^{n_2-1}e_{3\cdot(2^{2n_2-1}+2^{2n_1-1})+1}.$$

3. A branching function system on the infinite wedge

We review a representation of the fermion algebra which is called the *infinite wedge space*([**12**, **13**]) according to notation in [**13**]. In order to extend this representation to $\mathcal{O}_2$, we introduce the dual infinite wedge space at once and a branching function system on them.

3.1. Maya diagram. Denote $\mathbf{Z}+\frac{1}{2} \equiv \{n+\frac{1}{2} : n \in \mathbf{Z}\}$. Put

$$\mathbf{Z}_{+/2} \equiv \{n+1/2 : n \in \mathbf{Z}, n \geq 0\}, \quad \mathbf{Z}_{-/2} \equiv \{n-1/2 : n \in \mathbf{Z}, n \leq 0\}.$$

For a subset $S \subset \mathbf{Z}+\frac{1}{2}$, define $\Delta_\pm(S) \subset \mathbf{Z}+\frac{1}{2}$ by

$$(3.1)\qquad \Delta_\pm(S) \equiv (S \setminus \mathbf{Z}_{\mp/2}) \cup (\mathbf{Z}_{\mp/2} \setminus S).$$

Remark the sign of both sides.

Definition 3.1. *An element in* $\mathcal{M}_\pm \equiv \{S \subset \mathbf{Z} + \frac{1}{2} : \#\Delta_\pm(S) < \infty\}$ *is called a Maya diagram. Specially,* $\mathbf{Z}_{\mp/2} \in \mathcal{M}_\pm$ *is called the vacuum in* $\mathcal{M}_\pm$*.*

We see that $\mathcal{M}_+ \cap \mathcal{M}_- = \emptyset$ and $\mathcal{M}_\pm = \{-S : S \in \mathcal{M}_\mp\}$ where $-S \equiv \{-k : k \in S\}$. There are max$S$ for any $S \in \mathcal{M}_+$ and min$S$ for any $S \in \mathcal{M}_-$, and $\#S = \infty$ for any $S \in \mathcal{M}_\pm$. Therefore we can always parameterize as follows: $S = \{t_i : i \in \mathbf{N}\}$ such that $t_i > t_{i+1}$ for $i \geq 1$ when $S \in \mathcal{M}_+$, and $S = \{t_i : i \in \mathbf{N}\}$ such that $t_i < t_{i+1}$ for $i \geq 1$ when $S \in \mathcal{M}_-$.

We illustrate $S \in \mathcal{M}_\pm$ by a two-sided infinite sequence consisting of symbols $\circ$ and $\bullet$ along the lattice $\mathbf{Z} + \frac{1}{2}$ as follows: For $S \in \mathcal{M}_\pm$, put $\circ$ at $k \in \mathbf{Z} + \frac{1}{2}$ when $k \in S$, and put $\bullet$ at $k \in \mathbf{Z} + \frac{1}{2}$ when $k \notin S$. For example, if $\{-5/2, -1/2, 3/2, 7/2\} \subset S$ and $\{-7/2, -3/2, 1/2, 5/2\} \cap S = \emptyset$, then

$$\begin{array}{cccccccccc} \cdots & -7/2 & -5/2 & -3/2 & -1/2 & 1/2 & 3/2 & 5/2 & 7/2 & \cdots \\ \cdots & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \cdots \end{array}$$

By this illustration, $\mathcal{M}_\pm$ are the following sets:

$$\mathcal{M}_+ = \{\cdots \circ \circ * * * * * * * \bullet \bullet \cdots\}, \quad \mathcal{M}_- = \{\cdots \bullet \bullet * * * * * * * * \circ \circ \cdots\}$$

where $* * * * **$ is taken any finite sequence consisting of $\circ$ and $\bullet$. Specially,

vacuum	$\mathbf{Z}_{-/2} \in \mathcal{M}_+$	$\cdots \circ \circ \circ \circ \bullet \bullet \bullet \bullet \cdots$
dual vacuum	$\mathbf{Z}_{+/2} \in \mathcal{M}_-$	$\cdots \bullet \bullet \bullet \bullet \circ \circ \circ \circ \cdots$

3.2. A branching function system on the space of Maya diagrams. Put the space of all Maya diagrams

$$\mathcal{M} \equiv \mathcal{M}_+ \cup \mathcal{M}_-.$$

We give a branching function system on $\mathcal{M}$. Put $S_\pm \equiv S \cap \mathbf{Z}_{\pm/2}$, $S_{+1} \equiv \{k + 1 : k \in S\}$ and $S_{\pm,+1} \equiv (S_\pm)_{+1}$. For $S \in \mathcal{M}$, define

$$g_1(S) \equiv -(S_{+,+1} \cup S_- \cup \{1/2\}), \quad g_2(S) \equiv -(S_{+,+1} \cup S_-). \tag{3.2}$$

Then $g = \{g_1, g_2\}$ is a branching function system on $\mathcal{M}$, that is, g_1 and g_2 are injective maps on $\mathcal{M}$, $g_1(\mathcal{M}) \cap g_2(\mathcal{M}) = \emptyset$ and $g_1(\mathcal{M}) \cup g_2(\mathcal{M}) = \mathcal{M}$. Furthermore $g_1(\mathcal{M}) = \{S \in \mathcal{M} : -1/2 \in S\}$, $g_2(\mathcal{M}) = \{S \in \mathcal{M} : -1/2 \notin S\}$,

$$g_1^{-1}(S) = -\{(S_- \setminus \{-1/2\})_{+1} \cup S_+\}, \quad g_2^{-1}(S) = -(S_{-,+1} \cup S_+).$$

If $\theta(S) \equiv \mathbf{Z} + \frac{1}{2} \setminus S$, then $\theta^2 = id$, $\theta(\mathcal{M}_\pm) = \mathcal{M}_\mp$, $\theta(\mathbf{Z}_{\pm/2}) = \mathbf{Z}_{\mp/2}$ and $g_2 = \theta \circ g_1 \circ \theta$.

Lemma 3.2. *Denote* $S_{\pm n} \equiv \{k \pm n : k \in S\}$, $S_{\pm,+n} \equiv (S_\pm)_{+n}$*. Then the followings hold:*

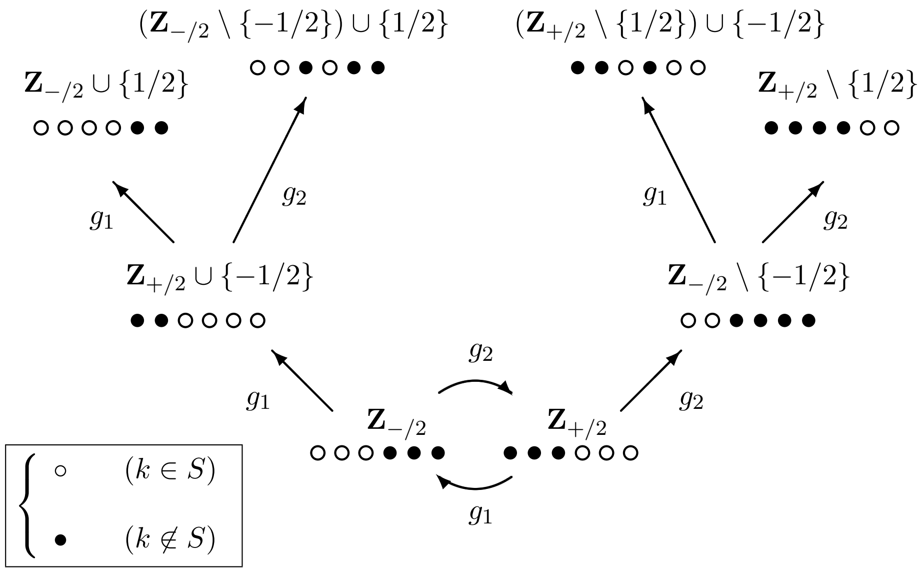
(i) $(g_1 \circ g_2)^n(S) = S_{-,-n} \cup S_{+,+n} \cup \{-1/2, \ldots, -(n-1) - 1/2\}$ *for* $n \in \mathbf{N}$.
(ii) $(g_1 \circ g_2)^{-n}(S) = (S_{-,+n})_- \cup S_{+,-n}$ *for* $n \in \mathbf{N}$.

(iii) *Put* $h_n(S) \equiv (g_{12}^{n-1} \circ g_1 \circ g_2^{-1} \circ (g_{12}^{n-1})^{-1})(S)$ *and* $k_n(S) \equiv (g_{12}^{n-1} \circ g_1 \circ g_1 \circ (g_{12}^{-1})^n)(S)$ *for* $n \in \mathbf{N}$. *Then*

$$h_n(S) = S \cup \{-(n-1) - 1/2\}, \quad k_n(S) = S \cup \{n - 1 + 1/2\} \quad (n \in \mathbf{N}).$$

We illustrate g by Maya diagrams:

S	$g_1(S)$	$g_2(S)$
$\mathbf{Z}_{-/2}$	$\mathbf{Z}_{+/2} \cup \{-1/2\}$	$\mathbf{Z}_{+/2}$
$\mathbf{Z}_{+/2}$	$\mathbf{Z}_{-/2}$	$\mathbf{Z}_{-/2} \setminus \{-1/2\}$
$\mathbf{Z}_{+/2} \cup \{-1/2\}$	$\mathbf{Z}_{-/2} \cup \{1/2\}$	$(\mathbf{Z}_{-/2} \setminus \{-1/2\}) \cup \{1/2\}$
$\mathbf{Z}_{-/2} \setminus \{-1/2\}$	$(\mathbf{Z}_{+/2} \setminus \{1/2\}) \cup \{-1/2\}$	$\mathbf{Z}_{+/2} \setminus \{1/2\}$
$\vdots$	$\vdots$	$\vdots$



3.3. The infinite wedge representation of CAR **and its dual.** We introduce the infinite wedge space by a Hilbert space of Maya diagrams. For a set Λ, $l_2(\Lambda)$ is a complex Hilbert space with a complete orthonormal basis $\{e_\lambda\}_{\lambda \in \Lambda}$ and $\dim l_2(\Lambda) = \#\Lambda$.

Definition 3.3. *For* $\mathcal{M}_\pm$ *in Definition* 3.3,

$$\Lambda^{\frac{\infty}{2}} V^{\#} \equiv l_2(\mathcal{M}), \quad \Lambda^{\frac{\infty}{2}} V \equiv l_2(\mathcal{M}_+), \quad \Lambda^{\frac{\infty}{2}} V^* \equiv l_2(\mathcal{M}_-)$$

are called the bi-infinite wedge space, the infinite wedge space and the dual infinite wedge space, respectively.

We see that $\Lambda^{\frac{\infty}{2}} V^{\#} = \Lambda^{\frac{\infty}{2}} V \oplus \Lambda^{\frac{\infty}{2}} V^*$. By definition, $\Lambda^{\frac{\infty}{2}} V^{\#}$, $\Lambda^{\frac{\infty}{2}} V$ and $\Lambda^{\frac{\infty}{2}} V^*$ have canonical basis $\{e_S : S \in \mathcal{M}\}$, $\{e_S : S \in \mathcal{M}_+\}$ and $\{e_S : S \in \mathcal{M}_-\}$, respectively. Usually, the symbol $\Lambda^{\frac{\infty}{2}} V$ means a subspace of $l_2(\mathcal{M}_+)$

consisting of finite linear combinations of $\{e_S : S \in \mathcal{M}_+\}$([**12**, **13**]). We denote

$$|\mathrm{vac}>_\pm \equiv e_{\mathbf{Z}_{\mp/2}}.$$

Since there are max$S$ for any $S \in \mathcal{M}_+$ and min$S$ for any $S \in \mathcal{M}_-$, and $\#S = \infty$ for any $S \in \mathcal{M}_\pm$, we can denote

$$t_1 \wedge t_2 \wedge \cdots = e_S \quad \text{when } S = \{t_i : {}^\forall i \in \mathbf{N},\, t_i > t_{i+1}\} \in \mathcal{M}_+,$$

$$t_1 \wedge t_2 \wedge \cdots = e_S \quad \text{when } S = \{t_i : {}^\forall i \in \mathbf{N},\, t_i < t_{i+1}\} \in \mathcal{M}_-.$$

Then we see that

$$|\mathrm{vac}>_+ = (-\tfrac{1}{2}) \wedge (-\tfrac{3}{2}) \wedge (-\tfrac{5}{2}) \wedge \cdots, \quad |\mathrm{vac}>_- = \tfrac{1}{2} \wedge \tfrac{3}{2} \wedge \tfrac{5}{2} \wedge \cdots.$$

For a permutation $\sigma \in \mathfrak{S}_k$ $k \geq 2$, define

$$t_{\sigma(1)} \wedge \cdots\cdots \wedge t_{\sigma(k)} \wedge t_{k+1} \wedge \cdots \equiv \mathrm{sgn}(\sigma) \cdot t_1 \wedge \cdots \wedge t_k \wedge t_{k+1} \wedge \cdots .$$

By these definitions, "$\wedge$" seems the exterior product of infinite vectors.

Define a family $\{\psi_k\}_{k \in \mathbf{Z}+\frac{1}{2}}$ of operators on $\Lambda^{\frac{\infty}{2}} V^\#$ by

$$\psi_k e_S \equiv \begin{cases} (-1)^{d_S(k)} \cdot e_{S \cup \{k\}} & (k \notin S), \\ & \\ 0 & (\text{otherwise}) \end{cases} \qquad (S \in \mathcal{M})$$

where $d_S(k) \equiv \min\{\#\{x \in S : x > k\}, \#\{x \in S : x < k\}\}$. We simply denote

$$\psi_k e_S = (-1)^{d_S(k)} \cdot \chi_{S^c}(k) \cdot e_{S \cup \{k\}}$$

where χ_{S^c} is the characteristic function on $S^c \equiv (\mathbf{Z} + \frac{1}{2}) \setminus S$. We can easily check that the definition of ψ_k coincides with the following ordinary definition:

$$\psi_k v = k \wedge v \quad (v \in \Lambda^{\frac{\infty}{2}} V^\#,\, k \in \mathbf{Z} + \tfrac{1}{2}).$$

Lemma 3.4. (i) *The adjoint ψ_k^* of ψ_k is given by*

$$\psi_k^* e_S = (-1)^{d_{S \setminus \{k\}}(k)} \cdot \chi_S(k) \cdot e_{S \setminus \{k\}} \quad (k \in \mathbf{Z} + \tfrac{1}{2},\, S \in \mathcal{M}).$$

(ii) $\psi_k \psi_k^* e_S = \chi_S(k) \cdot e_S$ *for* $k \in \mathbf{Z} + \frac{1}{2}$ *and* $S \in \mathcal{M}$.
(iii) $\psi_k \psi_l^* + \psi_l^* \psi_k = \delta_{kl} I$ *for* $k, l \in \mathbf{Z} + \frac{1}{2}$ *and other anticommutators vanish.*

We see that

$$\psi_{-k}|\mathrm{vac}>_+ = \psi_k^*|\mathrm{vac}>_+ = 0,$$

$$\psi_k|\mathrm{vac}>_+ = k \wedge |\mathrm{vac}>_+, \quad \psi_{-k}^*|\mathrm{vac}>_+ = (-1)^{k-\frac{1}{2}} \cdot e_{\mathbf{Z}_{-/2} \setminus \{-k\}}$$

for $k \in \mathbf{Z} + \frac{1}{2}$, $k > 0$. In the same way, we see that

	$\Lambda^{\frac{\infty}{2}} V$	$\Lambda^{\frac{\infty}{2}} V^*$
vacuum	$\|\mathrm{vac}>_+$	$\|\mathrm{vac}>_-$
creation	$\psi_{-k}^*,\, \psi_k$	$\psi_{-k},\, \psi_k^*$
annihilation	$\psi_{-k},\, \psi_k^*$	$\psi_{-k}^*,\, \psi_k$

where $k \in \mathbf{Z} + \frac{1}{2}$, $k > 0$.

Definition 3.5. *A representation* $(\Lambda^{\frac{\infty}{2}} V^{\#}, \pi_{\infty})$ *of* CAR *defined by*

$$\pi_{\infty}(a_{2n-1}) \equiv \psi_{-n+1/2}, \quad \pi_{\infty}(a_{2n}) \equiv \psi_{n-1/2} \quad (n \in \mathbf{N}) \tag{3.3}$$

is called the bi-infinite wedge representation of CAR.

On the other hand,

$$\psi_k = \pi_{\infty}(a_{2k+1}), \quad \psi_{-k} = \pi_{\infty}(a_{2k}) \quad (k \in \mathbf{Z} + \tfrac{1}{2},\ k > 0). \tag{3.4}$$

Proposition 3.6. (i) *The following irreducible decomposition of representations of* CAR *holds:*

$$\Lambda^{\frac{\infty}{2}} V^{\#} = \Lambda^{\frac{\infty}{2}} V \oplus \Lambda^{\frac{\infty}{2}} V^{*}.$$

(ii) *If we denote*

$$\pi_{\infty,+} \equiv \pi_{\infty}|_{\Lambda^{\frac{\infty}{2}} V}, \quad \pi_{\infty,-} \equiv \pi_{\infty}|_{\Lambda^{\frac{\infty}{2}} V^{*}},$$

then $(\Lambda^{\frac{\infty}{2}} V, \pi_{\infty,+})$ *is* $P[12]$ *and* $(\Lambda^{\frac{\infty}{2}} V^{*}, \pi_{\infty,-})$ *is* $P[21]$.

$(\Lambda^{\frac{\infty}{2}} V, \pi_{\infty,+})$ and $(\Lambda^{\frac{\infty}{2}} V^{*}, \pi_{\infty,-})$ are called the *infinite wedge representation* and the *dual-infinite wedge representation* of CAR, respectively.

4. Standard extensions of representations of CAR

In order to show extension theorems, we prepare a notion, "standard extension" of a representation of CAR to $\mathcal{O}_2$ as follows:

Definition 4.1. *Let* φ_S *be the standard embedding of CAR into* $\mathcal{O}_2$ *in (*2.1*). For a representation* $(\mathcal{H}, \pi)$ *of CAR,* $(\tilde{\mathcal{H}}, \tilde{\pi})$ *is the standard extension of* $(\mathcal{H}, \pi)$ *to* $\mathcal{O}_2$ *if* $\mathcal{H}$ *is a closed subspace of* $\tilde{\mathcal{H}}$ *such that*

$$(\tilde{\pi} \circ \varphi_S)|_{\mathcal{H}} = \pi. \tag{4.1}$$

4.1. Standard extension of the Fock representation.

Theorem 4.2. *Let* $(\mathcal{H}, \pi)$ *be the Fock representation of* CAR *with the vacuum* Ω *in Definition* 2.1*. Put two operators* $\tilde{\pi}(s_1), \tilde{\pi}(s_2)$ *on* $\mathcal{H}$ *by*

$$\tilde{\pi}(s_1)\Omega \equiv \Omega, \quad \tilde{\pi}(s_1)\pi(a^*_{n_1} \cdots a^*_{n_k})\Omega \equiv \pi(a^*_{n_1+1} \cdots a^*_{n_k+1})\Omega,$$

$$\tilde{\pi}(s_2)\Omega \equiv \pi(a^*_1)\Omega, \quad \tilde{\pi}(s_2)\pi(a^*_{n_1} \cdots a^*_{n_k})\Omega \equiv \pi(a^*_1 a^*_{n_1+1} \cdots a^*_{n_k+1})\Omega$$

for $n_1 < n_2 < \cdots < n_k$, $n_j \in \mathbf{N}$, $j = 1, \ldots, k$, $k \geq 1$*. Then the followings hold:*

(i) $(\mathcal{H}, \tilde{\pi})$ *is a representation of* $\mathcal{O}_2$.
(ii) $\tilde{\pi} \circ \varphi_S = \pi$.
(iii) $(\mathcal{H}, \tilde{\pi})$ *is* $P(1)$ *with the GP vector* Ω.

This proof is given by direct computation and Lemma 2.3. For more detail, see § 3.3 in [**1**]. Clearly, $(\tilde{\mathcal{H}} \equiv \mathcal{H}, \tilde{\pi})$ in Theorem 4.2 is the standard extension of the Fock representation. Theorem 1.1 (i) about an operator L follows from Theorem 4.2 as another expression of this extension.

4.2. Standard extension of the infinite wedge. For $g = \{g_1, g_2\}$ in (3.2), define a representation $(\Lambda^{\frac{\infty}{2}}V^{\#}, \Pi)$ of $\mathcal{O}_2$ by

$$\Pi(s_1)e_S \equiv (-1)^{d_+(S)}e_{g_1(S)}, \quad \Pi(s_2)e_S \equiv (-1)^{d_+(S)}e_{g_2(S)} \quad (S \in \mathcal{M}_+),$$

$$\Pi(s_1)e_S \equiv (-1)^{d'_-(S)}e_{g_1(S)}, \quad \Pi(s_2)e_S \equiv (-1)^{d_-(S)}e_{g_2(S)} \quad (S \in \mathcal{M}_-)$$

where $d_+(S) \equiv \#(S \cap \mathbf{Z}_{+/2}) + \#(\mathbf{Z}_{-/2} \setminus S)$ and $d'_-(S) \equiv \#(\mathbf{Z}_{+/2} \setminus S)$, $d_-(S) \equiv \#(\mathbf{Z}_{+/2} \setminus S) + \#(S \cap \mathbf{Z}_{-/2})$.

Lemma 4.3. *When $K = \{k_1, \ldots, k_n\}$ and $L = \{l_1, \ldots, l_m\} \subset \mathbf{Z}_{+/2}$ satisfy $k_1 > \cdots > k_n$ and $l_1 < \cdots < l_m$,*

$$\Pi(s_1)|\mathrm{vac}>_+ = \psi_{-1/2}|\mathrm{vac}>_- = e_{\mathbf{Z}_{+/2}\cup\{-1/2\}} \quad \Pi(s_2)|\mathrm{vac}>_+ = |\mathrm{vac}>_-,$$

$$\Pi(s_1)|\mathrm{vac}>_- = |\mathrm{vac}>_+, \quad \Pi(s_2)|\mathrm{vac}>_- = \psi^*_{-1/2}|\mathrm{vac}>_+ = e_{\mathbf{Z}_{-/2}\setminus\{-1/2\}}.$$

$$\Pi(s_1)e_{\mathbf{Z}_{-/2}\cup K\setminus(-L)} = (-1)^{n+m}e_{\mathbf{Z}_{+/2}\cup(-K^*_{+1})\setminus L},$$

$$\Pi(s_2)e_{\mathbf{Z}_{-/2}\cup K\setminus(-L)} = (-1)^{n+m}e_{\mathbf{Z}_{+/2}\cup(-K_{+1})\setminus L},$$

$$\Pi(s_1)e_{\mathbf{Z}_{+/2}\cup(-K)\setminus L} = (-1)^{m}e_{\mathbf{Z}_{-/2}\cup K\setminus(-L_{+1})},$$

$$\Pi(s_2)e_{\mathbf{Z}_{+/2}\cup(-K)\setminus L} = (-1)^{m+n}e_{\mathbf{Z}_{-/2}\cup K\setminus(-L^*_{+1})}$$

where $K_{+1} \equiv \{k+1 : k \in K\}$ and $K^*_{+1} \equiv K_{+1} \cup \{1/2\}$.

Proposition 4.4. (i) *$(\Lambda^{\frac{\infty}{2}}V^{\#}, \Pi)$ is $P(12)$.*
(ii) *If $\pi_\infty, \pi_{\infty,\pm}$ are in Proposition 3.6, then $\Pi \circ \varphi_S = \pi_\infty$. Specially,*

$$(\Lambda^{\frac{\infty}{2}}V, \pi_{\infty,+}) = (\Lambda^{\frac{\infty}{2}}V, (\Pi \circ \varphi_S)|_{\Lambda^{\frac{\infty}{2}}V}) \sim P[12],$$

$$(\Lambda^{\frac{\infty}{2}}V^*, \pi_{\infty,-}) = (\Lambda^{\frac{\infty}{2}}V^*, (\Pi \circ \varphi_S)|_{\Lambda^{\frac{\infty}{2}}V^*}) \sim P[21].$$

Proof. (i) By Lemma 4.3, $\Pi(s_1s_2)|\mathrm{vac}>_+ = |\mathrm{vac}>_+$. By definition of g_1, g_2, $(\Lambda^{\frac{\infty}{2}}V^{\#}, \Pi)$ is $P(12)$.
(ii) Identify $\varphi_S(a_n)$ and a_n for each $n \in \mathbf{N}$. By Lemma 3.2 and Lemma 4.3, we can check the followings:

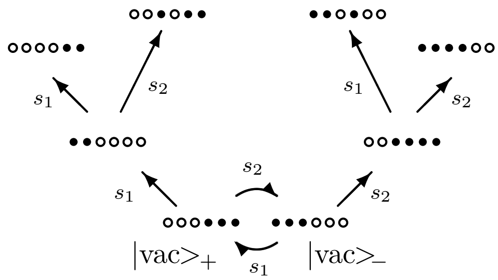
$$\Pi(a_{2n-1})|\mathrm{vac}>_+ = \Pi(a_{2n})|\mathrm{vac}>_- = \Pi(a^*_{2n})|\mathrm{vac}>_+ = \Pi(a_{2n-1})^*|\mathrm{vac}>_- = 0,$$

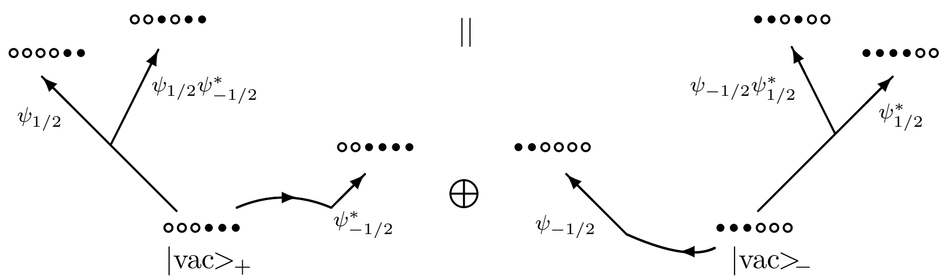
$$\Pi(a_{2n})|\mathrm{vac}>_+ = \psi_{n-1/2}|\mathrm{vac}>_+, \quad \Pi(a_{2n-1})|\mathrm{vac}>_- = \psi_{-n+1/2}|\mathrm{vac}>_-,$$

$$\Pi(a^*_{2n-1})|\mathrm{vac}>_+ = \psi^*_{-n+1/2}|\mathrm{vac}>_+, \quad \Pi(a^*_{2n})|\mathrm{vac}>_- = \psi^*_{n-1/2}|\mathrm{vac}>_-$$

for each $n \in \mathbf{N}$. By Lemma 4.3, $\Pi(a_n) = \pi_\infty(a_n)$ for each $n \in \mathbf{N}$. □

The branching law $\Pi|_{CAR} = \pi_{\infty,+} \oplus \pi_{\infty,-}$ is illustrated by Maya diagrams as follows:





We try to interpret this branching law from a physical standpoint. *Before* the symmetry breaking of $\mathcal{O}_2$ to CAR, the vacuum and the dual vacuum are coupled as a cycle: $|\mathrm{vac}\rangle_+ \stackrel{s_2}{\to} |\mathrm{vac}\rangle_- \stackrel{s_1}{\to} |\mathrm{vac}\rangle_+$. *After* the symmetry breaking, they are decomposed into two independent vacua of fermions. A $\mathbf{Z}_2$-symmetry between $\Lambda^{\frac{\infty}{2}}V$ and $\Lambda^{\frac{\infty}{2}}V^*$ are just a unitary U in (2.2) on $\Lambda^{\frac{\infty}{2}}V^{\#}$ which satisfies $Us_1U^* = s_2$.

4.3. Boson-fermion correspondence described by $\mathcal{O}_2$. By using the standard extension of the infinite wedge, we consider correspondence among boson, fermion and generators of the Cuntz algebra $\mathcal{O}_2$. Under identification of $\Pi(s_i)$ and s_i for $i = 1, 2$ in Proposition 4.4, we have the followings:

$$\psi_k = \zeta^{2k}(s_1 s_2^*), \quad \psi_{-k} = \zeta^{2k-1}(s_1 s_2^*) \quad (k \in \mathbf{Z} + \tfrac{1}{2},\, k > 0)$$

where ζ is in (1.4). From this, we have the following recurrence formulae:

Proposition 4.5.

$$\psi_{\frac{1}{2}} = \zeta(s_1 s_2^*), \quad \psi_{-\frac{1}{2}} = s_1 s_2^*,$$

$$\psi_{k+1} = \zeta^2(\psi_k), \quad \psi_{-k-1} = \zeta^2(\psi_{-k}) \quad (k \in \mathbf{Z} + \tfrac{1}{2},\, k > 0).$$

Intertwining relations are given as follows:

$$s_i\psi_k = (-1)^{i-1}\psi_{-(k+1)}s_i, \quad s_i\psi_{-k} = (-1)^{i-1}\psi_k s_i,$$

$$s_i\psi_k^* = (-1)^{i-1}\psi^*_{-(k+1)}s_i, \quad s_i\psi^*_{-k} = (-1)^{i-1}\psi^*_k s_i$$

for $i = 1, 2$ and $k \in \mathbf{Z} + \frac{1}{2}$, $k > 0$.

Proof of (1.7). If $n \geq 0$, then we can decompose

$$\alpha_n = A_n + B_n + C_n$$

where $A_n \equiv \sum_{k\in\mathbf{Z}+\frac{1}{2},\,k>n} \psi_{k-n}\psi_k^*$, $B_n \equiv \sum_{k\in\mathbf{Z}+\frac{1}{2},\,n>k>0} \psi_{k-n}\psi_k^*$ and $C_n \equiv \sum_{k\in\mathbf{Z}+\frac{1}{2},\,k<0} \psi_{k-n}\psi_k^*$. By Proposition 4.5,

$$A_n + C_n = \sum_{l\in\mathbf{N}} \rho^{2n-2}(X_n), \quad X_n \equiv \psi_{1/2}\psi_{n+1/2}^* + \psi_{-n-1/2}\psi_{-1/2}^*$$

where we use $\zeta(x)\zeta(y) = \rho(xy)$ for each $x, y \in \mathcal{O}_2$. This implies (1.8). Furthermore we have $B_1 = -s_1 s_2 s_1^* s_2^*$,

$$B_{2k} = -\sum_{1\leq l\leq k} \rho^{2(l-1)}\{\rho(s_2\zeta^{4(k-l)}(s_1 s_2^*)s_1^*) + s_1\zeta^{4(k-l)+2}(s_2 s_1^*)s_2^*\},$$

$$\begin{aligned} B_{2k+1} = \ & -\rho^{2k}(s_1 s_2 s_1^* s_2^*) \\ & -\sum_{1\leq l\leq k} \rho^{2(l-1)}\{\rho(s_2\zeta^{4(k-l)+2}(s_1 s_2^*)s_1^*) + s_1\zeta^{4(k-l)+4}(s_2 s_1^*)s_2^*\} \end{aligned}$$

for each $k \in \mathbf{N}$. Hence the recurrence formula (1.9) of B_n is obtained. $\square$

Remark that α_n is an unbounded operator on a Hilbert space $\Lambda^{\frac{\infty}{2}}V$ and above equations make sense on a dense domain in $\Lambda^{\frac{\infty}{2}}V$.

In the same way, the *energy* defined by

$$H \equiv \sum_{k\in\mathbf{Z}+\frac{1}{2}} k : \psi_k\psi_k^* : \; = \sum_{k\in\mathbf{Z}+\frac{1}{2}:k>0} k(\psi_k\psi_k^* + \psi_{-k}^*\psi_{-k})$$

is rewritten as follows:

$$H = \sum_{l\in\mathbf{N}} (l-1/2)\rho^{2l-2}(s_1 s_1 s_1^* s_1^* + s_2 s_1 s_1^* s_2^* + s_2 s_2^*).$$

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Appendix A. Inequivalences among $\mathcal{H}_{Fock}$, $P[12], P[21]$

Assume that $\mathcal{H}_{Fock}$ and $P[12]$ are equivalent. Then there is a cyclic representation $(\mathcal{H}, \pi)$ of CAR with two cyclic vectors Ω and Ω' such that $\pi(a_n)\Omega = 0$ and $\pi(a_{2n-1})\Omega' = \pi(a_{2n}^*)\Omega' = 0$ for each $n \in \mathbf{N}$. We identify $\pi(a_n)$ and a_n for each $n \in \mathbf{N}$. We see that $\mathcal{H}$ has a complete orthonormal basis $\{a_F^*\Omega : F \in \mathcal{F}(\mathbf{N})\}$ where $\mathcal{F}(\mathbf{N})$ is the set of all finite subsets of $\mathbf{N}$ and $a_\emptyset^* \equiv I$, $a_F^* \equiv a_{n_1}^* \cdots a_{n_k}^*$ when $F = \{n_1, \ldots, n_k\}$ and $n_1 < \cdots < n_k$. Hence we can denote $\Omega^{'} = \sum_F c_F a_F^* \Omega$ for suitable $c_F \in \mathbf{C}$. Then there are $n_0 \in \mathbf{N}$ and $F_0 \in \mathcal{F}(\mathbf{N})$ such that $2n_0 \notin F_0$ and $c_{F_0} \neq 0$. This implies

$$a_{2n_0} a_{F_0}^* \Omega = 0. \tag{A.1}$$

We see that

$$< a_n a_F^* \Omega | a_n a_{F'}^* \Omega >= \delta_{F,F'} \cdot \chi_F(n) \cdot \chi_{F'}(n) \tag{A.2}$$

for each $(n, F) \in \mathbf{N} \times \mathcal{F}(\mathbf{N})$. By assumption of $\Omega^{'}$ and anticommutation relations of a_n's,

$$\|a_{2n}\Omega^{'}\| = \|\Omega^{'}\| \quad ({}^{\forall} n \in \mathbf{N}). \tag{A.3}$$

By (A.1), (A.2) and (A.3),

$$\|\Omega^{'}\|^2 = \|a_{2n_0}\Omega^{'}\|^2 = \|\sum_F c_F a_{2n_0} a_F^* \Omega\|^2 \leq \sum_{F \neq F_0} |c_F|^2 < \|\Omega^{'}\|^2.$$

This is contradiction. Hence $\mathcal{H}_{Fock}$ and $P[12]$ are not equivalent. In the same way, inequivalences among $\mathcal{H}_{Fock}$, $P[21]$, $P[12]$ are shown. □

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