THE MODULI STACK OF GIESEKER-SL_2-BUNDLES ON A NODAL CURVE II

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1. Introduction

Let X_0 be an irreducible projective nodal curve with only one singular point, and let $\mathcal{P}_0$ be a line bundle on X_0. The moduli $SU_{X_0}(r;\mathcal{P}_0)$ of rank r vector bundles on X_0 with determinant $\mathcal{P}_0$ is not compact. In [A], using the technique of Kausz ([K1], [K2]), we constructed a compactification $GSL_2B(X_0;\mathcal{P}_0)$ of $SU_{X_0}(2;\mathcal{P}_0)$, and studied its structure. Surprisingly, despite its seemingly natural definition, $GSL_2B(X_0;\mathcal{P}_0)$ is not a good compactification. It has two components and one of them is non-reduced. This means that if $(X_0,\mathcal{P}_0)$ is a degeneration of $(X_b,\mathcal{P}_b)$, where $b\in B$ is a parameter and X_b ($b\neq 0$) is an irreducible smooth projective curve and $\mathcal{P}_b$ is a line bundle on X_b, $GSL_2B(X_0;\mathcal{P}_0)$ is not a semistable degeneration of $SU_{X_b}(2;\mathcal{P}_b)$. In this paper, we introduce a new compactification of $SU_{X_0}(2;\mathcal{P}_0)$, and prove that it gives a semistable reduction of the above degeneration. Moreover we prove a decomposition theorem for the generalized theta divisors on this new moduli space.

The contents of the sections are as follows. In section 2, we introduce basic definitions. In section 3, we introduce the new compactification of the moduli of vector bundles. In section 4, we study the local structure of the moduli space. In section 5, we study the global structure of the moduli space. The arguments in section 4 and 5 are quite similar to those in [A]. That is why we omitted some details. In section 6, we prove a decomposition theorem for the generalized theta divisors. In section 7, we collected some facts about the compactification KSL_2 of SL_2 that are used in the preceding sections.

2. Preliminaries

In this section, we fix notation and introduce basic definitions that are used throughout this paper.

Setting. As in [A], we put $B:=\mathrm{Spec}\mathbb{C}[[t]]$. $B_0\hookrightarrow B$ is the closed point and $B_\eta(\subset B)$ is the generic point of B. $\pi:\mathcal{X}\to B$ is a stable curve of genus $g\geq 2$ over B such that the generic fiber X_η is smooth, the special fiber X_0 is an irreducible curve with only one node Q. We assume that $\mathcal{X}$ is a regular scheme. Moreover we assume that $\pi:\mathcal{X}\to B$ is induced by an analytic family $\mathcal{X}^{an}\to B^{an}$, where B^{an} is a small open neighborhood of $0\in\mathbb{C}$. We fix a $\mathbb{C}[[t]]$-isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{X},Q}\simeq\mathbb{C}[[x_1,x_2,t]]/(x_1x_2-t) \tag{2.1}$$

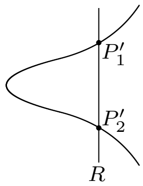
that is induced by an analytic isomorphism $\mathcal{O}^{an}_{\mathcal{X}^{an},Q}\simeq\mathbb{C}\{\{x_1,x_2,t\}\}/(x_1x_2-t)$. Let $\mathfrak{n}:\widetilde{X}_0\to X_0$ be the normalization, and put $\{P_1,P_2\}:=\mathfrak{n}^{-1}(Q)$ so that the local coordinate x_i gives the local coordinate at P_i.

Put $B':=\mathrm{Spec}\mathbb{C}[[t']]$ and let $B'\to B$ be given by $t'^2=t$. $B'_0\hookrightarrow B$ is the closed point and B'_η is the generic point. The point $Q\in\mathcal{X}$ gives, by base-change, the singular point Q' on $\mathcal{X}\times_B B'$. We have the isomorphism $\widehat{\mathcal{O}}_{\mathcal{X}\times_B B',Q'}\simeq\mathbb{C}[[x_1,x_2,t']]/(x_1x_2-t'^2)$. If $k:\mathcal{X}'\to\mathcal{X}\times_B B'$ denotes the blowing-up of $\mathcal{X}\times_B B'$

at the point Q', then $\mathcal{X}'$ is a regular scheme and $\pi' : \mathcal{X}' \to B'$ is a flat family of nodal curves over B'.

$$
\begin{array}{ccc}
\mathcal{X}' & & \\
\downarrow{\scriptstyle k} & & \\
\mathcal{X} \times_B B' & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B
\end{array}
\tag{2.2}
$$

Let R be the smooth rational curve $k^{-1}(Q')$ on $\mathcal{X}'$. Put$X_0' := \mathcal{X}' \times_{B'} B_0'$. Note that X_0' is a union of $\widetilde{X}_0$ and R. Let P_i' be the singular point of X_0' such that $\{P_i'\}|_{\widetilde{X}_0} = \{P_i\}$.

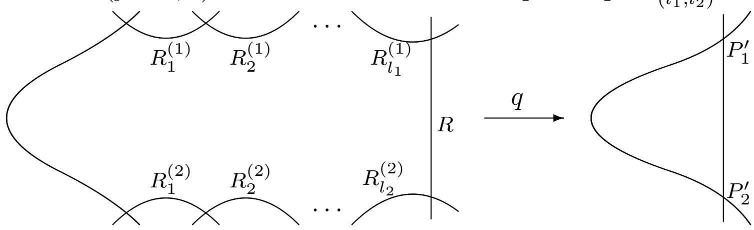


When the point P_i' on X_0' is regarded as a point on R, we also denote it by P_i'. The isomorphism (2.1) induces the isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{X}', P_i'} \simeq \mathbb{C}[[x_i, y_i, t']]/(x_i y_i - t'), \tag{2.3}$$

where $y_i = t'/x_i$.

Modifications. Let $R^{(j)} := R_1^{(j)} \cup \cdots \cup R_{l_j}^{(j)}$ $(l_j \geq 0, \quad j = 1, 2)$ be a chain of rational curves, where $R_i^{(j)} \simeq \mathbb{P}^1$, and $R_{i_1}^{(j)} \cap R_{i_2}^{(j)} \neq \emptyset$ if and only if $|i_1 - i_2| \leq 1$. Let a_j, b_j be closed points of $R_1^{(j)}, R_{l_j}^{(j)}$ respectively such that if $l_j = 1$ then $a_j \neq b_j$, and if $l_j > 1$ then $a_j \neq R_1^{(j)} \cap R_2^{(j)}$ and $b_j \neq R_{l_j-1}^{(j)} \cap R_{l_j}^{(j)}$. Let $X'_{(l_1,l_2)}$ be the nodal curve that is obtained by identifying the pair of points (P_j, P_j') on $\widetilde{X}_0 \sqcup R$ with (a_j, b_j) on $R^{(j)}$ $(j = 1, 2)$. We have the natural morphism $q : X'_{(l_1,l_2)} \to X_0'$.



Definition 2.1. (i) Let T be a B'-scheme and let $f : T \to B'$ denote the structure morphism. A modification of $\mathcal{X}'$ over T is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{h'} & \mathcal{X}' \times_{B'} T \\
& {\scriptstyle pr_2 \circ h'} \searrow \quad \swarrow {\scriptstyle pr_2} & \\
& T &
\end{array}
$$

such that $\mathcal{Y}'$ is flat, proper and of finite presentation over T, and that for any field K and any morphism $\mathrm{Spec}K \to T$ if $f(\mathrm{Spec}K)$ is B_η' then $h' \times \mathrm{id}_{\mathrm{Spec}K} : \mathcal{Y}' \times_T \mathrm{Spec}K \to \mathcal{X}' \times_{B'} \mathrm{Spec}K$ is an isomorphism, and if

$f(\mathrm{Spec}K)$ is B'_0 then for some $(l_1,l_2) \in (\mathbb{Z}_{\geq 0})^{\oplus 2}$ there is an isomorphism $g : X_{(l_1,l_2)} \times \mathrm{Spec}K \to \mathcal{Y}' \times_T \mathrm{Spec}K$ satisfying $(h' \times \mathrm{id}_{\mathrm{Spec}K}) \circ g = q \times \mathrm{id}_{\mathrm{Spec}K}$.

(ii) Let T be a B'_0-scheme. A modification of X'_0 over T is a modification of $\mathcal{X}'$ over T, where T is regarded as a B'-scheme by $B'_0 \hookrightarrow B'$.

(iii) If K is a field and $\mathrm{Spec}K \to B'_0$ is a morphism and $Y' \xrightarrow{h'} X'_0 \times \mathrm{Spec}K$ is a modification of X'_0 over $\mathrm{Spec}K$, then (l_1,l_2) that appears in (i) is called the length of the modification.

Definition 2.2. Let K be a field over $\mathbb{C}$.

(i) $h' := q \times \mathrm{id} : X'_{(l_1,l_2)} \times \mathrm{Spec}K \to X'_0 \times \mathrm{Spec}K$ is a modification of length $(l_1,l_2) \in (\mathbb{Z}_{\geq 0})^{\oplus 2}$ of X'_0 over $\mathrm{Spec}K$. A vector bundle E' on $X'_{(l_1,l_2)} \times \mathrm{Spec}K$ is said to be admissible iff all the conditions (a), (b) and (c) below hold;
 (a) $E'|_{R_i^{(j)}} \simeq \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus \mathrm{rank}E' - m}$ with $m > 0$ (m depends on j and i).
 (b) $E'|_R$ is globally generated.
 (c) If, by abuse of notation, $k : X'_0 \to X_0$ denotes the restriction of $k : \mathcal{X}' \to \mathcal{X} \times_B B'$ to the fiber over B'_0, then $((k \circ q) \times \mathrm{id}_{\mathrm{Spec}K})_* E'$ is a torsion-free sheaf on $X_0 \times \mathrm{Spec}K$.

(ii) Let $h : Y' \to X'_0 \times \mathrm{Spec}K$ be a modification of X'_0 over $\mathrm{Spec}K$ and let $g : X_{(l_1,l_2)} \times \mathrm{Spec}K \to Y'$ be as in (i) of Definition 2.1. A vector bundle E' on Y' is said to be admissible iff g^*E is admissible.

(iii) Let $f : T \to B'$ be a morphism and let $h' : \mathcal{Y}' \to \mathcal{X}' \times_{B'} T$ be a modification of $\mathcal{X}'$ over T. A vector bundle $\mathcal{E}'$ on $\mathcal{Y}'$ is said to be admissible iff for any $\mathrm{Spec}K \to T$, where K is a field, such that $f(\mathrm{Spec}K) = B'_0$, the pullback of $\mathcal{E}'$ to $\mathcal{Y}' \times_T \mathrm{Spec}K$ is admissible.

Notation for points on $X'_{(1,1)}$. In the sequel, we will deal with modifications of X'_0 of length $(1,1)$ many times. Therefore, we here prepare the notation for singular points of $X'_{(1,1)}$ as follows. (Note that $X'_{(1,1)} = \widetilde{X}_0 \cup R_1^{(1)} \cup R_1^{(2)} \cup R$.)

$$\begin{aligned} \{P_j\} &:= \widetilde{X}_0 \cap R_1^{(j)} \\ \{S_j\} &:= R \cap R_1^{(j)}. \end{aligned} \tag{2.4}$$

3. Gieseker-SL$_2$-bundles on $\mathcal{X}'/B'$

In the rest of this paper, we fix a line bundle $\mathcal{P}$ on $\mathcal{X}$, of degree d on the fibers over B. $\mathcal{P}'$ denotes the pullback of $\mathcal{P}$ to $\mathcal{X}'$. Put $\mathcal{P}_0 := \mathcal{P}|_{X_0}$ and $\mathcal{P}'_0 := \mathcal{P}'|_{X'_0}$.

Definition 3.1. Let S be a B'-scheme. A Gieseker-SL$_2$-bundle with determinant $\mathcal{P}'$ on $\mathcal{X}'$ over S, or a Gieseker-SL$_2$-bundle on $(\mathcal{X}';\mathcal{P}')$ over S, is the following data:

- a modification $h' : \mathcal{Y}' \to \mathcal{X}' \times_{B'} S$,
- an admissible 2-bundle $\mathcal{E}'$ on $\mathcal{Y}'$, of degree d on the fibers over S,
- an $\mathcal{O}_{\mathcal{Y}'}$-module homomorphism $\delta'^{(0)} : \det \mathcal{E}' \to (pr_1 \circ h')^* \mathcal{P}'$,
- an $\mathcal{O}_{\mathcal{Y}'}$-module homomorphism $\delta'^{(1)} : (pr_1 \circ h')^* \mathcal{P}'(-R) \to \det \mathcal{E}'$,

where we require the composite $\delta'^{(0)} \circ \delta'^{(1)}$ is the multiplication by $(pr_1 \circ h')^* \mathbf{1}_R$, here $\mathbf{1}_R \in \mathcal{O}_{\mathcal{X}'}(R)$ is the canonical section.

$GSL_2B'(\mathcal{X}'/B';\mathcal{P}')$ denotes the B'-groupoid that associates to an affine B'-scheme S the groupoid consisting of all the Gieseker-SL$_2$-bundles on $(\mathcal{X}';\mathcal{P}')$ over S. $GSL_2B'(X'_0;\mathcal{P}'_0)$ denotes the B'_0-groupoid $GSL_2B'(\mathcal{X}'/B';\mathcal{P}') \times_{B'} B'_0$.

Proposition 3.2. *$GSL_2B'(\mathcal{X}'/B';\mathcal{P}')$ and $GSL_2B'(X'_0;\mathcal{P}'_0)$ are algebraic stacks.*

Lemma 3.3. *Let K be a field extension of $\mathbb{C}$. Let* $(Y' \xrightarrow{h'} X'_0 \times_{B'_0} \mathrm{Spec}K, \mathcal{E}', \det\mathcal{E}' \xrightarrow{\delta'^{(0)}} (pr_1 \circ h')^*\mathcal{P}'_0, (pr_1 \circ h')^*\mathcal{P}'_0(-R) \xrightarrow{\delta'^{(1)}} \det\mathcal{E}')$ *be a Gieseker-SL_2-bundle on* $(\mathcal{X}'; \mathcal{P}')$ *over* $\mathrm{Spec}K$*. Then there are three possibilities:*

(Type $0'$) *h' is an isomorphism and $\mathcal{E}'|_R \simeq \mathcal{O}^{\oplus 2}$.*
(Type $1'$) *h' is an isomorphism and $\mathcal{E}'|_R \simeq \mathcal{O}(1)^{\oplus 2}$.*
(Type $2'$) *$Y' \xrightarrow{h'} X'_0 \times_{B'_0} \mathrm{Spec}K$ is a modification of length $(1,1)$, and $\mathcal{E}'|_R \simeq \mathcal{O}^{\oplus 2}$ and $\mathcal{E}'|_{R^{(i)}} \simeq \mathcal{O} \oplus \mathcal{O}(1)$ $(i = 1, 2)$.*

Proof. By (b) and (c) in Definition 2.2 (i), we have $E|_R \simeq \mathcal{O}^{2-a} \oplus \mathcal{O}(1)^a$ with $2 \geq a \geq 0$. Let $E|_{R_i^{(j)}} \simeq \mathcal{O}^{2-a_i^{(j)}} \oplus \mathcal{O}(1)^{a_i^{(j)}}$. By Lemma 3.3 of [K2], we have $A := a + \sum_{i=1}^{l_1} a_i^{(1)} + \sum_{i=1}^{l_2} a_i^{(2)} \leq 2$. If $A = 0$, then we have Type $0'$. Assume that $A > 0$. We have $A = 2$ because $\delta'^{(0)} \neq 0$. If $a = 2$, we have Type $1'$. If $a < 2$, we have $l_j > 0$ for $j = 1$ and 2 because $\delta'^{(1)}|_{\mathbf{X}_0} \neq 0$. Thus $(l_1, l_2) = (1, 1)$ and $a = 0$. We have Type $2'$. □

4. Local Structure

In this section, we shall investigate the local structure of $GSL_2B'(\mathcal{X}'/B'; \mathcal{P}')$.

Let us fix an object $\mathbb{E}'_0 := (Y' \xrightarrow{h'_0} X'_0, E'_0, \det E'_0 \xrightarrow{\delta_0'^{(0)}} h_0'^*\mathcal{P}'_0, h_0'^*(\mathcal{P}'_0(-R)) \xrightarrow{\delta_0'^{(1)}} \det E'_0)$ of $GSL_2B'(\mathcal{X}'/B'; \mathcal{P}')(B'_0)$. Put $L'_0 := (\det E'_0)^\vee \otimes h_0'^*\mathcal{P}'_0$ and let $\sigma_0'^{(0)}$ be the global section of L'_0 corresponding to $\delta_0'^{(0)}$ and let $\sigma_0'^{(1)}$ be the global section of $L_0'^\vee \otimes h_0'^*(\mathcal{O}_{\mathcal{X}'}(R)|_{X'_0})$ corresponding to $\delta_0'^{(1)}$. Put $\mathbb{L}'_0 := (Y' \xrightarrow{h'_0} X'_0, L'_0, \sigma_0'^{(0)}, \sigma_0'^{(1)})$.

As in [A, §5], we introduce deformation functors.

Definition 4.1. The functors $\mathcal{G}'$ and $\mathcal{F}'$ from $\mathcal{A}$(=the category of artinian local $\mathbb{C}[[t']]$- algebras) to the category of sets are defined as follows. For $A \in \mathcal{A}$,

$$\mathcal{G}'(A) := \left\{ \begin{array}{l} \mathbb{E}' \in GSL_2B'(\mathcal{X}'/B'; \mathcal{P}')(\mathrm{Spec}A) \\ \text{with isomorphism } \mathbb{E}' \times_{\mathrm{Spec}A} B'_0 \xrightarrow{\alpha'} \mathbb{E}'_0. \end{array} \right\} \Big/ \sim_{\mathcal{G}'},$$

$\mathcal{F}'(A)$

$$:= \left\{ \begin{array}{l|l} \mathbb{L}' := (\mathcal{Y}' \xrightarrow{h'} \mathcal{X}' \times_{B'} \mathrm{Spec}A, & \mathcal{Y}' \xrightarrow{h'} \mathcal{X}' \times_{B'} \mathrm{Spec}A \text{ is} \\ \mathcal{L}', \sigma'^{(0)}, \sigma'^{(1)}) & \text{a modification of } \mathcal{X}'/B' \text{ over } \mathrm{Spec}A. \\ \text{with isomorphism} & \mathcal{L}' \text{ is a line bundle on } \mathcal{Y}'. \\ \mathbb{L}' \times_{\mathrm{Spec}A} B'_0 \xrightarrow{\beta'} \mathbb{L}'_0 & \sigma'^{(0)} \text{ and } \sigma'^{(1)} \text{ are global sections of } \mathcal{L}' \text{ and} \\ & \mathcal{L}'^\vee \otimes h'^*\mathcal{O}(R) \text{ respectively} \\ & \text{such that } \sigma'^{(0)} \cdot \sigma'^{(1)} = h'^*(\mathbf{1}_R). \end{array} \right\} \Big/ \sim_{\mathcal{F}'},$$

where the equivalence relations $\sim_{\mathcal{G}'}$ and $\sim_{\mathcal{F}'}$ are defined obviously.

As we get $\mathbb{L}'_0$ out of $\mathbb{E}'_0$, we have the natural transformation $\Psi' : \mathcal{G}' \to \mathcal{F}'$. One can check Ψ' is smooth. By this, in order to understand the local structure of $GSL_2B'(\mathcal{X}'/B'; \mathcal{P}')$, we have only to investigate the versal deformation of $\mathcal{F}'$.

Theorem 4.2. *Let $h_{R'} \to \mathcal{F}'$ be a hull of $\mathcal{F}'$.*

($0'$ or $1'$) If $\mathbb{E}'_0$ is of Type $0'$ or of Type $1'$, then we have an isomorphism $R' \simeq \mathbb{C}[[t']]$ of $\mathbb{C}[[t']]$-algebras.

($2'$) If $\mathbb{E}'_0$ is of Type $2'$, then we have an isomorphism $R' \simeq \mathbb{C}[[t', t_0, t_1, u]]/(t' - t_0t_1)$ of $\mathbb{C}[[t']]$-algebras.

In what follows, we shall explicitly construct versal deformations of $\mathcal{F}'$ depending on the type of $\mathbb{E}'_0$. The verification that they are in fact versal deformations is analogous to the proof of Theorem 5.6 in [A] and is left to the reader.

Type $0'$: On $\mathcal{X}'$, put $\mathcal{L}' := \mathcal{O}_{\mathcal{X}'}$, $\sigma'^{(0)} := 1 \in \mathcal{L}'$, $\sigma'^{(1)} := \mathbf{1_R} \in \mathcal{L}'^{\vee} \otimes \mathcal{O}_{\mathcal{X}'}(R) = \mathcal{O}_{\mathcal{X}'}(R)$. Then the quadruple $(\mathcal{X}' \xrightarrow{\text{id}} \mathcal{X}', \mathcal{L}', \sigma'^{(0)}, \sigma'^{(1)})$ gives a versal deformation of $\mathcal{F}'$.

Type $1'$: On $\mathcal{X}'$, put $\mathcal{L}' := \mathcal{O}_{\mathcal{X}'}(R)$, $\sigma'^{(0)} := \mathbf{1}_R \in \mathcal{L}'$ and $\sigma'^{(1)} := 1 \in \mathcal{L}'^{\vee} \otimes \mathcal{O}_{\mathcal{X}'}(R) \simeq \mathcal{O}_{\mathcal{X}'}$. Then the quadruple $(\mathcal{X}' \xrightarrow{\text{id}} \mathcal{X}', \mathcal{L}', \sigma'^{(0)}, \sigma'^{(1)})$ gives a versal deformation of $\mathcal{F}'$.

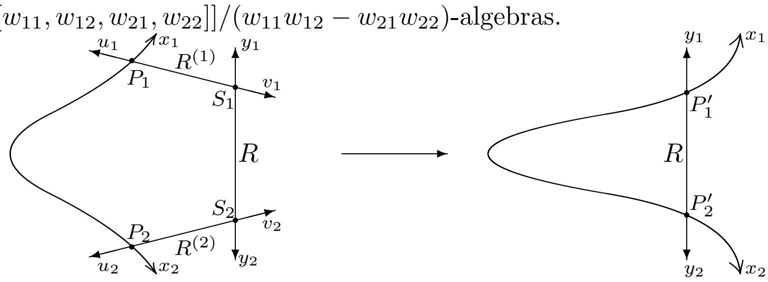
Type $2'$: In this case, the construction of the versal deformation is more involved than the previous two cases. Put $W := \mathrm{Spec}\mathbb{C}[[w_{11}, w_{12}, w_{21}, w_{22}]]/(w_{11}w_{12} - w_{21}w_{22})$ and let $f : W \to B'$ be given by $f^*(t') = w_{11}w_{12} (= w_{21}w_{22})$. At the point P'_i on the central fiber of $\mathcal{X}' \times_{B'} W$, we have an isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{X}' \times_{B'} W, P'_i} \simeq \frac{\mathbb{C}[[w_{11}, w_{12}, w_{21}, w_{22}, x_i, y_i]]}{(w_{11}w_{12} - w_{21}w_{22}, x_iy_i - w_{i1}w_{i2})} \tag{4.1}$$

as $\mathbb{C}[[w_{11}, w_{12}, w_{21}, w_{22}]]/(w_{11}w_{12} - w_{21}w_{22})$-algebra. Blowing up $\mathcal{X}' \times_{B'} W$ by the ideal (x_i, w_{i1}) (precisely speaking in the category of analytic spaces), we obtain the modification $h'_W : \mathcal{Y}' \to \mathcal{X}' \times_{B'} W$ over W, such that the central fiber is a modification of lengh $(1,1)$. If we put $u_i := w_{i1}/x_i$ and $v_i := x_i/w_{i1}$, then we have isomorphisms

$$\begin{aligned} \widehat{\mathcal{O}}_{\mathcal{Y}', P_i} &\simeq \frac{\mathbb{C}[[w_{11}, w_{12}, w_{21}, w_{22}, x_i, u_i]]}{(w_{11}w_{12} - w_{21}w_{22}, x_iu_i - w_{i1})} \\ \widehat{\mathcal{O}}_{\mathcal{Y}', S_i} &\simeq \frac{\mathbb{C}[[w_{11}, w_{12}, w_{21}, w_{22}, y_i, v_i]]}{(w_{11}w_{12} - w_{21}w_{22}, y_iv_i - w_{i2})} \end{aligned} \tag{4.2}$$

of $\mathbb{C}[[w_{11}, w_{12}, w_{21}, w_{22}]]/(w_{11}w_{12} - w_{21}w_{22})$-algebras.



Put

$$R' := \frac{\mathbb{C}[[w_{11}, w_{12}, w_{21}, w_{22}, q]]}{(w_{11}(1+q) - w_{21}, w_{12} - w_{22}(1+q))}$$

and $D := \mathrm{Spec}R'$. Let $\mathfrak{m}'$ be the maximal ideal of R' and put $R'_n := R'/\mathfrak{m}'^{n+1}$. By abuse of notation, the images of w_{ij} and $q \in R'$ into R'_n are also denoted by w_{ij} and q. If $h'_{R'_n} : \mathcal{Y}'_{R'_n} \to \mathcal{X}' \times_{B'} \mathrm{Spec}R'_n$ is the pullback of $h'_W : \mathcal{Y}' \to \mathcal{X}' \times_{B'} W$ by $\mathrm{Spec}R'_n \to W$, $\mathcal{Y}'_{R'_n}$ is isomorphic to $X'_{(1,1)}$ as a topological space, and (4.2) induces isomorphisms $(i = 1, 2)$

$$\begin{aligned} \widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n}, P_i} &\simeq \frac{R'_n[[x_i, u_i]]}{(x_iu_i - w_{i1})} \\ \widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n}, S_i} &\simeq \frac{R'_n[[y_i, v_i]]}{(y_iv_i - w_{i2})}. \end{aligned} \tag{4.3}$$

By $R'_n[[x_i, u_i]]/(x_iu_i - w_{i1}) \hookrightarrow R'_n((x_i)) \oplus R'_n((u_i))$, where $x_i \mapsto (x_i, w_{i1}/u_i)$ and $u_i \mapsto (w_{i1}/x_i, u_i)$, we have

$$\mathrm{H}^0\left(\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n}, P_i} - \{P_i\}, \mathcal{O}\right) \simeq R'_n((x_i)) \oplus R'_n((u_i)).$$

Similarly

$$\mathrm{H}^0\left(\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},S_i} - \{S_i\}, \mathcal{O}\right) \simeq R'_n((y_i)) \oplus R'_n((v_i)).$$

We put $U := \mathcal{Y}'_{R'_n} - \{P_1, P_2, S_1, S_2\}$. Let $j^{(P_i)}_{R'_n}$ and $j^{(S_i)}_{R'_n}$ denote the natural morphisms $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},P_i} \to \mathcal{Y}'_{R_n}$ and $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},S_i} \to \mathcal{Y}'_{R_n}$ respectively. Now let us construct a line bundle $\mathcal{L}'_n$ and sections $\sigma'^{(0)}_n \in \mathcal{L}'_n$ and $\sigma'^{(1)}_n \in \mathcal{L}'^{\vee}_n \otimes (pr_1 \circ h'_{R'_n})^*\mathcal{O}_{\mathcal{X}'}(R)$ on $\mathcal{Y}'_{R'_n}$. Let $\mathcal{L}'_n$ be the line bundle on $\mathcal{Y}'_{R'_n}$ that has the trivializations $\varphi^{(P_i)}_{R'_n} : j^{(P_i)*}_{R'_n}\mathcal{L}'_n \to \widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},P_i}$, $\varphi^{(S_i)}_{R'_n} : j^{(S_i)*}_{R'_n}\mathcal{L}'_n \to \widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},S_i}$ and $\psi^U_{R'_n} : \mathcal{L}'_n|_U \to (\mathcal{O}_{\mathcal{Y}'_{R'_n}}|_U)$ such that $\psi^U_{R'_n} \circ \varphi^{(P_1)-1}_{R'_n}$ is given, on $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},P_1} - \{P_1\}$, by $(1/x_1, u_1(1+q))$-multiplication, $\psi^U_{R'_n} \circ \varphi^{(P_2)-1}_{R'_n}$ is given, on $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},P_2} - \{P_2\}$, by $(1/x_2, u_2)$-multiplication, and $\psi^U_{R'_n} \circ \varphi^{(S_i)-1}_{R'_n}$ is identity on $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},S_i} - \{S_i\}$ $(i = 1, 2)$. Let $\sigma'^{(0)}_n$ be the global section of $\mathcal{L}'_n$ such that

$$\psi^U_{R'_n}(\sigma'^{(0)}_n) = \begin{cases} 1 & \text{on } \widetilde{X}_0 - \{P_1, P_2\} \\ w_{11}(1+q) = w_{21} & \text{on } R^{(1)} \cup R^{(2)} \cup R - \{P_1, P_2, S_1, S_2\} \end{cases} \tag{4.4}$$

and $\varphi^{(P_i)}_{R'_n}(j^{(P_i)*}_{R'_n}\sigma'^{(0)}_n) = x_i$ and $\varphi^{(S_i)}_{R'_n}(j^{(S_i)*}_{R'_n}\sigma'^{(0)}_n) = w_{11}(1+q) = w_{21}$. Before defining $\sigma'^{(1)}_n$, note that the line bundle $(pr_1 \circ h'_{R'_n})^*\mathcal{O}_{\mathcal{X}'}(R)$ has trivializations $\alpha^{(P_i)} : j^{(P_i)*}_{R'_n}(pr_1 \circ h'_{R'_n})^*\mathcal{O}_{\mathcal{X}'}(R) \to \mathcal{O}_{\mathcal{Y}'_{R_n},P_i}$, $\alpha^{(S_i)} : j^{(S_i)*}_{R'_n}(pr_1 \circ h'_{R'_n})^*\mathcal{O}_{\mathcal{X}'}(R) \to \mathcal{O}_{\mathcal{Y}'_{R_n},S_i}$ and $\alpha^U : (pr_1 \circ h'_{R'_n})^*\mathcal{O}_{\mathcal{X}'}(R)|_U \to \mathcal{O}_{\mathcal{Y}'_{R'_n}}|_U$ such that $\alpha^U \circ \alpha^{(P_i)-1}$ is given, on $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},P_i} - \{P_i\}$, by $(1/x_i, 1)$, $\alpha^U \circ \alpha^{(S_i)-1}$ is given, $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{Y}'_{R'_n},S_i} - \{S_i\}$, by $(y_i, 1)$ and

$$\alpha^U((pr_1 \circ h'_{R'_n})^*\mathbf{1}_R|_U) = \begin{cases} 1 & \text{on } \widetilde{X}_0 - \{P_1, P_2\} \\ \frac{w_{i1}}{u_i} = w_{i1}v_i = x_i & \text{on } R^{(i)} - \{P_i, S_i\} \\ w_{11}w_{12} = w_{21}w_{22} & \text{on } R - \{S_1, S_2\} \end{cases} \tag{4.5}$$

and $\alpha^{(P_i)}(j^{(P_i)*}_{R'_n}(pr_1 \circ h'_{R'_n})^*\mathbf{1}_R) = x_i$ and $\alpha^{(S_i)}(j^{(S_i)*}_{R'_n}(pr_1 \circ h'_{R'_n})^*\mathbf{1}_R) = w_{i1}v_i$. The trivializations of $\mathcal{L}'$ and $(pr_1 \circ h'_{R'_n})^*\mathcal{O}_{\mathcal{X}'}(R)$ give rise to the trivializations of $\mathcal{M}_n := \mathcal{L}'^{\vee} \otimes (pr_1 \circ h'_{R'_n})^*\mathcal{O}_{\mathcal{X}'}(R)$: $\beta^{(P_i)} : j^{(P_i)*}_{R'_n}\mathcal{M}_n \to \mathcal{O}_{\mathcal{Y}'_{R_n},P_i}$, $\beta^{(S_i)} : j^{(S_i)*}_{R'_n}\mathcal{M}_n \to \mathcal{O}_{\mathcal{Y}'_{R_n},S_i}$ and $\beta^U : \mathcal{M}_n|_U \to \mathcal{O}_{\mathcal{Y}'_{R'_n}}|_U$. We have

$$\begin{aligned} \beta^{(U)} \circ \beta^{(P_1)-1} &= (1, \frac{1}{u_1(1+q)}) \\ \beta^{(U)} \circ \beta^{(P_2)-1} &= (1, \frac{1}{u_2}) \\ \beta^{(U)} \circ \beta^{(S_i)-1} &= (y_i, 1). \end{aligned} \tag{4.6}$$

Let $\sigma'^{(1)}_n$ be the global section of $\mathcal{M}_n$ such that

$$\beta^U(\sigma'^{(1)}_n|_U) = \begin{cases} 1 & \text{on } \widetilde{X}_0 - \{P_1, P_2\} \\ \frac{1}{u_1(1+q)} = \frac{v_1}{1+q} & \text{on } R^{(1)} - \{P_1, S_1\} \\ \frac{w_{12}}{1+q} = w_{22} & \text{on } R - \{S_1, S_2\} \\ \frac{1}{u_2} = v_2 & \text{on } R^{(2)} - \{P_2, S_2\} \end{cases}$$

and $\beta^{(P_i)}(j^{(P_i)*}_{R'_n}\sigma'^{(1)}_n) = 1$, $\beta^{(S_1)}(j^{(S_1)*}_{R'_n}\sigma'^{(1)}_n) = v_1/(1+q)$ and $\beta^{(S_2)}(j^{(S_2)*}_{R'_n}\sigma'^{(1)}_n) = v_2$.

Then the projective system of the quadruples $\{(h'_{R'_n} : \mathcal{Y}'_{R'_n} \to \mathcal{X}' \times_{B'} \mathrm{Spec}R'_n, \mathcal{L}'_n, \sigma'^{(0)}_n, \sigma'^{(1)}_n)\}_{n\geq 0}$ gives a versal deformation of $\mathcal{F}'$.

5. Global Structure

Notation 5.1. Let K be a field extension of $\mathbb{C}$. Let $(h' : Y' \to X'_0\times\mathrm{Spec}K, E', \delta'^{(0)} : \det E' \to (pr_1\circ h')^*\mathcal{P}'_0, \delta'^{(1)} : (pr_1\circ h')^*\mathcal{P}'_0(-R) \to \det E')$ be an object of $GSL_2B'(X'_0;\mathcal{P}'_0)(\mathrm{Spec}K)$ that is of Type $0'$ or of Type $1'$. If it is of Type $0'$, then the point $h'^{-1}(\{P'_i\})$ is denoted by S_i. If it is of Type $1'$, then the point $h'^{-1}(\{P'_i\})$ is denoted by P_i.

Type $0'$ Type $1'$

Proposition 5.2. *Take a B'_0-scheme T and let $(h' : \mathcal{Y}' \to X'_0\times T, \mathcal{E}', \delta'^{(0)} : \det\mathcal{E}' \to (pr_1\circ h')^*\mathcal{P}'_0, \delta'^{(1)} : (pr_1\circ h')^*\mathcal{P}'_0(-R) \to \det\mathcal{E}')$ be an object of $GSL_2B'(X'_0;\mathcal{P}'_0)(T)$. Then there exist closed subsets $Z(P_i)$ and $Z(S_i)$ of $\mathcal{Y}'$ such that for any $t \in T$ $Z(P_i)|_{\mathcal{Y}'_t} = \{P_i\}$ and $Z(S_i)|_{\mathcal{Y}'_t} = \{S_i\}$ $(i = 1, 2)$.*

Proof. We may assume that T is of finite type over B'_0. By abuse of notation, the restriction of the morphism $k : \mathcal{X}' \to \mathcal{X} \times_B B'$ to the fiber over B'_0 is also denoted by $k : X'_0 \to X_0$. Let us consider $h : \mathcal{Y} := \mathrm{Proj}\oplus_{m\geq 0}((k\times \mathrm{id}_T)\circ h')_*((\det\mathcal{E}')^{\otimes m}) \to X_0 \times T$. Using Lemma 1.4 and Corollary 1.5 of [Kn], we can check that $\mathcal{Y}$ is flat over T and that there exists a natural morphism $contr : \mathcal{Y}' \to \mathcal{Y}$ with $(k\times\mathrm{id})\circ h' = h\circ contr$. Over $t \in T$, $contr$ collapses the rational curves on which $\mathcal{E}'$ is trivial. If we put $\mathcal{E} := contr_*\mathcal{E}'$, then we have the isomorphism $contr^*\mathcal{E} \simeq \mathcal{E}'$. $\delta'^{(0)}$ induces $\delta : \det\mathcal{E} \to (pr_1\circ h)^*\mathcal{P}_0$. The triple $(h : \mathcal{Y} \to X_0 \times T, \mathcal{E}, \delta : \det\mathcal{E} \to (pr_1\circ h)^*\mathcal{P}_0)$ is an object of $GSL_2B(X_0;\mathcal{P}_0)(T)$ (cf. §4 of [A]). Let Π_j $(j = 0, 1, 2)$ be the closed subset of $\mathcal{Y}$ defined in [A, Proposition 6.1]. Then the closed subset $contr^{-1}(\Pi_i)$ $(i = 1, 2)$ is $Z(P_i)$. The closed subset $contr^{-1}(\Pi_0) \cap h'^{-1}(\{P'_i\} \times T)$ $(i = 1, 2)$ is $Z(S_i)$. □

We endow each closed subset $Z(P_i)$ and $Z(Q_i)$ $(i = 1, 2)$ with the scheme structure defined by the first Fitting ideal of $\Omega_{\mathcal{Y}'/T}$. Then $(pr_2\circ h')|_{Z(P_i)} : Z(P_i) \to T$ and $(pr_2\circ h')|_{Z(S_i)} : Z(S_i) \to T$ $(i = 1, 2)$ are closed immersions. Moreover, by the description of the versal family of $GSL_2B'(X'_0;\mathcal{P}'_0)$, $Z(P_1)$ and $Z(P_2)$ (resp. $Z(S_1)$ and $Z(S_2)$) define the same closed subscheme of T. Let $\mathcal{I} \subset \mathcal{O}_T$ (resp. $\mathcal{J}$) be the ideal sheaf of the closed subscheme defined by $Z(P_i)$ (resp. $Z(S_i)$).

Definition 5.3. $GSL_2B'(X'_0;\mathcal{P}'_0)^{(0)}$ (resp. $GSL_2B'(X'_0;\mathcal{P}'_0)^{(1)}$ and $GSL_2B'(X'_0;\mathcal{P}'_0)^{(2)}$) is defined to be the closed substack defined by the ideal $\mathcal{J}$ (resp. $\mathcal{I}$ and $\mathcal{I}+\mathcal{J}$).

Theorem 5.4. *We have an isomorphism of B'_0-groupoids*

$$GSL_2B'(X'_0;\mathcal{P}'_0)^{(0)} \simeq GSL_2B(X_0;\mathcal{P}_0)^{(0)}.$$

(Rigorously speaking, the right-hand side should be written as $GSL_2B(X_0;\mathcal{P}_0)^{(0)}\times_{B_0} B'_0$. See §6 of [A] for the definition of $GSL_2B(X_0;\mathcal{P}_0)^{(0)}$.)

Proof. In the proof of Proposition 5.2, we constructed a morphism $GSL_2B'(X'_0;\mathcal{P}'_0) \to GSL_2B(X_0;\mathcal{P}_0)$, which induces a morphism $\Phi : GSL_2B'(X'_0;\mathcal{P}'_0)^{(0)} \to GSL_2B(X_0;\mathcal{P}_0)^{(0)}$. Let us construct the inverse of Φ. Let T be an affine B'_0-scheme. Given an object $(\mathcal{Y} \xrightarrow{h} X_0 \times T, \mathcal{E}, \delta : \det \mathcal{E} \to (prf \circ h)^* \mathcal{P}_0)$ of $GSL_2B(X_0; \mathcal{P}_0)^{(0)}(T)$. Let Π_0 be the closed subscheme of $\mathcal{Y}$ defined in the paragraph 6.2 in [A] and let $g : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be the blowing-up along Π_0. There exists a unique morphism $\tilde{h} : \widetilde{\mathcal{Y}} \to \widetilde{X}_0 \times T$ with $(\mathfrak{n} \times \mathrm{id}_T) \circ \tilde{h} = h \circ g$. $\widetilde{\mathcal{Y}}/T$ has two sections s_1 and s_2 such that $g^{-1}(\Pi_0) = s_1(T) \sqcup s_2(T)$ with $\tilde{h} \circ s_i = P_i \times \mathrm{id}_T$. Since $X'_0 \times T$ is obtained by gluing $\widetilde{X}_0 \times T$ and $R \times T$ along $\{P'_1\} \times T \sqcup \{P'_2\} \times T$, if $\mathcal{Y}'$ is the scheme obtained by gluing $\widetilde{\mathcal{Y}}$ and $R \times T$ along the two sections, we have a morphism $h' : \mathcal{Y}' \to \widetilde{X}_0 \times T$. We also have a morphism $contr : \mathcal{Y}' \to \mathcal{Y}$ by contracting $R \times T$ to Π_0. We have $h \circ contr = (k \times \mathrm{id}_T) \circ h'$.

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{h'} & X'_0 \times T \\ \Big\downarrow{\scriptstyle contr} & & \Big\downarrow{\scriptstyle k \times \mathrm{id}_T} \\ \mathcal{Y} & \xrightarrow[h]{} & X_0 \times T. \end{array}$$

Put $\mathcal{E}' := contr^* \mathcal{E}$. $\delta : \det \mathcal{E} \to (pr_1 \circ h)^* \mathcal{P}_0$ induces $\delta'^{(0)} : \det \mathcal{E}' \to (pr_1 \circ h')^* \mathcal{P}'_0$. By Lemma 6.11 in [A], the line bundle $(\det \mathcal{E}') \otimes h'^*(\mathcal{P}_0'^{\vee} \otimes \mathcal{O}(R))|_{\widetilde{\mathcal{Y}}}(-s_1(T) - s_2(T))$ on $\widetilde{\mathcal{Y}}$ is trivial. Since $\det \mathcal{E}' \otimes h'^*(\mathcal{P}_0'^{\vee} \otimes \mathcal{O}(R))|_{\widetilde{\mathcal{Y}}}(-s_1(T) - s_2(T)) \hookrightarrow \det \mathcal{E}' \otimes h'^*(\mathcal{P}_0'^{\vee} \otimes \mathcal{O}(R))$, we can find a morphism $\delta'^{(1)} : h'^*(\mathcal{P}'_0(-R)) \to \det \mathcal{E}'$ which is nonzero on every fiber over T. Adjusting $\delta'^{(1)}$ so that $\delta'^{(0)} \circ \delta'^{(1)}$ is $h'^*(\mathbf{1}_R)$-multiplication, we obtain an object of $GSL_2B'(X'_0; \mathcal{P}'_0)^{(0)}(T)$. □

5.5. Next we describe the global structure of $GSL_2B'(X'_0; \mathcal{P}'_0)^{(1)}$. We fix a line bundle L of degree one on the smooth rational curve R and we also fix an isomorphism

$$L^{\otimes 2} \simeq \mathcal{O}_R(-R) (:= \mathcal{O}_{\mathcal{X}'}(-R)|_R). \tag{5.1}$$

There exists a unique isomorphism

$$\mathcal{O}_{\mathcal{X}'}(R)|_{\widetilde{X}_0} \simeq \mathcal{O}_{\widetilde{X}_0}(P_1 + P_2) \tag{5.2}$$

such that $\mathbf{1}_R|_{\widetilde{X}_0} \in \mathcal{O}_{\mathcal{X}'}(R)|_{\widetilde{X}_0}$ corresponds to $1 \in \mathcal{O}_{\widetilde{X}_0}(P_1 + P_2)$. By (5.1) and (5.2), we have the isomorphism $(i = 1, 2)$

$$L^{\otimes 2}|_{P'_i} \simeq \mathcal{O}_{\widetilde{X}_0}(-P_i)|_{P_i}. \tag{5.3}$$

Let $SU_2(\widetilde{X}_0; \widetilde{\mathcal{P}}_0(-P_1 - P_2))$ be the moduli stack of 2-bundles on $\widetilde{X}_0$ with determinant $\widetilde{P}_0(-P_1 - P_2)$, that is, for an affine B'_0-scheme T, objects of the groupoid $SU_2(\widetilde{X}_0; \widetilde{\mathcal{P}}_0(-P_1 - P_2))(T)$ are 2-bundles $\mathcal{W}$ on $\widetilde{X}_0 \times T$ together with an isomorphism $\wedge^2 \mathcal{W} \simeq pr_1^* \widetilde{\mathcal{P}}_0(-P_1 - P_2)$. On $\widetilde{X}_0 \times SU_2(\widetilde{X}_0; \widetilde{\mathcal{P}}_0(-P_1 - P_2))$, we have the universal 2-bundle $\mathcal{W}_{univ}$ together with the isomorphism $\wedge^2 \mathcal{W}_{univ} \simeq pr_1^* \widetilde{\mathcal{P}}_0(-P_1 - P_2)$. Put $\tau_i := (P_i, \mathrm{id}) : SU_2(\widetilde{X}_0; \widetilde{\mathcal{P}}_0(-P_1 - P_2)) \to \widetilde{X}_0 \times SU_2(\widetilde{X}_0; \widetilde{\mathcal{P}}_0(-P_1 - P_2))$ $(i = 1, 2)$. Put $\mathcal{V}_i := (\tau_i^* \mathcal{W}_{univ}) \otimes (L|_{P'_i})^{\vee}$. We have isomorphisms

$$\begin{aligned} \det \mathcal{V}_i &\simeq (\det \tau_i^* \mathcal{W}_{univ}) \otimes_{\mathbb{C}} (L|_{P'_i})^{\otimes -2} \\ &\simeq \left(\tau_i^* pr_1^* \widetilde{\mathcal{P}}_0(-P_1 - P_2) \right) \otimes_{\mathbb{C}} (L|_{P'_i})^{\otimes -2} \\ &= \tau_i^* pr_1^* \widetilde{\mathcal{P}}_0(-P_i) \otimes_{\mathbb{C}} (L|_{P'_i})^{\otimes -2} \\ &\simeq \tau_i^* pr_1^* \widetilde{\mathcal{P}}_0(-P_i) \otimes_{\mathbb{C}} \left(\mathcal{O}_{\widetilde{X}_0}(P_i)|_{P_i} \right) \quad \text{by (5.3)} \\ &\simeq \tau_i^* pr_1^* \widetilde{\mathcal{P}}_0 = \mathcal{O}_{SU} \otimes_{\mathbb{C}} (\mathcal{P}_0|_Q). \end{aligned}$$

By this we have the canonical isomorphism

$$\theta : \det \mathcal{V}_1 \simeq \det \mathcal{V}_2. \tag{5.4}$$

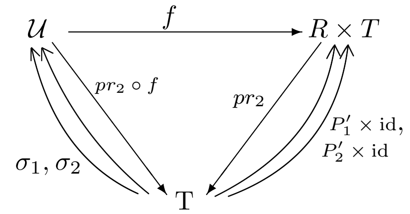
This allows us to consider $KSL_2\,(\mathcal{V}_1, \mathcal{V}_2)$.

Theorem 5.6. *We have an isomorphism of B_0'-groupoids*

$$GSL_2B'(X_0'; \mathcal{P}_0')^{(1)} \simeq KSL_2\,(\mathcal{V}_1, \mathcal{V}_2)\,.$$

Proof. Let T be an affine scheme of finite type over B_0'. Let $\mathbb{E}' := (\mathcal{Y}' \xrightarrow{h'} X_0' \times T, \mathcal{E}', \det\mathcal{E}' \xrightarrow{\delta'^{(0)}} (pr_1 \circ h')^*\mathcal{P}_0', (pr_1 \circ h')^*\mathcal{P}_0'(-R) \xrightarrow{\delta'^{(1)}} \det\mathcal{E}')$ be an object of $GSL_2B'(X_0'; \mathcal{P}_0')^{(1)}(T)$. We shall transform $\mathbb{E}'$ into equivalent data. By the definition of $GSL_2B'(X_0'; \mathcal{P}_0')^{(1)}$, the closed subschemes $Z(P_1)$ and $Z(P_2)$ of $\mathcal{Y}'$ are sections over T. Blowing up $\mathcal{Y}'$ along $Z(P_1) \sqcup Z(P_2)$, we obtain a morphism $(\widetilde{X}_0 \times T) \sqcup \mathcal{U} \xrightarrow{bl} \mathcal{Y}'$ such that $(h \circ bl)|_{\widetilde{X}_0 \times T} : \widetilde{X}_0 \times T \to X_0' \times T$ is a base-change of the inclusion $\widetilde{X}_0 \hookrightarrow X_0'$ and $(h' \circ bl)|_{\mathcal{U}} : \mathcal{U} \to X_0' \times T$ factors as $\mathcal{U} \xrightarrow{f} R \times T \hookrightarrow X_0' \times T$. The invers-image of $Z(P_i)$ to $\widetilde{X}_0 \times T$ is the section $\{P_i\} \times T$, and let $\sigma_i \subset \mathcal{U}$ (or $\sigma_i : T \to \mathcal{U}$) be the section over T that is the inverse-image of $Z(P_i)$ to $\mathcal{U}$. Then $f : \mathcal{U} \to R \times T$ is a bi-simple modification of the two-pointed curve $(R; P_1', P_2')$. (See [A, Definition 6.6] for the definition of a bi-simple modification.) Put $\mathcal{F} := (bl|_{\widetilde{X}_0 \times T})^*\mathcal{E}'$. The pullback of $\delta'^{(0)}$ by $bl|_{\widetilde{X}_0 \times T}$ gives $\det\mathcal{F} \to pr_1^*\widetilde{\mathcal{P}}_0$, which is the composite of the isomorphism $\alpha : \det\mathcal{F} \to pr_1^*(\widetilde{\mathcal{P}}_0(-P_1 - P_2))$ and the natural inclusion $pr_1^*(\widetilde{\mathcal{P}}_0(-P_1 - P_2)) \hookrightarrow pr_1^*\widetilde{\mathcal{P}}_0$. Note that the pullback of $\delta'^{(1)}$ by $bl|_{\widetilde{X}_0 \times T}$ gives also an isomorphism $pr_1^*(\widetilde{\mathcal{P}}_0 \otimes (\mathcal{O}_{\mathcal{X}'}(-R)|_{\widetilde{X}_0})) \to \det\mathcal{F}$, and it is the inverse of α, where $\mathcal{O}_{\mathcal{X}'}(-R)|_{\widetilde{X}_0} \simeq \mathcal{O}_{\widetilde{X}_0}(-P_1 - P_2)$ by (5.2). Put $\mathcal{G} := (bl|_{\mathcal{U}})^*\mathcal{E}'$. Let $\beta : (f^*pr_1^*\mathcal{O}_R(-R)) \otimes_{\mathbb{C}} \mathcal{P}_Q \to \det\mathcal{G}$ be the pullback of $\delta'^{(1)}$ by $bl_{\mathcal{U}}$. It is easy to see that $\sigma_i^*(\beta) : \mathcal{O}_T \otimes_{\mathbb{C}} (\mathcal{O}_R(-R)|_{P_i'}) \otimes_{\mathbb{C}} \mathcal{P}_Q \to \det\sigma_i^*\mathcal{G}$ is an isomorphism. By the definition of $\mathcal{F}$ and $\mathcal{G}$, we have a canonical isomorphism $\varphi_i : (P_i, \mathrm{id}_T)^*\mathcal{F} \xrightarrow{\sim} \sigma_i^*\mathcal{G}$ $(i = 1, 2)$. After all, we obtained the following data:

(i) A 2-bundle $\mathcal{F}$ on $\widetilde{X}_0 \times T$,
(ii) An isomorphism $\alpha : \det\mathcal{F} \xrightarrow{\sim} pr_1^*(\mathcal{P}_0(-P_1 - P_2))$,
(iii) A bi-simple modification of the two-pointed curve $(R; P_1', P_2')$ over T



(iv) A 2-bundle $\mathcal{G}$ on $\mathcal{U}$ satisfying (#),
(v) A homomorphism $\beta : (f^*pr_1^*\mathcal{O}_R(-R)) \otimes_{\mathbb{C}} \mathcal{P}_Q \to \det\mathcal{G}$,

where (#) is the following condition:

> (#) For $t \in T$, if $\mathcal{U} \times_T \mathrm{Spec}\kappa(t) \xrightarrow{f \times \mathrm{id}} R \times \mathrm{Spec}\kappa(t)$ is an isomorphism, then $\mathcal{G}|_{\mathcal{U} \times_T \mathrm{Spec}\kappa(t)} \simeq \mathcal{O}(1)^{\oplus 2}$, and if $\mathcal{U} \times_T \mathrm{Spec}\kappa(t) \simeq R \cup R^{(1)} \cup R^{(2)} \xrightarrow{f \times \mathrm{id}} R \times \mathrm{Spec}\kappa(t)$, then $\mathcal{G}|_R \simeq \mathcal{O}^{\oplus 2}$, $\mathcal{G}|_{R^{(i)}} \simeq \mathcal{O} \oplus \mathcal{O}(1)$ $(i = 1, 2)$ and $\mathrm{H}^0((\mathcal{G}|_{\mathcal{U} \times \mathrm{Spec}\kappa(t)})(-\sigma_1(t) - \sigma_2(t))) = 0$.

Moreover the diagram

$$(\clubsuit)\qquad \begin{array}{ccc} (P_i, \mathrm{id})^* \det\mathcal{F} & \xrightarrow{(P_i, \mathrm{id})^*} & (\mathcal{P}_Q \otimes_{\mathbb{C}} P_i^*\mathcal{O}_{\widetilde{X}_0}(-P_i)) \otimes_{\mathbb{C}} \mathcal{O}_T \\ \wr\big\downarrow \det(\varphi_i) & & \wr\big\downarrow \\ \sigma_i^* \det\mathcal{G} & \xleftarrow{\sigma_i^*(\beta)} & (\mathcal{P}_Q \otimes_{\mathbb{C}} P_i'^*\mathcal{O}_R(-R)) \otimes_{\mathbb{C}} \mathcal{O}_T \end{array}$$

commutes ($i = 1, 2$). These data are equivalent to $\mathbb{E}'$, that is, we can reconstruct $\mathbb{E}'$ from these data. Let us see that the above data (i) $\sim$ (v) give an object of $KSL_2(\mathcal{V}_1, \mathcal{V}_2)(T)$ and vice versa. Put $\mathcal{N} := \mathcal{O}_{\mathcal{U}}(-\sigma_1 - \sigma_2) \otimes (pr_1 \circ f)^*\mathcal{O}_R(P_1' + P_2')$ and let $\nu \in \mathcal{N}$ be the canonical section. Put $\mathcal{H} := f_*(\mathcal{G} \otimes \mathcal{N})$. Then by Proposition 7.4 of [K2], we have bf-morphisms of rank one

$$(N_i, \nu_i, \sigma_i^*\mathcal{G} \to (P_i', \mathrm{id})^*\mathcal{H}, N_i \otimes (P_i', \mathrm{id})^*\mathcal{H} \to \sigma_i^*\mathcal{G}, 1), \tag{5.5}$$

where $N_i := \sigma_i^*\mathcal{N}$ and $\nu_i := \sigma_i^*\nu$ ($i = 1, 2$).

Claim 5.6.1. For any $t \in T$, $\mathcal{H}|_{R \times \mathrm{Spec}\kappa(t)} \simeq \mathcal{O}(1)^{\oplus 2}$.

Proof of Claim 5.6.1. If f is an isomorphism over $t \in T$, then the claim is obvious by (#). Assume that, over $t \in T$, f is of the form $\mathcal{U} \times_T \mathrm{Spec}\kappa(t) \simeq R \cup R^{(1)} \cup R^{(2)} \to R \times \mathrm{Spec}\kappa(t)$. Then $\mathrm{H}^0((\mathcal{H}|_{R\times\mathrm{Spec}\kappa(t)})(-P_1' - P_2')) = 0$ by (#). Since $\deg(\mathcal{H}|_{R\times\mathrm{Spec}\kappa(t)})(-P_1' - P_2') = -2$, we have $(\mathcal{H}|_{R\times\mathrm{Spec}\kappa(t)})(-P_1' - P_2') \simeq \mathcal{O}(-1)^{\oplus 2}$. $\square$

Put $\mathcal{T} := pr_{2*}(\mathcal{H} \otimes pr_1^*L^{-1})$. Then we have an isomorphism $pr_2^*\mathcal{T} \simeq \mathcal{H} \otimes pr_1^*L^{-1}$, which induces an isomorphism ($i = 1, 2$)

$$\mathcal{T} \otimes_{\mathbb{C}} (L|_{P_i'}) \simeq (P_i' \times \mathrm{id})^*\mathcal{H}. \tag{5.6}$$

(5.5) and (5.6) imply

$$\begin{aligned} \sigma_i^* \det \mathcal{G} &\simeq (\det \mathcal{T}) \otimes N_i^{-1} \otimes_{\mathbb{C}} (L|_{P_i'})^{\otimes 2} \\ &\simeq (\det \mathcal{T}) \otimes N_i^{-1} \otimes_{\mathbb{C}} (\mathcal{O}_R(-R)|_{P_i'}). \end{aligned} \tag{5.7}$$

This and β imply $N_i \simeq (\det \mathcal{T})^\vee \otimes_{\mathbb{C}} \mathcal{P}_Q$. Let v be the composite $N_1 \to (\det \mathcal{T})^\vee \otimes_{\mathbb{C}} \mathcal{P}_Q \to N_2$.

Claim 5.6.2. $v(\nu_1) = \nu_2$.

Proof of Claim 5.6.2. Let $\overline{\mathcal{U}}$ be the family of nodal curves that is obtained from $\mathcal{U}$ by gluing σ_1 and σ_2, and let $\overline{R}$ be the nodal rational curve obtained from R by gluing P_1' and P_2'. We have the commutative diagram:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\ f\ } & R \times T \\ \big\downarrow & & \big\downarrow \\ \overline{\mathcal{U}} & \xrightarrow{\ \overline{f}\ } & \overline{R} \times T. \end{array} \tag{5.8}$$

Let $\overline{N}$ be the line bundle obtained from $\mathcal{N}$ gluing along σ_1^* and σ_2^* by the isomorphism v. By [A, Lemma 6.11], $\det \mathcal{G} \simeq f^*(\det \mathcal{H}) \otimes \mathcal{N}^{-1}$, so we have $\det \mathcal{G} \simeq (pr_2 \circ f)^* \det \mathcal{T} \otimes L^{\otimes 2} \otimes \mathcal{N}^{-1} \simeq (pr_2 \circ f)^* \det \mathcal{T} \otimes (pr_1 \circ f)^*\mathcal{O}_R(-R) \otimes \mathcal{N}^{-1}$. The existence of β implies that we have a morphism $(pr_2 \circ \overline{f})^*(\det \mathcal{T} \otimes_{\mathbb{C}} \mathcal{P}_Q^\vee) \to \overline{\mathcal{N}}^{-1}$ that is fiber-wisely nonzero. Therefore we have the isomorphism $\det \mathcal{T} \otimes_{\mathbb{C}} \mathcal{P}_Q^\vee \simeq (pr_2 \circ \overline{f})_*(\overline{\mathcal{N}}^{-1})$. This implies that $(pr_2 \circ \overline{f})_*(\overline{\mathcal{N}}^{-1})$ is a line bundle on T and commutes with base-change. This is equivalent to that both $(pr_2 \circ \overline{f})_*(\overline{\mathcal{N}}^{-1})$ and $R^1(pr_2 \circ \overline{f})_*(\overline{\mathcal{N}}^{-1})$ are line bunldes.

In order to prove the claim, we may assume that $T = \mathrm{Spec}A$ with A an artinian local $\mathbb{C}$-algebra. By [NS, Appendix III (iv)], we have $\omega_{\overline{\mathcal{U}}/T} \simeq \overline{f}^*\omega_{\overline{R}\times T}$. Since $p_a(\overline{R}) = 0$, $\omega_{\overline{\mathcal{U}}/T} \simeq \mathcal{O}_{\overline{\mathcal{U}}}$. Then by Theorem 11.1 of [H] for a vector bundle $\mathcal{M}$ on $\overline{\mathcal{U}}$, we have a spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_T}(R^{-q}(pr_2 \circ \overline{f})_*\mathcal{M}, \mathcal{O}_T) \Rightarrow R^{p+q+1}(pr_2 \circ \overline{f})_*(\mathcal{M}^\vee). \tag{5.9}$$

Therefore $R^j(pr_2\circ\overline{f})_*(\overline{\mathcal{N}}^\vee)$ is a line bundle for $j=0,1$ if and only if $R^j(pr_2\circ\overline{f})_*(\overline{\mathcal{N}})$ is a line bundle for $j=0,1$. The latter condition is equivalent to $v(\nu_1)=\nu_2$ This completes the proof of Claim 5.6.2. □

The bf-morphism (5.5) together with the isomorphisms$\varphi : (P_i,\mathrm{id}_T)^*\mathcal{F}\to\sigma_i^*\mathcal{G}$ and (5.6) gives rise to a bf-morphism

$$(P_i,\mathrm{id}_T)^*\mathcal{F}\otimes_{\mathbb{C}}(L|_{P_i'})^\vee \overset{\otimes N_i}{\curvearrowleft\!\longrightarrow} \mathcal{T}, \tag{5.10}$$

or equivalently

$$\mathcal{T}\otimes N_i \overset{\otimes N_i}{\curvearrowleft\!\longrightarrow} (P_i,\mathrm{id}_T)^*\mathcal{F}\otimes_{\mathbb{C}}(L|_{P_i'})^\vee \tag{5.11}$$

By (#) and the commutative diagram (♣), these data satisfy the conditions (a) and (b) in Definition 3.1 of [A]. Thus we obtained an object of $KSL_2(\mathcal{V}_1,\mathcal{V}_2)(T)$ The reconstruction of the data (i)∼ (v) from an these data is straightforward. □

6. Decomposition Theorem

Definition 6.1. The generalized theta line bundle Θ on $GSL_2B'(\mathcal{X}'/B';\mathcal{P}')$ is defined by associating to each object $(h':\mathcal{Y}'\to\mathcal{X}'\times_{B'}T,\mathcal{E}',\delta'^{(0)},\delta'^{(1)})\in GSL_2B'(\mathcal{X}'/B';\mathcal{P}')(T)$ the line bundle $(\det\mathbf{R}(pr_2\circ h')_*\mathcal{E}')^\vee$ on T. By abuse of notation, the restrictions of Θ to substacks are also denoted by Θ. Let V_m (resp. $V_m^{(i)}$) be the vector space $\mathrm{H}^0(GSL_2B'(X_0';\mathcal{P}_0'),\Theta^m)$ (resp. $\mathrm{H}^0(GSL_2B'(X_0';\mathcal{P}_0')^{(i)},\Theta^m)$) ($m\geq 1$, $i=0,1,2$).

Note that we have the natural homomorphisms $V\to V^{(i)}$, $V^{(0)}\to V^{(2)}$ and $V^{(1)}\to V^{(2)}$.

Proposition 6.2. *The natural homomorphism $V_m\to V_m^{(0)}\times_{V_m^{(1)}}V_m^{(1)}$ is an isomorphism.*

Proof. We just mention the exact sequence

$$0\to R/(xy)\to R/(x)\oplus R/(y)\to R/(x,y)\to 0, \tag{6.1}$$

where $R=\mathbb{C}[[x,y,z_1,z_2,\dots]]$, and leave the details to the reader. □

Proposition 6.3. *The natural homomorphism $V_m^{(1)}\to V_m^{(2)}$ is an isomorphism.*

Proof. We use the notation in the paragraph 5.5. By Theorem 5.6, we have the commutative diagram

(6.2)

$$\begin{array}{ccccc} GSL_2B'(X_0';\mathcal{P}_0')^{(1)} & \simeq & KSL_2(\mathcal{V}_1,\mathcal{V}_2) & \underset{pr}{\longrightarrow} & SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0(-P_1-P_2)) \\ \bigcup & & \bigcup & & \| \\ GSL_2B'(X_0';\mathcal{P}_0')^{(2)} & \simeq & \mathbb{B}(\mathcal{V}_1,\mathcal{V}_2) & \xrightarrow{pr|_{\mathbb{B}}} & SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0(-P_1-P_2)). \end{array}$$

(See §7 for $\mathbb{B}(\mathcal{V}_1,\mathcal{V}_2)$.)

Claim 6.3.1. The line bundle Θ on $GSL_2B'(X_0';\mathcal{P}_0')^{(1)}$ is isomorphic to $pr^*(\det\mathbf{R}pr_{2*}\mathcal{W}_{univ})^\vee$, where $pr_2:\widetilde{X}_0\times SU_2\to SU_2$.

Proof of Claim 6.3.1. We use the notation used in the proof of Theorem 5.6. The claim follows from the isomorphisms

$$\begin{aligned}\det\mathbf{R}(pr_2\circ h')_*\mathcal{E}' &\simeq (\det\mathbf{R}pr_{2*}\mathcal{F}')\otimes\det\mathbf{R}(pr_2\circ f)_*\mathcal{G}\\ &\otimes\det(\sigma_1^*\mathcal{G})^\vee\otimes\det(\sigma_2^*\mathcal{G})^\vee\\ &\simeq(\det\mathbf{R}pr_{2*}\mathcal{F}')\otimes\det\mathbf{R}pr_{2*}(\mathcal{H}\otimes pr_1^*\mathcal{O}_R(-P_1'-P_2'))\\ &\simeq\det\mathbf{R}pr_{2*}\mathcal{F}'.\end{aligned} \tag{6.3}$$

□

Since we have the isomorphisms

$$\begin{aligned}\mathrm{H}^0\left(SU_2\left(\widetilde{X}_0;\widetilde{\mathcal{P}}_0(-P_1-P_2)\right),\mathcal{M}\right) &\simeq \mathrm{H}^0\left(KSL_2\left(\mathcal{V}_1,\mathcal{V}_2\right),pr^*\mathcal{M}\right)\\ &\simeq \mathrm{H}^0\left(\mathbb{B}\left(\mathcal{V}_1,\mathcal{V}_2\right),(pr|_{\mathbb{B}})^*\mathcal{M}\right)\end{aligned} \tag{6.4}$$

for any line bundle $\mathcal{M}$ on $SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0(-P_1-P_2))$, the proposition is proved. □

Corollary 6.4. *The natural homomorphism $V_m \to V_m^{(0)}$ is an isomorphism.*

Proof. This is immediate from Proposition 6.2 and Proposition 6.3. □

6.5. Let $SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0)$ be the moduli stack of 2-bundles on $\widetilde{X}_0$ with determinant $\widetilde{\mathcal{P}}_0$. Put $\sigma_i := (P_i,\mathrm{id}) : SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0) \to \widetilde{X}_0 \times SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0)$ $(i=1,2)$. On $\widetilde{X}_0 \times SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0)$, we have the universal 2-bundle $\mathcal{F}_{univ}$ together with the isomorphism $\det\mathcal{F}_{univ} \simeq pr_1^*\widetilde{\mathcal{P}}_0$. Put

$$PB := \mathbb{P}(\sigma_1^*\mathcal{F}_{univ}) \times_{SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0)} \mathbb{P}(\sigma_2^*\mathcal{F}_{univ}).$$

Put $q_i : PB \to \mathbb{P}(\sigma_i^*\mathcal{F}_{univ})$, $\pi_i : \mathbb{P}(\sigma_i^*\mathcal{F}_{univ}) \to SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0)$ and $\pi := q_i \circ \pi_i$. Let $\Theta_{PB}^m(j)$ be the line bundle $\pi^*(\det\mathbf{R}pr_{2*}\mathcal{F}_{univ})^{\otimes(-m)} \otimes q_1^*\mathcal{O}(j) \otimes q_2^*\mathcal{O}(j) \otimes_{\mathbb{C}} (\mathcal{P}_0)_Q^{\otimes(m-j)}$ on PB.

Proposition 6.6. *We have the canonical isomorphism $V_m^{(0)} \simeq \oplus_{j=0}^m \mathrm{H}^0\left(PB,\Theta_{PB}^m(j)\right)$.*

Proof. By Theorem 5.4 and [A, Theorem 6.4] and Proposition 7.2, we have the diagram

$$\begin{array}{ccccc} GSL_2B'(X_0';\mathcal{P}_0')^{(0)} & \simeq & KSL_2(\sigma_1^*\mathcal{F}_{univ},\sigma_2^*\mathcal{F}_{univ}) & \xrightarrow[pr]{} & SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0) \\ \bigcup & & \bigcup & & \| \\ GSL_2B'(X_0';\mathcal{P}_0')^{(2)} & \simeq & \mathbb{B}(\sigma_1^*\mathcal{F}_{univ},\sigma_2^*\mathcal{F}_{univ}) & \xrightarrow{pr|_{\mathbb{B}}} & SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0) \\ & & \wr\downarrow & & \| \\ & & PB & \xrightarrow{\pi} & SU_2(\widetilde{X}_0;\widetilde{\mathcal{P}}_0). \end{array} \tag{6.5}$$

Claim 6.6.1. The line bundle Θ on $GSL_2B'(X_0';\mathcal{P}_0')^{(0)}$ is isomorphic to the line bundle $\mathcal{O}_{KSL_2}(\mathbb{B}) \otimes pr^*(\det\mathbf{R}pr_{2*}\mathcal{F}_{univ})^\vee \otimes_{\mathbb{C}} (\mathcal{P}_0)_Q$ on $KSL_2(\sigma_1^*\mathcal{F}_{univ},\sigma_2^*\mathcal{F}_{univ})$, where $pr_2 : \widetilde{X}_0 \times SU_2 \to SU_2$.

Proof of Claim 6.6.1. Analogous to the proof of Claim 6.3.1. □

Therefore, by Proposition 7.5 and Proposition 7.3, we have the isomorphisms

(6.6)

$$\begin{aligned} V_m^{(0)} &\simeq \mathrm{H}^0(KSL_2(\sigma_1^*\mathcal{F}_{univ},\sigma_2^*\mathcal{F}_{univ}),\mathcal{O}_{KSL_2}(m\mathbb{B}) \otimes pr^*(\det\mathbf{R}pr_{2*}\mathcal{F}_{univ})^{\otimes(-m)} \otimes_{\mathbb{C}} (\mathcal{P}_0)_Q^{\otimes m})\\ &\simeq \bigoplus_{j=0}^m \mathrm{H}^0(\mathbb{B}(\sigma_1^*\mathcal{F}_{univ},\sigma_2^*\mathcal{F}_{univ}),\mathcal{O}_{\mathbb{B}}(j\mathbb{B}) \otimes (pr|_{\mathbb{B}})^*(\det\mathbf{R}pr_{2*}\mathcal{F}_{univ})^{\otimes(-m)} \otimes_{\mathbb{C}} (\mathcal{P}_0)_Q^{\otimes m})\\ &\simeq \bigoplus_{j=0}^m \mathrm{H}^0(PB,\Theta_{PB}^m(j)). \end{aligned}$$

□

Corollary 6.7. *We have the canonical isomorphism*

$$\mathrm{H}^0(GSL_2B'(X_0';\mathcal{P}_0'),\Theta^m) \simeq \bigoplus_{j=0}^m \mathrm{H}^0(PB,\Theta_{PB}^m(j)). \tag{6.7}$$

7. Appendix

In this appendix, we gather some facts on KSL_2.

Universal family. Let S be a scheme over Spec$\mathbb{C}$, and let $\mathcal{E}$ and $\mathcal{F}$ be trivial 2-bundles on S. Let $\theta : \det \mathcal{E} \simeq \mathcal{O} \to \mathcal{O} \simeq \det \mathcal{F}$ be the identity map. Let $\mathbb{P}$ be the S-scheme $\mathrm{Proj}\mathcal{O}_S[x_{11}, x_{12}, x_{21}, x_{22}, x_{00}]$, and Q be the closed subscheme of $\mathbb{P}$ defined by $x_{11}x_{22} - x_{12}x_{21} - x_{00}^2 = 0$. Put $\mathbb{B} := Q \cap \{x_{00} = 0\}$. Let π be the projection to S. Let $\mathbf{x} : \pi^*\mathcal{E} \to \pi^*\mathcal{F} \otimes \mathcal{O}_Q(\mathbb{B})$ be given by the matrix $(x_{ij}/x_{00})_{1\leq i,j\leq 2}$. Put $\mathcal{E}_1 := \mathbf{x}^{-1}(\pi^*\mathcal{F})$ and $\mathcal{F}_1 := \mathbf{x}(\mathcal{E}_1) \subset \pi^*\mathcal{F}$. We have natural morphisms

$$\begin{aligned} \pi^*\mathcal{E} &\hookrightarrow \mathcal{E}_1 \otimes \mathcal{O}(\mathbb{B}) \\ \pi^*\mathcal{F} &\hookrightarrow \mathcal{F}_1 \otimes \mathcal{O}(\mathbb{B}). \end{aligned} \tag{7.1}$$

Hence, on Q, we have the diagram of isomorphisms and bf-morphisms:

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\mathbf{x}} & \mathcal{F}_1 \\ {\scriptstyle \otimes\mathcal{O}(B)}\upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright{\scriptstyle \otimes\mathcal{O}(B)} \\ \pi^*\mathcal{E} & & \pi^*\mathcal{F}\,. \end{array}$$

This gives the universal family of $KSL_2(\mathcal{E}, \mathcal{F})$.

Degenerate locus.

Definition 7.1. Let $\mathbb{B}(\mathcal{E}, \mathcal{F})$ be the subfunctor of $\mathcal{KSL}_2(\mathcal{E}, \mathcal{F})$ defined by the additional condition $\mu_1 = \mu_2 = 0$ in [A, Definition 3.1].

Clearly the functor $\mathbb{B}(\mathcal{E}, \mathcal{F})$ is represented by $\mathbb{B}$ in the above.

Proposition 7.2. *We have an S-isomorphism*

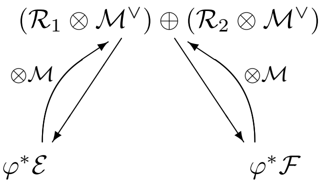
$$\mathbb{B} \simeq \mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{F}). \tag{7.2}$$

Proof. We construct a bijection on T-valued points for an S-scheme $T \xrightarrow{\varphi} S$. For a T-valued point of $\mathbb{B}(\mathcal{E}, \mathcal{F})$

$$\begin{array}{ccc} \mathcal{U}_1 & \longrightarrow & \mathcal{U}_2 \\ {\scriptstyle \otimes\mathcal{M}_1}\upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright{\scriptstyle \otimes\mathcal{M}_2} \\ \varphi^*\mathcal{E} & & \varphi^*\mathcal{F}\,, \end{array}$$

put $\mathcal{R}_1 := \mathrm{Im}(\varphi^*\mathcal{E} \to \mathcal{U}_1 \otimes \mathcal{M}_1)$ and $\mathcal{R}_2 := \mathrm{Im}(\varphi^*\mathcal{F} \to \mathcal{U}_2 \otimes \mathcal{M}_2)$. Then the pair $(\varphi^*\mathcal{E} \twoheadrightarrow \mathcal{R}_1, \varphi^*\mathcal{F} \twoheadrightarrow \mathcal{R}_2)$ gives a T-valued point of $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{F})$. Thus we have $\mathbb{B} \to \mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{F})$.

Conversely, if we are given a T-valued point $(\alpha : \varphi^*\mathcal{E} \twoheadrightarrow \mathcal{R}_1, \beta : \varphi^*\mathcal{F} \twoheadrightarrow \mathcal{R}_2)$ of $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{F})$, put $\mathcal{M} := \mathcal{R}_1 \otimes (\mathrm{Ker}\beta)^\vee$. The isomorphism $\det \varphi^*\mathcal{E} \xrightarrow{\varphi^*\theta} \det \varphi^*\mathcal{F}$ induces the isomorphism $\theta' : \mathcal{R}_1 \otimes (\mathrm{Ker}\beta)^\vee \to \mathcal{R}_2 \otimes (\mathrm{Ker}\alpha)^\vee$. Let $g_1^\# : (\mathcal{R}_1 \otimes \mathcal{M}^\vee) \oplus (\mathcal{R}_2 \otimes \mathcal{M}^\vee) \to \varphi^*\mathcal{E}$ be the composite $(\mathcal{R}_1 \otimes \mathcal{M}^\vee) \oplus (\mathcal{R}_2 \otimes \mathcal{M}^\vee) \xrightarrow{pr_2} \mathcal{R}_2 \otimes \mathcal{M}^\vee = \mathcal{R}_2 \otimes \mathcal{R}_1^\vee \otimes (\mathrm{Ker}\beta) \xrightarrow{\theta'} \mathrm{Ker}\alpha \hookrightarrow \varphi^*\mathcal{E}$. $g_2^\# : (\mathcal{R}_1 \otimes \mathcal{M}^\vee) \oplus (\mathcal{R}_2 \otimes \mathcal{M}^\vee) \to \varphi^*\mathcal{F}$ be the composite $(\mathcal{R}_1 \otimes \mathcal{M}^\vee) \oplus (\mathcal{R}_2 \otimes \mathcal{M}^\vee) \xrightarrow{pr_1} \mathcal{R}_1 \otimes \mathcal{M}^\vee \simeq \mathrm{Ker}\beta \hookrightarrow \varphi^*\mathcal{F}$. Then the diagram



gives a T-valued point of $\mathbb{B}(\mathcal{E}, \mathcal{F})$. Thus we have $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{F}) \to \mathbb{B}$. These morphisms are inverses to each other. □

Line bundle.

Proposition 7.3. *By identifying $\mathbb{B}$ and $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{F})$ by the isomorphism (7.2), we have an isomorphism of line bundles*

$$\mathcal{O}_{\mathbb{B}}(\mathbb{B}) \simeq \big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \boxtimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)\big) \otimes \pi^*(\det \mathcal{F})^\vee. \tag{7.3}$$

Proof. This follows from the proof of Proposition 7.2. □

Remark 7.4. Put $G := SL(\mathcal{E}) \times_S SL(\mathcal{F})$. We have the natural (left) G-action on $KSL_2(\mathcal{E}, \mathcal{F})$ and $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{F})$. G also induces the natural G-linearlizations on the line bundles $\mathcal{O}_{\mathbb{B}}(\mathbb{B})$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \boxtimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. The isomorphisms (7.2) and (7.3) are G-equivariant.

Proposition 7.5. *We have the canonical equivariant isomorphism ($m \geq 0$)*

$$\pi_*\mathcal{O}_Q(m\mathbb{B}) \simeq \bigoplus_{i=0}^{m} \pi_*\mathcal{O}_{\mathbb{B}}(i\mathbb{B}). \tag{7.4}$$

Proof. Since S is a scheme over $\mathrm{Spec}\mathbb{C}$, we may assume that $S = \mathrm{Spec}\mathbb{C}$. We prove this proposition by induction. If $m = 0$, it is trivial. Assume that $m > 0$. We have an exact sequence

$$0 \to \mathcal{O}_Q((m-1)\mathbb{B}) \to \mathcal{O}_Q(m\mathbb{B}) \to \mathcal{O}_{\mathbb{B}}(m\mathbb{B}) \to 0. \tag{7.5}$$

Since $\mathrm{H}^1(Q, \mathcal{O}_Q((m-1)\mathbb{B})) = 0$, we have

$$0 \to \mathrm{H}^0\left(Q, \mathcal{O}_Q((m-1)\mathbb{B})\right) \to \mathrm{H}^0\left(Q, \mathcal{O}_Q(m\mathbb{B})\right) \to \mathrm{H}^0\left(\mathbb{B}, \mathcal{O}_{\mathbb{B}}(m\mathbb{B})\right) \to 0. \tag{7.6}$$

If U_i denotes the irreducible SL_2-representation with $\dim U_i = i+1$, then we have, by induction and Proposition 7.3,

$$\mathrm{H}^0\left(Q, \mathcal{O}_Q\left((m-1)\mathbb{B}\right)\right) \simeq \bigoplus_{i=0}^{m-1} U_i \otimes U_i \tag{7.7}$$

as $SL_2 \times SL_2$-modules. Therefore $\mathrm{H}^0(Q, \mathcal{O}_Q(m\mathbb{B})) \simeq \oplus_{i=0}^m U_i \otimes U_i$ as $SL_2 \times SL_2$-modules, and the exact sequence (7.6) has a canonical splitting. This proves the proposition. □

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References

[A] T. Abe: *The moduli stack of Gieseker-SL_2-bundles on a nodal curve*, preprint RIMS-1460.

[H] R. Hartshorne: *Residues and duality*, Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York (1966)

[K1] I. Kausz: *A modular compactification of the general linear group*, Doc. Math. Vol. 5, 553-594 (2000)

[K2] I. Kausz: *A Gieseker type degeneration of moduli stacks of vector bundles on curves*, arXiv:math.AG/0201197

[Kn] Finn F. Knudsen: *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. 52 (1983), no. 2, 161–199.

[NS] D. S. Nagaraj and C. S. Seshadri: *Degenerations of the moduli spaces of vector bundles on curves II (Generalized Gieseker moduli spaces)*, Proc. Indian Acad. Sci. Vol. 109, 165-201 (1999)

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