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A Two-Sided Discrete-Concave Market with Possibly Bounded Side Payments: An Approach by Discrete Convex Analysis

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Abstract

The marriage model due to Gale and Shapley and the assignment model due to Shapley and Shubik are standard in the theory of two-sided matching markets. We give a common generalization of these models by utilizing discrete concave functions and considering possibly bounded side payments. We show the existence of a pairwise stable outcome in our model. Our present model is a further natural extension of the model examined in our previous paper (Fujishige and Tamura [12]), and the proof of the existence of a pairwise stable outcome is even simpler than the previous one.

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1. Introduction

The marriage model due to Gale and Shapley [14] and the assignment model due to Shapley and Shubik [28] are standard in the theory of two-sided matching markets. The largest difference between these two models is that the former does not allow side payments or transferable utilities whereas the latter does (see Roth and Sotomayor [26]).

Since Gale and Shapley's paper a large number of variations and extensions have been proposed. Recently, the marriage model was extended to frameworks in combinatorial optimization. Fleiner [9] extended the marriage model to the framework of matroids, and Eguchi, Fujishige and Tamura [5] extended this formulation to a more general one in terms of discrete convex analysis which was developed by Murota [20, 21, 22]. Alkan and Gale [2] and Fleiner [10] also generalized the marriage model to another wide frameworks. The existence of stable matchings in these models are guaranteed.

For the other standard model, the assignment model, Kelso and Crawford [18] proposed a seminal one-to-many variation in which a payoff function of each worker is strictly increasing (not necessarily linear) in a side payment, and a payoff function of each firm satisfies gross substitutability and is linear in a side payment. They showed the existence of a stable outcome.

On the other hand, progress has been made toward unifying the marriage model and the assignment model. Crawford and Knoer [3] extended Gale and Shapley's deferred acceptance algorithm for the marriage model to the assignment model. Kaneko [17] formulated a general model that includes the two by means of characteristic functions, and proved the nonemptiness of the core. Roth and Sotomayor [27] proposed a general model that also encompasses both and investigated the lattice property for payoffs. Eriksson and Karlander [6] proposed a hybrid model of the marriage model and the assignment model. In the Eriksson-Karlander model, the set of agents is partitioned into two categories, one for "rigid" agents and the other for "flexible" agents. Rigid agents do not get side payments, that is, they behave like agents in the marriage model, while flexible agents behave like ones in the assignment model. Sotomayor [31] also further investigated this hybrid model and gave a non-constructive proof of the existence of a pairwise stable outcome. Fujishige and Tamura [12] proposed a generalization of the hybrid model due to Eriksson and Karlander [6] and Sotomayor [31] by utilizing $M^\natural$-concave functions which play a central role in discrete convex analysis.

The model in [12] motivates us to consider a more natural common generalization of the marriage model and the assignment model by utilizing discrete convex analysis. Our goal is to propose such a model which includes models due to Gale and Shapley [14], Shapley and Shubik [28], Eriksson and Karlander [6], Sotomayor [31], Fleiner [9], Eguchi et al. [5], and Fujishige and Tamura [12] as special cases, and to verify the existence of a pairwise stable outcome. The characteristic idea of our present model is to adopt a range of a side payment for each pair of agents instead of using the concept of rigid and flexible pairs. Our model can deal with rigidity and flexibility of pairs as ranges $[0, 0]$ and $(-\infty, +\infty)$ of side payments respectively as well as any ranges of side payments. This approach is more natural and adaptable than that adopting rigidity and flexibility. Furthermore, our proof for the existence of a pairwise stable outcome is simpler than that in our previous paper [12].

As we will discuss in Section 2, gross substitutability and $M^\natural$-concavity are equivalent for set functions. It is our contribution in contrast to the results of Kelso and Crawford [18] that the existence of pairwise stable outcome is preserved in a many-to-many variation with quasi-linear workers' payoff functions as well as its extensions with multi-units of labor time and possibly bounded side payments. Moreover, we give not only a general mathematical model but also a new concrete common generalization of the marriage and assignment models. We call it the assignment model with possibly bounded side payments, which is the simplest common generalization. It seems that this model has not been studied in the literature. The existence of a pairwise stable outcome of this model is a direct consequence of our main result.

The present paper is organized as follows. Section 2 explains $M^\natural$-concavity together with some examples and gives its nice properties and several useful lemmas from the viewpoint of mathematical economics. Section 3 describes our general model and two concepts of stability, namely "pairwise stability" and "pairwise strict stability," discusses relations between these two concepts, and gives our main theorem about the existence of pairwise stable outcomes. Proofs of preliminary lemmas and theorems are put in Section 6, and a proof of our main theorem is given in Section 5. Section 4 discusses relations between several existing models and our general model. In Section 5 we present an algorithm for finding a pairwise strictly stable outcome and prove its correctness, which shows our main theorem about the existence of a pairwise stable outcome. In Section 6 we give proofs of the lemma and theorems appearing in Section 3. Section 7 gives future work and open problems.

2. M$^\natural$-concavity

In this section we explain the concept of M$^\natural$-concave function, which plays a central role in discrete convex analysis (see [22] for details). Let E be a nonempty finite set, and let 0 be a new element not in E. We denote by $\mathbf{Z}$ the set of integers, and by $\mathbf{Z}^E$ the set of integral vectors $x = (x(e) \mid e \in E)$ indexed by E, where $x(e)$ denotes the e-component of vector x. Also, $\mathbf{R}$ and $\mathbf{R}^E$ denote the set of reals and of real vectors indexed by E, respectively. Let $\mathbf{0}$ and $\mathbf{1}$ be vectors of all zeros and all ones of an appropriate dimension. We define the positive support $\mathrm{supp}^+(x)$ and the negative support $\mathrm{supp}^-(x)$ of $x \in \mathbf{Z}^E$ by

$$\mathrm{supp}^+(x) = \{e \in E \mid x(e) > 0\}, \quad \mathrm{supp}^-(x) = \{e \in E \mid x(e) < 0\}.$$

For each $S \subseteq E$, we denote by χ_S the characteristic vector of S defined by: $\chi_S(e) = 1$ if $e \in S$ and $\chi_S(e) = 0$ otherwise, and write simply χ_e instead of $\chi_{\{e\}}$ for all $e \in E$. We also define χ_0 as the zero vector in $\mathbf{Z}^E$, where we assume $0 \notin E$. For $S \subseteq E$ and $x \in \mathbf{Z}^E$, let $x(S) = \sum_{e \in S} x(e)$. For a vector $p \in \mathbf{R}^E$ and a function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$, we define functions $\langle p, x \rangle$ and $f[p](x)$ in $x \in \mathbf{Z}^E$ by

$$\langle p, x \rangle = \sum_{e \in E} p(e)x(e), \qquad f[p](x) = f(x) + \langle p, x \rangle \qquad (\forall x \in \mathbf{Z}^E).$$

We also define arg max, the set of maximizers, of f on $U \subseteq \mathbf{Z}^E$ and the *effective domain* of f by

$$\arg\max\{f(y) \mid y \in U\} = \{x \in U \mid \forall y \in U : \ f(x) \geq f(y)\},$$
$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^E \mid f(x) > -\infty\}.$$

We abbreviate $\arg\max\{f(y) \mid y \in \mathbf{Z}^E\}$ to $\arg\max f$.

A function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-concave* (Murota [22] and Murota and Shioura [23]) if it satisfies

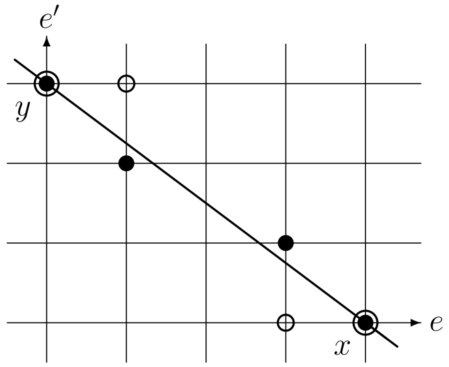

Figure 1: M$^\natural$-concavity for two dimensional case: the sum of function values of black points or that of white points is greater than or equal to that of x and y.

(M$^\natural$) $\forall x, y \in \operatorname{dom} f$, $\forall e \in \operatorname{supp}^+(x-y)$, $\exists e' \in \operatorname{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_e + \chi_{e'}) + f(y + \chi_e - \chi_{e'}).$$

((M$^\natural$) is denoted by ($-$M$^\natural$-EXC) in Murota [22].) Condition (M$^\natural$) says that the sum of the function values at two points does not decrease as the points symmetrically move one or two step closer to each other on the set of integral lattice points of $\mathbf{Z}^E$ (see Figure 1). This is a discrete analogue of the fact that for an ordinary concave function the sum of the function values at two points does not decrease as the points symmetrically move closer to each other on the straight line segment between the two points.

By the definition of M$^\natural$-concavity, if f is M$^\natural$-concave, then $f[p]$ is also M$^\natural$-concave for any $p \in \mathbf{R}^E$. Here are two simple examples of M$^\natural$-concave functions.

Example 1: For the independence family $\mathcal{I} \subseteq 2^E$ of a matroid on E and $w \in \mathbf{R}^E$, the function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ defined by

$$f(x) = \begin{cases} \sum_{e \in X} w(e) & \text{if } x = \chi_X \text{ for some } X \in \mathcal{I} \\ -\infty & \text{otherwise} \end{cases} \qquad (\forall x \in \mathbf{Z}^E)$$

is M$^\natural$-concave (see Murota [22]). ∎

Example 2: We call a nonempty family $\mathcal{T}$ of subsets of E a *laminar family* if $X \cap Y = \emptyset$, $X \subseteq Y$ or $Y \subseteq X$ holds for every $X, Y \in \mathcal{T}$. For a laminar family

$\mathcal{T}$ and a family of univariate concave functions $f_Y : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ indexed by $Y \in \mathcal{T}$, the function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ defined by

$$f(x) = \sum_{Y \in \mathcal{T}} f_Y\left(x(Y)\right) \qquad (\forall x \in \mathbf{Z}^E)$$

is M$^\natural$-concave if $\operatorname{dom} f \neq \emptyset$ (see Murota [22]). ∎

An M$^\natural$-concave function has nice features as a value function from the point of view of mathematical economics. For any M$^\natural$-concave function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$, there exists an ordinary concave function $\bar{f} : \mathbf{R}^E \to \mathbf{R} \cup \{-\infty\}$ such that $\bar{f}(x) = f(x)$ for all $x \in \mathbf{Z}^E$ (Murota [20]). That is, any M$^\natural$-concave function on $\mathbf{Z}^E$ has a concave extension on $\mathbf{R}^E$. An M$^\natural$-concave function f also satisfies submodularity (Murota and Shioura [24]): $f(x)+f(y) \geq f(x \wedge y)+f(x \vee y)$ for all $x, y \in \operatorname{dom} f$, where $x \wedge y$ and $x \vee y$ are the vectors whose e-components $(x \wedge y)(e)$ and $(x \vee y)(e)$ are, respectively, $\min\{x(e), y(e)\}$ and $\max\{x(e), y(e)\}$ for all $e \in E$.

An M$^\natural$-concave function satisfies the following two properties which are natural generalizations of the gross substitutability and single improvement property discussed in Kelso and Crawford [18] and Gul and Stacchetti [15].

(GS) For any $p, q \in \mathbf{R}^E$ and any $x \in \arg\max f[-p]$ such that $p \leq q$ and $\arg\max f[-q] \neq \emptyset$, there exists $y \in \arg\max f[-q]$ such that $y(e) \geq x(e)$ for all $e \in E$ with $p(e) = q(e)$.

(SI) For any $p \in \mathbf{R}^E$ and any $x, y \in \operatorname{dom} f$ with $f[-p](x) < f[-p](y)$,

$$f[-p](x) < \max_{e \in \operatorname{supp}^+(x-y) \cup \{0\}} \max_{e' \in \operatorname{supp}^-(x-y) \cup \{0\}} f[-p](x - \chi_e + \chi_{e'}).$$

Here E denotes the set of indivisible commodities, $p \in \mathbf{R}^E$ a price vector of commodities, $x \in \mathbf{Z}^E$ a consumption of commodities, and $f(x)$ a monetary valuation for x. The above conditions are interpreted as follows. Condition (GS) says that when each price increases or remains the same, the consumer wants a consumption such that the numbers of the commodities whose prices remain the same do not decrease. Condition (SI) guarantees that the consumer can bring consumption x closer to any better consumption y by changing the consumption of one or two commodities. The equivalence between gross substitutability and the single improvement condition for set functions was first pointed out by Gul and Stacchetti [15], and the equivalence between the single improvement condition and M$^\natural$-concavity for set functions was by Fujishige and Yang [13]. Moreover, M$^\natural$-concavity can be characterized by these properties or their extensions under a natural assumption (see Danilov, Koshevoy and Lang [4] and Murota and Tamura [25] for details).

Fujishige and Tamura [12] showed that an M$^\natural$-concave function satisfies the following properties.

(S1) Let $z_1, z_2 \in \mathbf{Z}^E$ be such that $z_1 \geq z_2$, $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$, and $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$. For any $x_1 \in \arg\max\{f(y) \mid y \leq z_1\}$, there exists x_2 such that

$$x_2 \in \arg\max\{f(y) \mid y \leq z_2\} \quad \text{and} \quad z_2 \wedge x_1 \leq x_2.$$

(S2) Let $z_1, z_2 \in \mathbf{Z}^E$ be such that $z_1 \geq z_2$, $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$, and $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$. For any $x_2 \in \arg\max\{f(y) \mid y \leq z_2\}$, there exists x_1 such that

$$x_1 \in \arg\max\{f(y) \mid y \leq z_1\} \quad \text{and} \quad z_2 \wedge x_1 \leq x_2.$$

Suppose that E denotes a set of workers, $y \in \mathbf{Z}^E$ a labor allocation representing labor times of the workers, $f(y)$ a valuation of a firm for labor allocation y, and $z_1, z_2 \in \mathbf{Z}^E$ vectors representing capacities of labor times. Property (S1) says that when each capacity decreases or remains the same, there exists an optimal labor allocation such that for every worker, if his/her original labor time is less than or equal to the new capacity, then the labor time increases or remains the same, and if the original labor time is greater than the new capacity, then the labor time becomes equal to the new capacity. On the other hand, (S2) says that when each capacity increases or remains the same, there exists an optimal labor allocation such that for every worker, if his/her original labor time is less than its original capacity, then the labor time decreases or remains the same. Hence, (S1) and (S2) imply that a choice function $C : \mathbf{Z}^E \to 2^{\mathrm{dom}\, f}$ defined by $C(z) = \arg\max\{f(y) \mid y \leq z\}$ satisfies "substitutability," where $2^{\mathrm{dom}\, f}$ denotes the set of all subsets of $\mathrm{dom}\, f$. In fact, if $\mathrm{dom}\, f \subseteq \{0,1\}^E$ then (S1) and (S2) are equivalent to conditions of substitutability in Sotomayor [30, Definition 4], and if C always gives a singleton (in this case (S1) and (S2) are equivalent), then (S1) and (S2) are equivalent to persistence (substitutability) in Alkan and Gale [2].

Farooq and Tamura [8] showed that $f : \{0,1\}^E \to \mathbf{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if $f[-p]$ satisfies (S1) for all $p \in \mathbf{R}^E$, and that f is M$^\natural$-concave if and only if $f[-p]$ satisfies (S2) for all $p \in \mathbf{R}^E$. Farooq and Shioura [7] extended these characterizations to the case where $\mathrm{dom}\, f$ is bounded.

The maximizers of an M$^\natural$-concave function have a good characterization as follows.

Theorem 2.1 (Murota [20, 21], Murota and Shioura [23]): *For an M$^\natural$-concave function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ and $x \in \mathrm{dom}\, f$, we have $x \in \arg\max f$ if and only if $f(x) \geq f(x - \chi_e + \chi_{e'})$ for all $e, e' \in \{0\} \cup E$.*

The set of all maximizers of an M$^\natural$-concave function is called a g-polymatroid in $\mathbf{Z}^E$ (see [11]), which is also called an M$^\natural$-convex set in [22]. M$^\natural$-convex sets have the following property.

Lemma 2.2 (Lemma 4.5 in Fujishige [11]): *Let B be an M$^\natural$-convex set. For any $x \in B$ and any distinct elements $e_1,\ e'_1,\ e_2,\ e'_2, \cdots, e_r, e'_r \in \{0\} \cup E$, if $x - \chi_{e_i} + \chi_{e'_i} \in B$ for all $i = 1, \cdots, r$ and $x - \chi_{e_i} + \chi_{e'_j} \notin B$ for all i, j with $i < j$, then $y = x - \sum_{i=1}^{r}(\chi_{e_i} - \chi_{e'_i}) \in B$.*

The following lemmas also show some basic properties of M$^\natural$-concave functions, which will be useful in Sections 5 and 6.

Lemma 2.3 (Fujishige and Tamura [12]): *Let $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function. For an element $e \in E$ let $z_1, z_2 \in (\mathbf{Z} \cup \{+\infty\})^E$ be vectors such that $z_1 = z_2 + \chi_e$, $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$, and $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$. Then, the following two statements hold:*

(a) *For each $x \in \arg\max\{f(y) \mid y \leq z_1\}$ there exists $e' \in \{0\} \cup E$ (possibly $e' = e$) such that*

$$x - \chi_e + \chi_{e'} \in \arg\max\{f(y) \mid y \leq z_2\}.$$

(b) *For each $x \in \arg\max\{f(y) \mid y \leq z_2\}$ there exists $e' \in \{0\} \cup E$ (possibly $e' = e$) such that*

$$x + \chi_e - \chi_{e'} \in \arg\max\{f(y) \mid y \leq z_1\}.$$

Lemma 2.3 says that when the capacity of the labor time of one worker decreases (or increases) by one, an optimal allocation can be obtained from the current optimal allocation by changing the labor times of at most two workers.

Lemma 2.4 (Fujishige and Tamura [12]): *For an* $M^\natural$*-concave function* $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ *and a vector* $z_2 \in (\mathbf{Z} \cup \{+\infty\})^E$ *suppose that* $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$*. For any* $x \in \arg\max\{f(y) \mid y \leq z_2\}$ *and any* $z_1 \in (\mathbf{Z} \cup \{+\infty\})^E$ *such that* (i) $z_1 \geq z_2$ *and* (ii) $x(e) = z_2(e) \Longrightarrow z_1(e) = z_2(e)$*, we have* $x \in \arg\max\{f(y) \mid y \leq z_1\}$*.*

Lemma 2.4 says that any capacity larger than the corresponding labor time can be made arbitrarily large without destroying the optimality of the given optimal labor allocation.

3. Model description

We consider a two-sided market consisting of disjoint sets P and Q of agents, in which an agent in P may be called a worker and one in Q a firm. Each worker $i \in P$ can supply multi-units of labor time, and each firm $j \in Q$ can employ workers with multi-units of labor time and pay a salary to worker i if j hires i. We assume possibly bounded side payments, i.e., each pair (i, j) may have lower and upper bounds on a salary per unit of labor time. We also assume that the valuation of each agent $k \in P \cup Q$ on labor allocations is described by a function in monetary terms. We will examine two concepts of stability, namely, *pairwise stability* and *pairwise strict stability*, in a market where the payoff function of each agent is quasi-linear. We will give precise definitions of the two concepts later.

First we describe our model mathematically. Let $E = P \times Q$, i.e., the set of all ordered pairs (i, j) of agents $i \in P$ and $j \in Q$. Also define $E_{(i)} = \{i\} \times Q$ for all $i \in P$ and $E_{(j)} = P \times \{j\}$ for all $j \in Q$. Denoting by $x(i, j)$ the number of units of labor time for which j hires i, we represent a labor allocation by vector $x = (x(i, j) \mid (i, j) \in E) \in \mathbf{Z}^E$. We express lower and upper bounds of salaries per unit of labor time by two vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$. For each $y \in \mathbf{R}^E$ and $k \in P \cup Q$, we denote by $y_{(k)}$ the restriction of y on $E_{(k)}$. For example, for a labor allocation $x \in \mathbf{Z}^E$, $x_{(k)}$ represents the labor allocation of agent k with respect to x. We assume that the valuation of each worker on a labor allocation is determined only by how many units of labor time he/she works in the firms, and that the valuation of each firm is determined only by how many units of labor time it hires the workers. That is, the value function f_k of each $k \in P \cup Q$ is defined on $E_{(k)}$ as $f_k : \mathbf{Z}^{E_{(k)}} \to \mathbf{R} \cup \{-\infty\}$. We assume that each value function f_k satisfies the following assumption:

(A) $\mathrm{dom}\, f_k$ is bounded and hereditary, and has $\mathbf{0}$ as the minimum point,

where heredity means that for any $y, y' \in \mathbf{Z}^{E_{(k)}}$, $\mathbf{0} \leq y' \leq y \in \mathrm{dom}\, f_k$ implies $y' \in \mathrm{dom}\, f_k$. The boundedness of effective domains implies that each value function is implicitly imposed on firm's budget constraint or worker's constraint on labor time. The heredity of effective domains implies that each agent can arbitrarily decrease related labor time (before contract) without any permission from the partner.

A vector $x \in \mathbf{Z}^E$ is called a *feasible allocation* if $x_{(k)} \in \mathrm{dom}\, f_k$ for all $k \in P \cup Q$, and a vector $s \in \mathbf{R}^E$ is called a *feasible salary vector* if $\underline{\pi}(i,j) \leq s(i,j) \leq \overline{\pi}(i,j)$ for all $(i,j) \in E$. We call a pair (x, s) of a feasible allocation $x \in \mathbf{Z}^E$ and a feasible salary vector $s \in \mathbf{R}^E$ an *outcome*.

The payoff functions of agents on outcomes are defined as follows: the payoff of worker $i \in P$ on (x, s) is given by $f_i[+s_{(i)}](x_{(i)}) = f_i(x_{(i)}) + \sum_{j \in Q} s(i,j)x(i,j)$, i.e., the value of i on x plus the income from the firms that hire worker i, and the payoff of firm $j \in Q$ on (x, s) is given by $f_j[-s_{(j)}](x_{(j)}) = f_j(x_{(j)}) - \sum_{i \in P} s(i,j)x(i,j)$, i.e., the value of firm j on x minus the payments to the workers that firm j hires.

An outcome (x, s) is said to satisfy *incentive constraints* if each agent has no incentive to unilaterally decrease the current units x of labor time at the current salary agreements s, that is, if it satisfies

$$f_i[+s_{(i)}](x_{(i)}) = \max\{f_i[+s_{(i)}](y) \mid y \leq x_{(i)}\} \qquad (\forall i \in P), \tag{3.1}$$

$$f_j[-s_{(j)}](x_{(j)}) = \max\{f_j[-s_{(j)}](y) \mid y \leq x_{(j)}\} \qquad (\forall j \in Q). \tag{3.2}$$

Next we define pairwise (un)stability formally. For any $s \in \mathbf{R}^E$, $\alpha \in \mathbf{R}$, $i \in P$, and $j \in Q$, let $(s_{(i)}^{-j}, \alpha)$ be defined as the vector obtained from $s_{(i)}$ by replacing its (i,j)-component by α, and $(s_{(j)}^{-i}, \alpha)$ be similarly defined. We say that an outcome (x, s) is *pairwise unstable* if it does not satisfy incentive constraints or there exist

$i \in P$, $j \in Q$, $\alpha \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E_{(i)}}$ and $y'' \in \mathbf{Z}^{E_{(j)}}$ such that

$$f_i[+s_{(i)}](x_{(i)}) < f_i[+(s_{(i)}^{-j}, \alpha)](y'), \tag{3.3}$$

$$y'(i,j') \leq x(i,j') \qquad (\forall j' \in Q \setminus \{j\}), \tag{3.4}$$

$$f_j[-s_{(j)}](x_{(j)}) < f_j[-(s_{(j)}^{-i}, \alpha)](y''), \tag{3.5}$$

$$y''(i',j) \leq x(i',j) \qquad (\forall i' \in P \setminus \{i\}), \tag{3.6}$$

$$y'(i,j) = y''(i,j). \tag{3.7}$$

For some feasible salary α between i and j, conditions (3.3) and (3.4) say that worker i can strictly increase his/her payoff by changing the current units of labor time with j without increasing units of labor time with other firms, and (3.5) and (3.6) say that firm j can also strictly increase its payoff by changing the current units of labor time with i without increasing units of labor time with other workers. Moreover, condition (3.7) requires that i and j agree on units of labor time between them. An outcome (x, s) is called *pairwise stable* if it is not pairwise unstable.

We also consider a stronger pairwise stability, which might be regarded as artificial but plays an important role in showing the existence of a pairwise stable outcome. We say that an outcome (x, s) is *pairwise quasi-unstable* if it does not satisfy incentive constraints or there exist $i \in P$, $j \in Q$, $\alpha \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E_{(i)}}$ and $y'' \in \mathbf{Z}^{E_{(j)}}$ satisfying (3.3)$\sim$(3.6) (but not necessarily (3.7)). Without requirement (3.7), conditions (3.3)$\sim$(3.6) mean that i and j have an incentive to deviate from (x, s) without consent to possible labor time between them. An outcome (x, s) is called *pairwise strictly stable* if it is not pairwise quasi-unstable. Since a pairwise unstable outcome is pairwise quasi-unstable, a pairwise strictly stable outcome is pairwise stable. An outcome (x, s) is pairwise strictly stable if and only if (3.1) and (3.2) hold and for all $i \in P$, $j \in Q$ and $\alpha \in \mathbf{R}$ with $\underline{\pi}(i,j) \leq \alpha \leq \overline{\pi}(i,j)$,

$$f_i[+s_{(i)}](x_{(i)}) \geq \max\{f_i[+(s_{(i)}^{-j}, \alpha)](y) \mid y(i,j') \leq x(i,j'),\ \forall j' \neq j\}, \tag{3.8}$$

or

$$f_j[-s_{(j)}](x_{(j)}) \geq \max\{f_j[-(s_{(j)}^{-i}, \alpha)](y) \mid y(i',j) \leq x(i',j),\ \forall i' \neq i\}. \tag{3.9}$$

Conditions (3.8) and (3.9) is equivalent to that for each pair $(i, j) \in E$ and each feasible salary between them, both i and j cannot strictly increase their payoffs without increasing labor times with other partners.

The next example illustrates a gap between pairwise stability and pairwise strict stability.

Example 3: Let us consider the case where $E = \{(i,j)\}$ (a singleton),

$$f_i(x) = \begin{cases} x & \text{if } x \in \{0,1,2\} \\ -\infty & \text{otherwise} \end{cases} \qquad (\forall x \in \mathbf{Z}),$$

$$f_j(x) = \begin{cases} x & \text{if } x \in \{0,1,2,3\} \\ -\infty & \text{otherwise} \end{cases} \qquad (\forall x \in \mathbf{Z}),$$

and $\underline{\pi}(i,j) = 0$ and $\overline{\pi}(i,j) = 1/4$. In this case, an outcome $(x,s) = (2,0)$ is not pairwise strictly stable, because $f_i(2) < f_i[+\epsilon](2)$ and $f_j(2) < f_j[-\epsilon](3)$ for all $\epsilon \in (0, 1/4]$. However, the outcome is pairwise stable. On the other hand, an outcome $(x,s) = (2, 1/4)$ is pairwise strictly stable (and hence, pairwise stable). ■

The concept of a pairwise strictly stable outcome may be regarded as artificial but, as can be seen from Lemma 3.1, pairwise strict stability coincides with pairwise stability in some useful special cases: (i) salaries are constant, and (ii) each worker-firm pair can be matched at most once. These two cases comprise many known existing models such as the marriage model due to Gale and Shapley [14], the assignment model due to Shapley and Shubik [28], and an extension [5] of the marriage model with M$^\natural$-concave value functions on $\mathbf{Z}^E$ (also see the assignment model with possibly bounded side payments to be considered in Section 4).

Lemma 3.1: *If f_k $(k \in P \cup Q)$ are M$^\natural$-concave functions satisfying* (A) *and if one of the following conditions*

(i) $\underline{\pi} = \overline{\pi}$,

(ii) $\operatorname{dom} f_k \subseteq \{0,1\}^{E_{(k)}}$ *for all* $k \in P \cup Q$,

(iii) *there exists a vector* $u \in \mathbf{Z}^E$ *such that for each* $k \in P \cup Q$ *we have* $\operatorname{dom} f_k = \{y \in \mathbf{Z}^{E_{(k)}} \mid \mathbf{0} \le y \le u_{(k)}\}$ *and* f_k *is linear over* $\operatorname{dom} f_k$

holds, then any pairwise stable outcome is pairwise strictly stable.

Proof. See Section 6.1. ■

Although the concepts of pairwise stability and pairwise strict stability are different in our general model, we have the following theorem.

Theorem 3.2: *Assume that f_k is an M$^\natural$-concave function satisfying* (A) *for each $k \in P \cup Q$. If (x, s) is a pairwise stable outcome in our model, then there exists a feasible salary vector s' such that (x, s') is a pairwise strictly stable outcome.*

Proof. See Section 6.3. ∎

Hence, if we call a feasible allocation x *pairwise* (*strictly*) *stable* if there exists a feasible salary vector s such that (x, s) is pairwise (strictly) stable, then there is no gap between the two concepts of pairwise stability and pairwise strict stability in terms of allocations.

The following is our main theorem that for M$^\natural$-concave value functions there exists a pairwise strictly stable outcome and hence a pairwise stable outcome in our model. (It should be noted that due to Theorem 3.2, there exists a pairwise stable outcome if and only if there exists a pairwise strictly stable outcome in our model.)

Theorem 3.3: *For M$^\natural$-concave functions f_k $(k \in P \cup Q)$ satisfying* (A) *and for vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$, there exists a pairwise strictly stable outcome (x, s), and hence, there exists a pairwise stable outcome. Moreover, if f_k $(k \in P \cup Q)$ are integer-valued on their effective domains, $\underline{\pi} \in (\mathbf{Z} \cup \{-\infty\})^E$, and $\overline{\pi} \in (\mathbf{Z} \cup \{+\infty\})^E$, then the above s can be chosen from $\mathbf{Z}^E$.*

To show Theorem 3.3, we give an alternative characterization of a pairwise strictly stable outcome. (Note that by Theorem 3.2, the following theorem also gives a characterization of a pairwise stable allocation.)

Theorem 3.4: *Assume that f_k is an M$^\natural$-concave function satisfying* (A) *for each $k \in P \cup Q$. Let x be a feasible allocation. There exists a feasible salary vector s forming a pairwise strictly stable outcome (x, s) if and only if there exist $p \in \mathbf{R}^E$, $z_P = (z_{(i)} \mid i \in P) \in (\mathbf{Z} \cup \{+\infty\})^E$, and $z_Q = (z_{(j)} \mid j \in Q) \in (\mathbf{Z} \cup \{+\infty\})^E$ such that*

$$x_{(i)} \in \arg\max\{f_i[+p_{(i)}](y) \mid y \leq z_{(i)}\} \qquad (\forall i \in P), \tag{3.10}$$

$$x_{(j)} \in \arg\max\{f_j[-p_{(j)}](y) \mid y \leq z_{(j)}\} \qquad (\forall j \in Q), \tag{3.11}$$

$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{3.12}$$

$$e \in E,\ z_P(e) < +\infty \ \Rightarrow\ p(e) = \underline{\pi}(e),\ z_Q(e) = +\infty, \tag{3.13}$$

$$e \in E,\ z_Q(e) < +\infty \ \Rightarrow\ p(e) = \overline{\pi}(e),\ z_P(e) = +\infty. \tag{3.14}$$

Moreover, for any x, p, z_P, and z_Q satisfying the above conditions, (x, p) is a pairwise strictly stable outcome.

Proof. See Section 6.2. ∎

We note that M$^\natural$-concavity in Theorem 3.4 is not required to show the if part, while it is required to show the only-if part.

Consider the case where $z_P(i,j) = +\infty$ and $z_Q(i,j) < +\infty$. Condition (3.10) implies that worker i has no incentive to increase $x(i,j)$ at the current salary. If firm j could strictly increase its payoff by increasing $x(i,j)$ at the current salary, then j would try to increase the salary of worker i to give worker i incentive to increase $x(i,j)$. Condition (3.14), however, implies that firm j is in an extreme situation where firm j cannot increase the current i's salary any more, i.e., $p(i,j) = \overline{\pi}(i,j)$, and that firm j must give up increasing $x(i,j)$ (and hence $z_Q(i,j)$ is put to be a finite value). Analogously, when $z_P(i,j) < +\infty$ and $z_Q(i,j) = +\infty$, Conditions (3.11) and (3.13) imply that if worker i must give up increasing $x(i,j)$, then firm j has no incentive to increase $x(i,j)$ at the current salary and i is in an extreme situation where worker i cannot decrease his/her current salary to give firm j incentive to hire more units of labor time $x(i,j)$. It is of importance that (3.10)∼(3.14) give a decentralized characterization of a pairwise (strictly) stable allocation. That is, given appropriate vectors p, z_P and z_Q, a pairwise (strictly) stable allocation can be obtained by individually maximizing each agent's payoff.

To prove our main theorem (Theorem 3.3) in Section 5, it is convenient to use two aggregated M$^\natural$-concave functions on $\mathbf{Z}^E$, one for each of P and Q. Let us define f_P and f_Q by

$$f_P(x) = \sum_{i \in P} f_i(x_{(i)}), \quad f_Q(x) = \sum_{j \in Q} f_j(x_{(j)}) \qquad (\forall x \in \mathbf{Z}^E). \tag{3.15}$$

Since $E_{(i)}$ and $E_{(i')}$ are disjoint for all $i, i' \in P$ with $i \neq i'$, function f_P is M$^\natural$-concave if all functions f_i $(i \in P)$ are M$^\natural$-concave. Similarly, f_Q is M$^\natural$-concave if all functions f_j $(j \in Q)$ are. Moreover, the following lemma obviously holds.

Lemma 3.5: *Condition* (3.10) *holds if and only if* $x \in \arg\max\{f_P[+p](y) \mid y \leq z_P\}$. *Condition* (3.11) *holds if and only if* $x \in \arg\max\{f_Q[-p](y) \mid y \leq z_Q\}$.

Furthermore, Assumption (A) is rewritten in terms of f_P and f_Q as:

(A′) Effective domains $\mathrm{dom}\, f_P$ and $\mathrm{dom}\, f_Q$ are bounded and hereditary, and have the common minimum point $\mathbf{0} \in \mathbf{Z}^E$.

By Theorem 3.4 and Lemma 3.5, Theorem 3.3 is a direct consequence of the following theorem.

Theorem 3.6: *For $M^\natural$-concave functions $f_P, f_Q : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ satisfying* (A′) *and for vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$, there exist $x \in \mathbf{Z}^E$, $p \in \mathbf{R}^E$, and $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ such that*

$$x \in \arg\max\{f_P[+p](y) \mid y \leq z_P\}, \tag{3.16}$$

$$x \in \arg\max\{f_Q[-p](y) \mid y \leq z_Q\}, \tag{3.17}$$

$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{3.18}$$

$$e \in E,\ z_P(e) < +\infty \ \Rightarrow\ p(e) = \underline{\pi}(e),\ z_Q(e) = +\infty, \tag{3.19}$$

$$e \in E,\ z_Q(e) < +\infty \ \Rightarrow\ p(e) = \overline{\pi}(e),\ z_P(e) = +\infty. \tag{3.20}$$

Moreover, if f_P and f_Q are integer-valued on their effective domains, $\underline{\pi} \in (\mathbf{Z} \cup \{-\infty\})^E$, and $\overline{\pi} \in (\mathbf{Z} \cup \{+\infty\})^E$, then the above p can be chosen from $\mathbf{Z}^E$.

We also say that a pair (x, p) of $x \in \mathbf{Z}^E$ and $p \in \mathbf{R}^E$ is a *pairwise strictly stable outcome* if there exist $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ satisfying (3.16)∼(3.20).

In Section 5 we will give an algorithm for finding a pairwise strictly stable outcome (x, p) and prove its validity, which will complete the proof of Theorem 3.6 and hence Theorem 3.3.

Remark 1: We briefly discuss a time scheduling problem of a feasible labor allocation. A solution of this problem can be given by a famous result on graph coloring, namely, "any bipartite graph can be edge-colorable with the maximum degree colors." That is, given a feasible labor allocation $x \in \mathbf{Z}^E$, if workers hired by firm j can simultaneously work at j for every $j \in Q$, then there exists a scheduling of the feasible labor allocation within time horizon $\max_{i \in P}\{\sum_{j \in Q} x(i,j)\}$; also, if each firm can join at most one worker at each unit time, then there exists a scheduling within time horizon $\max\left\{\max_{i \in P}\{\sum_{j \in Q} x(i,j)\}, \max_{j \in Q}\{\sum_{i \in P} x(i,j)\}\right\}$. Here, for simplicity we neglect time required for moving from one firm to another.

4. Related models

In this section we discuss models that are closely related to our model.

4.1. Marriage model and assignment model

We briefly explain that our model includes the marriage model due to Gale and Shapley [14] and the assignment model due to Shapley and Shubik [28] as special cases. In these models, we are given pairs $(a_{ij}, b_{ij}) \in (\mathbf{R} \cup \{-\infty\})^2$ for all $(i, j) \in E = P \times Q$. Here, in the assignment model a_{ij} and b_{ij} are interpreted as profits of i and j when i and j are matched, while in the marriage model a_{ij} and b_{ij} define preferences as: $i \in P$ *prefers* j_1 to j_2 if $a_{ij_1} > a_{ij_2}$, and i is *indifferent* between j_1 and j_2 if $a_{ij_1} = a_{ij_2}$ (similarly, a preference of $j \in Q$ over P is defined by $\{b_{ij} \mid i \in P\}$). We assume that $a_{ij} > 0$ if j is *acceptable* to i, and $a_{ij} = -\infty$ otherwise, and $b_{ij} > 0$ if i is *acceptable* to j, and $b_{ij} = -\infty$ otherwise. A *matching* is a subset of E such that every agent appears at most once. Given a matching X, $i \in P$ (respectively $j \in Q$) is called *unmatched in* X if there exists no $j \in Q$ (resp. $i \in P$) with $(i, j) \in X$. In the marriage model, a matching X is called pairwise stable if there exist $q \in \mathbf{R}^P$ and $r \in \mathbf{R}^Q$ such that

(m1) $q_i = a_{ij}$ and $r_j = b_{ij}$ for all $(i, j) \in X$,

(m2) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X,

(m3) $q_i \geq a_{ij}$ or $r_j \geq b_{ij}$ for all $(i, j) \in E$.

In the assignment model, an outcome which is a triple $(q, r; X)$ consisting of payoff vectors $q = (q_i \mid i \in P) \in \mathbf{R}^P$, $r = (r_j \mid j \in Q) \in \mathbf{R}^Q$, and a subset $X \subseteq E$, is called pairwise stable if

(a1) X is a matching,

(a2) $q_i + r_j = a_{ij} + b_{ij}$ for all $(i, j) \in X$,

(a3) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X,

(a4) $q_i + r_j \geq a_{ij} + b_{ij}$ for all $(i, j) \in E$.

Define functions f_i for all $i \in P$ and f_j for all $j \in Q$ by

$$f_i(x) = \begin{cases} a_{ij} & \text{if } x = \chi_{(i,j)} \text{ for some } j \in Q \\ 0 & \text{if } x = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases} \qquad (\forall x \in \mathbf{Z}^{E_{(i)}}), \qquad (4.1)$$

$$f_j(x) = \begin{cases} b_{ij} & \text{if } x = \chi_{(i,j)} \text{ for some } i \in P \\ 0 & \text{if } x = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases} \qquad (\forall x \in \mathbf{Z}^{E_{(j)}}). \qquad (4.2)$$

It can easily be shown that the above functions are M$^\natural$-concave. We can show that, by putting $\underline{\pi} = \overline{\pi} = \mathbf{0}$, pairwise stability in our model coincides with pairwise stability in the marriage model for functions defined by (4.1) and (4.2). On the other hand, by putting $\underline{\pi} = (-\infty, \cdots, -\infty)$ and $\overline{\pi} = (+\infty, \cdots, +\infty)$, pairwise stability in our model coincides with pairwise stability in the assignment model for these functions. Furthermore, by Lemma 3.1, in these special cases, pairwise strict stability is identical with pairwise stability.

4.2. The assignment model with possibly bounded side payments

In the assignment model, for each $(i, j) \in X$, $s_{ij} = q_i - a_{ij} = b_{ij} - r_j$ denotes a transfer (or a side payment) from j to i. In a labor allocation case, it would not be practical to consider that a firm can pay an arbitrarily large amount of money to a worker as a salary or that a worker receives a negative salary (of arbitrary large absolute value). Hence we introduce possible bounds on side payments like in our general model.

Let us consider an extension of the assignment model in which, given two vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$, a transfer s_{ij} from j to i is bounded as $\underline{\pi}_{ij} \leq s_{ij} \leq \overline{\pi}_{ij}$ for all $(i, j) \in E$. We say that an outcome $(q, r; X)$ is pairwise stable if

(b1) X is a matching,

(b2) $q_i = a_{ij} + s_{ij}$, $r_j = b_{ij} - s_{ij}$, and $\underline{\pi}_{ij} \leq s_{ij} \leq \overline{\pi}_{ij}$ for all $(i, j) \in X$,

(b3) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X, (b4) $q_i \geq a_{ij} + \alpha$ or $r_j \geq b_{ij} - \alpha$ for all $(i,j) \in E$ and α with $\underline{\pi}_{ij} \leq \alpha \leq \overline{\pi}_{ij}$.

We call this extended model *the assignment model with possibly bounded side payments*. Obviously, if $\underline{\pi} = \overline{\pi} = \mathbf{0}$, then pairwise stability of this model coincides with pairwise stability of the marriage model. Furthermore, we can easily show that if $\underline{\pi} = (-\infty, \cdots, -\infty)$ and $\overline{\pi} = (+\infty, \cdots, +\infty)$, then pairwise stability in this model coincides with pairwise stability in the assignment model. Hence, the assignment model with possibly bounded side payments is a common generalization of the marriage and the assignment model. Even though the present model is the simplest common generalization, it seems that it has not been studied in the literature on the two-sided matching market.

By defining value functions of agents by (4.1) and (4.2), Theorem 3.3 and Lemma 3.1 immediately imply the existence of a pairwise stable outcome in the assignment model with possibly bounded side payments. We remark that the central assignment model due to Kaneko [17] also includes the assignment model with possibly bounded side payments but not a many-to-many variation whose value functions are defined by

$$f_i(x) = \begin{cases} \sum_{j \in Q} a_{ij} x_{ij} & \text{if } x \in \{0,1\}^{E_{(i)}}, \sum_{j \in Q} x_{ij} \leq \lambda_i \\ -\infty & \text{otherwise} \end{cases} \quad (\forall x \in \mathbf{Z}^{E_{(i)}}) \quad (4.3)$$

for each $i \in P$ and

$$f_j(x) = \begin{cases} \sum_{i \in P} b_{ij} x_{ij} & \text{if } x \in \{0,1\}^{E_{(j)}}, \sum_{i \in P} x_{ij} \leq \mu_j \\ -\infty & \text{otherwise} \end{cases} \quad (\forall x \in \mathbf{Z}^{E_{(j)}}) \quad (4.4)$$

for each $j \in Q$, where λ_i and μ_j denote capacities on labor times of agents. As is seen in Example 2, the functions defined by (4.3) and (4.4) are M$^\natural$-concave. Hence, our model also includes the many-to-many variation of the assignment model with possibly bounded side payments.

4.3. A labor allocation model with several categories of workers

We consider a labor allocation model without bounds on side payments in which each worker can supply one unit of labor-time and each firm can employ several workers, i.e., a one-to-many case. We further assume that there are several categories of workers, e.g., engineers, cashiers, secretaries, and so on. Mathematically, the set P of workers is partitioned into $P^1, P^2, \cdots, P^n$. Each firm j can employ at most μ_j^t workers of category $t \in \{1, 2, \cdots, n\}$ and at most μ_j workers in total. It seems that existing models in the literature cannot deal with such a situation even if a valuation of each firm j on each category t can be described by a linear function $f_j^t : \mathbf{Z}^{P^t \times \{j\}} \to \mathbf{R} \cup \{-\infty\}$ defined in the same way as (4.4), because the effective domain of a value function of a firm is not a simplex. M$^\natural$-concavity enables us to deal with such a situation.

For any $x \in \mathbf{Z}^E$, $t \in \{1, 2, \cdots, n\}$, and $j \in Q$ let $x_{(j)}^{(t)}$ denote the restriction of x on $P^t \times \{j\}$ and δ_j be the function defined by

$$\delta_j(x_{(j)}) = \begin{cases} 0 & \text{if } x_{(j)} \in \{0,1\}^{E_{(j)}}, \sum_{i \in P} x_{ij} \leq \mu_j \\ -\infty & \text{otherwise.} \end{cases}$$

Then define functions f_j $(j \in Q)$ by

$$f_j(x_{(j)}) = \sum_{t=1}^{n} f_j^t(x_{(j)}^{(t)}) + \delta_j(x_{(j)}) \qquad (\forall x \in \mathbf{Z}^E).$$

Here each f_j is M$^\natural$-concave as shown in Example 2, and gives an appropriate valuation of j satisfying its total capacity of workers.

By the discussion in Section 2, the model due to Kelso and Crawford [18] includes the one-to-many labor allocation model with several categories of workers without bounds on side payments, where a payoff function of each firm satisfies gross substitutability and is linear in salary and a payoff function of each worker is strictly increasing (not necessarily linear). On the other hand, our model can deal with the many-to-many variation in which a payoff function of each agent satisfies gross substitutability and is linear in salary, and furthermore, its extension with multiplicity of units of labor time and with possibly bounded side payments. This is one of the merits of our model in contrast to the seminal model by Kelso and Crawford [18].

While the above-mentioned model is an extension of the assignment model, similar extensions of the marriage model have been discussed in [1, 5, 9]. Our general model also includes these models as special cases.

4.4. Hybrid models

Eriksson and Karlander [6] and Sotomayor [31] proposed a hybridization of the marriage and assignment models. Their idea is to partition agents into two categories: rigid agents and flexible agents. Rigid agents do not get side payments, that is, they behave like agents in the marriage model, while flexible agents behave like ones in the assignment model. Fujishige and Tamura [12] generalized these models by using M$^\natural$-concave functions. In their model, two M$^\natural$-concave functions $f_P, f_Q : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ satisfying (A$'$), and an arbitrary partition (F, R) of E are given. For a vector d on E and $S \subseteq E$, let $d|_S$ denote the restriction of d on S. A vector $x \in \mathrm{dom}\, f_P \cap \mathrm{dom}\, f_Q$ is called an *$f_P f_Q$-pairwise stable solution with respect to* (F, R) if there exist $p \in \mathbf{R}^E$, disjoint subsets R_P and R_Q of R, $\hat{z}_P \in \mathbf{Z}^{R_P}$, and $\hat{z}_Q \in \mathbf{Z}^{R_Q}$ such that

$$p|_R = \mathbf{0}, \tag{4.5}$$

$$x \in \arg\max\{f_P[+p](y) \mid y|_{R_P} \leq \hat{z}_P\}, \tag{4.6}$$

$$x \in \arg\max\{f_Q[-p](y) \mid y|_{R_Q} \leq \hat{z}_Q\}. \tag{4.7}$$

We can show that $f_P f_Q$-pairwise stability is equivalent to our pairwise strict stability in the case where $\underline{\pi}(e) = \overline{\pi}(e) = 0$ for all $e \in R$, and $\underline{\pi}(e) = -\infty$ and $\overline{\pi}(e) = +\infty$ for all $e \in F$. Thus, Theorem 3.6 implies the existence of an $f_P f_Q$-pairwise stable outcome in the hybrid model in [12]. This means that our model also includes many existing models (see [12] for details).

5. An algorithm for finding a pairwise strictly stable outcome

We assume that given M$^\natural$-concave functions $f_P, f_Q : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ satisfy Assumption (A$'$). The problem of finding a pairwise strictly stable outcome is rewritten as that of finding $x_P, x_Q \in \mathbf{Z}^E$, $p \in \mathbf{R}^E$ and $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ such that

$$x_P = x_Q, \tag{5.1}$$

$$x_P \in \arg\max\{f_P[+p](y) \mid y \leq z_P\}, \tag{5.2}$$

$$x_Q \in \arg\max\{f_Q[-p](y) \mid y \leq z_Q\}, \tag{5.3}$$

$$\underline{\pi} \le p \le \overline{\pi}, \tag{5.4}$$

$$e \in E,\ z_P(e) < +\infty \Rightarrow p(e) = \underline{\pi}(e),\ z_Q(e) = +\infty, \tag{5.5}$$

$$e \in E,\ z_Q(e) < +\infty \Rightarrow p(e) = \overline{\pi}(e),\ z_P(e) = +\infty. \tag{5.6}$$

In this section, we give an algorithm that finds x_P, x_Q, p, z_P, and z_Q satisfying (5.1)∼(5.6). The algorithm may be recognized as one of auction algorithms. Roughly speaking, its strategy is to initially put p as large as possible so that there exist x_P, x_Q, z_P and z_Q satisfying (5.2)∼(5.6) and several extra conditions such as $x_Q \le x_P$, and then to monotonically decrease p preserving these conditions so that (5.1) is eventually satisfied. A characteristic feature of the algorithm is to use a sophisticated procedure for decreasing p, due to the technique of network flow algorithms (see (Case 2) below). On the other hand, when specialized to a marriage model, the algorithm can find a pairwise stable matching of the marriage model. This means that the algorithm also retains an essence of the deferred acceptance algorithm of Gale and Shapley [14]. The deferred acceptance algorithm is generalized as a procedure of updating z_P and z_Q, which relies on Lemmas 2.3 and 2.4 (see (Case 1) below).

Now, we describe our algorithm. Initially, we put p as

$$p(e) := \begin{cases} \overline{\pi}(e) & \text{if } \overline{\pi}(e) < +\infty \\ b & \text{otherwise} \end{cases} \qquad (\forall e \in E),$$

where b is a sufficiently large positive integer to be specified later. Furthermore, put $z_P := (+\infty, \cdots, +\infty)$ and choose any $x_P \in \arg\max f_P[+p]$. Obviously, (5.2), (5.4), and (5.5) are satisfied. We put z_Q as

$$z_Q(e) := \begin{cases} x_P(e) & \text{if } \overline{\pi}(e) < +\infty \\ +\infty & \text{otherwise} \end{cases} \qquad (\forall e \in E),$$

and choose an x_Q satisfying (5.3). Condition (5.6) evidently holds. Moreover, by setting b to be a large enough integer so that $x_Q(e) = 0$ for all $e \in E$ with $\overline{\pi}(e) = +\infty$, we have

$$x_Q(e) \le x_P(e) \qquad (\forall e \in E). \tag{5.7}$$

From Assumption (A′), such a b exists. By Lemma 2.4, (5.3) is preserved even if $z_Q(e)$ is set to $+\infty$ for every $e \in E$ with $p(e) = \overline{\pi}(e)$ and $x_Q(e) < x_P(e)$. Thus we can assume that the following condition is satisfied:

$$e \in E,\ z_Q(e) < +\infty \quad \Longrightarrow \quad x_Q(e) = x_P(e) = z_Q(e). \tag{5.8}$$

Our aim is to modify vectors x_P, x_Q, p, z_P, and z_Q preserving (5.2)∼(5.8) and eventually to attain (5.1).

We now assume that we are given vectors x_P, x_Q, p, z_P and z_Q satisfying (5.2)∼(5.8) but not (5.1). Let L and U be subsets of E defined by

$$L = \{e \in E \mid p(e) = \underline{\pi}(e)\}, \tag{5.9}$$

$$U = \{e \in E \mid z_Q(e) < +\infty\}. \tag{5.10}$$

It follows from (5.6) and (5.8) that for all $e \in U$ we have $p(e) = \overline{\pi}(e)$ and $x_Q(e) = x_P(e) = z_Q(e)$. Note that L and U may have a common element e with $\underline{\pi}(e) = \overline{\pi}(e)$.

We divide our argument into two parts that treat: (Case 1) there exists $e \in L$ with $x_Q(e) < x_P(e)$; (Case 2) the other case.

In (Case 1), we will modify x_P, x_Q, z_P and z_Q while keeping p the same. Let e be an element of L with $x_Q(e) < x_P(e)$. From (5.8), we have $z_Q(e) = +\infty$, and hence, we can assume $z_P(e) = x_P(e)$ while preserving (5.2) and (5.5). We replace $z_P(e)$ by $x_P(e) - 1$. By (a) of Lemma 2.3, there exists $e' \in \{0\} \cup E \setminus \{e\}$ such that $x_P := x_P - \chi_e + \chi_{e'}$ satisfies (5.2) for the modified z_P. If $e' = 0$ or $z_Q(e') = +\infty$ then x_Q and z_Q are kept the same, and Conditions (5.2)∼(5.8) are preserved. In the case where $z_Q(e') < +\infty$ (i.e., $e' \in U$), we modify x_Q and z_Q as follows. By (5.8), for the updated x_P we have $x_P(e') = x_Q(e') + 1$. Thus, (b) of Lemma 2.3 guarantees the existence of $e'' \in E$, which may coincide with e', such that $x_Q := x_Q + \chi_{e'} - \chi_{e''}$ and $z_Q := z_Q + \chi_{e'}$ satisfy (5.3). At this point, (5.2)∼(5.7) are satisfied. If $e' \neq e''$, then we have $z_Q(e') = x_Q(e') = x_P(e')$, which implies (5.8) for e'. In order to ensure (5.8) for e'', we replace $z_Q(e'')$ by $+\infty$ if $e'' \in U$. Since we have $x_Q(e'') < x_P(e'')$, this modification does not destroy (5.3), by Lemma 2.4. Hence, in (Case 1), the modified vectors satisfy all the required conditions.

Next we consider (Case 2), where $x_Q(e) = x_P(e)$ for all $e \in L$. In this case, we modify p as well as x_P, x_Q, and z_Q (while we keep z_P the same). The procedure given below is based on a successive shortest path algorithm for finding a maximizer of the sum of two M$^\natural$-concave functions (Moriguchi and Murota [19], also see Iwata et al. [16]). We deal with two functions defined by

$$\begin{aligned} f_P^{\leqq}(y) &= \begin{cases} f_P(y) & \text{if } y \leq z_P \\ -\infty & \text{otherwise} \end{cases} \qquad (\forall y \in \mathbf{Z}^E), \\ f_Q^{\leqq}(y) &= \begin{cases} f_Q(y) & \text{if } y \leq z_Q \\ -\infty & \text{otherwise} \end{cases} \qquad (\forall y \in \mathbf{Z}^E). \end{aligned} \tag{5.11}$$

Obviously, $f_P^{\leqq}$ and $f_Q^{\leqq}$ are also M$^\natural$-concave, and $x_P \in \arg\max f_P^{\leqq}[+p]$ and $x_Q \in \arg\max f_Q^{\leqq}[-p]$ hold.

We construct a directed graph $G = (\{0\} \cup E, A)$ and an arc length function $\ell : A \to \mathbf{R}$ as follows. Arc set A consists of two disjoint parts:

$$\begin{aligned} A_P &= \{(e, e') \mid e, e' \in \{0\} \cup E,\ e \neq e',\ x_P - \chi_e + \chi_{e'} \in \operatorname{dom} f_P^{\leqq}\}, \\ A_Q &= \{(e, e') \mid e, e' \in \{0\} \cup E,\ e \neq e',\ x_Q + \chi_e - \chi_{e'} \in \operatorname{dom} f_Q^{\leqq}\}, \end{aligned} \tag{5.12}$$

and $\ell \in \mathbf{R}^A$ is defined by

$$\ell(a) = \begin{cases} f_P^{\leqq}[+p](x_P) - f_P^{\leqq}[+p](x_P - \chi_e + \chi_{e'}) & \text{if } a = (e, e') \in A_P \\ f_Q^{\leqq}[-p](x_Q) - f_Q^{\leqq}[-p](x_Q + \chi_e - \chi_{e'}) & \text{if } a = (e, e') \in A_Q. \end{cases} \tag{5.13}$$

Length function ℓ is nonnegative due to (5.2), (5.3) and Theorem 2.1.

Let $S = \operatorname{supp}^+(x_P - x_Q)$ and $T = \{0\} \cup L \cup U$. We note that $S \cap T = \emptyset$ because $0 \notin S$ and because $x_Q(e) = x_P(e)$ for all $e \in L \cup U$ by (5.8) and the assumption in (Case 2). Let $d : \{0\} \cup E \to \mathbf{R} \cup \{+\infty\}$ denote the shortest distances from S to all vertices in G with respect to ℓ. For any $a = (e, e') \in A$ with $d(e) < +\infty$ we have

$$\ell(a) + d(e) - d(e') \geq 0. \tag{5.14}$$

We note that there always exists a path from S to T because $(e, 0) \in A_P$ for all $e \in S$ by (A$'$). Let $\mathbf{P}$ be a shortest path from S to T with the minimum number of arcs. We define δ by

$$\delta = \min\{\ell(\mathbf{P}), \min\{p(e) - \underline{\pi}(e) + d(e) \mid e \in E\}\}, \tag{5.15}$$

where $\ell(\mathbf{P}) = \sum_{a \in \mathbf{P}} \ell(a)$, and define a vector $\Delta p \in \mathbf{R}^E$ by

$$\Delta p(e) = \min\{d(e) - \delta, 0\} \qquad (\forall e \in E). \tag{5.16}$$

For convenience, we define $\Delta p(0)$ by $\min\{d(0) - \delta, 0\} = 0$. Obviously $\Delta p \leq \mathbf{0}$ holds. Because of $\delta \leq \ell(\mathbf{P})$, we have

$$\Delta p(e) = 0 \qquad (\forall e \in T). \tag{5.17}$$

It follows from (5.14)∼(5.16) and the nonnegativity of ℓ that

$$\ell(a) + \Delta p(e) - \Delta p(e') \geq 0 \qquad (\forall a = (e, e') \in A).$$

The above system of inequalities is equivalent to

$$\left.\begin{array}{l} f_P^{\lessgtr}[+p](x_P) - f_P^{\lessgtr}[+p](x_P - \chi_e + \chi_{e'}) + \Delta p(e) - \Delta p(e') \geq 0 \\ f_Q^{\lessgtr}[-p](x_Q) - f_Q^{\lessgtr}[-p](x_Q + \chi_e - \chi_{e'}) + \Delta p(e) - \Delta p(e') \geq 0 \end{array}\right. \qquad (\forall e, e' \in \{0\} \cup E),$$

which is further equivalent to

$$x_P \in \arg\max f_P^{\lessgtr}[+(p + \Delta p)], \quad x_Q \in \arg\max f_Q^{\lessgtr}[-(p + \Delta p)],$$

due to Theorem 2.1. We show that $p + \Delta p$ satisfies (5.4). Since $\Delta p \leq \mathbf{0}$, it is enough to show that $\underline{\pi}(e) \leq p(e) + \Delta p(e)$ for all $e \in E$. It follows from (5.15) that for all $e \in E$ we have

$$\begin{aligned} p(e) + \Delta p(e) &= \min\{p(e) + d(e) - \delta,\ p(e)\} \\ &\geq \min\{p(e) + d(e) - (p(e) + d(e) - \underline{\pi}(e)),\ p(e)\} \\ &= \underline{\pi}(e). \end{aligned}$$

Thus, x_P, x_Q, $p + \Delta p$, z_P, and z_Q satisfy Conditions (5.2)∼(5.8).

The above calculation shows that if $\delta < \ell(\mathbf{P})$, then there exists $e \in E$ with $p(e) > p(e) + \Delta p(e) = \underline{\pi}(e)$, that is, L is enlarged. We next deal with the case where $\delta = \ell(\mathbf{P})$.

Suppose $\delta = \ell(\mathbf{P})$. Note that for each arc $a = (e, e') \in A$, $\ell'(a) = \ell(a) + \Delta p(e) - \Delta p(e')$ is the length of a in the directed graph defined in the same way as above for $f_P^{\lessgtr}[+(p + \Delta p)]$, $f_Q^{\lessgtr}[-(p + \Delta p)]$, x_P, and x_Q. Since $\delta = \ell(\mathbf{P})$, we have $\ell'(a) = 0$ for all arcs $a \in \mathbf{P}$. Therefore, we have

$$\begin{array}{ll} x_P - \chi_e + \chi_{e'} \in \arg\max f_P^{\lessgtr}[+(p + \Delta p)] & (\forall (e, e') \in \mathbf{P} \cap A_P), \\ x_Q + \chi_e - \chi_{e'} \in \arg\max f_Q^{\lessgtr}[-(p + \Delta p)] & (\forall (e, e') \in \mathbf{P} \cap A_Q). \end{array} \tag{5.18}$$

Since $\mathbf{P}$ has the minimum number of arcs, we have

$$\begin{array}{l} x_P - \chi_e + \chi_{e''} \notin \arg\max f_P^{\lessgtr}[+(p + \Delta p)], \\ x_Q + \chi_e - \chi_{e''} \notin \arg\max f_Q^{\lessgtr}[-(p + \Delta p)] \end{array} \tag{5.19}$$

for all vertices e and e'' of $\mathbf{P}$ such that $(e, e'') \notin \mathbf{P}$ and e appears earlier than e'' in $\mathbf{P}$. Furthermore, arcs of A_P and A_Q appear alternately in $\mathbf{P}$. For otherwise, assume that two consecutive arcs $(e,e'),(e',e'') \in \mathbf{P}$ belong to A_P and then, by repeatedly using (M$^\natural$) we have

$$\begin{aligned}
& f_P^{\lesseqgtr}[+p](x_P-\chi_{e'}+\chi_{e''}) + f_P^{\lesseqgtr}[+p](x_P-\chi_e+\chi_{e'}) \\
\leq\ & \max\left\{\begin{array}{l} f_P^{\lesseqgtr}[+p](x_P-\chi_e+\chi_{e''}) + f_P^{\lesseqgtr}[+p](x_P) \\ f_P^{\lesseqgtr}[+p](x_P-\chi_e-\chi_{e'}+\chi_{e''}) + f_P^{\lesseqgtr}[+p](x_P+\chi_{e'}) \end{array}\right\} \\
\leq\ & \max\left\{\begin{array}{l} f_P^{\lesseqgtr}[+p](x_P-\chi_e+\chi_{e''}) + f_P^{\lesseqgtr}[+p](x_P) \\ f_P^{\lesseqgtr}[+p](x_P-\chi_e) + f_P^{\lesseqgtr}[+p](x_P+\chi_{e''}) \end{array}\right\} \\
\leq\ & f_P^{\lesseqgtr}[+p](x_P-\chi_e+\chi_{e''}) + f_P^{\lesseqgtr}[+p](x_P),
\end{aligned}$$

which yields

$$\ell(e,e') + \ell(e',e'') \geq \ell(e,e''). \tag{5.20}$$

This contradicts the minimality of the number of arcs in $\mathbf{P}$. Consequently, we have

$$\begin{aligned}
a_1{=}(e_1,e_1'), a_2{=}(e_2,e_2') \in \mathbf{P}{\cap}A_P, a_1 \neq a_2 \quad &\Longrightarrow \quad \{e_1,e_1'\} \cap \{e_2,e_2'\} = \emptyset, \\
a_1{=}(e_1,e_1'), a_2{=}(e_2,e_2') \in \mathbf{P}{\cap}A_Q, a_1 \neq a_2 \quad &\Longrightarrow \quad \{e_1,e_1'\} \cap \{e_2,e_2'\} = \emptyset.
\end{aligned} \tag{5.21}$$

From Lemma 2.2 together with (5.18), (5.19) and (5.21), we have

$$x_P' = x_P - \sum_{(e,e')\in\mathbf{P}\cap A_P} (\chi_e - \chi_{e'}) \quad \in \quad \arg\max f_P^{\lesseqgtr}[+(p+\Delta p)], \tag{5.22}$$

$$x_Q' = x_Q + \sum_{(e,e')\in\mathbf{P}\cap A_Q} (\chi_e - \chi_{e'}) \quad \in \quad \arg\max f_Q^{\lesseqgtr}[-(p+\Delta p)]. \tag{5.23}$$

We replace x_P, x_Q and p by x_P', x_Q' and $p+\Delta p$, respectively. Modifications (5.22) and (5.23) guarantee that (5.2), (5.3) and (5.7) hold for modified vectors. We have already shown that (5.4) holds. Since z_P and z_Q remain the same, Condition (5.17) implies (5.5) and (5.6). Let e' be the terminal vertex of $\mathbf{P}$ and let a^* be the last arc of $\mathbf{P}$. If $e' \notin U$, then (5.8) trivially holds. Hence we assume that $e' \in U$ in the sequel.

If $a^* \in A_Q$, then we have $x_Q(e') < z_Q(e') = x_P(e')$ and (5.3). Hence, it follows from Lemma 2.4 that we can put $z_Q(e') := +\infty$ while preserving (5.3) and (5.8).

Suppose that a^* belongs to A_P. Thus, $x_P(e') = x_Q(e') + 1 = z_Q(e') + 1$ holds at this point. We increase $z_Q(e')$ by one. By (b) of Lemma 2.3, there exists $e'' \in \{0\} \cup E$ such that $x_Q := x_Q + \chi_{e'} - \chi_{e''}$ satisfy (5.3) for the updated z_Q. If $e'' \in U$, then we put $z_Q(e'') := +\infty$. In the same way as in the argument for (Case 1), this modification yields (5.8) while preserving the other conditions.

Summarizing the above argument, we describe an algorithm Pairwise_Stable. We will show that it terminates in a finite number of iterations and finds a pairwise strictly stable outcome.

Algorithm Pairwise_Stable

Step 0. Find x_P, x_Q, p, z_P, and z_Q satisfying (5.2)∼(5.8).

Step 1. If $x_P = x_Q$ then stop.

Step 2. Set L and U as (5.9) and (5.10). If there exists $e \in L$ with $x_Q(e) < x_P(e)$ then go to Step 3.a; else go to Step 4.a.

Step 3.a. Set $z_P(e) := x_P(e) - 1$ and $x_P := x_P - \chi_e + \chi_{e'}$, where e' is an element such that (5.2) is satisfied by the updated x_P and z_P.

3.b. If $e' \notin U$ then go to Step 1; else go to Step 5.

Step 4.a. Construct G and compute ℓ for $f_P^{\leq}[+p]$, $f_Q^{\leq}[-p]$, x_P and x_Q by (5.12) and (5.13). Set $S := \mathrm{supp}^+(x_P - x_Q)$ and $T := \{0\} \cup L \cup U$. Compute the shortest distances $d(e)$ from S to all $e \in \{0\} \cup E$ in G with respect to ℓ. Find a shortest path $\mathbf{P}$ from S to T with the minimum number of arcs.

4.b. Compute δ by (5.15). For each $e \in E$, set $p(e) := p(e) + \min\{d(e) - \delta, 0\}$. If $\delta < \ell(\mathbf{P})$ then go to Step 1.

4.c. Update x_P and x_Q by (5.22) and (5.23). If the terminal vertex e' of $\mathbf{P}$ is not in U then go to Step 1.

4.d. If the last arc of $\mathbf{P}$ is in A_Q then put $z_Q(e') := +\infty$ and go to Step 1; else go to Step 5.

Step 5. Set $z_Q := z_Q + \chi_{e'}$ and $x_Q := x_Q + \chi_{e'} - \chi_{e''}$, where e'' is an element such that (5.3) holds. If $e'' \in U$ then set $z_Q(e'') := +\infty$. Go to Step 1.

We have already shown the following lemma.

Lemma 5.1: *Conditions* (5.2)∼(5.8) *are satisfied at* Step 1 *in each iteration of* Pairwise_Stable.

The following two lemmas show the termination of Pairwise_Stable.

Lemma 5.2: *In each iteration of* PAIRWISE_STABLE*, the following statements hold.*

(a) *L enlarges or remains the same.*

(b) *z_P decreases or remains the same.*

(c) *z_Q increases or remains the same.*

(d) *$\sum_{e \in E}(x_P(e) - x_Q(e))$ decreases or remains the same.*

Proof. Since $L = \{e \in E \mid p(e) = \underline{\pi}(e)\}$ and p does not increase during PAIRWISE_STABLE, we have (a). Statements (b) and (c) trivially hold. Obviously $\sum_{e \in E}(x_P(e) - x_Q(e))$ is nonnegative by (5.7). Vectors x_P and/or x_Q are modified at Step 3.a, Step 4.c or Step 5. At Step 3.a or Step 4.c, if $e' = 0$, then $\sum_{e \in E}(x_P(e) - x_Q(e))$ is decreased by one; otherwise it remains the same. At Step 5, if $e'' = 0$, then $\sum_{e \in E}(x_P(e) - x_Q(e))$ is decreased by one; otherwise it remains the same. Hence (d) holds. ∎

We denote by [StepXX→Step1] the case where we go from Step XX to Step 1 after execution of Step XX in PAIRWISE_STABLE.

Lemma 5.3: PAIRWISE_STABLE *has the following features.*

(a) *In* [Step3.b→Step1]*, some component of z_P strictly decreases.*

(b) *In* [Step4.b→Step1]*, L strictly enlarges.*

(c) *In* [Step4.c→Step1]*, either $\sum_{e \in E}(x_P(e) - x_Q(e))$ strictly decreases or some component of z_P strictly decreases at* Step3.b *in the next iteration.*

(d) *In* [Step4.d→Step1]*, some component of z_Q strictly increases.*

(e) *In* [Step5→Step1]*, some component of z_Q strictly increases.*

Proof. (a): At the beginning of Step 3.a, we have $z_P(e) \geq x_P(e)$. Hence, $z_P(e)$ strictly decreases at Step 3.a.

(b): As we have already shown, L strictly enlarges when $\delta < \ell(\mathbf{P})$.

(c): In this case, we have either $e' = 0$ or $e' \in L \setminus U$. In the former case $\sum_{e \in E}(x_P(e) - x_Q(e))$ is decreased by one. Since $x_Q(e) = x_P(e)$ for all $e \in L \cup U$ at the beginning of Step 4, we have $x_P(e') > x_Q(e')$ for $e'(\in L)$ at the end of Step 4.c, which results in (a) in the next iteration.

(d): In this case, we have $e' \in U$ at the beginning of Step 4.d. Hence, $z_Q(e')$ strictly increases.

(e): If $e' \neq e''$, then $z_Q(e')$ increases by one; otherwise $z_Q(e') = +\infty$. ∎

Lemma 5.4: Pairwise_Stable *terminates in a finite number of iterations if f_P and f_Q satisfy* (A′).

Proof. Since f_P and f_Q satisfy (A′), we have

- $\sum_{e \in E}(x_P(e) - x_Q(e))$ is nonnegative and bounded from above,
- if $z_P(e) < +\infty$ then it is nonnegative and bounded from above,
- if $z_Q(e) < +\infty$ then it is bounded from above by (5.8).

Hence, Lemmas 5.2 and 5.3 guarantee the termination of Pairwise_Stable in finitely many steps. ∎

Lemma 5.5: *If f_P and f_Q are integer-valued on their effective domains, $\underline{\pi} \in (\mathbf{Z} \cup \{-\infty\})^E$, and $\overline{\pi} \in (\mathbf{Z} \cup \{+\infty\})^E$, then p is preserved to be integer-valued in* Pairwise_Stable.

Proof. Because $\overline{\pi} \in (\mathbf{Z} \cup \{+\infty\})^E$, p is initially defined to be integer-valued in Pairwise_Stable. Since f_P and f_Q are integer-valued on their effective domains, ℓ defined in (5.13) is integer-valued. Furthermore, as $\underline{\pi} \in (\mathbf{Z} \cup \{-\infty\})^E$, δ defined in (5.15) is also integer-valued. Hence, $p(e) := p(e) + \min\{d(e) - \delta, 0\}$ modified at Step 4.b is preserved to be integer for each $e \in E$. ∎

By Lemmas 5.1 and 5.4, Pairwise_Stable always finds x_P, x_Q, p, z_P, and z_Q satisfying (5.1)∼(5.6) under (A′). By Lemma 5.5, if f_P and f_Q are integer-valued on their effective domains, $\underline{\pi} \in (\mathbf{Z} \cup \{-\infty\})^E$, and $\overline{\pi} \in (\mathbf{Z} \cup \{+\infty\})^E$, then p is preserved to be integer-valued. This completes the proof of Theorem 3.6.

We briefly discuss the oracle complexity of algorithm Pairwise_Stable, by assuming that the function value $f(x)$ of a given M$^\natural$-concave function f can be calculated in constant time for each input x. Pairwise_Stable initially solves the maximization problem of an M$^\natural$-concave function. It is known that a maximizer of an M$^\natural$-concave function f on E can be found in polynomial time in m and $\log D$, where $m = |E|$ and $D = \max\{||x-y||_\infty \mid x, y \in \mathrm{dom}\, f\}$. For example, $\mathrm{O}(m^3 \log D)$-time algorithms are proposed in [29, 32]. In our case we can define $D = \max\{||x-y||_\infty \mid x, y \in \mathrm{dom}\, f_P\}$, because (5.7) is preserved. Each iteration of PAIRWISE_STABLE can be executed in polynomial time in m, because construction of network (G, ℓ) and finding a shortest path can be done in polynomial time in m. Moreover, Lemma 5.3 guarantees that PAIRWISE_STABLE terminates in polynomial time in m and D, which says that if D is constant, or $\mathrm{dom}\, f_P \subseteq \{0,1\}^E$ in particular, then PAIRWISE_STABLE is an oracle polynomial time algorithm for finding a pairwise strictly stable outcome.

6. Proofs

In this section we give proofs of Lemma 3.1, Theorems 3.4 and 3.2 in this order.

6.1. A proof of Lemma 3.1

It is enough to show that if an outcome (x, s) is pairwise quasi-unstable, then it is also pairwise unstable. Let (x, s) be a pairwise quasi-unstable outcome. We may assume that (x, s) satisfies incentive constraints (3.1) and (3.2). Then there exist $i \in P$, $j \in Q$, $\alpha \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E_{(i)}}$ and $y'' \in \mathbf{Z}^{E_{(j)}}$ satisfying (3.3)$\sim$(3.6).

We first deal with Case (i). In this case, $s(i,j) = \alpha$ holds. Because (x, s) satisfies incentive constraints, we have $y'(i,j) > x(i,j)$ and $y''(i,j) > x(i,j)$. In addition, we assume that $y'(i,j)$ and $y''(i,j)$, respectively, are as small as possible among vectors satisfying (3.3)$\sim$(3.6). By (M$^\natural$) for y' and $x_{(i)}$, there exists $e' \in \mathrm{supp}^-(y' - x_{(i)}) \cup \{0\}$ such that, putting $e = (i,j)$,

$$\begin{aligned} & f_i[+(s_{(i)}^{-j}, \alpha)](y') + f_i[+s_{(i)}](x_{(i)}) \\ = \;& f_i[+(s_{(i)}^{-j}, \alpha)](y') + f_i[+(s_{(i)}^{-j}, \alpha)](x_{(i)}) \\ \leq \;& f_i[+(s_{(i)}^{-j}, \alpha)](y' - \chi_e + \chi_{e'}) + f_i[+(s_{(i)}^{-j}, \alpha)](x_{(i)} + \chi_e - \chi_{e'}). \end{aligned}$$

Since $f_i[+(s_{(i)}^{-j}, \alpha)](y') > f_i[+(s_{(i)}^{-j}, \alpha)](y' - \chi_e + \chi_{e'})$ by the choice of y', we obtain $f_i[+s_{(i)}](x_{(i)}) < f_i[+(s_{(i)}^{-j}, \alpha)](x_{(i)} + \chi_e - \chi_{e'})$. This implies that $y'(i,j) = x(i,j)+1$, since $x_{(i)} + \chi_e - \chi_{e'}$ is a possible candidate for y'. Analogously, we can show that $y''(i,j) = x(i,j) + 1$. Hence y' and y'' also satisfy (3.7).

We next consider Cases (ii) and (iii). We assume that $s(i,j) < \alpha$ (The case where $s(i,j) = \alpha$ can be treated similarly as in Case (i) and we can also deal with the case where $s(i,j) > \alpha$ by interchanging the roles of workers and firms in the following argument). If $x(i,j) = 0$, then we can show the assertion in the same way as in Case (i). Hence we assume that $x(i,j) > 0$. In this case, the following relations

$$\begin{aligned} f_i[+s_{(i)}](x_{(i)}) &< f_i[+(s_{(i)}^{-j}, \alpha)](x_{(i)}), \\ f_j[-s_{(j)}](y) &\geq f_j[-(s_{(j)}^{-i}, \alpha)](y) \qquad (\forall y \leq x_{(j)}) \end{aligned}$$

hold. Hence, $y''(i,j)$ must be greater than $x(i,j)$, so that $y''(i,j) \geq 2$. Therefore, it is sufficient to deal with Case (iii). Replace y' by $x_{(i)} + (y''(i,j) - x(i,j))\chi_{(i,j)}$, which belongs to $\mathrm{dom}\, f_i$. We then have $f_i[+(s_{(i)}^{-j}, \alpha)](y') > f_i[+s_{(i)}](y')$. Since $x(i,j) > 0$, f_i is linear over $\mathrm{dom}\, f_i$, and (x,s) satisfies incentive constraints, we also have $f_i[+s_{(i)}](y') \geq f_i[+s_{(i)}](x_{(i)})$. Hence (3.3), (3.4), and (3.7) are satisfied by the new y'.

6.2. A proof of Theorem 3.4

THE ONLY-IF PART: Let (x,s) be a pairwise strictly stable outcome. For each pair $(i,j) \in E$ with $x(i,j) = 0$, we define $r(i,j)$ as the supremum of the set of αs satisfying (3.8) without the constraint $\alpha \leq \overline{\pi}(i,j)$. (We have $r(i,j) \neq +\infty$ if there exists $y \in \mathrm{dom}\, f_i$ such that $y(i,j) > 0$ and $y(i,j') \leq x(i,j')$ for all $j' \in Q \setminus \{j\}$.) If $r(i,j) = +\infty$, then we redefine $r(i,j)$ as the infimum of the set of αs satisfying (3.9) without the constraint $\underline{\pi}(i,j) \leq \alpha$. (We have $r(i,j) \neq -\infty$ if there exists $y \in \mathrm{dom}\, f_j$ such that $y(i,j) > 0$ and $y(i',j) \leq x(i',j)$ for all $i' \in P \setminus \{i\}$.) If $r(i,j) = -\infty$, then we redefine $r(i,j) = -b$ for a sufficiently large positive number b. Then, we define $p \in \mathbf{R}^E$ by

$$p(i,j) = \begin{cases} s(i,j) & \text{if } x(i,j) > 0 \\ r(i,j) & \text{else if } \underline{\pi}(i,j) \leq r(i,j) \leq \overline{\pi}(i,j) \\ \underline{\pi}(i,j) & \text{else if } r(i,j) < \underline{\pi}(i,j) \\ \overline{\pi}(i,j) & \text{else if } \overline{\pi}(i,j) < r(i,j) \end{cases} \qquad (\forall (i,j) \in E). \qquad (6.1)$$

Condition (3.12) is satisfied by p because s is feasible.

We also define z_P and z_Q by

$$z_P(i,j) = \begin{cases} x(i,j) & \text{if (3.8) does not hold for } \alpha = p(i,j) \\ +\infty & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E), \quad (6.2)$$

$$z_Q(i,j) = \begin{cases} x(i,j) & \text{if (3.9) does not hold for } \alpha = p(i,j) \\ +\infty & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E). \quad (6.3)$$

It follows from pairwise strict stability of (x, s) that $z_P(i,j) = +\infty$ or $z_Q(i,j) = +\infty$ holds. We consider the case where $z_P(i,j) < +\infty$. In this case, there exists $y' \in \mathbf{Z}^{E_{(i)}}$ such that $f_i[+p_{(i)}](x_{(i)}) < f_i[+p_{(i)}](y')$ and $y'(i,j') \leq x(i,j')$ for all $j' \in Q \setminus \{j\}$, where note that $f_i[+p_{(i)}](x_{(i)}) = f_i[+s_{(i)}](x_{(i)})$ and $f_i[+p_{(i)}](y') = f_i[+(s_{(i)}^{-j}, p(i,j))](y')$. We show (3.13). Suppose, to the contrary, that $p(i,j) > \underline{\pi}(i,j)$. If $x(i,j) > 0$, then for a sufficiently small number $\epsilon > 0$ we have

$$\begin{aligned} f_i[+p_{(i)}](x_{(i)}) &< f_i[+(p_{(i)} - \epsilon\chi_{(i,j)})](y'), \\ f_j[-p_{(j)}](x_{(j)}) &< f_j[-(p_{(j)} - \epsilon\chi_{(i,j)})](x_{(j)}), \end{aligned}$$

which imply that neither (3.8) nor (3.9) holds for $\alpha = p(i,j) - \epsilon \geq \underline{\pi}(i,j)$, a contradiction. Hence, assume $x(i,j) = 0$. Since $p(i,j) > \underline{\pi}(i,j)$, we have $p(i,j) \leq r(i,j)$ by (6.1). As the right-hand side of (3.8) is nondecreasing in α and $p(i,j) \leq r(i,j)$, the definition of $r(i,j)$ guarantees that (3.8) holds for $\alpha = p(i,j)$. However, this contradicts $z_P(i,j) < +\infty$. We thus have (3.13).

We next consider the case where $z_Q(i,j) < +\infty$ and show (3.14). Suppose, to the contrary, that $p(i,j) < \overline{\pi}(i,j)$. If $x(i,j) > 0$, then we reach a contradiction in the same way as above. Hence assume $x(i,j) = 0$. By (6.1) and the assumption we have $p(i,j) \geq r(i,j)$. If $r(i,j)$ is defined by (3.8), then for a sufficiently small number $\epsilon > 0$, neither (3.8) nor (3.9) holds for $\alpha = p(i,j) + \epsilon \leq \overline{\pi}(i,j)$, which is a contradiction. In the other case (where $r(i,j)$ is either defined by (3.9) or set to be sufficiently small $-b$), (3.9) holds for $\alpha = p(i,j)$ since $p(i,j) \geq r(i,j)$. However, this contradicts the assumption that $z_Q(i,j) < +\infty$. Hence, we have (3.14).

Finally, we show (3.10) ((3.11) can be shown similarly). Suppose that (3.10) does not hold, i.e., for some $i \in P$ there exists $y' \in \arg\max\{f_i[+p_{(i)}](y) \mid y \leq z_{(i)}\}$ with $f_i[+p_{(i)}](x_{(i)}) < f_i[+p_{(i)}](y')$. We choose $y' \in \arg\max\{f_i[+p_{(i)}](y) \mid y \leq z_{(i)}\}$ with $f_i[+p_{(i)}](x_{(i)}) < f_i[+p_{(i)}](y')$ that minimizes $\sum\{y'(e) - x_{(i)}(e) \mid e \in \mathrm{supp}^+(y' -$

$x_{(i)})\}$. Since $f_i[+s_{(i)}](y) = f_i[+p_{(i)}](y)$ holds for all $y \in \mathbf{Z}^{E_{(i)}}$ with $\mathbf{0} \leq y \leq x_{(i)}$, (3.1) implies the existence of $e \in E_{(i)}$ with $y'(e) > x_{(i)}(e)$. By (M$^\natural$), there exists $e' \in \mathrm{supp}^-(y' - x_{(i)}) \cup \{0\}$ such that

$$f_i[+p_{(i)}](y') + f_i[+p_{(i)}](x_{(i)}) \leq f_i[+p_{(i)}](y'-\chi_e+\chi_{e'}) + f_i[+p_{(i)}](x_{(i)}+\chi_e-\chi_{e'}).$$

By the definition of y', we have

$$f_i[+p_{(i)}](y') > f_i[+p_{(i)}](y' - \chi_e + \chi_{e'}).$$

The above two inequalities imply $f_i[+p_{(i)}](x_{(i)}) < f_i[+p_{(i)}](x_{(i)} + \chi_e - \chi_{e'})$, which yields that $z_P(e) = x_{(i)}(e)$, by (6.2). This contradicts $y' \leq z_{(i)}$. Hence (3.10) holds.

THE IF PART: Let $p \in \mathbf{R}^E$ and $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ be vectors satisfying (3.10)$\sim$(3.14). We put $s = p$. We show that (x, s) is pairwise strictly stable. Since $x_{(k)} \leq z_{(k)}$ for all $k \in P \cup Q$, Conditions (3.1) and (3.2) are direct consequences of (3.10) and (3.11). Suppose, to the contrary, that there exist $i \in P$, $j \in Q$, $\alpha \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E_{(i)}}$ and $y'' \in \mathbf{Z}^{E_{(j)}}$ such that

$$\begin{aligned} & f_i[+s_{(i)}](x_{(i)}) < f_i[+(s_{(i)}^{-j}, \alpha)](y'), \\ & y'(i,j') \leq x(i,j') \qquad (\forall j' \in Q \setminus \{j\}), \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} & f_j[-s_{(j)}](x_{(j)}) < f_j[-(s_{(j)}^{-i}, \alpha)](y''), \\ & y''(i',j) \leq x(i',j) \qquad (\forall i' \in P \setminus \{i\}). \end{aligned} \tag{6.5}$$

By (3.10) and since $y' \geq \mathbf{0}$, Condition (6.4) implies that either (Case 1) $y'(i,j) > z_{(i)}(i,j)$ or (Case 2) $y'(i,j) \leq z_{(i)}(i,j)$ and $p(i,j) < \alpha$. Similarly, by (3.11) and (6.5), we have either (Case 3) $y''(i,j) > z_{(j)}(i,j)$ or (Case 4) $y''(i,j) \leq z_{(j)}(i,j)$ and $\alpha < p(i,j)$. Trivially, (Case 2) and (Case 4) are inconsistent. By (3.13) or (3.14), (Case 1) and (Case 3) do not hold simultaneously. Also, (Case 1) together with (3.13) implies $p(i,j) = \underline{\pi}(i,j)$, which is irreconcilable with (Case 4). Analogously, (Case 2) is irreconcilable with (Case 3), due to (3.14). This means that (6.4) and (6.5) do not hold simultaneously, a contradiction. Hence (x, s) is pairwise strictly stable.

6.3. A proof of Theorem 3.2

Let x be a pairwise stable allocation and s a feasible salary vector such that (x, s) is pairwise stable. By Theorem 3.4 and Lemma 3.5, it is enough to show that there exists $p \in \mathbf{R}^E$, and $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ satisfying (3.16)~(3.20) with x. Let $N = (G, \ell)$ be a network with a directed graph $G = (\{0\} \cup E, A)$ and an arc length function $\ell : A \to \mathbf{R}$, where the arc set A of G is the union of the following two sets

$$\begin{aligned} A_P &= \{(e, e') \mid e, e' \in \{0\} \cup E,\ e \neq e',\ x - \chi_e + \chi_{e'} \in \mathrm{dom}\, f_P\}, \\ A_Q &= \{(e, e') \mid e, e' \in \{0\} \cup E,\ e \neq e',\ x + \chi_e - \chi_{e'} \in \mathrm{dom}\, f_Q\} \end{aligned} \tag{6.6}$$

and the length function ℓ is defined by

$$\ell(a) = \begin{cases} f_P[+s](x) - f_P[+s](x - \chi_e + \chi_{e'}) & \text{if } a = (e, e') \in A_P \\ f_Q[-s](x) - f_Q[-s](x + \chi_e - \chi_{e'}) & \text{if } a = (e, e') \in A_Q \end{cases} \tag{6.7}$$

with f_P and f_Q being defined by (3.15).

By the pairwise stability of (x, s) we have the following claim.

Claim 6.3.A: Consider any two consecutive arcs $a = (e, e') \in A_P$ and $a' = (e', e'') \in A_Q$. Then we have $\ell(a) \geq 0$ or $\ell(a') \geq 0$. Furthermore, if $\ell(a) + \ell(a') < 0$, then $\ell(a) < 0$ implies $\ell(a') - (s(e') - \underline{\pi}(e')) \geq 0$, and $\ell(a') < 0$ implies $\ell(a) - (\overline{\pi}(e') - s(e')) \geq 0$.

(Proof) Suppose to the contrary that $\ell(a) < 0$ and $\ell(a') < 0$. Since (x, s) satisfies incentive constraints, the lengths of arcs of A_P entering vertex 0 and of arcs of A_Q leaving vertex 0 are nonnegative. Hence we have $e' \neq 0$. Let $e' = (i, j) \in E$, $y' = x - \chi_e + \chi_{e'}$, and $y'' = x + \chi_{e'} - \chi_{e''}$.

If $e = 0$ or $e = (i, j')$ for some $j' \in Q$, then $\ell(a) < 0$ means

$$f_i[+s_{(i)}](x_{(i)}) < f_i[+s_{(i)}](y'_{(i)}). \tag{6.8}$$

Also consider the other case where $e \neq 0$ and $e = (i', j')$ for some $i' (\neq i) \in P$ and $j' \in Q$. By the incentive constraints for (x, s), we have

$$f_{i'}[+s_{(i')}](x_{(i')}) \geq f_{i'}[+s_{(i')}](y'_{(i')}),$$

which together with $\ell(a) < 0$ implies (6.8).

Similarly, it follows from $\ell(a') < 0$ that

$$f_j[-s_{(j)}](x_{(j)}) < f_j[-s_{(j)}](y''_{(j)}). \tag{6.9}$$

Inequalities (6.8) and (6.9) contradict the pairwise stability of (x, s).

Next we show that $\ell(a') - (s(e') - \underline{\pi}(e')) \geq 0$ if $\ell(a) + \ell(a') < 0$ and $\ell(a) < 0$ (we can similarly show the second part of the last assertion). Suppose to the contrary that $\ell(a') - (s(e') - \underline{\pi}(e')) < 0$. Then there exists $\alpha \in \mathbf{R}$ such that $\ell(a) + \alpha < 0$, $\ell(a') - \alpha < 0$, and $0 \leq \alpha \leq s(e') - \underline{\pi}(e')$. In the same way as the argument showing (6.8) and (6.9), inequalities $\ell(a) + \alpha < 0$ and $\ell(a') - \alpha < 0$ yield

$$\begin{aligned} f_i[+s_{(i)}](x_{(i)}) &< f_i[+(s_{(i)}^{-j}, s(i,j) - \alpha)](y'_{(i)}), \\ f_j[-s_{(j)}](x_{(j)}) &< f_j[-(s_{(j)}^{-i}, s(i,j) - \alpha)](y''_{(j)}), \end{aligned}$$

where $y' = x - \chi_e + \chi_{e'}$ and $y'' = x + \chi_{e'} - \chi_{e''}$. This contradicts the pairwise stability of (x, s). ∎

We initially put $p = s$, $z_P = z_Q = (+\infty, \cdots, +\infty)$, and modify them as follows. For each pair of consecutive arcs $a = (e, e') \in A_P$ and $a' = (e', e'') \in A_Q$ with $\ell(a) + \ell(a') < 0$, if $\ell(a) < 0$ then we set $z_P(e') := x(e')$ and $p(e') := \underline{\pi}(e')$, and if $\ell(a') < 0$ then we set $z_Q(e') := x(e')$ and $p(e') := \overline{\pi}(e')$. We define subsets L and U of E by

$$L = \{e \in E \mid z_P(e) < +\infty\}, \qquad U = \{e \in E \mid z_Q(e) < +\infty\}.$$

It follows from Claim 6.3.A that L and U are disjoint. We update network $N = (G, \ell)$ by (6.6) and (6.7) with f_P, f_Q and s being replaced by $f_P^{\lesssim}$, $f_Q^{\lesssim}$ and p, where $f_P^{\lesssim}$ and $f_Q^{\lesssim}$ are defined by (5.11). Let $S = \{0\} \cup L \cup U$. For the updated network N and S, we show the following three claims.

Claim 6.3.B: For any two consecutive arcs $a = (e, e') \in A_P$ and $a' = (e', e'') \in A_Q$, we have $\ell(a) + \ell(a') \geq 0$.
(Proof) One of the two consecutive arcs $a \in A_P$ and $a' \in A_Q$ that do not satisfy the present claim for the original network N disappears in the updated network. Hence the present claim holds. ∎

Claim 6.3.C: All arcs between vertices in S have nonnegative lengths.
(Proof) By the definitions of L, U, and updated A_P and A_Q, no arc of A_P enters a vertex in L and no arc of A_Q leaves a vertex in U. Furthermore, by the definitions of L and U and by Claim 6.3.A, all arcs of A_Q leaving a vertex in L and all arcs of A_P entering a vertex in U have nonnegative lengths. Hence, it suffices to consider arcs a of the following four types:

(1) $a \in A_P$ entering vertex 0 from L,

(2) $a \in A_Q$ entering L from 0,

(3) $a \in A_P$ entering 0 from U, and

(4) $a \in A_Q$ entering U from 0.

First, we deal with any arc $a = (0, e')$ of type (2). Since (x, s) satisfies incentive constraints, we have

$$f_Q[-s](x) - f_Q[-s](x - \chi_{e'}) \geq 0.$$

This inequality is preserved when s is replaced by p, because $p(e') \leq s(e')$, and hence, $\ell(a) \geq 0$. Next, consider any arc $a = (e', 0)$ of type (1). Since $e' \in L$, there exists a vertex $e \in \{0\} \cup E$ such that

$$f_P[+s](x - \chi_e + \chi_{e'}) - f_P[+s](x) > s(e') - \underline{\pi}(e') \geq 0, \tag{6.10}$$

where the first inequality follows from the definition of L and Claim 6.3.A. By ($\mathrm{M}^\natural$) and incentive constraints for (x, s), we have

$$\begin{aligned} f_P[+s](x - \chi_e + \chi_{e'}) - f_P[+s](x) &\leq f_P[+s](x - \chi_e) - f_P[+s](x - \chi_{e'}) \\ &\leq f_P[+s](x) - f_P[+s](x - \chi_{e'}). \end{aligned} \tag{6.11}$$

From (6.10) and (6.11) we obtain $\ell(a) = f_P[+s](x) - f_P[+s](x - \chi_{e'}) - (s(e') - \underline{\pi}(e')) \geq 0$. Similarly, we can also show the nonnegativity of the lengths of arcs of types (3) and (4). ∎

Claim 6.3.D: All arcs of A_P entering S and all arcs of A_Q leaving S have non-negative lengths.
(Proof) By the first part of the proof of Claim 6.3.C, it is sufficient to deal with the arcs of A_P entering 0 and arcs of A_Q leaving 0. The nonnegativity of lengths of such arcs follows from incentive constraints for (x, s), because $p(e) = s(e)$ for all $e \in E \setminus S$. ∎

Preserving (3.18)∼(3.20) and keeping Claims 6.3.B, 6.3.C and 6.3.D valid, we repeat modifying p, z_P, z_Q and S as described below. Since each modification results in enlarging $S \subseteq \{0\} \cup E$, after at most $|E|$ repetitions we eventually get an S such that (3.16) and (3.17) hold with x , due to Claim 6.3.C, and this will complete the proof of Theorem 3.2.

For each $e' \in E \setminus S$ we consider the following three cases:

(1) there exists an arc $a = (e, e') \in A_P$ with $\ell(a) < 0$,

(2) there exists an arc $a' = (e', e'') \in A_Q$ with $\ell(a') < 0$, and

(3) $\ell(a) \geq 0$ for any arc $a = (e, e') \in A_P$ and $\ell(a') \geq 0$ for any arc $a' = (e', e'') \in A_Q$.

By Claim 6.3.B, these three cases are exclusive.

First, we deal with Case (3). In this case, we put $S := S \cup \{e'\}$ and leave p, z_P and z_Q the same. Obviously, Claim 6.3.B remains valid. In Case (3) there is no arc of A_P, entering e', of negative length and no arc of A_Q, leaving e', of negative length. This implies Claim 6.3.D for updated S, and this together with Claim 6.3.D for old S implies Claim 6.3.C for updated S.

Next, we deal with Case (1). In this case, we modify $p(e')$ as

$$p(e') := p(e') + \max\{\min\{\ell(a) \mid a = (e, e') \in A_P\}, \underline{\pi}(e') - p(e')\},$$

and, if $\min\{\ell(a) \mid a = (e, e') \in A_P\} < \underline{\pi}(e') - p(e')$ then we set $z_P(e') := x(e')$ and update network N for new z_P. This modification of $p(e')$ increases the lengths of the arcs entering e' and decreases those of the arcs leaving e'. Though the lengths of arcs of A_Q leaving e' get decreased, they remain nonnegative for updated N because of Claim 6.3.B for old N, and hence, Case (3) applies for the present e' in updated N, so that we put $S := S \cup \{e'\}$. In order to show Claim 6.3.C for updated N, it is enough to verify $\ell(a_1) \geq 0$ for any arc $a_1 = (e', e_1) \in A_P$ with $e_1 \in S$ in updated N. Let $a^* = (e_0, e') \in A_P$ be an arc that attains $\min\{\ell(a) \mid a = (e, e') \in A_P\}$ in old N. It follows from $\mathrm{M}^\natural$-concavity that $\ell(a^*) + \ell(a_1) \geq \ell(e_0, e_1)$ and $(e_0, e_1) \in A_P$ for $a^*, a_1 \in A_P$ in old N (see (5.20)). By Claims 6.3.C and 6.3.D, we have $\ell(e_0, e_1) \geq 0$ in old N. The length $\ell(a_1)$ in updated N is greater than or equal to $\ell(a^*) + \ell(a_1)$ in old N, and hence, Claim 6.3.D holds for updated N. In order to show that Claim 6.3.B holds for updated N, it is enough to verify that in updated N we have $\ell(a_1) + \ell(a_2) \geq 0$ for any consecutive two arcs $a_1 = (e', e_1) \in A_P$ and $a_2 = (e_1, e_2) \in A_Q$. By $\mathrm{M}^\natural$-concavity, we have

$$\ell(a^*) + \ell(a_1) + \ell(a_2) \geq \ell(e_0, e_1) + \ell(a_2) \geq 0 \tag{6.12}$$

in old N, where the second inequality follows from Claim 6.3.B for old N. The sum $\ell(a_1) + \ell(a_2)$ in updated N is greater than or equal to the left-hand side of (6.12). We thus see that Claim 6.3.B holds for updated N. Also, Claim 6.3.D obviously holds for updated N.

Finally, we deal with Case (2). In this case, we modify $p(e')$ as

$$p(e') := p(e') - \max\{\min\{\ell(a') \mid a' = (e', e'') \in A_Q\}, p(e') - \overline{\pi}(e')\},$$

and, if $\min\{\ell(a') \mid a' = (e', e'') \in A_Q\} < p(e') - \overline{\pi}(e')$ then we set $z_Q(e') := x(e')$ and update network N for new z_Q. After this modification, we put $S := S \cup \{e'\}$. In the same way as the proof for Case (1), we can show that updated N and S satisfy Claims 6.3.B, 6.3.C and 6.3.D.

7. Concluding remarks

We have proposed a general two-sided matching market model with possibly bounded side payments and have verified the existence of a pairwise stable outcome by utilizing discrete convex analysis.

However, we have not discussed a structure of the set of all pairwise stable outcomes. It is well-known that the set of all payoff vectors in the assignment model and the set of all stable matchings in the marriage model without indifference have lattice structures.

In our model without bounds on side payments, namely in the case where $\underline{\pi} = (-\infty, \cdots, -\infty)$ and $\overline{\pi} = (+\infty, \cdots, +\infty)$, we can show, by using the duality in discrete convex analysis, that the set of all feasible salary vectors that form pairwise stable outcomes with certain feasible allocations has a lattice structure (more precisely, L$^\natural$-convexity) (see [22] for details). Investigations on structures of pairwise stable outcomes are left for future work.

Both the Kelso-Crawford model [18] and our model assume gross substitutability of value functions of agents. Kelso and Crawford showed the existence of a pairwise stable outcome in a one-to-many case in which a payoff function of each worker is strictly increasing (not necessarily linear) in salary. On the other hand, we showed the existence of a pairwise stable outcome in a many-to-many case with multi-units of labor time and possibly bounded side payments under the hypothesis that a payoff function of each agent is linear in salary. It is open to determine whether there exist pairwise stable outcomes in a many-to-many case where a payoff function of each worker (resp. firm) is strictly increasing (resp. decreasing) in salary under gross substitutability of value functions of agents.

In Section 5 we have proposed Algorithm PAIRWISE_STABLE for finding a pairwise strictly stable outcome and have shown that the time complexity is polynomial in $m = |E|$ and $D = \max\{||x - y||_\infty \mid x, y \in \mathrm{dom}\, f_P\}$. Unfortunately, we know that there exist a series of instances for which PAIRWISE_STABLE requires time proportional to D even if $\underline{\pi} = \overline{\pi} = \mathbf{0}$. It is an open problem to devise an algorithm to compute a pairwise (strictly) stable outcome in time polynomial in m and $\log D$.

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References

[1] D. J. Abraham, R. W. Irving and D. F. Manlove, *Two algorithms for the student-project allocation problem*, J. Discrete Algorithms, to appear.

[2] A. Alkan and D. Gale, *Stable schedule matching under revealed preference*, J. Econom. Theory **112** (2003), 289–306.

[3] V. P. Crawford and E. M. Knoer, *Job matching with heterogeneous firms and workers*, Econometrica **49** (1981), 437–450.

[4] V. Danilov, G. Koshevoy, and C. Lang, *Gross substitution, discrete convexity, and submodularity*, Discrete Appl. Math. **131** (2003), 283–298.

[5] A. Eguchi, S. Fujishige, and A. Tamura, *A generalized Gale-Shapley algorithm for a discrete-concave stable-marriage model*, Algorithms and Computation: 14th International Symposium, ISAAC2003, LNCS 2906 (T. Ibaraki, N. Katoh, and H. Ono, eds.), Springer-Verlag, Berlin, 2003, pp. 495–504.

[6] K. Eriksson, and J. Karlander, *Stable matching in a common generalization of the marriage and assignment models*, Discrete Math. **217** (2000), 135–156.

[7] R. Farooq and A. Shioura, *A note on the equivalence between substitutability and M$^\natural$-convexity*, Pacific J. Optim. **1** (2005), 243–252.

[8] R. Farooq and A. Tamura, *A new characterization of M$^\natural$-convex set functions by substitutability*, J. Oper. Res. Soc. Japan **47** (2004), 18–24.

[9] T. Fleiner, *A matroid generalization of the stable matching polytope*, Integer Programming and Combinatorial Optimization: 8th International IPCO Conference, LNCS 2081 (B. Gerards and K. Aardal, eds.), Springer-Verlag, Berlin, 2001, pp. 105–114.

[10] T. Fleiner, *A fixed point approach to stable matchings and some applications*, Math. Oper. Res. **28** (2003), 103–126.

[11] S. Fujishige, *Submodular Functions and Optimization*, Annals of Discrete Mathematics 47, North-Holland, Amsterdam, 1991.

[12] S. Fujishige and A. Tamura, *A general two-sided matching market with discrete concave utility functions*, RIMS Preprint 1401, Kyoto University, 2003, `http://www.kurims.kyoto-u.ac.jp/home_page/preprint/list/`.

[13] S. Fujishige and Z. Yang, *A note on Kelso and Crawford's gross substitutes condition*, Math. Oper. Res. **28** (2003), 463–469.

[14] D. Gale and L. S. Shapley, *College admissions and the stability of marriage*, Amer. Math. Monthly **69** (1962), 9–15.

[15] F. Gul and F. Stacchetti, *Walrasian equilibrium with gross substitutes*, J. Econom. Theory **87** (1999), 95–124.

[16] S. Iwata, S. Moriguchi, and K. Murota, *A capacity scaling algorithm for M-convex submodular flow*, Math. Programming, to appear.

[17] M. Kaneko, *The central assignment game and the assignment markets*, J. Math. Econom. **10** (1982), 205–232.

[18] J. A. S. Kelso and V. P. Crawford, *Job matching, coalition formation, and gross substitutes*, Econometrica **50** (1982), 1483–1504.

[19] S. Moriguchi and K. Murota, *Capacity scaling algorithm for scalable M-convex submodular flow problems*, Optim. Methods Softw. **18** (2003), 207–218.

[20] K. Murota, *Convexity and Steinitz's exchange property*, Adv. Math. **124** (1996), 272–311.

[21] K. Murota, *Discrete convex analysis*, Math. Programming **83** (1998), 313–371.

[22] K. Murota, *Discrete Convex Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[23] K. Murota and A. Shioura, *M-convex function on generalized polymatroid*, Math. Oper. Res. **24** (1999), 95–105.

[24] K. Murota and A. Shioura *Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati–Tardella*, Discrete Appl. Math. **115** (2001), 151–176.

[25] K. Murota and A. Tamura, *New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities*, Discrete Appl. Math. **131** (2003), 495–512.

[26] A. E. Roth and M. A. O. Sotomayor, *Two-Sided Matching — A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press, Cambridge, 1990.

[27] A. E. Roth and M. A. O. Sotomayor, *Stable outcomes in discrete and continuous models of two-sided matching: A unified treatment*, Rev. Econom. **16** (1996), 1–24.

[28] L. S. Shapley and M. Shubik, *The assignment game I: The core*, Internat. J. Game Theory **1** (1972), 111–130.

[29] A. Shioura, *Fast scaling algorithms for M-convex function minimization with application to the resource allocation problem*, Discrete Appl. Math. **134** (2004), 303–316.

[30] M. Sotomayor, *Three remarks on the many-to-many stable matching problem: Dedicated to David Gale on his 75th birthday*, Math. Social Sci. **38** (1999), 55–70.

[31] M. Sotomayor, *Existence of stable outcomes and the lattice property for a unified matching market*, Math. Social Sci. **39** (2000), 119–132.

[32] A. Tamura, *Coordinatewise domain scaling algorithm for M-convex function minimization*, Math. Programming **102** (2005), 339–354.