WKB analysis of higher order Painlevé equations with a large parameter — Local reduction of 0-parameter solutions for Painlevé hierarchies (P_J) $(J = \mathrm{I}, \mathrm{II}\text{-}1 \text{ or } \mathrm{II}\text{-}2)$

To be dedicated to Professor H. Komatsu on his seventieth birthday

by

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§0. Introduction

This paper is the second of a series of articles on WKB analysis of higher order Painlevé equations with a large parameter. In the first of the series ([KKNT]) we studied the geometric aspect of the Painlevé hierarchy (P_J) $(J = \mathrm{I}, \mathrm{II}\text{-}1 \text{ or } \mathrm{II}\text{-}2)$ with a large parameter, and in this article we begin to analyze the WKB-theoretic structure of each member $(P_J)_m$ $(J = \mathrm{I}, \mathrm{II}\text{-}1 \text{ or } \mathrm{II}\text{-}2; m = 1, 2, 3, \ldots)$ of the hierarchy. To be concrete, we show that a 0-parameter solution of $(P_J)_m$ $(J = \mathrm{I}, \mathrm{II}\text{-}1 \text{ or } \mathrm{II}\text{-}2; m = 1, 2, 3, \ldots)$ constructed in [KKNT] and [N] can be reduced to a 0-parameter solution of $(P_\mathrm{I})_1$, the traditional (i.e., second order) Painlevé-I equation (P_I) with a large parameter, i.e.,

$$\frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t), \tag{0.1}$$

with the aid of a formal transformation defined near a turning point of $(P_J)_m$ of the first kind in the sense of [KKNT]. (See Theorem 3.2.1 in Section 3 for the precise statement.) Throughout this paper we use the same notions and notations used in [KKNT]. A résumé of this paper is given in [KT2].

An important step of our reasoning in this paper is to derive a pair of Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$ from the Lax pair $(L_J)_m$ associated with the m-th member $(P_J)_m$ of the Painlevé hierarchy in question. Here we make essential use of the fact that $(L_J)_m$ consists of 2×2 systems. (See Section 1 for the details.) Once we obtain the simultaneous equations $(SL_J)_m$ and $(D_J)_m$ for one unknown function, we can employ the techniques used in [KT1]; we first establish some analyticity properties of the odd part of a solution of the Riccati equation attached to $(SL_J)_m$ (Theorem 2.1) and then in Proposition 3.2.1 we construct a semi-global transformation that brings $(SL_J)_m$ to (SL_I), the Schrödinger equation underlying (P_I) (cf. [KT1]). In constructing the semi-global transformation we need some matching conditions, and the constructed semi-global transformation together with the matching conditions is used to reduce the 0-parameter solution in question to a 0-parameter solution of (P_I). (Theorem 3.2.1.) We note that the actual reduction is divided into two steps: we first solve an algebraic equation of degree m whose coefficients are defined in terms of a 0-parameter solution of $(P_J)_m$ to find some formal series $b_j(t, \eta)$ $(j = 1, \ldots, m)$, and we then employ the analytic machinery of semi-globally transforming $(SL_J)_m$ to (SL_I) so that we may reduce b_j that is relevant to the turning point of

$(P_J)_m$ in question to a 0-parameter solution of (P_{I}), with the help of the constructed semi-global transformation. We discuss the geometric meaning of b_j in Section 1.

In ending this introduction we want to repeat the same comment as that given in [KT1]: it is probably worth emphasizing that the above reduction is attained through the study of $(SL_J)_m$, a differential equation on the extended (x,t)-space, despite the fact that the required relation is relevant only to the t-variable.

§1. Derivation of a Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$

§1.1. The case $J = \mathrm{I}$

For the convenience of the reader, we first recall the definition of $(P_{\mathrm{I}})_m$ and the underlying Lax pair $(L_{\mathrm{I}})_m$. See [KKNT] and [S] for their backgrounds.

Definition 1.1.1. The m-th member of P_{I}-hierarchy with a large parameter η is the following system of non-linear differential equations:

$$(1.1.1) \qquad (P_{\mathrm{I}})_m : \begin{cases} \dfrac{du_j}{dt} = 2\eta v_j \quad (j = 1, \ldots, m) & (1.1.1.\mathrm{a}) \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) \quad (j = 1, \ldots, m) & (1.1.1.\mathrm{b}) \\ u_{m+1} = 0 & (1.1.1.\mathrm{c}) \end{cases}$$

where w_j is a polynomial of u_k and v_l $(1 \le k, l \le j)$ that is determined by the following recursive relation:

$$(1.1.2) \qquad \begin{aligned} w_j = & \frac{1}{2}\left(\sum_{k=1}^{j} u_k u_{j+1-k}\right) + \sum_{k=1}^{j-1} u_k w_{j-k} \\ & - \frac{1}{2}\left(\sum_{k=1}^{j-1} v_k v_{j-k}\right) + c_j + \delta_{jm} t \quad (j = 1, \ldots, m). \end{aligned}$$

Here c_j is a constant and δ_{jm} stands for Kronecker's delta.

Definition 1.1.2. The Lax pair $(L_{\mathrm{I}})_m$ underlying $(P_{\mathrm{I}})_m$ is the following pair of linear differential equations on (x,t)-space:

$$(1.1.3)\qquad (L_{\mathrm{I}})_m : \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\vec{\psi} = 0, & (1.1.3.a)\\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\vec{\psi} = 0, & (1.1.3.b)\end{cases}$$

where $\vec{\psi} = {}^t(\psi_1, \psi_2)$,

$$(1.1.4)\qquad A = \begin{pmatrix} V(x)/2 & U(x) \\ (2x^{m+1} - xU(x) + 2W(x))/4 & -V(x)/2 \end{pmatrix}$$

and

$$(1.1.5)\qquad B = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 & 0 \end{pmatrix}$$

with

$$(1.1.6)\qquad U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j},$$

$$(1.1.7)\qquad V(x) = \sum_{j=1}^{m} v_j x^{m-j}$$

and

$$(1.1.8)\qquad W(x) = \sum_{j=1}^{m} w_j x^{m-j}.$$

Remark 1.1.1. As is proved in Proposition 1.1.1 of [KKNT], $(P_{\mathrm{I}})_m$ states the compatibility condition for $(L_{\mathrm{I}})_m$.

Remark 1.1.2. Combining (1.1.1.a), (1.1.1.b) and (1.1.2), we find that u_{j+1} $(j \le m-1)$ is a polynomial of $u_1, \ldots, u_j$, $du_1/dt, \ldots, du_j/dt$ and d^2u_j/dt^2. Hence u_{j+1} $(j \le m-1)$ is a polynomial of $u_1, du_1/dt, \ldots, d^{2j}u_1/dt^{2j}$. Substituting these polynomials into

$$(1.1.9)\qquad \frac{d^2u_m}{dt^2} = 4\eta^2(u_1u_m + w_m),$$

we obtain a $2m$-th order differential equation for u_1. It is also clear that, once a solution u_1 of the $2m$-th order differential equation is given, we can find $(u_1, \ldots, u_m; v_1, \ldots, v_m; w_1, \ldots w_m)$ so that they satisfy (1.1.1) and (1.1.2). Thus $(P_{\mathrm{I}})_m$ is equivalent to a single $2m$-th order differential equation. The explicit form of the resulting equation for $m = 1$ is

$$d^2u_1/dt^2 = \eta^2(6u_1^2 + 4c_1 + 4t), \tag{1.1.10}$$

and that for $m = 2$ is

$$\begin{aligned} d^4u_1/dt^4 = {}& \eta^2(20u_1 d^2u_1/dt^2 + 10(du_1/dt)^2) \\ & + \eta^4(-40u_1^3 - 16c_1u_1 + 16c_2 + 16t). \end{aligned} \tag{1.1.11}$$

It is clear that the scaling

$$\tilde{t} = \alpha(t + c_1) \quad \text{and} \quad \lambda = \alpha^3 u_1/4 \quad \text{with} \quad \alpha = 4^{1/5} \tag{1.1.12}$$

brings (1.1.10) into

$$d^2\lambda/d\tilde{t}^2 = \eta^2(6\lambda^2 + \tilde{t}), \tag{1.1.13}$$

the traditional Painlevé-I equation (P_{I}) with a large parameter η. These facts explain why (1.1.1) is called (P_{I})-hierarchy, or often with some abuse of language, a higher order Painlevé-I equation.

Let us first write down the equation that the first component ψ_1 of a solution $\vec{\psi}$ of (1.1.3.a) satisfies:

(1.1.14)

$$\left(\frac{\partial^2}{\partial x^2} - \frac{U_x}{U}\frac{\partial}{\partial x} - \frac{\eta^2}{4}((2x^{m+1} - xU + 2W)U + V^2) + \frac{\eta}{2}\left(\frac{U_xV}{U} - V_x\right)\right)\psi_1 = 0.$$

Next we eliminate the term $-U_xU^{-1}\partial\psi_1/\partial x$ by introducing ψ by

$$\exp(-\int^x \frac{U_x}{2U}dx)\psi_1 = \frac{1}{\sqrt{U}}\psi_1; \tag{1.1.15}$$

the resulting equation for ψ is

$$\frac{\partial^2\psi}{\partial x^2} = \eta^2 Q_{(\mathrm{I},m)}\psi, \tag{1.1.16}$$

where

$$Q_{(\mathrm{I},m)} = \frac{1}{4}(2x^{m+1} - xU + 2W)U + \frac{1}{4}V^2 - \frac{\eta^{-1}U_xV}{2U} + \frac{\eta^{-1}V_x}{2} + \frac{3\eta^{-2}U_x^2}{4U^2} - \frac{\eta^{-2}U_{xx}}{2U}. \tag{1.1.17}$$

On the other hand, (1.1.3.b) implies

$$\frac{\partial\psi_1}{\partial t} = 2\eta\psi_2, \tag{1.1.18}$$

and it also follows from (1.1.3.a) that

$$\frac{\partial\psi_1}{\partial x} = \frac{\eta V}{2}\psi_1 + \eta U\psi_2. \tag{1.1.19}$$

Hence we find

$$\frac{\partial\psi_1}{\partial x} = \frac{\eta V}{2}\psi_1 + \frac{U}{2}\frac{\partial\psi_1}{\partial t}. \tag{1.1.20}$$

Therefore we obtain

$$\psi_x = \frac{1}{2}U\psi_t + (\frac{1}{4}U_t + \frac{1}{2}\eta V - \frac{1}{2}U_xU^{-1})\psi. \tag{1.1.21}$$

It also follows from (1.1.1), (1.1.6) and (1.1.7) that

$$U_t = -2\eta V. \tag{1.1.22}$$

Thus we conclude

$$\frac{\partial\psi}{\partial t} = \mathfrak{a}_{(\mathrm{I},m)}\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial\mathfrak{a}_{(\mathrm{I},m)}}{\partial x}\psi, \tag{1.1.23}$$

where

$$\mathfrak{a}_{(\mathrm{I},m)} = \frac{2}{U(x)}. \tag{1.1.24}$$

Thus we have arrived at simultaneous equations (1.1.16) and (1.1.23) for one unknown function ψ. This is the setting that [KT1] used to establish a reduction theorem for 0-parameter solutions of the traditional Painlevé equations. In what follows, the equation (1.1.16) (resp., (1.1.23)) is referred to as $(SL_{\mathrm{I}})_m$ (resp., $(D_{\mathrm{I}})_m$), and we analyze these equations by substituting a 0-parameter solution $(\hat{u}_j, \hat{v}_j)_{1\le j\le m}$ of $(P_{\mathrm{I}})_m$ into their coefficients. The existence and basic properties of a 0-parameter solution of $(P_{\mathrm{I}})_m$ are shown in [KKNT]; it is a formal series in η^{-1} of the following form:

$$\hat{u}_j(t,\eta) = \hat{u}_{j,0}(t) + \hat{u}_{j,1}(t)\eta^{-1} + \cdots, \tag{1.1.25}$$

$$\hat{v}_j(t,\eta) = \hat{v}_{j,0}(t) + \hat{v}_{j,1}(t)\eta^{-1} + \cdots. \tag{1.1.26}$$

We note that

$$\hat{u}_{j+1,0} + \hat{u}_{1,0}\hat{u}_{j,0} + \widehat{w}_{j,0} = 0, \quad j = 1, \ldots, m, \tag{1.1.27}$$

$$\hat{v}_{j,0} = 0, \quad j = 1, \ldots, m, \tag{1.1.28}$$

and
(1.1.29)

$$\widehat{w}_{j,0} = \frac{1}{2}\left(\sum_{k=1}^{j} \hat{u}_{k,0}\hat{u}_{j+1-k,0}\right) + \sum_{k=1}^{j-1} \hat{u}_{k,0}\widehat{w}_{j-k,0} + c_j + \delta_{jm}t, \quad j = 1, \ldots, m$$

follow from (1.1.1) and (1.1.2) and that these relations together with (1.1.1.c) determine $\hat{u}_{j,0}$ algebraically. (See [KKNT, §2.1] for the details.) If we substitute the expansions (1.1.25) and (1.1.26) into the coefficients of U, V and W, they are accordingly expanded in powers of η^{-1}; we let U_l, V_l and W_l respectively denote the coefficient of η^{-l} in the expansion. Using the 0-parameter solution we define series $b_j(t,\eta)$ $(j = 1, \ldots, m)$ as solutions of the equation

$$U(b_j) = 0, \quad j = 1, 2, \ldots, m, \tag{1.1.30}$$

that is,

$$b_j^m - \sum_{j=1}^{m} \hat{u}_j b_j^{m-j} = 0. \tag{1.1.31}$$

It is clear that $b_j(t,\eta)$ is also expanded as

$$b_j = b_{j,0}(t) + b_{j,1}(t)\eta^{-1} + \cdots. \tag{1.1.32}$$

Although we have started our discussion with the equation (1.1.1) with unknown functions $(u_j, v_j(= (du_j/dt)/2\eta))$, the quantities $(u_1, \ldots, u_m)$ were first introduced as the elementary symmetric polynomials of $(b_1, \ldots, b_m)$ in [S].

Since b_j $(j = 1, \ldots, m)$ is determined by $(\hat{u}_1, \ldots, \hat{u}_m)$ through the algebraic equation (1.1.31), we try to find a transformation that brings b_j to a 0-parameter solution of (P_{I}), i.e., (1.1.13), in a neighborhood of a turning point of $(P_{\mathrm{I}})_m$ that is relevant to b_j. This task is accomplished in Section 3 with the essential use of the results in Section 2. Before proceeding further, we note two important geometric properties of the function $b_{j,0}(t)$ $(j = 1, \ldots, m)$, the top order term of the expansion of b_j in η^{-1}.

First, $x = b_{j,0}(t)$ is, as a zero of $U_0(x)$, a singular point of $Q_{(\mathrm{I},m)}$ and $\mathfrak{a}_{(\mathrm{I},m)}$. Hence their expansions in $\eta^{-1}$are considered outside the point; their coefficients of η^{-l} are denoted respectively by $Q_{(\mathrm{I},m),l}$ and $\mathfrak{a}_{(\mathrm{I},m),l}$. Second, $x = b_{j,0}(t)$ is a double turning point of $(SL_{\mathrm{I}})_m$. In fact, (1.1.27) together with the definition of U_0, V_0 and W_0 entails

$$\begin{aligned}
(1.1.33) \qquad & 2x^{m+1} - xU_0(x) + 2W_0(x) \\
&= x^{m+1} + \sum_{j=1}^{m} \hat{u}_{j,0} x^{m+1-j} - 2\sum_{j=1}^{m} (\hat{u}_{j+1,0} + \hat{u}_{1,0}\hat{u}_{j,0}) x^{m-j} \\
&= x^{m+1} + 2\hat{u}_{1,0}x^m - \hat{u}_{1,0}x^m - \sum_{j=2}^{m} \hat{u}_{j,0} x^{m+1-j} \\
&\quad - 2\hat{u}_{1,0} \sum_{j=1}^{m} \hat{u}_{j,0} x^{m-j} \\
&= (x + 2\hat{u}_{1,0}) U_0(x),
\end{aligned}$$

and hence (1.1.28) and (1.1.33) imply

$$(1.1.34) \qquad Q_{(\mathrm{I},m),0} = \frac{1}{4}(x + 2\hat{u}_{1,0}) U_0(x)^2.$$

As we will see below, similar facts are observed for $(P_J)_m$ ($J =$ I-1 or II-2), and they play critically important roles in Section 2 and Section 3.

§1.2. The case $J =$ II-1

Let us begin our discussions by briefly recalling the definition of $(P_{\text{II-1}})_m$ and the underlying Lax pair $(L_{\text{II-1}})_m$. See [KKNT] and [GP] for the details. We refer the reader to [GP] for their relevance to the non-isospectral scattering problem.

Definition 1.2.1. The m-th member of $P_{\text{II-1}}$-hierarchy with a large parameter η is, by definition, the following differential equation for v:

$$(P_{\text{II-1}})_m : (\eta^{-1}\frac{\partial}{\partial t} + 2v)K_m + g(2tv + \eta^{-1}) + c = 0, \tag{1.2.1}$$

where g and c are constants and K_m is a polynomial of v and its derivatives defined by the following recursive relation:

$$\begin{aligned}\eta^{-1}\partial_t K_{p+1} = (\eta^{-3}\partial_t^3 + 4\eta^{-1}(v^2 - \eta^{-1}v_t)\partial_t \\ + 2(2vv_t - \eta^{-1}v_{tt}))K_p, \quad p = 0, 1, 2, \dots\end{aligned} \tag{1.2.2}$$

with

$$K_0 = 1/2. \tag{1.2.3}$$

Remark 1.2.1. The above recursive relation allows K_p to contain integrated terms like $\partial_t^{-1}v$. However, we can choose K_p so that it is a polynomial of v and its derivatives. (See [KKNT, Appendix A] for the proof.) One can then easily confirm that such preferred K_p has the form

$$\frac{(-1)^p 2^{p-1}(2p-1)!!}{p!}v^{2p} + \sum_{l=1}^{2(p-1)} \eta^{-l}K_{p,l} + \eta^{-(2p-1)}\frac{d^{2p-1}v}{dt^{2p-1}}. \tag{1.2.4}$$

Hence $(P_{\text{II-1}})_m$ is a $2m$-th order non-linear differential equation. The explicit form of the first two preferred K_p is as follows:

$$K_1 = -v^2 + \eta^{-1}v_t, \tag{1.2.5}$$

$$K_2 = 3v^4 - 6\eta^{-1}v^2v_t + \eta^{-2}((v_t)^2 - 2vv_{tt}) + \eta^{-3}v_{ttt}. \tag{1.2.6}$$

Hence we find

$$(P_{\text{II-1}})_1 : \eta^{-2}\frac{d^2v}{dt^2} = v^3 - 2gtv - (c + g\eta^{-1}), \tag{1.2.7}$$

and

$$\begin{aligned}(P_{\text{II-1}})_2 : \eta^{-4}\frac{d^4v}{dt^4} = \eta^{-2}(10v^2\frac{d^2v}{dt^2} + 10v\left(\frac{dv}{dt}\right)^2) \\ - 6v^5 - 2gtv - (c + g\eta^{-1}).\end{aligned} \tag{1.2.8}$$

As (1.2.7) is the traditional Painlevé-II equation (P_{II}) with a large parameter η, it is reasonable to call the totality of these equations the Painlevé-II hierarchy with a large parameter η. As we will see in Section 1.3 another hierarchy whose first member is (P_{II}), in order to distinguish these two hierarchies, we coin the terminology $P_{\mathrm{II}\text{-}1}$-hierarchy to call the equations discussed in this section. The equations discussed in Section 1.3 will be called $P_{\mathrm{II}\text{-}2}$-hierarchy.

Remark 1.2.2. We sometimes allow constants c and g to contain powers of η^{-1} like $c = c_0 + c_1\eta^{-1}$. For example, we usually assume that g is a genuine constant (i.e., free from η) and that

$$c = c_0 - g\eta^{-1} \tag{1.2.9}$$

so that $c + g\eta^{-1}$ is free from η^{-1}. In what follows we also assume that g is different from 0.

Definition 1.2.2. The Lax pair $(L_{\mathrm{II}\text{-}1})_m$ underlying $(P_{\mathrm{II}\text{-}1})_m$ is the following pair of linear differential equations on (x, t)-space:

$$(1.2.10) \qquad (L_{\mathrm{II}\text{-}1})_m : \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\vec{\psi} = 0, & (1.2.10.a) \\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\vec{\psi} = 0, & (1.2.10.b) \end{cases}$$

where $\vec{\psi} = {}^t(\psi_1, \psi_2)$,

$$A = \frac{1}{4gx}\begin{pmatrix} -\eta^{-1}\partial_t T_m & 2T_m \\ 2qT_m - \eta^{-2}\partial_t^2 T_m & \eta^{-1}\partial_t T_m \end{pmatrix} \tag{1.2.11}$$

and

$$B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \tag{1.2.12}$$

with T_m and q being given respectively by

$$T_m = gt + \sum_{k=0}^{m} (4x)^k K_{m-k} \tag{1.2.13}$$

and

$$q = x - K_1 \tag{1.2.14}$$

Remark 1.2.3. As is discussed in [KKNT, §1.2] and [GP], $(P_{\text{II-1}})_m$ states the compatibility condition of $(L_{\text{II-1}})_m$.

As in Section 1.1, we begin our discussion by writing down the equation that the first component ψ_1 of $\vec{\psi}$ should satisfy:

$$\Big(\frac{\partial^2}{\partial x^2}+\Big(\frac{1}{x}-\frac{T_{m,x}}{T_m}\Big)\frac{\partial}{\partial x}-\frac{1}{16g^2x^2}((T_{m,t})^2+4\eta^2qT_m^2-2T_mT_{m,tt}) + \frac{1}{4gx}\Big(T_{m,tx}-\frac{T_{m,x}T_{m,t}}{T_m}\Big)\Big)\psi_1=0. \tag{1.2.15}$$

Here $T_{m,tx}$ etc. designate $\partial^2 T_m/\partial t\partial x$ etc. By introducing ψ by

$$\exp(\frac{1}{2}\int^x\Big(\frac{1}{x}-\frac{T_{m,x}}{T_m}\Big)dx)\psi_1=x^{1/2}T_m^{-1/2}\psi_1, \tag{1.2.16}$$

we find the required Schrödinger equation for ψ:

$$\frac{\partial^2}{\partial x^2}\psi=\eta^2Q_{(\text{II-1},m)}\psi, \tag{1.2.17}$$

where

$$\begin{aligned}Q_{(\text{II-1},m)}=&\frac{1}{4g^2x^2}qT_m^2+\frac{\eta^{-2}}{16g^2x^2}(T_{m,t}^2-2T_mT_{m,tt})\\&+\frac{\eta^{-2}}{4gx}(\frac{T_{m,x}T_{m,t}}{T_m}-T_{m,tx})+\frac{3\eta^{-2}T_{m,x}^2}{4T_m^2}-\frac{\eta^{-2}T_{m,xx}}{2T_m}\\&-\frac{\eta^{-2}T_{m,x}}{2xT_m}-\frac{\eta^{-2}}{4x^2}\,.\end{aligned} \tag{1.2.18}$$

On the other hand, (1.2.10.b) implies

$$\frac{\partial\psi_1}{\partial t}=\eta\psi_2, \tag{1.2.19}$$

and (1.2.10.a) entails

$$\frac{\partial\psi_1}{\partial x}=\frac{1}{4gx}(-T_{m,t}\psi_1+2\eta T_m\psi_2). \tag{1.2.20.a}$$

Hence we find

$$\frac{\partial\psi_1}{\partial x}=\frac{1}{4gx}(-T_{m,t}\psi_1+2T_m\frac{\partial\psi_1}{\partial t}). \tag{1.2.21}$$

Then, combining (1.2.16) and (1.2.21), we obtain

$$4gx(-\frac{1}{2}x^{-3/2}T_m^{1/2}\psi+\frac{1}{2}x^{-1/2}T_m^{-1/2}T_{m,x}\psi + x^{-1/2}T_m^{1/2}\psi_x) = -T_{m,t}x^{-1/2}T_m^{1/2}\psi + 2T_m x^{-1/2}(\frac{1}{2}T_m^{-1/2}T_{m,t}\psi + T_m^{1/2}\psi_t), \tag{1.2.22}$$

that is,

$$\psi_t = \frac{2gx}{T_m}\psi_x + \frac{1}{T_m}(gxT_m^{-1}T_{m,x} - g)\psi. \tag{1.2.23}$$

Therefore, by setting

$$\mathfrak{a}_{(\text{II-1},m)} = \frac{2gx}{T_m}, \tag{1.2.24}$$

we find

$$\frac{\partial\psi}{\partial t} = \mathfrak{a}_{(\text{II-1},m)}\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial\mathfrak{a}_{(\text{II-1},m)}}{\partial x}\psi. \tag{1.2.25}$$

Thus we obtain simultaneous equations (1.2.17) and (1.2.25) for the unknown function ψ; equation (1.2.17) (resp., (1.2.25)) is referred to as $(SL_{\text{II-1}})_m$ (resp., $(D_{\text{II-1}})_m$). In parallel with the case of $(SL_{\text{I}})_m$ and$(D_{\text{I}})_m$, we first construct a 0-parameter solution

$$\hat{v}(t,\eta) = \hat{v}_0(t) + \hat{v}_1(t)\eta^{-1} + \cdots \tag{1.2.26}$$

of $(P_{\text{II-1}})_m$, and then substitute it into the coefficients of $(SL_{\text{II-1}})_m$ and $(D_{\text{II-1}})_m$ to analyze their structure. We note that (1.2.4) implies

$$\frac{(-1)^m 2^m(2m-1)!!}{m!}\,\hat{v}_0^{2m+1} + 2gt\hat{v}_0 + c_0 = 0. \tag{1.2.27}$$

In what follows we let $T_{m,0}, K_{m,0}, q_0$, etc. respectively denote the top order term of the expansion obtained by substituting the 0-parameter solution into the coefficients of T_m, K_m, q, etc; for example, $q_0 = x + \hat{v}_0^2$.

Using the 0-parameter solution $\hat{v}$, we introduce another set of formal series

$$b_j(t,\eta) = b_{j,0}(t) + \eta^{-1}b_{j,1}(t) + \cdots \quad (j = 1, \ldots, m) \tag{1.2.28}$$

as solutions of the equation

$$T_m(x,t,\eta)\Big|_{x=b_j(t,\eta)} = 0. \tag{1.2.29}$$

We then immediately find that $x = b_{j,0}(t)$ is a singular point of $Q_{(\text{II-1},m)}$ and $\mathfrak{a}_{(\text{II-1},m)}$. It is also clear from (1.2.18) that $x = b_{j,0}(t)$ is a double turning point of $(SL_{\text{II-1}})_m$. These observations are exactly the same as those for the series $b_j(t,\eta)$ introduced in the previous subsection.

§1.3. The case $J = \text{II-2}$

Let us first recall the definition of $P_{\text{II-2}}$-hierarchy with a large parameter η and its underlying Lax pair $(L_{\text{II-2}})$. We refer the reader to [GJP] and [N] for the detailed discussions concerning $P_{\text{II-2}}$-hierarchy.

Definition 1.3.1. The m-th member of $P_{\text{II-2}}$-hierarchy with a large parameter η is, by definition, the following differential equations for the unknown functions u and v:

$$(P_{\text{II-2}})_m : \begin{cases} K_{m+1} + \sum_{j=1}^{m-1} c_j K_j + gt = 0 \\ L_{m+1} + \sum_{j=1}^{m-1} c_j L_j = \delta \end{cases} \tag{1.3.1}$$

Here c_j, g and δ are constants, and K_j and L_j are polynomials of u, v and their derivatives, which are defined by the following recursive relations:

$$\eta^{-1}\partial_t \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \eta^{-1}\partial_t u - \eta^{-2}\partial_t^2 & 2\eta^{-1}\partial_t \\ 2\eta^{-1} v\partial_t + \eta^{-1} v_t & \eta^{-1} u\partial_t + \eta^{-2}\partial_t^2 \end{pmatrix} \begin{pmatrix} K_j \\ L_j \end{pmatrix} \tag{1.3.2}$$

$$(j \geq 0)$$

with $K_0 = 2$ and $L_0 = 0$.

Remark 1.3.1. See [N] for the proof of the existence of such preferred K_j and L_j, that is, those which are polynonials of u, v and their derivatives. The first three terms of such preferred K_j and L_j are as follows:

$$\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.3.3}$$

$$\begin{pmatrix} K_2 \\ L_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u^2 + 2v - \eta^{-1} u_t \\ 2uv + \eta^{-1} v_t \end{pmatrix} \tag{1.3.4}$$

$$\begin{pmatrix} K_3 \\ L_3 \end{pmatrix} = \left(\frac{1}{2}\right)^2 \begin{pmatrix} u^3 + 6uv - 3\eta^{-1} u u_t + \eta^{-2} u_{tt} \\ 3u^2 v + 3v^2 + 3\eta^{-1} u v_t + \eta^{-2} v_{tt} \end{pmatrix} \tag{1.3.5}$$

Remark 1.3.2. In what follows we assume

$$c_j = 0, \quad j = 1, 2, \ldots, m-1. \tag{1.3.6}$$

We also assume that g is a non-zero genuine constant and that δ has the form $\delta_0 + \eta^{-1}\delta_1$ with

$$\delta_1 = -g/2. \tag{1.3.7}$$

Remark 1.3.3. (i) $(P_{\text{II-2}})_1$ is reduced to

$$\eta^{-2} \frac{d^2 u}{dt^2} = 2u^3 + 2g(2tu + \eta^{-1}) + 4\delta. \tag{1.3.8}$$

(ii) $(P_{\text{II-2}})_2$ is reduced to

$$\begin{aligned} \eta^{-4} \frac{d^4 u}{dt^4} = \frac{1}{2u^2} (\eta^{-4} \Big(& -4\Big(\frac{du}{dt}\Big)^2 \frac{d^2 u}{dt^2} + 3u\Big(\frac{d^2 u}{dt^2}\Big)^2 \\ & + 4u \frac{du}{dt}\frac{d^3 u}{dt^3}\Big) + \eta^{-2}(16gu\frac{du}{dt} - 16gt\Big(\frac{du}{dt}\Big)^2 \\ & + 5u^3\Big(\frac{du}{dt}\Big)^2 + 16gtu\frac{d^2u}{dt^2} + 10u^4\frac{d^2u}{dt^2}) \\ & - 24\eta^{-1} g u^3 + (16g^2t^2u - 48\delta u^3 - 16gtu^4 - 5u^7)). \end{aligned} \tag{1.3.9}$$

Definition 1.3.2. The Lax pair $(L_{\text{II-2}})_m$ underlying $(P_{\text{II-2}})_m$ is the following pair of linear differential equations on (x, t)-space:

$$(L_{\text{II-2}})_m : \begin{cases} \Big(\dfrac{\partial}{\partial x} - \eta A\Big)\vec{\psi} = 0, & \text{(1.3.10.a)} \\ \Big(\dfrac{\partial}{\partial t} - \eta B\Big)\vec{\psi} = 0, & \text{(1.3.10.b)} \end{cases} \tag{1.3.10}$$

where $\vec{\psi} = {}^t(\psi_1, \psi_2)$,

(1.3.11) A

$$= \frac{1}{g}\begin{pmatrix} -(2x-u)T_m - \eta^{-1}T_{m,t} & 2T_m \\ -2vT_m - \eta^{-1}\partial_t((2x-u)T_m + \eta^{-1}T_{m,t} + K_{m+1}) & (2x-u)T_m + \eta^{-1}T_{m,t} \end{pmatrix}$$

with

$$T_m = \frac{1}{2}\sum_{j=0}^{m} x^{m-j}K_j, \tag{1.3.12}$$

and

$$B = \begin{pmatrix} -x+u/2 & 1 \\ -v & x-u/2 \end{pmatrix}. \tag{1.3.13}$$

In parallel with the discussions in the preceding subsections, we first write down the differential equation that the first component ψ_1 of the solution $\vec{\psi}$ of the equation (1.3.10) should satisfy:

$$\begin{aligned} \Big[\frac{\partial^2}{\partial x^2} &- \frac{T_{m,x}}{T_m}\frac{\partial}{\partial x} + \frac{\eta^2}{g^2}(-(2x-u)^2T_m^2 + 4vT_m^2) \\ &+ \frac{\eta}{g^2}(-2(2x-u)T_mT_{m,t} + 2T_m\partial_t((2x-u)T_m \\ &+ \eta^{-1}T_{m,t} + K_{m+1}) + 2gT_m) \\ &- \frac{T_{m,t}^2}{g^2} - \frac{T_{m,t}T_{m,x}}{gT_m} + \frac{T_{m,tx}}{g}\Big]\psi_1 = 0. \end{aligned} \tag{1.3.14}$$

To eliminate the first order differential operator part, we introduce

$$\psi = \exp(-\frac{1}{2}\int^x \frac{T_{m,x}}{T_m}dx)\psi_1 = T_m^{-1/2}\psi_1 \tag{1.3.15}$$

and find the Schrödinger equation for ψ:

$$\frac{\partial^2\psi}{\partial x^2} = \eta^2 Q_{(\text{II-2},m)}\psi, \tag{1.3.16}$$

where

$$Q_{(\text{II-2},m)} = \frac{1}{g^2}((2x-u)^2-4v)T_m^2 + \frac{\eta^{-1}}{g^2}(2u_tT_m^2 - 2T_mK_{m+1,t} - 2gT_m) + \eta^{-2}\Big(\frac{3}{4}\frac{T_{m,x}^2}{T_m^2} - \frac{T_{m,xx}}{2T_m} + \frac{T_{m,t}^2}{g^2} - 2\frac{T_mT_{m,tt}}{g^2} + \frac{T_{m,t}T_{m,x}}{gT_m} - \frac{T_{m,tx}}{g}\Big). \tag{1.3.17}$$

On the other hand, (1.3.10.b) implies

$$\frac{\partial\psi_1}{\partial t} = \eta(-x+\frac{u}{2})\psi_1 + \eta\psi_2, \tag{1.3.18}$$

and (1.3.10.a) implies

$$\frac{\partial\psi_1}{\partial x} = -\frac{\eta}{g}((2x-u)T_m + \eta^{-1}T_{m,t})\psi_1 + \frac{2\eta}{g}T_m\psi_2. \tag{1.3.19}$$

Combining these relations, we obtain

$$\frac{\partial\psi_1}{\partial x} = -\frac{1}{g}T_{m,t}\psi_1 + \frac{2T_m}{g}\frac{\partial\psi_1}{\partial t}. \tag{1.3.20}$$

Substituting (1.3.15) into (1.3.20), we find

$$\frac{\partial\psi}{\partial t} = \frac{g}{2T_m}\frac{\partial\psi}{\partial x} + \frac{gT_{m,x}}{4T_m^2}\psi. \tag{1.3.21}$$

Therefore, by setting

$$\mathfrak{a}_{(\text{II-2},m)} = \frac{g}{2T_m}, \tag{1.3.22}$$

we arrive at

$$\frac{\partial\psi}{\partial t} = \mathfrak{a}_{(\text{II-2},m)}\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial\mathfrak{a}_{(\text{II-2},m)}}{\partial x}\psi. \tag{1.3.23}$$

In what follows, (1.3.17) (resp., (1.3.23)) is referred to as $(SL_{\text{II-2}})_m$ and $(D_{\text{II-2}})_m$, and our aim is to analyze them by substituting a 0-parameter solution $(\hat{u},\hat{v})$ of $(P_{\text{II-2}})_m$ in their coefficients. We refer the reader to [N] concerning the existence proof of a 0-parameter solution and its basic properties. In parallel with the preceding subsections, we introduce another set of formal series

$$b_j(t,\eta) = b_{j,0}(t) + b_{j,1}(t)\eta^{-1} + \cdots \quad (j = 1,\ldots,m) \tag{1.3.24}$$

as solutions of the equation

$$T_m(x,t,\eta)\Big|_{(u,v)=(\hat{u},\hat{v}),x=b_j(t,\eta)} = 0. \tag{1.3.25}$$

It is then clear that $x = b_{j,0}(t)$ is a singular point of $Q_{(\text{II-2},m)}$ and $\mathfrak{a}_{(\text{II-2},m)}$, and that it is a double turning point of $(SL_{\text{II-2}})_m$. These facts are completely in parallel with the results we obtained in the preceding subsections.

§2. Regularity of S_{odd} near $x = b_{j,0}(t)$

In Section 1 we have derived a pair of Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$ from the Lax pair $(L_J)_m(J = \text{I}, \text{II-1}, \text{II-2})$. We have also confirmed that all of them share the following important property: the point $x = b_{j,0}(t)$ $(j = 1,\cdots,m)$ is a double turning point of the Schrödinger equation we obtained, where $b_{j,0}(t)$ is the top order term of the formal series $b_j(t,\eta)$ which is determined algebraically by a 0-parameter solution of $(P_J)_m$ $(J = \text{I}, \text{II-1}, \text{II-2})$. In the subsequent section (Section 3) we will construct a formal transformation that reduces $b_j(t,\eta)$ to a 0-parameter solutin of the traditional Painlevé-I equation (i.e., $(P_\text{I})_1$) near an appropriate turning point of $(P_J)_m$, and in this section we prepare some results needed for the construction. As our reasoning in this section applies uniformly to every $(SL_J)_m(J = \text{I}, \text{II-1}, \text{II-2})$, we omit the suffix (J,m) of $Q_{(J,m)}$ and $\mathfrak{a}_{(J,m)}$. In what follows we let $S^{\pm}$ denote the solution of the Riccati equation associated with $(SL_J)_m$, i.e.,

$$(S^{\pm})^2 + \frac{\partial S^{\pm}}{\partial x} = \eta^2 Q, \tag{2.1}$$

that begins with $\pm\eta\sqrt{Q_0}$ (with an appropriate choice of the branch of $\sqrt{Q_0}$). We also use the symbol S_{odd} to denote

$$\frac{1}{2}(S^+ - S^-). \tag{2.2}$$

We note that the definition of S_{odd} given here is different from that given in [KT1], although they coincide in the situation discussed in [KT1]. As a matter of fact, they are also coincident for $(SL_{\rm I})_m$ by a result on the structure of a 0-parameter solution (cf. Appendix); in general, if Q has the form

$$\sum_{l\geq 0}\eta^{-2l}Q_{2l}, \tag{2.3}$$

then the two definitions coincide. When Q contains odd degree terms in η, the definition given in [KT1] does not work; then we should use the definition (2.2). Making use of the reasoning in [AKT,§2], we can readily deduce the following relation (2.4) from $(D_J)_m$:

$$\frac{\partial S_{\rm odd}}{\partial t}=\frac{\partial}{\partial x}(\mathfrak{a}S_{\rm odd}) \tag{2.4}$$

for $S_{\rm odd}$ thus defined.

Remark 2.1. We note that the denominator of $\mathfrak{a}$ is a polynomial of degree m in x; in the analysis of the traditional Painlevé equations ([O], [KT1]), the corresponding function was linear in x.

Now, using the relation (2.4) we prove the following.

Theorem 2.1. *Assume that $x=b_{j,0}(t)$ is an exactly double zero of $Q_0(x,t)$ near $(x,t)=(b_{j,0}(t_0),t_0)$. Then the series $S_{\rm odd}$ and $\mathfrak{a}S_{\rm odd}$ are holomorphic on a neighborhood of $x=b_{j,0}(t)$ in the sense that each of their coefficients as formal power series in η^{-1} is holomorphic on the neighborhood of $x=b_{j,0}(t)$.*

Proof. For the sake of the uniformity of the presentation we use the symbol U also to denote $T_m/(gx)$ if $J=$ II-1 and $4T_m/g$ if $J=$ II-2. Let us substitute a 0-parameter solution into the coefficients of $\mathfrak{a}$ and U and expand them in powers of η^{-1} as follows:

$$\mathfrak{a}=\sum_{j\geq 0}\mathfrak{a}_j(x,t)\eta^{-j}, \tag{2.5}$$

$$U=\sum_{j\geq 0}U_j(x,t)\eta^{-j}. \tag{2.6}$$

To simplify the notation we let R denote $S_{\rm odd}$; in accordance with this convention, R_l stands for the coefficient of η^{-l} in the expansion of $S_{\rm odd}$. It then follows from (2.4) that

$$\frac{\partial R_{m-1}}{\partial t}=\frac{\partial}{\partial x}\left(\sum_{k=0}^{m}\mathfrak{a}_kR_{m-1-k}\right). \tag{2.7}$$

It also follows from the definition of $\mathfrak{a}$ that

$$U_0\mathfrak{a}_k + \sum_{l=1}^{k} U_l\mathfrak{a}_{k-l} = 0 \text{ for } k \geq 1. \tag{2.8}$$

Since $U_l (l \geq 0)$ is a polynomial in x, (2.8) shows that $\mathfrak{a}_k$ has the form $N_k U_0^{-k-1}$ with some polynomial N_k in x.

Now, combining (2.7) and (2.8), we find

$$\begin{aligned} \frac{\partial R_{m-1}}{\partial t} &= \frac{\partial}{\partial x}(\mathfrak{a}_0 R_{m-1}) - \frac{\partial}{\partial x}\left[\frac{1}{U_0}\sum_{k=1}^{m}\left(\left(\sum_{l=1}^{k} U_l\mathfrak{a}_{k-l}\right)R_{m-1-k}\right)\right] \\ &= \frac{\partial}{\partial x}(\mathfrak{a}_0 R_{m-1}) - \frac{\partial}{\partial x}\left[\frac{1}{U_0}\left(\sum_{l=1}^{m} U_l\left(\sum_{k=l}^{m}\mathfrak{a}_{k-l}R_{m-1-k}\right)\right)\right] \\ &= \frac{\partial}{\partial x}(\mathfrak{a}_0 R_{m-1}) - \frac{\partial}{\partial x}\left[\frac{1}{U_0}\left(\sum_{s=0}^{m-1} U_{m-s}\left(\sum_{r=0}^{s}\mathfrak{a}_r R_{s-1-r}\right)\right)\right]. \end{aligned} \tag{2.9}$$

Making use of (2.9) we show that there exists an open neighborhood ω of $x = b_{j,0}(t)$ on which the following assertion $(\mathcal{A})_n$ is validated for $n = 0, 1, 2, \cdots$:

$$(\mathcal{A})_n : \begin{cases} \text{(i)} & R_{n-1} \textit{ is holomorphic,} \\ \text{(ii)} & \displaystyle\sum_{l=0}^{n}\mathfrak{a}_l R_{n-1-l} \textit{ is holomorphic.} \end{cases}$$

We prove this by the induction on n. But, before embarking on proving this, we make some preparatory study on the structure of the function $R_l = (S_l^+ - S_l^-)/2$. By solving the Riccati equation (2.1) we can find a neighborhood ω of $x = b_{j,0}(t)$ on which $S_l^\pm$ has the following form:

$$\frac{C_l^\pm P_l^\pm}{(S_{-1}^\pm)^{p_{l,\pm}} U_0^{q_{l,\pm}}}, \tag{2.10}$$

where $p_{l,\pm}$ and $q_{l,\pm}$ are some non-negative integers, $C_l^\pm$ is an analytic function that does not vanish on ω and $P_l^\pm$ is a polynomial in x that depends analytically on t on ω. Since $S_{-1}^\pm$ has the form $\pm\alpha U_0$ with a non-vanishing analytic factor α on ω, we may assume every $S_l^\pm$, in particular $S_n^\pm$, has the form $\widetilde{C}^\pm P^\pm U_0^{p_\pm}$ with an integer $p_\pm$, a polynomial $P_\pm$ in x and a non-vanishing analytic factor $\widetilde{C}^{\pm}$ on ω. Hence we find R_n has the form $\widetilde{C}PU_0^p$ with an integer p, a polynomial P in x and a non-vanishing analytic factor $\widetilde{C}$ on ω. Here P is assumed not to vanish identically on $\{(x,t); x = b_{j,0}(t)\}$. Having this structure of R_n in mind, we embark on the confirmation of $(\mathcal{A})_n$ by the induction on n. First of all, $(\mathcal{A})_0$ is clear, because R_{-1} has the form αU_0 with an analytic factor α on ω and $\mathfrak{a}_0 = 2/U_0$ (resp., $2x/U_0$) for $J = $ I or $J = $ II-2 (resp., $J = $ II-1). Let us next assume that $(\mathcal{A})_m$ is validated for $m = 0, 1, \cdots, n$. Then this induction hypothesis guarantees that

$$\sum_{s=0}^{n} U_{n+1-s}\left(\sum_{r=0}^{s} \mathfrak{a}_r R_{s-1-r}\right) \tag{2.11}$$

is holomorphic on ω, and hence the second term in the right-hand side of (2.9) with $m = n+1$, namely,

$$-\frac{\partial}{\partial x}\left[\frac{1}{U_0}\left(\sum_{s=0}^{n} U_{n+1-s}\left(\sum_{r=0}^{s} \mathfrak{a}_r R_{s-1-r}\right)\right)\right], \tag{2.12}$$

has an at most double pole at $x = b_{j,0}(t)$ that originates from the simple pole factor U_0^{-1}. On the other hand, our preparatory study on the structure of R_n shows that $\partial R_n/\partial t$ has the form βU_0^{p-1} with an analytic factor β on ω and that $\partial(\mathfrak{a}_0 R_n)/\partial x = \partial(2\widetilde{C}PU_0^{p-1})/\partial x = \widetilde{\beta}U_0^{p-2}$ with another non-vanishing analytic factor $\widetilde{\beta}$ on ω. Therefore (2.9) with $m = n+1$ implies $p \geq 0$, i.e., R_n should be holomorphic. This validates the first part of the assertion $(\mathcal{A})_{n+1}$. It also entails that $\partial R_n/\partial t$ is holomorphic on ω, and hence the relation (2.7) with $m = n+1$ shows that

$$\frac{\partial}{\partial x}\left(\sum_{k=0}^{n+1} \mathfrak{a}_k R_{n-k}\right) \tag{2.13}$$

is holomorphic on ω. But, then, in view of the structure of $\mathfrak{a}_k$ and R_{n-k}, that is, the fact that their singularities, if any, are of the form U_0^{-r} for some non-negative integer r, we conclude that

$$\sum_{r=0}^{n+1} \mathfrak{a}_k R_{n-k} \tag{2.14}$$

should be holomorphic on ω. This is nothing but the second part of the assertion $(\mathcal{A})_{n+1}$. Thus the induction proceeds.

It is clear that the validity of $(\mathcal{A})_n$ for every $n (n = 0, 1, 2, \cdots)$ means that $R = S_{\rm odd}$ and $\mathfrak{a} S_{\rm odd}$ are holomorphic on ω. This completes the proof of the theorem.

§3. Reduction of $b_j(t, \eta)$ $(j = 1, \ldots, m)$ to a 0-parameter solution of $(P_{\rm I})_1$

§3.1. Some preparation of notions and notations about the Stokes geometry of $(P_J)_m$ and that of $(SL_J)_m$ ($J =$ I, II-1, II-2).

Before entering the analysis of $(SL_J)_m$ we clarify the geometric setting on which we consider the problem. To begin with, let us fix a turning point $t = \tau$ of the first kind of $(P_J)_m$ ($J =$ I, II-1, II-2) in the sense of [KKNT, §2], that is, there exist two solutions $\nu_\pm(t)$ of the characteristic equation of the linearization of $(P_J)_m$ at a 0-parameter solution (often called the Fréchet derivative of $(P_J)_m$) which merge at $t = \tau$ and whose values $\nu_\pm(\tau)$ are 0. Then it follows from the explicit form of the characteristic equation of the Fréchet derivative (cf. [KKNT, (2.1.23), (2.2.13), (2.3.8)]) that some $b_{j,0}(t)$, a double turning point of $(SL_J)_m$, and a simple turning point, say $a(t)$, of $(SL_J)_m$ merge at $t = \tau$. Note that every turning point of $(P_J)_m$ is of the first kind if $m = 1$. This explains why the turning point is not assumed to be of the first kind in [KT1]. We further assume, as in [KT1], that the turning point is simple: unlike the situation discussed in [KT1], we want to impose the condition without using the explicit form of the equation and employ the general definition given in [AKKT]. However, the characteristic equation written in t-variable has singularities at turning points and an immediate application of [AKKT] is not possible. Hence we use a local parameter u of the Riemann surface $\mathcal{R}$ associated with the 0-parameter solution as the independent variable that replaces t. Note that the Stokes geometry of $(P_J)_m$ is described on $\mathcal{R}$ (cf. [KKNT] and [NT]). Thus we require that the characteristic polynomial $P(u, \nu)$ of the Fréchet derivative of $(P_J)_m$ should satisfy the following conditions at $\hat{u}_0 = u(\tau)$:

$$P(\hat{u}_0, 0) = \frac{\partial P_0}{\partial \nu}(\hat{u}_0, 0) = 0 \tag{3.1.1}$$

$$\frac{\partial P}{\partial u}(\hat{u}_0, 0) \neq 0, \quad \frac{\partial^2 P}{\partial \nu^2}(\hat{u}_0, 0) \neq 0. \tag{3.1.2}$$

These conditions guarantee that τ is a square-root branch point of $\mathcal{R}$, and hence they imply that

$$\nu_{\pm}(t) \quad \text{is of exactly order} \quad (t-\tau)^{1/4}. \tag{3.1.3}$$

The results in [KKNT, §2] tell us then that

$$\nu_{-} = -\nu_{+} \tag{3.1.4}$$

and

$$\int_{\tau}^{t} \nu_{+}(s)ds = 2\int_{a(t)}^{b_{j,0}(t)} \sqrt{Q_{(J,m),0}(x,t)}dx \tag{3.1.5}$$

hold. Note that a Stokes curve of $(P_J)_m$ that emanates from τ is, by definition, given by

$$\mathrm{Im}\int_{\tau}^{t} \nu_{+}(s)ds = 0. \tag{3.1.6}$$

Since $a(\tau)$ and $b_{j,0}(\tau)$ coincide by their definition, (3.1.5) guarantees that $a(t)$ and $b_{j,0}(t)$ are connected by a Stokes curve (or, rather a Stokes segment) of $(SL_J)_m$ if t is a point in a Stokes curve of $(P_J)_m$ that is sufficiently close to τ. Note, however, that Stokes curves of $(P_J)_m$ cross for $m \geq 2$, and that the so-called Nishikawa phenomena ([N]) are observed at crossing points. Hence we cannot expect, in general, that $a(t)$ and $b_{j,0}(t)$ are connected by a Stokes curve of $(SL_J)_m$ even if t lies in a Stokes curve of $(P_J)_m$. Thus we consider the problem near a point $\sigma (\neq \tau)$ in a Stokes curve of $(P_J)_m$ that emanates from τ and that satisfies the following condition:

(3.1.7) $a(\sigma)$ and $b_{j,0}(\sigma)$ are connected by a Stokes curve of $(SL_J)_m$.

In this geometric setting we try to reduce $b_j(t,\eta)$ to a 0-parameter solution of $(P_{\mathrm{I}})_1$ on a neighborhood of σ. This is what we will achieve in the next subsection.

§3.2. Construction of formal transformations

In the setting described in Section 3.1 we construct appropriate formal transformations $\tilde{x}(x,t,\eta)$ and $\tilde{t}(t,\eta)$ for which the following relation holds:

$$\tilde{x}(x,t,\eta)\mid_{x=b_j(t,\eta)} = \lambda_{\mathrm{I}}(\tilde{t}(t,\eta),\eta), \tag{3.2.1}$$

where $\lambda_{\rm I}(\tilde{t},\eta)$ stands for a 0-parameter solution of the traditional Painlevé-I equation, that is,

$$\frac{d^2\lambda_{\rm I}}{d\tilde{t}^2}=\eta^2(6\lambda_{\rm I}^2+\tilde{t}). \tag{3.2.2}$$

Note that a 0-parameter solution is uniquely fixed once we fix the branch of its highest degree term $\lambda_0(t)=\sqrt{-t/6}$. In what follows we also use symbols $\nu_{\rm I}(\tilde{t},\eta)$ and $\widetilde{Q}(\tilde{x},\tilde{t},\eta)$ to denote respectively

$$\eta^{-1}d\lambda_{\rm I}/d\tilde{t} \tag{3.2.3}$$

and

$$4\tilde{x}^3+2\tilde{t}\tilde{x}+\nu_{\rm I}^2-4\lambda_{\rm I}^3-2\tilde{t}\lambda_{\rm I}-\eta^{-1}\frac{\nu_{\rm I}}{\tilde{x}-\lambda_{\rm I}}+\eta^{-2}\frac{3}{4(x-\lambda_{\rm I})^2}. \tag{3.2.4}$$

We note that $\widetilde{Q}$ is the potential of the Schrödinger equation $(SL_{\rm I})$ that is associated with the traditional Painlevé-I equation in the notation of [KT1]. Hence we use the symbol $\widetilde{S}_{\rm I,odd}\ (\tilde{x},\tilde{t})$ to denote the odd part of a solution $\widetilde{S}$ of the Riccati equation associated with $(SL_{\rm I})$, that is,

$$\widetilde{S}^2+\frac{\partial\widetilde{S}}{\partial\tilde{x}}=\eta^2\widetilde{Q}. \tag{3.2.5}$$

Using these symbols we first prove the following

Proposition 3.2.1. *Let τ be a simple turning point of the first kind of $(P_J)_m$ ($J=\mathrm{I},\mathrm{II}\text{-}1,\mathrm{II}\text{-}2; m=1,2,3,\cdots$), and let $\sigma(\neq\tau)$ be a point that is sufficiently close to τ (that is, σ satisfies the assumption* (3.1.7) *) and that lies in a Stokes curve of $(P_J)_m$ which emanates from τ. Let γ denote the Stokes segment which connects turning points $b_{j,0}(t)$ and $a(t)$ of $(SL_J)_m$ that are fixed in terms of τ in Section 3.1. Then there exist a neighborhood Ω of γ, a neighborhood ω of σ and holomorphic functions $\tilde{x}_j(x,t)$ ($j=0,1,2,\cdots$) on $\Omega\times\omega$ and $\tilde{t}_j(t)$ ($j=0,1,2,\cdots$) on ω so that they satisfy the following relations:*
(i) *The function $\tilde{t}_0(t)$ satisfies the following relation*

$$\int_\tau^t\nu_+(s)ds=\Big(\int_0^{\tilde{t}}\sqrt{12\lambda_0(\tilde{s})}d\tilde{s}\Big)\Big|_{\tilde{t}=\tilde{t}_0(t)}, \tag{3.2.6}$$

where ν_+ *denotes the solution of the characteristic equation of the Fréchet derivative of* $(P_J)_m$ *which is fixed in terms of* τ *in Section 3.1.*
(ii) $\tilde{x}_0(b_{j,0}(t),t) = \lambda_0(\tilde{t}_0(t))$ *and* $\tilde{x}_0(a(t),t) = -2\lambda_0(\tilde{t}_0(t))$.
(iii) $d\tilde{t}_0/dt \neq 0$ *on* ω *and* $\partial\tilde{x}_0/\partial x \neq 0$ *on* $\Omega \times \omega$.
(iv) *Letting* $\tilde{x}(x,t,\eta)$ *and* $\tilde{t}(t,\eta)$ *respectively denote* $\sum_{j\geq 0} \tilde{x}_j(x,t)\eta^{-j}$ *and* $\sum_{j\geq 0} \tilde{t}_j(t)\eta^{-j}$, *we find the following relation*:

$$(3.2.7)\qquad Q_{(J,m)}(x,t,\eta) = \left(\frac{\partial\tilde{x}}{\partial x}\right)^2 \widetilde{Q}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta) - \frac{1}{2}\eta^{-2}\{\tilde{x}(x,t,\eta);x\},$$

where $\{\tilde{x};x\}$ *denotes the Schwarzian derivative*

$$(3.2.8)\qquad \frac{\partial^3\tilde{x}/\partial x^3}{\partial\tilde{x}/\partial x} - \frac{3}{2}\left(\frac{\partial^2\tilde{x}/\partial x^2}{\partial\tilde{x}/\partial x}\right)^2.$$

Proof. To begin with, we note that the relation (3.2.7) follows from the following relation (3.2.9) together with the relevant Riccati equations (cf. [AKT]):

$$(3.2.9)\qquad S_{(J,m),\mathrm{odd}}(x,t,\eta) = \left(\frac{\partial\tilde{x}}{\partial x}\right)\widetilde{S}_{\mathrm{I,odd}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta),$$

where $S_{(J,m),\mathrm{odd}}$ stands for the odd part of a solution of the Riccati equation (2.1) with $Q = Q_{(J,m)}$. To simplify the notations, we use the symbol R and $\widetilde{R}$ respectively to denote $S_{(J,m),\mathrm{odd}}$ and $\widetilde{S}_{\mathrm{I,odd}}$; accordingly R_l and $\widetilde{R}_l$ respectively stand for the coefficient of η^{-l} $(l = -1,0,1,2,\cdots)$ of R and $\widetilde{R}$.

In constructing $\tilde{x}_j(x,t)$ and $\tilde{t}_j(t)$ in an inductive manner, we make use of the following assertion $(\mathcal{C})_n$ $(n = 0,1,2,\cdots)$ to make the argument run smoothly:

$(\mathcal{C})_n$ We can construct $\{\tilde{x}_j(x,t)\}_{0\leq j\leq n}$ and $\{\tilde{t}_j(t)\}_{0\leq j\leq n}$ so that (3.2.9) holds modulo terms of order equal to or at most η^{-n}.

Let us first show $(\mathcal{C})_0$; the way of our reasoning is exactly the same as that used in [KT1], but for the sake of completeness we repeat it here. [The only difference is the usage of $\sim$ in the rotations (x,t) etc. and $(\tilde{x},\tilde{t})$ etc.; it is reversed here.] The construction of the function $\tilde{t}_0(t)$ is attained by solving the implicit relation (3.2.6); we readily find it is a constant multiple of

$$(3.2.10)\qquad \left(\int_\tau^t \nu_+(s)ds\right)^{4/5},$$

which is holomorphic on a neighborhood of τ by the relation (3.1.3). If we define $\tilde{\sigma}$ by $\tilde{t}_0(\sigma)$, the relation (3.2.6) implies that $\tilde{\sigma}$ lies on a Stokes curve of (P_{I}), and hence a double turning point $\tilde{x} = \lambda_0(\tilde{\sigma})$ and a simple turning point $x = \tilde{a}(\tilde{\sigma}) = -2\lambda_0(\tilde{\sigma})$ of (SL_{I}) are connected by a Stokes segment $\tilde{\gamma}$ of (SL_{I}). Here we note that (SL_{I}) has one double turning point and one simple turning point if $\tilde{t} \neq 0$; in fact we know

$$\widetilde{Q}_0 = 4(\tilde{x} - \lambda_0(\tilde{t}))^2(\tilde{x} + 2\lambda_0(\tilde{t})). \tag{3.2.11}$$

Now we note

$$\int_0^{\tilde{t}} \sqrt{12\lambda_0(\tilde{s})}d\tilde{s} = 2\int_{-2\lambda_0(\tilde{t})}^{\lambda_0(\tilde{t})} \sqrt{\widetilde{Q}_0(\tilde{x}, \tilde{t})}d\tilde{x} \tag{3.2.12}$$

holds as a special case of (3.1.5). Hence combining (3.1.5), (3.2.6) and (3.2.12) we find

$$\int_{a(t)}^{b_{j,0}(t)} \sqrt{Q_{(J,m),0}(x,t)}dx = \int_{-2\lambda_0(\tilde{t}_0(t))}^{\lambda_0(\tilde{t}_0(t))} \sqrt{\widetilde{Q}_0(\tilde{x}, \tilde{t})}d\tilde{x}. \tag{3.2.13}$$

Furthermore it is a real number when t lies in the Stokes curve of $(P_J)_m$ in question; we may assume without loss of generality that the number is negative. We let $\rho = \rho(t)$ denote the number multiplied by (-1). Let us now introduce the following functions $z_1(x,t)$ and $z_2(\tilde{x}, t)$:

$$z_1(x,t) = \int_{b_{j,0}(t)}^{x} \sqrt{Q_{(J,m),0}(y,t)}dy, \tag{3.2.14}$$

$$z_2(\tilde{x}, t) = 2\int_{\lambda_0(\tilde{t}_0(t))}^{\tilde{x}} (\tilde{y} - \lambda_0(\tilde{t}_0(t)))\sqrt{\tilde{y} + 2\lambda_0(\tilde{t}_0(t))}d\tilde{y}. \tag{3.2.15}$$

We then try to construct $\tilde{x}_0(x,t)$ that satisfies

$$z_1(x,t) = z_2(\tilde{x}_0(x,t), t). \tag{3.2.16}$$

It is clear that (3.2.16) guarantees (3.2.9) at the level of η^{-1}. Hence the construction of $\tilde{x}_0(x,t)$ satisfying (3.2.16) will show $(\mathcal{C})_0$.

Now, the following assertions immediately follow from the definitions of

z_1, z_2 and ρ:

(3.2.17) $z_1(\gamma, t)$, i.e., the image of the segment γ by the map z, is a closed interval $[0, \rho]$,

(3.2.18) $\partial z_1/\partial x \neq 0$ on γ except for its endpoints,

(3.2.19) $z_1^{1/2}$ is holomorphic at $x = b_{j,0}(t)$ and $(\partial z_1^{1/2}/\partial x)\ |_{x=b_{j,0}(t)} \neq 0$,

(3.2.20) $(z_1-\rho)^{2/3}$ is holomorphic at $x = a(t)$ and $\dfrac{\partial}{\partial x}(z_1-\rho)^{2/3}\ |_{x=a(t)} \neq 0$,

(3.2.21) $z_2(\tilde{\gamma}, t) = [0, \rho]$,

(3.2.22) $\partial z_2/\partial \tilde{x} \neq 0$ on $\tilde{\gamma}$ except for its endpoints,

(3.2.23) $z_2^{1/2}$ is holomorphic at $\tilde{x} = \lambda_0(\tilde{t}_0(t))$ and $\dfrac{\partial}{\partial \tilde{x}} z_2^{1/2}\ |_{\tilde{x}=\lambda_0(\tilde{t}_0(t))} \neq 0$,

(3.2.24) $(z_2 - \rho)^{2/3}$ is holomorphic at $\tilde{x} = -2\lambda_0(\tilde{t}_0(t))$.

We next consider the composition of maps z_1 and z_2^{-1}, the inverse map of z_2, and we denote it by x_0, that is,

$$x_0 = z_2^{-1} \circ z_1 : \gamma \to \tilde{\gamma}. \tag{3.2.25}$$

It is then clear that

$$x_0(b_{j,0}(t), t) = \lambda_0(\tilde{t}_0(t)) \tag{3.2.26}$$

and

$$x_0(a(t), t) = -2\lambda_0(\tilde{t}_0(t)) \tag{3.2.27}$$

hold. It also follows from (3.2.18) and (3.2.22) that x_0 is holomorphic on γ except for its endpoints and that $\partial x_0/\partial \tilde{x} \neq 0$ holds there. To confirm its analyticity at $b_{j,0}(t)$ and $a(t)$, first say at $b_{j,0}(t)$, let us consider the following equation for $\tilde{x}_0^\dagger(x, t)$ near $x = b_{j,0}(t)$:

$$z_1(x, t)^{1/2} = z_2(\tilde{x}_0^\dagger(x, t), t)^{1/2}, \tag{3.2.28}$$

where the branch of $z_1^{1/2}$ (resp., $z_2^{1/2}$) is chosen so that it may be positive in γ (resp., $\tilde{\gamma}$). It then follows from (3.2.19) and (3.2.23) that (3.2.28) has a unique holomorphic solution $\tilde{x}_0^\dagger(x, t)$ near $x = b_{j,0}(t)$ that satisfies

$$\tilde{x}_0^\dagger(b_{j,0}(t), t) = \lambda_0(\tilde{t}_0(t)) \quad \text{and} \quad \frac{\partial \tilde{x}_0^\dagger}{\partial x}(b_{j,0}(t), t) \neq 0. \tag{3.2.29}$$

It is clear that $\tilde{x}_0^\dagger$ and x_0 coincide on their common domain of definition. Hence $\tilde{x}_0$ is holomorphic at $x = b_{j,0}(t)$ and $\partial\tilde{x}_0/\partial x$ does not vanish there. The holomorphy of $\tilde{x}_0(x,t)$ at $x = a(t)$ is also confirmed by a similar reasoning if we start with the following equation (3.2.30) instead of (3.2.28):

$$(z_1(x,t) - \rho)^{2/3} = (z_2(\tilde{x}(x,t),t) - \rho)^{2/3}. \tag{3.2.30}$$

Thus we have proved $(\mathcal{C})_0$. In the course of the proof we have also confirmed properties (i), (ii) and (iii) in the statement of the proposition.

We now embark on the proof of $(\mathcal{C})_n$ $(n \geq 1)$. Our method of the proof is essentially the same as that given in [KT1]. There is, however, one important difference: we have to construct non-zero $\tilde{x}_j$ and $\tilde{t}_j$ even for odd j. (As we show in Appendix, a 0-parameter solution of $(P_{\mathrm{I}})_m$ enjoys a nice property which guarantees (2.3); in this case we may assume $\tilde{x}_j = \tilde{t}_j = 0$ for odd j. But a 0-parameter solution of $(P_{\text{II-1}})_m$ or $(P_{\text{II-2}})_m$ does not have the property.) Our strategy of the proof is to construct a solution of the equation (3.2.31.n) below globally on $\Omega \times \omega$ by matching a solution holomorphic near $x = b_{j,0}(t)$ with another solution holomorphic near $x = a(t)$ with an appropriate choice of the "parameter" $\tilde{t}_n(t)$. One technical problem in putting this idea into practice is the non-analyticity of the coefficients of (3.2.32.n) at $x = a(t)$; we circumvent this problem by considering another defining equation (3.2.33.n) of x_n as a replacement of (3.2.32.n).

Now the actual task in proceeding from $(\mathcal{C})_{n-1}$ to $(\mathcal{C})_n$ $(n \geq 1)$ is to construct $\tilde{x}_n(x,t)$ and $\tilde{t}_n(t)$, the coefficients of η^{1-n} of (3.2.9), so that the following relation (3.2.31.n) may be satisfied globally on $\Omega \times \omega$:

$$\begin{aligned} R_{n-1}(x,t) &= \widetilde{R}_{-1}(\tilde{x}_0(x,t),\tilde{t}_0(x,t))\frac{\partial \tilde{x}_n}{\partial x}(x,t) \\ &+ \frac{\partial \tilde{x}_0}{\partial x}(x,t)\left\{\frac{\partial \widetilde{R}_{-1}}{\partial \tilde{x}}(\tilde{x}_0(x,t),\tilde{t}_0(x,t))\tilde{x}_n(x,t)\right. \\ &\left. + \frac{\partial \widetilde{R}_{-1}}{\partial \tilde{t}}(\tilde{x}_0(x,t),\tilde{t}_0(x,t))\tilde{t}_n(x,t)\right\} + \tilde{\rho}_n \quad (n \geq 1), \end{aligned} \tag{3.2.31.n}$$

where $\tilde{\rho}_n$ is a function of $\{\tilde{x}_j, \tilde{t}_k\}_{0 \leq j,k \leq n-1}$. Note that Theorem 2.1 guarantees that (3.2.31.n) is a differential equation for $\tilde{x}_n(x,t)$ with analytic coefficients near $x = b_{j,0}(t)$. To make the computation run smoothly we introduce a new variable z by defining it to be $\tilde{x}_0(x,t)$. Then (3.2.31.n) can be rewritten as follows:

$$(3.2.32.\mathrm{n})\qquad \left(\widetilde{R}_{-1}\frac{\partial}{\partial z}+\frac{\partial \widetilde{R}_{-1}}{\partial \tilde{x}}\right)\tilde{x}_n=\left(\frac{\partial \tilde{x}_0}{\partial x}\right)^{-1}(R_{n-1}-\tilde{\rho}_n)-\frac{\partial \widetilde{R}_{-1}}{\partial \tilde{t}}\tilde{t}_n.$$

We also find the following relation (3.2.33.n) through the comparison of the coefficients of η^{-n} of (3.2.7) (divided by $(\partial\tilde{x}_0/\partial x)^2$):

$$(3.2.33.\mathrm{n})\qquad \begin{aligned}&\left(2\widetilde{Q}_0\frac{\partial}{\partial z}+\frac{\partial \widetilde{Q}_0}{\partial \tilde{x}}\right)\tilde{x}_n\\ &\quad=\left(\frac{\partial \tilde{x}_0}{\partial x}\right)^{-2}(Q_n-\tilde{r}_n)-\frac{\partial \widetilde{Q}_0}{\partial \tilde{t}}\tilde{t}_n,\end{aligned}$$

where $\tilde{r}_n$ is a holomorphic function of $\{\tilde{x}_j,\tilde{t}_k\}_{0\le j,k\le n-1}$. In what follows we let $L_{\widetilde{R}}$ and $L_{\widetilde{Q}}$ denote respectively the differential operator

$$(3.2.34)\qquad \widetilde{R}_{-1}\frac{\partial}{\partial z}+\frac{\partial \widetilde{R}_{-1}}{\partial \tilde{x}}$$

and another differential operator

$$(3.2.35)\qquad 2\widetilde{Q}_0\frac{\partial}{\partial z}+\frac{\partial \widetilde{Q}_0}{\partial \tilde{x}}.$$

Clearly they satisfy

$$(3.2.36)\qquad 2\sqrt{\widetilde{Q}_0}L_{\widetilde{R}}=L_{\widetilde{Q}}.$$

It also follows immediately from the induction hypothesis that

$$(3.2.37)\qquad 2\sqrt{\widetilde{Q}_0}\left(\frac{\partial \tilde{x}_0}{\partial x}\right)^{-1}(R_{n-1}-\tilde{\rho}_n)=\left(\frac{\partial \tilde{x}_0}{\partial x}\right)^{-2}(Q_{n-1}-\tilde{r}_n).$$

Therefore (3.2.32.n) and (3.2.33.n) are equivalent; in what follows we make full use of this fact. Let us first note that the differential equation $L_{\widetilde{R}}u=f$ (resp., $L_{\widetilde{Q}}v=g$) has a unique holomorphic solution u (resp., v) near $z=\lambda_0(\tilde{t}_0(t))$ (resp., $z=-2\lambda_0(\tilde{t}_0(t))$) if f (resp., g) is holomorphic there, because the characteristic exponent of $L_{\widetilde{R}}$ (resp., $L_{\widetilde{Q}}$) at $z=\lambda_0$ (resp., $z=-2\lambda_0$) is equal to -1 (resp., $-1/2$). Now let f_1 and f_2 respectively denote $(\partial\tilde{x}_0/\partial x)^{-1}(R_{n-1}-\tilde{\rho}_n)$ and $\partial\widetilde{R}_{-1}/\partial\tilde{t}$. Then Theorem 2.1 together with the induction hypothesis guarantees that f_1 and f_2 are holomorphic near $z = \lambda_0$. Hence we find a unique holomorphic solution ϕ_j of the equation

$$L_{\widetilde{R}}\phi_j = f_j \quad (j = 1, 2) \tag{3.2.38}$$

near $z = \lambda_0$. Since (3.2.37) entails the holomorphy of $2\sqrt{\widetilde{Q}_0}f_1$ at $z = -2\lambda_0$ and since $2\sqrt{\widetilde{Q}_0}f_2 = \partial\widetilde{Q}_0/\partial\tilde{t}$ is clearly holomorphic at $z = -2\lambda_0$, we find a unique holomorphic solution $\hat{\phi}_j$ of the equation

$$L_{\widetilde{Q}}\hat{\phi}_j = 2\sqrt{\widetilde{Q}_0}f_j \quad (j = 1, 2) \tag{3.2.39}$$

near $z = -2\lambda_0$. Let now ϕ denote a non-zero multi-valued analytic solution of $L_{\widetilde{R}}\phi = 0$ on a neighborhood of γ; it is unique up to a constant multiple. On the other hand, (3.2.36) implies

$$L_{\widetilde{Q}}\phi_j = 2\sqrt{\widetilde{Q}_0}f_j \quad (j = 1, 2) \tag{3.2.40}$$

near $z = -2\lambda_0$ after the analytic continuation of ϕ_j along γ. Therefore we find

$$\phi_j - \hat{\phi}_j = c_j\phi \quad (j = 1, 2) \tag{3.2.41}$$

for some constants c_j $(j = 1, 2)$. If we can choose a constant $\tilde{t}_n$ so that

$$c_1 - \tilde{t}_n c_2 = 0 \tag{3.2.42}$$

holds, then, by choosing

$$\tilde{x}_n = \phi_1 - \tilde{t}_n\phi_2, \tag{3.2.43}$$

we find that all the required conditions are satisfied. Thus what remains to be done is the confirmation of the non-vanishing of the constant c_2. It follows from the definition of the operator $L_{\widetilde{R}}$ and the function f_2 together with the explicit form of (SL_{I}) (cf. (3.2.11)) that ϕ_2 satisfies the following relation near $z = \lambda_0$:

(3.2.44)
$$\begin{aligned}2(z+2\lambda_0)^{1/2}(z-\lambda_0)\frac{\partial\phi_2}{\partial z} + \{(z-\lambda_0)(z+2\lambda_0)^{-1/2} + 2(z+2\lambda_0)^{1/2}\}\phi_2 \\ = -2(z+2\lambda_0)^{1/2}\frac{d\lambda_0}{d\tilde{t}} + 2(z-\lambda_0)(z+2\lambda_0)^{-1/2}\frac{d\lambda_0}{d\tilde{t}},\end{aligned}$$

that is,

$$(z+2\lambda_0)(z-\lambda_0)\frac{\partial\phi_2}{\partial z}+\frac{3}{2}(z+\lambda_0)\phi_2=-3\lambda_0\frac{d\lambda_0}{d\tilde{t}}. \tag{3.2.45}$$

Since we know (cf. (3.2.2))

$$6\lambda_0(\tilde{t})^2+\tilde{t}=0, \tag{3.2.46}$$

the right-hand side of (3.2.45) is equal to 1/4. Hence by integrating (3.2.45) we find

$$\phi_2=\frac{1}{4(z-\lambda_0)\sqrt{(z+2\lambda_0)}}\int_{\lambda_0}^{z}\frac{dw}{\sqrt{w+2\lambda_0}}. \tag{3.2.47}$$

Then we analytically continue ϕ_2 near $z=-2\lambda_0$ to find

$$\frac{1}{4(z-\lambda_0)\sqrt{(z+2\lambda_0)}}\left(\int_{\lambda_0}^{-2\lambda_0}\frac{dw}{\sqrt{w+2\lambda_0}}+\int_{-2\lambda_0}^{z}\frac{dw}{\sqrt{w+2\lambda_0}}\right). \tag{3.2.48}$$

As it is evident that

$$L_{\tilde{R}}\left(\frac{1}{4(z-\lambda_0)\sqrt{(z+2\lambda_0)}}\right)=0, \tag{3.2.49}$$

the expression (3.2.48) implies

$$\begin{aligned} c_2\phi &= \frac{1}{4(z-\lambda_0)\sqrt{(z+2\lambda_0)}}\int_{\lambda_0}^{-2\lambda_0}\frac{dw}{\sqrt{w+2\lambda_0}} \\ &= -\frac{\sqrt{3\lambda_0}}{2(z-\lambda_0)\sqrt{(z+2\lambda_0)}}. \end{aligned} \tag{3.2.50}$$

Thus we see that c_2 is different from 0 on the condition that λ_0 is different from 0. Since we are considering the problem near $\sigma\neq\tau$, we may assume that $\lambda_0(\tilde{t}_0(t))$ is different from 0 for t in ω. Thus we have constructed $(\tilde{x}_n,\tilde{t}_n)$ that satisfy (3.2.31.n), that is, the induction proceeds, completing the proof of Proposition 3.2.1.

Using the formal series $\tilde{x}(x,t,\eta)$ and $\tilde{t}(t,\eta)$ which satisfy (3.2.7), and hence (3.2.9), we obtain the following reduction theorem.

Theorem 3.2.1. *In the geometric setting of Proposition 3.2.1, the series* $\tilde{x}(x,t,\eta)$ *and* $\tilde{t}(t,\eta)$ *constructed there satisfy the following relation :*

$$\tilde{x}(x,t,\eta)\mid_{x=b_j(t,\eta)}=\lambda_{\mathrm{I}}(\tilde{t}(t,\eta),\eta), \tag{3.2.51}$$

where $\lambda_{\mathrm{I}}(\tilde{t},\eta)$ *designates a* 0*-parameter solution of the traditional Painlevé*-I *equation, namely,*

$$\frac{d^2\lambda_{\mathrm{I}}}{d\tilde{t}^2}=\eta^2(6\lambda_{\mathrm{I}}^2+\tilde{t}). \tag{3.2.52}$$

Proof. First we note that every $\mathfrak{a}_{(J,m)}$ ($J=$ I, II-1, II-2; $m=1,2,\cdots$) has the form

$$\frac{\mathfrak{c}_{(J,m)}(x,t,\eta)}{(x-b_j(t,\eta))}, \tag{3.2.53}$$

where $\mathfrak{c}_{(J,m)}$ has the form

$$\sum_{l\geq 0}c_l(x,t)\eta^{-l} \tag{3.2.54}$$

with

$$c_0(x,t)\mid_{x=b_{j,0}(t)}\neq 0. \tag{3.2.55}$$

In what follows we say, as is always the case in this paper, that a series in η^{-1} is holomorphic if the coefficient of η^{-l} is holomorphic on a fixed open set for every l. Using this wording, we know by Theorem 2.1 that

(3.2.56)

$$\frac{\mathfrak{c}}{(x-b_j(t,\eta))}S_{(J,m),\mathrm{odd}} \text{ is holomorphic on a neighborhood of } x=b_{j,0}(t).$$

Since c_0 is different from 0 at $x=b_{j,0}(t)$, the series $\mathfrak{c}$ is invertible as a formal series in η^{-1}. Hence (3.2.56) implies

$$\frac{S_{(J,m),\mathrm{odd}}}{x-b_j(t,\eta)} \text{ is holomorphic on a neighborhood of } x=b_{j,0}(t). \tag{3.2.57}$$

On the other hand, (3.2.9) implies

$$\begin{aligned}\frac{S_{(J,m),\mathrm{odd}}}{x-b_j(t,\eta)} &\\ =\frac{\tilde{x}(x,t,\eta)-\lambda_{\mathrm{I}}(\tilde{t}(t,\eta),\eta)}{x-b_j(t,\eta)}&\frac{\partial\tilde{x}(x,t,\eta)}{\partial x}\frac{\tilde{S}_{\mathrm{I,odd}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta)}{\tilde{x}(x,t,\eta)-\lambda_{\mathrm{I}}(\tilde{t}(t,\eta),\eta)}.\end{aligned} \tag{3.2.58}$$

Since $\partial \tilde{x}_0/\partial x$ is different from 0 at $x = b_{j,0}(t)$, the series $\partial \tilde{x}/\partial x$ is invertible there. We also find by an explicit computation that

$$\frac{\tilde{S}_{\mathrm{I},-1}(\tilde{x}_0(x,t), \tilde{t}_0(t))}{\tilde{x}_0(x,t) - \lambda_{\mathrm{I},0}(\tilde{t}_0(t))} = 2\sqrt{\tilde{x}_0(x,t) + 2\lambda_{\mathrm{I},0}(\tilde{t}_0(\tilde{t}))}, \tag{3.2.59}$$

which is clearly different from 0 at $x = b_{j,0}(t)$. Hence

$$\frac{\widetilde{S}_{\mathrm{I,odd}}(\tilde{x}(x,t,\eta), \tilde{t}(t,\eta), \eta)}{\tilde{x}(x,t,\eta) - \lambda_{\mathrm{I}}(\tilde{t}(t,\eta), \eta)} \tag{3.2.60}$$

is also invertible near $x = b_{j,0}(t)$. Therefore (3.2.57) and (3.2.58) imply that

$$\tilde{x}(x,t,\eta) - \lambda_{\mathrm{I}}(\tilde{t}(t,\eta), \eta) = (x - b_j(t,\eta))d(x,t,\eta) \tag{3.2.61}$$

holds for some holomorphic series $d(x,t,\eta)$ near $x = b_{j,0}(t)$. Setting $x = b_j(t,\eta)$ in (3.2.61), we obtain the required relation (3.2.51).

Appendix

The purpose of this Appendix is to prove the following Proposition A.1 concerning the structure of a 0-parameter solution of $(P_{\mathrm{I}})_m$, which guarantees that $Q_{(\mathrm{I},m)}$ satisfies the condition (2.3).

Proposition A.1. *Let $(\hat{u}, \hat{v}) = (\hat{u}_1, \cdots, \hat{u}_m, \hat{v}_1, \cdots, \hat{v}_m)$ be a 0-parameter solution of $(P_{\mathrm{I}})_m$ defined near $t = t_0$ and $\hat{w} = (\hat{w}_1, \cdots, \hat{w}_m)$ be the formal series determined by $(\hat{u}, \hat{v})$ through the relation* (1.1.2). *Assume that the simple turning point of $(SL_{\mathrm{I}})_m$, namely $x = -2\hat{u}_{1,0}(t)$, does not coincide with any double turning point of $(SL_{\mathrm{I}})_m$ at $t = t_0$. Then all the odd degree (in η^{-1}) terms of $\hat{u}, \hat{v}$ and $\hat{w}$ vanish.*

*Remark A.*1. It is evident from (1.1.33) that the above assumption of non-coincidence of the simple turning point and a double turning point can be summarized as follows:

$$U_0(-2\hat{u}_{1,0}(t_0)) \neq 0. \tag{A.1}$$

To make the logical structure of the proof of Proposition A.1 lucid, we divide the proof into several steps; each step is summarized as a sublemma, and the proof of the Proposition is completed after Sublemma A.3.

Sublemma A.1. *We find*

$$\hat{w}_{j,1} = \hat{u}_{1,0}\hat{u}_{j,1} \tag{A.2}$$

holds for $j = 1, 2, \cdots, m$.

Proof. As it follows from the definition of $\hat{w}_j$ (cf. (1.1.2)) that

$$\hat{w}_1 = \frac{1}{2}\hat{u}_1^2 + c_1 + \delta_{1m}t, \tag{A.3}$$

we find

$$\hat{w}_{1,1} = \hat{u}_{1,0}\hat{u}_{1,1}. \tag{A.4}$$

Thus (A.2) holds for $j = 1$. We now use the induction on j; let us suppose that (A.2) holds for $j = 1, 2, \cdots, j_0$. The definition of $\hat{w}_j$ implies

$$\begin{aligned} \hat{w}_{j_0+1,1} =& \frac{1}{2}\left(\sum_{k=1}^{j_0+1}\hat{u}_{k,0}\hat{u}_{j_0+2-k,1} + \sum_{k=1}^{j_0+1}\hat{u}_{k,1}\hat{u}_{j_0+2-k,0}\right) \\ &+ \sum_{k=1}^{j_0}(\hat{u}_{k,0}\hat{w}_{j_0+1-k,1} + \hat{u}_{k,1}\hat{w}_{j_0+1-k,0}) \end{aligned} \tag{A.5}$$

because we know by (1.1.28)

$$\hat{v}_{j,0} = 0, \quad j = 1, \cdots, m. \tag{A.6}$$

Then the induction hypothesis entails

$$\begin{aligned} \hat{w}_{j_0+1,1} =& \sum_{l=1}^{j_0+1}\hat{u}_{j_0+2-l,0}\hat{u}_{l,1} + \sum_{k=1}^{j_0}\hat{u}_{k,0}\hat{u}_{1,0}\hat{u}_{j_0+1-k,1} + \sum_{k=1}^{j_0}\hat{u}_{k,1}\hat{w}_{j_0+1-k,0} \\ =& \hat{u}_{1,0}\hat{u}_{j_0+1,1} + \sum_{l=1}^{j_0}(\hat{u}_{j_0+2-l,0} + \hat{u}_{1,0}\hat{u}_{j_0+1-l,0} + \hat{w}_{j_0+1-l,0})\hat{u}_{l,1}. \end{aligned} \tag{A.7}$$

Hence (1.1.27) with $j = j_0 + 1 - l$ proves

$$\hat{w}_{j_0+1,1} = \hat{u}_{1,0}\hat{u}_{j_0+1,1}. \tag{A.8}$$

Thus the induction proceeds, completing the proof of Sublemma A.1.

Sublemma A.2. *The coefficient of the degree one (in η^{-1}) part of $\hat{u} = (\hat{u}_1, \cdots, \hat{u}_m)$, i.e., $(\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1})$, is zero.*

Proof. First, the comparison of the coefficients of η^0 in (1.1.1.b) with the help of (A.6) entails

$$\hat{u}_{j+1,1} + \hat{u}_{1,0}\hat{u}_{j,1} + \hat{u}_{1,1}\hat{u}_{j,0} + \hat{w}_{j,1} = 0, \ j = 1, \cdots, m. \tag{A.9}$$

Then Sublemma A.1 implies

$$\hat{u}_{j+1,1} + 2\hat{u}_{1,0}\hat{u}_{j,1} + \hat{u}_{j,0}, \hat{u}_{1,1} = 0, \ j = 1, \cdots, m. \tag{A.10}$$

Since $\hat{u}_{m+1}$, and in particular $\hat{u}_{m+1,1}$, vanishes by its definition, the relation (A.10) can be re-written as a matrix equation for the unknown vector ${}^t(\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1})$:

$$\begin{pmatrix} 3\hat{u}_{1,0} & 1 & 0 & & & \\ \hat{u}_{2,0} & 2\hat{u}_{1,0} & 1 & & & \mathbf{0} \\ \hat{u}_{3,0} & 0 & 2\hat{u}_{1,0} & 1 & & \\ \vdots & & & \ddots & \ddots & \\ \hat{u}_{m-1,0} & & & & 2\hat{u}_{1,0} & 1 \\ \hat{u}_{m,0} & & \mathbf{0} & & & 2\hat{u}_{1,0} \end{pmatrix} \begin{pmatrix} \hat{u}_{1,1} \\ \hat{u}_{2,1} \\ \vdots \\ \vdots \\ \hat{u}_{m-1,1} \\ \hat{u}_{m,1} \end{pmatrix} = 0. \tag{A.11}$$

Then the determinant $\triangle$ of the matrix in the left-hand side of (A.11) is

(A.12)

$$\begin{aligned} &3\hat{u}_{1,0}(2\hat{u}_{1,0})^{m-1} - \{\hat{u}_{2,0}(2\hat{u}_{1,0})^{m-2} - \hat{u}_{3,0}(2\hat{u}_{1,0})^{m-3} \\ &\quad + \hat{u}_{4,0}(2\hat{u}_{1,0})^{m-4} - \cdots + (-1)^{m-1}(\hat{u}_{m-1,0}(2\hat{u}_{1,0}) - \hat{u}_{m,0})\} \\ =&(-1)^m\{(-2\hat{u}_{1,0})^m - \hat{u}_{1,0}(-2\hat{u}_{1,0})^{m-1} - \hat{u}_{2,0}(-2\hat{u}_{1,0})^{m-2} \\ &\quad - \hat{u}_{3,0}(-2\hat{u}_{1,0})^{m-3} - \hat{u}_{4,0}(-2\hat{u}_{1,0})^{m-4} - \cdots - \hat{u}_{m-1,0}(-2\hat{u}_{1,0}) - \hat{u}_{m,0}\} \\ =&(-1)^m U_0(-2\hat{u}_{1,0}). \end{aligned}$$

Hence Remark A.1 guarantees that the determinant $\triangle$ does not vanish at $t = t_0$. Therefore the solution ${}^t(\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1})$ of the homogeneous equation (A.11) should be 0. This completes the proof of Sublemma A.2.

Sublemma A.3. *Suppose that*

(A.13.p_0) $\quad \hat{u}_{j,2p-1} = \hat{w}_{j,2p-1} = \hat{v}_{j,2p-2} = 0 \quad (j = 1, 2, \cdots, m)$
hold for $p = 1, 2, \cdots, p_0$.

Then we find

$$\hat{w}_{j,2p_0+1} = \hat{u}_{1,0}\hat{u}_{j,2p_0+1} \quad (j = 1, 2, \cdots, m). \tag{A.14.p_0+1}$$

Proof. We can use essentially the same reasoning as in the proof of Sublemma A.1. First we note that (A.3) together with (A.13.p_0) entails

$$\hat{w}_{1,2p_0+1} = \hat{u}_{1,0}\hat{u}_{1,2p_0+1}. \tag{A.15}$$

Hence we use the induction on j to prove (A.14.p_0+1), starting with (A.15): let us suppose

$$\hat{w}_{j,2p_0+1} = \hat{u}_{1,0}\hat{u}_{j,2p_0+1} \tag{A.16}$$

holds for $j = 1, 2, \cdots, j_0$. Since

$$\hat{v}_{j,2p_0} = \frac{1}{2}\frac{d\hat{u}_{j,2p_0-1}}{dt} \tag{A.17}$$

holds by (1.1.1.a), (A.13.p_0) implies

$$\hat{v}_{j,2p_0} = 0 \quad (j = 1, 2, \cdots, m). \tag{A.18}$$

Then, in parallel with (A.5), we find

$$\begin{aligned} \hat{w}_{j_0+1,2p_0+1} =& \frac{1}{2}\left(\sum_{k=1}^{j_0+1} \hat{u}_{k,0}, \hat{u}_{j_0+2-k,2p_0+1} + \sum_{k=1}^{j_0+1} \hat{u}_{k,2p_0+1}\hat{u}_{j_0+2-k,0}\right) \\ &+ \sum_{k=1}^{j_0} \left(\hat{u}_{k,0}\hat{w}_{j_0+1-k,2p_0+1} + \hat{u}_{k,2p_0+1}\hat{w}_{j_0+1-k,0}\right). \end{aligned} \tag{A.19}$$

Hence the induction hypothesis together with (1.1.27) (with $j = j_0 + 1 - k$) proves

$$\hat{w}_{j_0+1,2p_0+1} = \hat{u}_{1,0}\hat{u}_{j_0+1,2p_0+1}. \tag{A.20}$$

Thus the induction on j proceeds, proving (A.14.p_0+1).

Proof of Proposition A.1. Sublemma A.2, Sublemma A.1 and (A.6) imply that (A.13.p_0) is true for $p_0 = 1$. We now prove by induction on p_0 that (A.13.p_0) holds for every $p_0 = 1, 2, \cdots$; it clearly proves Proposition A.1. In view of Sublemma A.3 and (A.18), it suffices to prove that (A.13.p_0) implies

$$\hat{u}_{j,2p_0+1} = 0 \text{ for } j = 1, \cdots, m. \tag{A.21}$$

Now, with the help of the induction hypothesis supplemented by (A.18), the comparison of the coefficients of η^{-2p_0} in (1.1.1.b) gives us

$$\hat{u}_{j+1,2p_0+1} + (\hat{u}_{1,0}\hat{u}_{j,2p_0+1} + \hat{u}_{1,2p_0+1}\hat{u}_{j,0}) + \hat{w}_{j,2p_0+1} = 0 \tag{A.22}$$

for every $j = 1, 2, \cdots, m$. Then, applying Sublemma A.3 to (A.22), we find

$$\hat{u}_{j+1,2p_0+1} + 2\hat{u}_{1,0}\hat{u}_{j,2p_0+1} + \hat{u}_{j,0}\hat{u}_{1,2p_0+1} = 0, \ j = 1, 2, \cdots, m. \tag{A.23}$$

Since $\hat{u}_{m+1,2p_0+1} = 0$ by (1.1.1.c), (A.23) leads to the same matrix equation as (A.11) with the replacement of the unknown vector ${}^t(\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1})$ by ${}^t(\hat{u}_{1,2p_0+1}, \hat{u}_{2,2p_0+1}, \cdots, \hat{u}_{m,2p_0+1})$, in exactly the same manner as (A.10) has led to (A.11). We have already confirmed in the proof of Sublemma A.2 that the determinant $\triangle$ of the coefficient matrix in (A.11) is different from 0 at $t = t_0$ by the assumption of Proposition A.1. Therefore we conclude that ${}^t(\hat{u}_{1,2p_0+1}, \hat{u}_{2,2p_0+1}, \cdots, \hat{u}_{m,2p_0+1})$ should vanish. Thus the induction on p_0 proceeds, and we have completed the proof of Proposition A.1.

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