SEMI-GRAPHS OF ANABELIOIDS

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Abstract. In this paper, we discuss various "general nonsense" aspects of the geometry of semi-graphs of profinite groups [cf. [Mzk3], Appendix], by applying the language of *anabelioids* introduced in [Mzk16]. After proving certain basic properties concerning various *commensurators* associated to a *semi-graph of anabelioids*, we show that the geometry of a semi-graph of anabelioids may be recovered from the category-theoretic structure of certain naturally associated *categories* — e.g., *"temperoids"* [in essence, the analogue of a Galois category for the "tempered fundamental groups" of [André]] and *"categories of localizations"*. Finally, we apply these techniques to obtain certain results in the *absolute anabelian geometry* [cf. [Mzk3], [Mzk8]] of *tempered fundamental groups* associated to hyperbolic curves over p-adic local fields.

Introduction

In this paper, we continue to pursue the theme of *categorical representation of scheme-theoretic geometries*, which played a central role in [Mzk6], [Mzk7], as well as in the previous *anabelian* work of the author [e.g., [Mzk2], [Mzk3], [Mzk5], [Mzk8]]. The original motivation of the present work lies in the problem of finding an appropriate and efficient way of representing, via categories, the geometry of *"formal localizations"* of *hyperbolic curves over p-adic local fields*. Here, we use the term "formal localizations" to refer to the localizations of the p-adic formal

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completion of a pointed stable log curve over the ring of integers of a p-adic local field obtained by *completing* along the irreducible components and nodes of the geometric logarithmic special fiber specified by some *sub-semi-graph* of the the *"dual semi-graph with compact supports"* [cf. [Mzk3], Appendix] associated to this geometric logarithmic special fiber. Since the geometry of such formal localizations is substantially reflected in the geometry of localizations of the *semi-graph of profinite groups* [cf. [Mzk3], Appendix] associated to this geometric logarithmic special fiber, it is thus natural, from the point of view of the goal of categorical representation of this geometry of formal localizations, to study the geometry of this semi-graph of profinite groups. Moreover, when working with profinite groups as "geometric objects", it is natural to apply the language of *anabelioids* introduced in [Mzk4].

The *main results* of this paper may be summarized as follows:

(1) In §1, we study the *geometry of semi-graphs*, and in particular, expose a proof related to the author by M. Matsumoto of a sort of analogue for certain types of morphisms of finite semi-graphs of *"Zariski's main theorem"* in scheme theory [cf. Theorem 1.2]. This result has some interesting *group-theoretic consequences* related to the author by A. Tamagawa [cf. Corollary 1.6]; in addition, it admits an interesting interpretation from a more "arithmetic" point of view [cf. Remark 1.5.1].

(2) In §2, we begin our study of the *geometry of semi-graphs of anabeloids*. Our main result [cf. Corollary 2.7] concerns certain properties of the *commensurator* in the profinite fundamental group associated to a graph of anabelioids of the various subgroups associated to subgraphs of the given graph of anabelioids.

(3) In §3, we take up the study of *tempered fundamental groups* [cf. [André]], by working with *"temperoids"*, i.e., the analogue for tempered fundamental groups of Galois categories [in the case of profinite groups]. Our main result [cf. Theorem 3.7; Corollary 3.9] states that for certain kinds of graphs of anabelioids, the vertices (respectively, edges) of the underlying graph may be recovered from the associated tempered fundamental group as the [conjugacy classes of] *maximal compact subgroups* (respectively, *nontrivial intersections of distinct maximal compact subgroups*) of this tempered fundamental group. We then apply this result to show, in the case of hyperbolic curves over p-adic local fields, that the *entire dual semi-graph with compact supports* may be recovered solely from the *geometric tempered fundamental group* of such a curve [cf. Corollary 3.11].

(4) Although the tempered fundamental group furnishes perhaps the most efficient way of reconstructing a graph of anabelioids from a naturally associated category, in §4, we examine another natural approach to this problem, via *categories of localizations*. After studying various basic properties of such categories of localizations [including some interesting properties that follow from "Zariski's main theorem for semi-graphs" — cf. Proposition 4.4, (i), (ii)], we show that, given a graph of anabelioids satisfying certain properties, the original graph of anabelioids may be recovered functorially from its associated category of localizations [cf. Theorem 4.8].

(5) In §3, 4, we considered semi-graphs of anabelioids that are *not* equipped with *"Galois actions"*. Thus, in §5, we generalize the [more efficient] theory of §3 [instead of the theory of §4, since this becomes somewhat cumbersome] to the "arithmetic" situation that arises in the case of a hyperbolic curve over a p-adic local field, i.e., of a semi-graph of anabelioids equipped with an "arithmetic action" by a profinite group. The translation of the theory of §3 into its "arithmetic analogue" in §5 is essentially routine, once one replaces, for instance, "maximal compact subgroups" by *"arithmetically maximal compact subgroups"* [cf. Theorem 5.4].

(6) In §6, we consider the *tempered analogue* of the *absolute anabelian geometry* developed in [Mzk8]. In particular, we show that in many respects, this tempered analogue is essentially equivalent to the original profinite version [cf. Theorem 6.6], and, moreover, that the various absolute anabelian results of [Mzk8] concerning *decomposition groups* of closed points — in particular, a sort of *"weak section conjecture"* — also hold in the tempered case [cf. Theorem 6.8; Corollaries 6.9, 6.11]. This is particularly interesting in that the tempered version exhibits, in a very explicit way, the geometry of this "weak section conjecture" in a fashion that is quite reminisicent of the *"discrete real section conjecture"* of [Mzk5], §3.2 [cf. Remark 6.9.1], i.e., relative to the well-known analogy between *geodesics on trees* [cf., e.g., Lemma 1.8, (ii); [Serre]] and *geodesics in Riemannian "straight line spaces"* [i.e., Riemannian spaces satisfying the condition (*) of [Mzk5], §3.2].

Finally, in the Appendix, we discuss a slight generalization of the notions of "temperoids" and "anabelioids" that sometimes appears in practice [cf., e.g., Remark 4.8.4].

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Section 0: Notations and Conventions

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall \ h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \bigcap H \text{ has finite index in } H, \ g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions*

$$H, \ Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$$

and (iii) H is *normal* in $N_G(H)$.

Note that $Z_G(H)$, $N_G(H)$ are *always closed in* G, while $C_G(H)$ is *not necessarily closed in* G.

Categories:

Let $\mathcal{C}$ be a *category*. We shall denote the collection of *objects* of $\mathcal{C}$ by:

$$\text{Ob}(\mathcal{C})$$

If $A \in \text{Ob}(\mathcal{C})$ is an *object* of $\mathcal{C}$, then we shall denote by

$$\mathcal{C}_A$$

the category whose *objects* are morphisms $B \to A$ of $\mathcal{C}$ and whose morphisms (from an object $B_1 \to A$ to an object $B_2 \to A$) are A-morphisms $B_1 \to B_2$ in $\mathcal{C}$. Thus, we have a *natural functor*

$$(j_A)_! : \mathcal{C}_A \to \mathcal{C}$$

(given by forgetting the structure morphism to A). Similarly, if $f : A \to B$ is a *morphism* in $\mathcal{C}$, then f defines a *natural functor*

$$f_! : \mathcal{C}_A \to \mathcal{C}_B$$

by mapping an arrow (i.e., an object of $\mathcal{C}_A$) $C \to A$ to the object of $\mathcal{C}_B$ given by the composite $C \to A \to B$ with f. Also, we shall denote by

$$\mathcal{C}[A] \subseteq \mathcal{C}$$

the *full subcategory* determined by the objects of $\mathcal{C}$ that admit a morphism to A.

If the category $\mathcal{C}$ *admits finite products*, then $(j_A)_!$ is *left adjoint* to the *natural functor*

$$j_A^* : \mathcal{C} \to \mathcal{C}_A$$

given by taking the *product with* A, and $f_!$ is *left adjoint* to the *natural functor*

$$f^* : \mathcal{C}_B \to \mathcal{C}_A$$

given by taking the *fibered product over* B *with* A. We shall call an object $A \in \mathrm{Ob}(\mathcal{C})$ *terminal* if for every object $B \in \mathrm{Ob}(\mathcal{C})$, there exists a unique arrow $B \to A$ in $\mathcal{C}$.

We shall refer to a *natural transformation* between functors all of whose component morphisms are *isomorphisms* as an *isomorphism between the functors* in question. A functor $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ between categories $\mathcal{C}_1$, $\mathcal{C}_2$ will be called *rigid* if ϕ has no nontrivial automorphisms. A category $\mathcal{C}$ will be called *slim* if the natural functor $\mathcal{C}_A \to \mathcal{C}$ is *rigid*, for every $A \in \mathrm{Ob}(\mathcal{C})$.

If G is a profinite group, then we shall denote by

$$\mathcal{B}(G)$$

the category of finite sets with continuous G-action. Thus, $\mathcal{B}(G)$ is a *Galois category*, or, in the terminology of [Mzk4], a *connected anabelioid*. Moreover, $\mathcal{B}(G)$ is *slim* if and only if, for every open subgroup $H \subseteq G$, we have $Z_G(H) = \{1\}$ [cf. [Mzk4], Corollary 1.1.6, Definition 1.2.4].

A diagram of functors between categories will be called 1-*commutative* if the various composite functors in question are *isomorphic*. When such a diagram "commutes in the literal sense" we shall say that it 0-*commutes*. Note that when a diagram in which the various composite functors are all *rigid* "1-commutes", it follows from the *rigidity* hypothesis that any isomorphism between the composite functors in question is necessarily *unique*. Thus, to state that such a diagram 1-commutes does not result in any "loss of information" by comparison to the datum of a *specific isomorphism* between the various composites in question.

Given two functors $\Phi_i : \mathcal{C}_i \to \mathcal{D}_i$ (where $i = 1, 2$) between categories $\mathcal{C}_i$, $\mathcal{D}_i$, we shall refer to a 1-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\sim} & \mathcal{C}_2 \\ \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2} \\ \mathcal{D}_1 & \xrightarrow{\sim} & \mathcal{D}_2 \end{array}$$

— where the horizontal arrows are equivalences of categories — as an *abstract equivalence* from Φ_1 to Φ_2. If there exists an abstract equivalence from Φ_1 to Φ_2, then we shall say that Φ_1, Φ_2 are *abstractly equivalent*.

We shall say that a nonempty [i.e., non-initial] object $A \in \mathrm{Ob}(\mathcal{C})$ is *connected* if it is not isomorphic to the coproduct of two nonempty objects of $\mathcal{C}$. We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *mobile* if there exists an object $B \in \mathrm{Ob}(\mathcal{C})$ such that the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has *cardinality* ≥ 2 [i.e., the diagonal from this set to the product of this set with itself is not bijective]. We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *quasi-connected* if it is either *immobile* [i.e., not mobile] or *connected*. Thus, connected objects are always quasi-connected. If *every* object of a category $\mathcal{C}$ is *quasi-connected*, then we shall say that $\mathcal{C}$ is a *category of quasi-connected objects*. We shall say that a category $\mathcal{C}$ is *totally* (respectively, *almost totally*) *epimorphic* if every morphism in $\mathcal{C}$ whose domain is *arbitrary* (respectively, *nonempty*) and whose codomain is *quasi-connected* is an *epimorphism*.

We shall say that $\mathcal{C}$ is *of finitely (respectively, countably) connected type* if it is closed under formation of finite (respectively, countable) coproducts; every object of $\mathcal{C}$ is a coproduct of a finite (respectively, countable) collection of connected objects; and, moreover, all finite (respectively, countable) coproducts $\coprod A_i$ in the category satisfy the condition that the natural map

$$\coprod \mathrm{Hom}_{\mathcal{C}}(B, A_i) \to \mathrm{Hom}_{\mathcal{C}}(B, \coprod A_i)$$

is *bijective*, for all connected $B \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C}$ is *of finitely or countably connected type*, then every nonempty object of $\mathcal{C}$ is *mobile*; in particular, a nonempty object of $\mathcal{C}$ is connected *if and only if* it is quasi-connected.

If a *mobile* object $A \in \mathrm{Ob}(\mathcal{C})$ satisfies the condition that every morphism in $\mathcal{C}$ whose domain is nonempty and whose codomain is A is an *epimorphism*, then A is *connected*. [Indeed, $C_1 \coprod C_2 \overset{\sim}{\to} A$ implies that the composite map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(C_1, B) \times \mathrm{Hom}_{\mathcal{C}}(C_2, B) \\ = \mathrm{Hom}_{\mathcal{C}}(C_1 \coprod C_2, B) \overset{\sim}{\to} \mathrm{Hom}_{\mathcal{C}}(A, B)\end{aligned}$$

is *bijective*, for all $B \in \mathrm{Ob}(\mathcal{C})$.]

If $\mathcal{C}$ is a *category of finitely or countably connected type*, then we shall write

$$\mathcal{C}^0 \subseteq \mathcal{C}$$

for the *full subcategory* of connected objects. [Note, however, that in general, objects of $\mathcal{C}^0$ are *not necessarily connected* — or even *quasi-connected* — as objects of $\mathcal{C}^0$!] On the other hand, if, in addition, $\mathcal{C}$ is *almost totally epimorphic*, then $\mathcal{C}^0$ is *totally epimorphic*, and, moreover, an object of $\mathcal{C}^0$ is *connected* [as an object of $\mathcal{C}^0$!] if and only if [cf. the argument of the preceding paragraph!] it is *mobile* [as an object of $\mathcal{C}^0$]; in particular, every object of $\mathcal{C}^0$ is *quasi-connected* [as an object of $\mathcal{C}^0$].

If $\mathcal{C}$ is a *category*, then we shall write

$$\mathcal{C}^{\perp} \text{ (respectively, } \mathcal{C}^{\top})$$

for the category formed from $\mathcal{C}$ by taking *arbitrary "formal" [possibly empty] finite (respectively, countable) coproducts* of objects in $\mathcal{C}$. That is to say, we define the "Hom" of $\mathcal{C}^{\perp}$ (respectively, $\mathcal{C}^{\top}$) by the following formula:

$$\mathrm{Hom}(\coprod_i A_i, \coprod_j B_j) \stackrel{\mathrm{def}}{=} \prod_i \coprod_j \mathrm{Hom}_{\mathcal{C}}(A_i, B_j)$$

[where the A_i, B_j are objects of $\mathcal{C}$]. Note that objects of $\mathcal{C}$ define *connected* objects of $\mathcal{C}^\perp$ or $\mathcal{C}^\top$. Moreover, there are natural [up to isomorphism] *equivalences of categories*

$$(\mathcal{C}^\perp)^0 \xrightarrow{\sim} \mathcal{C}; \quad (\mathcal{C}^\top)^0 \xrightarrow{\sim} \mathcal{C}; \quad (\mathcal{D}^0)^\perp \xrightarrow{\sim} \mathcal{D}; \quad (\mathcal{E}^0)^\top \xrightarrow{\sim} \mathcal{E}$$

if $\mathcal{D}$ (respectively, $\mathcal{E}$) is a *category of finitely connected type* (respectively, *category of countably connected type*). If $\mathcal{C}$ is a *totally epimorphic category of quasi-connected objects*, then $\mathcal{C}^\perp$ (respectively, $\mathcal{C}^\top$) is an *almost totally epimorphic category of finitely (respectively, countably) connected type.*

In particular, the operations "0", "$\perp$" (respectively, "$\top$") define *one-to-one correspondences* [up to equivalence] between the *totally epimorphic categories of quasi-connected objects* and the *almost totally epimorphic categories of finitely (respectively, countably) connected type.*

If $\mathcal{C}$ is a *category*, then we shall write $\mathbb{G}(\mathcal{C})$ for the *graph associated to* $\mathcal{C}$. This graph is the graph with precisely one *vertex* for each object of $\mathcal{C}$ and precisely one *edge* for each arrow of $\mathcal{C}$ [joining the vertices corresponding to the domain and codomain of the arrow]. We shall refer to the full subcategories of $\mathcal{C}$ determined by the objects and arrows that compose a connected component of the graph $\mathbb{G}(\mathcal{C})$ as a *connected component* of $\mathcal{C}$. In particular, we shall say that $\mathcal{C}$ is *connected* if $\mathbb{G}(\mathcal{C})$ is connected.

If $\mathcal{C}$ is a *category*, and $\{\phi_i : A_i \to A\}_{i \in I}$ is a collection of arrows in $\mathcal{C}$ all of which have codomain equal to A and nonempty domains, then we shall say that this collection *strongly* (respectively, *weakly*) *dissects* A if, for every pair of *distinct* elements $i, j \in I$, $\mathcal{C}$ fails to contain a pair of arrows $\psi_i : B \to A_i$, $\psi_j : B \to A_j$ (respectively, pair of arrows $\psi_i : B \to A_i$, $\psi_j : B \to A_j$ such that $\phi_i \circ \psi_i = \phi_j \circ \psi_j$), where B is a nonempty object of $\mathcal{C}$. We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *strongly dissectible* (respectively, *weakly dissectible*) if it admits a strongly (respectively, weakly) dissecting pair of arrows. If A is not weakly (respectively, strongly) dissectible, then we shall say that it is *strongly indissectible* (respectively, *weakly indissectible*). Thus, if A is strongly dissectible (respectively, indissectible), then it is weakly dissectible (respectively, indissectible).

If $\mathcal{C}$ if a *category* and $\mathcal{S}$ is a *collection of arrows in* $\mathcal{C}$, then we shall say that an arrow $A \to B$ is *minimal-adjoint to* $\mathcal{S}$ if every factorization $A \to C \to B$ of this arrow $A \to B$ in $\mathcal{C}$ such that $A \to C$ lies in $\mathcal{S}$ satisfies the property that $A \to C$ is, in fact, an *isomorphism.* Often, the collection $\mathcal{S}$ will be taken to be the collection of arrows satisfying a *particular property* $\mathcal{P}$; in this case, we shall refer to the property of being "minimal-adjoint to $\mathcal{S}$" as the *minimal-adjoint notion to* $\mathcal{P}$.

Curves:

Suppose that $g \geq 0$ is an *integer.* Then if S is a scheme, a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism of schemes $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. We shall denote the *moduli stack of r-pointed stable curves of genus g* (where we assume the points to be *unordered*) by $\overline{\mathcal{M}}_{g,r}$ [cf. [DM], [Knud] for an exposition of the theory of such curves; strictly speaking, [Knud] treats the finite étale covering of $\overline{\mathcal{M}}_{g,r}$ determined by *ordering* the marked points]. The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth r-pointed stable curves of genus g* or, alternatively, as the *moduli stack of hyperbolic curves of type* (g, r). The *divisor at infinity* $\overline{\mathcal{M}}_{g,r} \backslash \mathcal{M}_{g,r}$ of $\overline{\mathcal{M}}_{g,r}$ determines a *log structure* on $\overline{\mathcal{M}}_{g,r}$; denote the resulting log stack by $\overline{\mathcal{M}}_{g,r}^{\log}$.

A *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism*. (Indeed, when S is the spectrum of a field, this fact is well-known from the elementary theory of algebraic curves. Next, we consider an arbitrary *connected normal* S on which a prime l is *invertible* (which, by Zariski localization, we may assume without loss of generality). Denote by $S' \to S$ the finite étale covering parametrizing *orderings of the marked points* and *trivializations of the l-torsion points of the Jacobian of* Y. Note that $S' \to S$ is *independent* of the choice of (Y, D), since (by the normality of S), S' may be constructed as the *normalization* of S in the function field of S' (which is independent of the choice of (Y, D) since the restriction of (Y, D) to the generic point of S has already been shown to be unique). Thus, the uniqueness of (Y, D) follows by considering the classifying morphism (associated to (Y, D)) from S' to the finite étale covering of $(\mathcal{M}_{g,r})_{\mathbb{Z}[\frac{1}{l}]}$ parametrizing orderings of the marked points and trivializations of the l-torsion points of the Jacobian [since this covering is well-known to be a scheme, for l sufficiently large].) We shall refer to Y (respectively, D; D; D) as the *compactification* (respectively, *divisor at infinity*; *divisor of cusps*; *divisor of marked points*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above.

Write

$$\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$$

for the *tautological curve* over $\overline{\mathcal{M}}_{g,r}$; $\overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ for the corresponding *tautological divisor of marked points*. The divisor given by the union of $\overline{\mathcal{D}}_{g,r}$ with the inverse image in $\overline{\mathcal{C}}_{g,r}$ of the divisor at infinity of $\overline{\mathcal{M}}_{g,r}$ determines a *log structure* on $\overline{\mathcal{C}}_{g,r}$; denote the resulting log stack by $\overline{\mathcal{C}}_{g,r}^{\log}$. Thus, we obtain a morphism of log stacks

$$\overline{\mathcal{C}}_{g,r}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$$

which we refer to as the *tautological log curve* over $\overline{\mathcal{M}}_{g,r}^{\log}$. If $S^{\log}$ is *any log scheme*, then we shall refer to a morphism

$$C^{\log} \to S^{\log}$$

which is obtained as the pull-back of the tautological log curve via some [necessarily *uniquely determined* — cf., e.g., [Mzk1], §3] *classifying morphism* $S^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ as a *stable log curve*. If C has *no nodes*, then we shall refer to $C^{\log} \to S^{\log}$ as a *smooth log curve*.

If X_K (respectively, Y_L) is a *hyperbolic curve over a field* K (respectively, L), then we shall say that X_K is *isogenous* to Y_L if there exists a hyperbolic curve Z_M over a field M together with *finite étale morphisms* $Z_M \to X_K$, $Z_M \to Y_L$.

Section 1: Zariski's Main Theorem for Semi-graphs

In this §, we prove an analogue [cf. Theorem 1.2 below] for semi-graphs of *"Zariski's main theorem"* (for schemes).

We begin with some general remarks concerning *semi-graphs* [a notion defined in [Mzk3], Appendix].

For the definition of a *semi-graph*, we refer to [Mzk3], Appendix. In *loc. cit.*, one may also find the definition of a *morphism of semi-graphs*. Here, we remark that, in this definition, the "injections $e \hookrightarrow e'$", where e (respectively, e') is an edge of the domain (respectively, codomain) of the morphism, are necessarily *bijections*, since both e and e' are sets of cardinality 2. A semi-graph will be called *finite* (respectively, *countable*) if both its set of vertices and its set of edges are finite (respectively, countable). A *component* of a semi-graph is defined to be the datum of either an *edge* or a *vertex* of the semi-graph.

A semi-graph $\mathbb{G}$ may be thought of as a *topological space* as follows: We regard each *vertex* v as a point $[v]$. If e is an *edge*, consisting of branches b_1, b_2, then we regard e as the "interval" given by the set of formal sums $\lambda_1 \cdot [b_1] + \lambda_2 \cdot [b_2]$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ [here, $\mathbb{R}$ denotes the topological field of real numbers]; $\lambda_1 + \lambda_2 = 1$; for $i = 1, 2$, $\lambda_i \leq 1$ (respectively, $\lambda_i < 1$) if b_i abuts (respectively, does not abut) to a vertex; moreover, if b_i abuts to a vertex v, then we identify the formal sum $1 \cdot [b_i] + 0 \cdot [b_{3-i}]$ with $[v]$. Thus, relative to this point of view, it is natural to think of the *branch* b_i as the portion of the interval just defined consisting of formal sums such that $\lambda_i > \frac{1}{2}$. Also, we observe that this construction of an associated topological space is *functorial*: Every morphism of semi-graphs induces a continuous morphism of the corresponding topological spaces. In the following discussion, we shall often invoke this point of view without further explanation.

A *sub-semi-graph* $\mathbb{H}$ of a semi-graph $\mathbb{G}$ is a semi-graph satisfying the following properties: (a) the set of *vertices* (respectively, *edges*) of $\mathbb{H}$ is a subset of the set of vertices (respectively, edges) of $\mathbb{G}$; (b) every branch of an edge of $\mathbb{H}$ that abuts, relative to $\mathbb{G}$, to a vertex v of $\mathbb{G}$ lying in $\mathbb{H}$ also abuts to v, relative to $\mathbb{H}$; (c) the coincidence maps of $\mathbb{H}$ are those induced by the coincidence maps of $\mathbb{G}$. A morphism of semi-graphs will be called an *embedding* if it induces an isomorphism of the domain onto a sub-semi-graph of the codomain.

Let $\mathbb{G}$ be a *semi-graph*. Then we shall refer to an edge of $\mathbb{G}$ that is of verticial cardinality 2 (respectively, < 2; 0) as *closed* (respectively, *open*; *isolated*). We shall say that two closed edges e and e' of $\mathbb{G}$ are *coverticial* if the following condition holds: the edge e abuts to a vertex v of $\mathbb{G}$ if and only if the edge e' abuts to v. We shall say that $\mathbb{G}$ is *locally finite* if, for every vertex v of $\mathbb{G}$, the set of edges that abut to v is *finite*. We shall say that $\mathbb{G}$ is *untangled* if every closed edge of $\mathbb{G}$ abuts to two *distinct* vertices. We shall refer to a connected semi-graph that has precisely one vertex and precisely two edges, both of which are open, as a *joint*. If a sub-semi-graph of a given semi-graph is a joint, then we shall refer to this sub-semi-graph as a *subjoint* of the given semi-graph. We shall refer to the sub-semi-graph of $\mathbb{G}$ obtained by omitting all of the open edges as the *maximal* *subgraph* of the semi-graph. We shall refer to as the *compactification* of $\mathbb{G}$ the graph obtained from $\mathbb{G}$ by *appending* to $\mathbb{G}$, for each branch b of an edge of $\mathbb{G}$ that does not abut to a vertex, a new vertex v_b to which b is to abut. Thus, $\mathbb{G}$ forms a *sub-semi-graph* of its compactification. Moreover, any morphism of semi-graphs *induces* a unique morphism between the respective compactifications. Finally, we observe that every connected component of the topological space associated to the maximal subgraph of $\mathbb{G}$ (respectively, $\mathbb{G}$) is a *deformation retract* [in the sense of algebraic topology] of the corresponding connected component of the topological space associated to $\mathbb{G}$ (respectively, the compactification of $\mathbb{G}$). A semi-graph whose associated topological space is *contractible* [in the sense of algebraic topology] will be referred to as a *tree.*

Let v (respectively, e; b) be a(n) vertex (respectively, edge; branch of an edge) of $\mathbb{G}$. Then we define morphisms of semi-graphs

$$\mathbb{G}[v] \to \mathbb{G}; \quad \mathbb{G}[e] \to \mathbb{G}; \quad \mathbb{G}[b] \to \mathbb{G}$$

as follows: $\mathbb{G}[v]$ consists of a single vertex v', which maps to v, and, for each branch b_v of an edge e_v of $\mathbb{G}$ that abuts to v, an edge e'_{b_v} of verticial cardinality 1 that maps to e_v in such a way that the branch of e'_{b_v} lying over b_v abuts to v'. $\mathbb{G}[e]$ consists of a single edge e', which maps to e, and, for each branch b_e of e abutting to a vertex v_{b_e} of $\mathbb{G}$, a vertex v'_{b_e} [of $\mathbb{G}[e]$] that maps to v_{b_e} and is the abutment of the branch $b'_{e'}$ of e' that lies over b_e. If b is a branch of an edge e_b that abuts to a vertex v_b [of $\mathbb{G}$], then $\mathbb{G}[b]$ is the sub-semi-graph of $\mathbb{G}[e_b]$ consisting of the unique edge of $\mathbb{G}[e_b]$ and the vertex of $\mathbb{G}[e_b]$ which is the abutment of the branch of this unique edge that lies over b. Thus, $\mathbb{G}[v]$, $\mathbb{G}[e]$, $\mathbb{G}[b]$ are all *trees* [even if $\mathbb{G}$ fails to be untangled]; we have natural morphisms $\mathbb{G}[b] \to \mathbb{G}[v]$, $\mathbb{G}[b] \to \mathbb{G}[e]$ over $\mathbb{G}$.

A morphism

$$\phi : \mathbb{G}_A \to \mathbb{G}_B$$

between semi-graphs will be called an *immersion* [or as an *immersive morphism*] (respectively, *excision* [or as an *excisive morphism*]) if it satisfies the condition that, for every vertex v_A of $\mathbb{G}_A$ that maps to a vertex v_B of $\mathbb{G}_B$, the induced map from branches abutting to v_A to branches abutting to v_B is *injective* (respectively, *bijective*). Thus, if we think of $\mathbb{G}_A$ and $\mathbb{G}_B$ as topological spaces, then an immersion $\phi : \mathbb{G}_A \to \mathbb{G}_B$ is *locally* [in some small neighborhood of every point of $\mathbb{G}_A$] an *embedding* (respectively, a *homeomorphism*) of topological spaces.

Observe that: the five classes of morphisms $\mathbb{G}[v] \to \mathbb{G}$, $\mathbb{G}[b] \to \mathbb{G}[e]$, $\mathbb{G}[e] \to \mathbb{G}$, $\mathbb{G}[b] \to \mathbb{G}$, $\mathbb{G}[b] \to \mathbb{G}[v]$, are all *immersive*; the first two of these classes are always *excisive*; the last three of these classes are *not* excisive in general.

Also, we observe that a morphism of sub-semi-graphs $\mathbb{G}_A \subseteq \mathbb{G}_B$ is *immersive* (respectively, *excisive*) if and only if, for every vertex v_A of $\mathbb{G}_A$ mapping to a vertex v_B of $\mathbb{G}_B$, the induced morphism of semi-graphs $\mathbb{G}_A[v_A] \to \mathbb{G}_B[v_B]$ is an *embedding* (respectively, *isomorphism*).

A morphism of semi-graphs

$$\phi : \mathbb{G}_A \to \mathbb{G}_B$$

will be called *proper* if it preserves verticial cardinalities of edges. A proper excision will be referred to as a *graph-covering*. A graph-covering with finite fibers will be referred to as a *finite graph-covering*. Note that if $\phi : \mathbb{G}_A \to \mathbb{G}_B$ is a graph-covering, with $\mathbb{G}_A$, $\mathbb{G}_B$ connected, then the associated map of topological spaces will be a *covering in the sense of algebraic topology*. Conversely, every covering, in the sense of algebraic topology, of the topological space associated to $\mathbb{G}_B$ arises in this way. Also, we observe that, just as in the case of coverings of topological spaces, it makes sense to speak of a graph-covering as *Galois* [i.e., "arising from a *normal* subgroup of the fundamental group"] and to speak of the *pull-back* of a graph-covering by an arbitrary morphism of semi-graphs.

Proposition 1.1. *Any immersion from a connected graph into a tree is, in fact, an embedding.*

Proof. Indeed, suppose that we are given an immersion $\phi : \mathbb{G}_A \to \mathbb{G}_B$ into a tree $\mathbb{G}_B$ which is *not* an embedding. If ϕ is injective on vertices, then it follows from the definition of an immersion that ϕ is injective on edges, hence that ϕ is an embedding. Thus, it suffices to show that ϕ is *injective on vertices*.

Suppose that there exist distinct vertices v_1, v_2 of $\mathbb{G}_A$ that map to the same vertex w of $\mathbb{G}_B$. Write γ_A for a *path* on $\mathbb{G}_A$ that connects v_1 to v_2. Without loss of generality, we may assume that γ_A has *minimal length* among paths on $\mathbb{G}_A$ that join distinct vertices of $\mathbb{G}_A$ that map to the same vertex of $\mathbb{G}_B$. Write $\gamma_B \stackrel{\text{def}}{=} \phi(\gamma_A)$. Then note that the minimality condition (together with the fact that ϕ is an *immersion*) implies that γ_B does not intersect itself. Thus, γ_B is a *loop*, starting and ending at w, and defined by a sequence of edges, all of which are *distinct*. But this contradicts the fact that $\mathbb{G}_B$ is a tree. This completes the proof. ○

Thus, in particular, if we start with an *arbitrary immersion* of connected graphs [which are not necessarily trees]

$$\phi : \mathbb{G}_A \to \mathbb{G}_B$$

then Proposition 1.1 implies that the induced morphism

$$\widetilde{\mathbb{G}}_A \to \widetilde{\mathbb{G}}_B$$

on *universal graph-coverings* [i.e., the associated topological coverings are universal coverings of $\mathbb{G}_A$, $\mathbb{G}_B$, respectively, in the sense of algebraic topology] is an *embedding* [since it is an immersion into a *tree*]. More generally, given an arbitrary *graph-covering*

$$\mathbb{G}_{B'} \to \mathbb{G}_B$$

one can ask when the base-changed immersion

$$\phi' : \mathbb{G}_{A'} \to \mathbb{G}_{B'}$$

is an *embedding on each connected component of* $\mathbb{G}_{A'}$. Proposition 1.1 implies that the universal graph-covering $\widetilde{\mathbb{G}}_B \to \mathbb{G}_B$ is *sufficient* to realize this condition.

In fact, however, when $\mathbb{G}_A$, $\mathbb{G}_B$ are *finite*, this condition may be realized by a *finite graph-covering* $\mathbb{G}_{B'} \to \mathbb{G}_B$:

Theorem 1.2. **("Zariski's Main Theorem for Semi-graphs")** *Let*

$$\phi : \mathbb{G}_A \to \mathbb{G}_B$$

be an **immersion** *of finite semi-graphs. Then:*

(i) The morphism ϕ *factors as the composite of an* **embedding**

$$\mathbb{G}_A \hookrightarrow \mathbb{G}_{B'}$$

and a **finite graph-covering** $\mathbb{G}_{B'} \to \mathbb{G}_B$.

(ii) There exists a **finite graph-covering** $\mathbb{G}_{B'} \to \mathbb{G}_B$ *such that the restriction of the base-changed morphism*

$$\phi' : \mathbb{G}_{A'} \to \mathbb{G}_{B'}$$

to each connected component of $\mathbb{G}_{A'}$ *is an* **embedding**.

Remark 1.2.1. The author is indebted to M. Matsumoto for the following elegant graph-theoretic proof of Theorem 1.2.

Remark 1.2.2. The general form of Theorem 1.2 is reminiscent of the well-known result in algebraic geometry ("Zariski's Main Theorem" — cf., e.g., [Milne], Chapter I, Theorem 1.8) that any *separated quasi-finite morphism*

$$f : X \to Y$$

between noetherian schemes factors as the composite of an open immersion $X \hookrightarrow Y'$ and a finite morphism $Y' \to Y$ — cf. also Lemma 1.5 below.

Proof. First, we observe that (ii) follows formally from (i) [by taking the finite graph-covering of (ii) to be a *Galois* finite graph-covering of $\mathbb{G}_B$ that dominates the graph-covering of (i)]. Thus, it suffices to prove (i).

Next, let us observe that:

(a) Any immersion of semi-graphs for which the induced morphism between the respective compactifications is an embedding is itself an embedding (of semi-graphs).

(b) Restriction from the compactification of $\mathbb{G}_B$ to $\mathbb{G}_B$ induces an equivalence of categories between the respective categories of finite graph-coverings. Moreover, the compactification of a finite graph-covering of $\mathbb{G}_B$ is naturally isomorphic to the corresponding finite graph-covering of the compactification of $\mathbb{G}_B$.

In particular, by replacing the semi-graphs involved by their compactifications, it suffices to prove (i) in the case where all of the semi-graphs are, in fact, graphs. Thus, for the remainder of the proof, we assume that $\mathbb{G}_A$, $\mathbb{G}_B$ are *graphs*.

Let us write

$$\mathbb{H}_n$$

(where $n \geq 1$ is an integer) for the graph consisting of one vertex $v_{\mathbb{H}}$ and n edges $e_{\mathbb{H},1}; \ldots ; e_{\mathbb{H},n}$ (all of which run from $v_{\mathbb{H}}$ to $v_{\mathbb{H}}$).

Next, let us observe that by Lemma 1.4 below, there exists an *immersion*

$$\zeta : \mathbb{G}_B \to \mathbb{H}_n$$

which we may compose with ϕ to form an *immersion*:

$$\psi : \mathbb{G}_A \to \mathbb{H}_n$$

Moreover, since pull-backs of finite graph-coverings of $\mathbb{H}_n$ via ζ form finite graph-coverings of $\mathbb{G}_B$, it follows that in order to prove that the assertion of Theorem 1.2, (i), is true for ϕ, it suffices to prove that it is true for ψ. On the other hand, Theorem 1.2, (i), follows for ψ by Lemma 1.5 below. ○

Note that, relative to the *topological space* point of view discussed above, the the vertex $v_{\mathbb{H}}$ of the graph $\mathbb{H}_n$ meets precisely $2n$ branches.

Lemma 1.3. *Let $\mathbb{G}$ be a finite graph. Then:*

(i) To give a morphism

$$\phi : \mathbb{G} \to \mathbb{H}_n$$

is equivalent to assigning an **orientation** *and a* **"color"** $\in \{1, \ldots , n\}$ *to each edge of* $\mathbb{G}$.

(ii) The morphism ϕ is an **immersion** *if and only if for each color $i \in \{1, \ldots , n\}$, and at each vertex v of $\mathbb{G}$, the number of branches of color i that enter (respectively, leave) v — i.e., relative to the assigned orientations — is ≤ 1.*

(iii) The morphism ϕ is an **excision** *[or, equivalently, a* **finite graph-covering***] if and only if for each color $i \in \{1, \ldots , n\}$, and at each vertex v of $\mathbb{G}$, the number of branches of color i that enter (respectively, leave) v — i.e., relative to the assigned orientations — is $= 1$.*

Proof. First, we fix an *orientation* on each edge $e_{\mathbb{H},i}$ of $\mathbb{H}_n$, and regard the edge $e_{\mathbb{H},i}$ as being of *color* i.

Now let us prove (i). Given a morphism $\phi : \mathbb{G} \to \mathbb{H}_n$, we obtain orientations and colors on the edges of $\mathbb{G}$ by pulling back the orientations and colors of $\mathbb{H}_n$ via ϕ. Conversely, given a choice of orientations and colors on the edges of $\mathbb{G}$, we obtain a morphism $\phi : \mathbb{G} \to \mathbb{H}_n$ by sending all the vertices of $\mathbb{G}$ to $v_{\mathbb{H}}$ and mapping the edges of $\mathbb{G}$ to the edges of $\mathbb{H}_n$ in the unique way which preserves orientations and colors.

Assertions (ii) and (iii) follow immediately by considering the local structure of $\mathbb{H}_n$ at $v_{\mathbb{H}}$. Note that in general, a morphism of finite graphs is *always* proper, hence is a finite graph-covering if and only if it is excisive. ○

Lemma 1.4. *Every finite graph* $\mathbb{G}$ *admits an immersion* $\mathbb{G} \to \mathbb{H}_n$ *for some integer* $n \geq 1$.

Proof. Indeed, if we take n to be the number of edges of $\mathbb{G}$ and assign distinct colors to distinct edges of $\mathbb{G}$, then it is immediate from Lemma 1.3, (ii), that (for any assignment of orientations) the resulting morphism $\mathbb{G} \to \mathbb{H}_n$ is an immersion. ○

Lemma 1.5. *Let* $\phi : \mathbb{G} \to \mathbb{H}_n$ *be an immersion of finite graphs. Then* ϕ *extends to a finite graph-covering* $\phi' : \mathbb{G}' \to \mathbb{H}_n$ *for some inclusion* $\mathbb{G} \hookrightarrow \mathbb{G}'$.

Proof. We construct $(\mathbb{G}', \phi')$ from $(\mathbb{G}, \phi)$ by adding edges (equipped with orientations and colors) to $\mathbb{G}$ until the resulting ϕ' is *excisive*, i.e., satisfies the condition of Lemma 1.3, (iii). Suppose that there exists a vertex v of $\mathbb{G}$ that does not satisfy this condition. This means that there is some color i such that *either* there does not exist a branch of color i entering v *or* there does not exist a branch of color i leaving v (or *both*). If there do not exist any branches of color i meeting v, then we simply add an edge of color i to $\mathbb{G}$ that runs from v to v. Now suppose (without loss of generality) that there exists a branch of color i *leaving* v, but that there does *not* exist a branch of color i *entering* v. Then we follow the i-colored edge leaving $v_1 \stackrel{\mathrm{def}}{=} v$ to a *new vertex* v_2 (necessarily distinct from v_1). Now there are *two* possibilities:

(1) There exists an i-colored edge leaving this vertex.

(2) There does not exist an i-colored edge leaving this vertex.

If (1) holds, then we repeat the above procedure — i.e., we follow this i-colored edge out of v_2 to another vertex v_3, which is necessarily distinct from v_2 since the *unique* (by Lemma 1.3, (ii)) i-colored edge entering v_2 originated from a vertex which is distinct from v_2. Thus, continuing in this way, we obtain a sequence

$$v_1, v_2, v_3, \ldots$$

of *distinct* (by Lemma 1.3, (ii)) vertices of $\mathbb{G}$. Since $\mathbb{G}$ is *finite*, this sequence must eventually *terminate* at some vertex v_k satisfying (2). Then we add an i-colored edge to $\mathbb{G}$ running from v_k to v_1 to form a pair $(\mathbb{G}[2], \phi[2])$ extending the original $(\mathbb{G}[1], \phi[1]) \stackrel{\text{def}}{=} (\mathbb{G}, \phi)$.

Note that $\phi[2]$ is still an *immersion*, that $\mathbb{G}[2]$ has the *same set of vertices* as $\mathbb{G}[1]$, and that the *extent* [in an evidently defined sense] to which this set of vertices violates the condition of Lemma 1.3, (iii), relative to $\phi[2]$ is $<$ the extent to which this set of vertices violates the condition of Lemma 1.3, (iii), relative to $\phi[1]$. Thus, if we apply the procedure

$$(\mathbb{G}[1], \phi[1]) \mapsto (\mathbb{G}[2], \phi[2])$$

to $(\mathbb{G}[2], \phi[2])$ to obtain some $(\mathbb{G}[3], \phi[3])$, and so on, we obtain a sequence of pairs

$$(\mathbb{G}[1], \phi[1]); \ (\mathbb{G}[2], \phi[2]); \ (\mathbb{G}[3], \phi[3]); \ \ldots$$

[where we note that the codomain of every $\phi[-]$ is equal to $\mathbb{H}_n$, for the *same* n] which — by the *finiteness of the degree of violability* of the condition of Lemma 1.3, (iii) — necessarily terminates in a pair $(\mathbb{G}', \phi')$ such that ϕ' is a finite graph-covering, as desired. ○

Remark 1.5.1. Consider the case of an *immersion*

$$\phi : \mathbb{G} \to \mathbb{H}_1$$

where $\mathbb{G}$ is a finite connected graph. Since the (topological) fundamental group of $\mathbb{H}_1$ is equal to $\mathbb{Z}$, a (connected) finite graph-covering $\mathbb{G}' \to \mathbb{H}_1$ of $\mathbb{H}_1$ is determined by its *degree* d (a positive integer). Then one can ask what conditions one must place on d for the corresponding finite graph-covering to satisfy the property of Theorem 1.2, (ii). In some sense, there are essentially *two phenomena* that may occur:

(1) The case where ϕ itself is a *finite graph-covering*, of degree n. In this case, the resulting condition on d is *nonarchimedean*, i.e.:

$$d \equiv 0 \pmod{n}$$

(2) The case where $\mathbb{G}$ consists of n vertices $v_1, \ldots, v_n$, and precisely one edge joining v_j to v_{j+1}, for $j = 1, \ldots, n-1$ (and no other edges). In this case, the resulting condition on d is *archimedean*, i.e.:

$$d \geq n$$

The above analysis suggests that there is some *interesting arithmetic* "hidden" in Theorem 1.2.

The following interesting consequence of Theorem 1.2 — which asserts, in effect, that *finitely generated subgroups of finite rank [discrete] free groups admit bases with properties reminiscent of their* **abelian** *counterparts* — was pointed out to the author by A. Tamagawa:

Corollary 1.6. (Finitely generated Subgroups of Finite Rank Free Groups) *Let F be a finitely generated subgroup of a free group G of finite rank (so F is also free of finite rank). Then:*

(i) There exists an **immersion** *of finite graphs $\phi : \mathbb{G}_A \to \mathbb{G}_B$ whose induced morphism on (topological) fundamental groups is isomorphic to the inclusion $F \hookrightarrow G$.*

(ii) There exists a **finite index** *subgroup $H \subseteq G$ such that H contains F, and, moreover, there exists a set of free generators $\gamma_1, \ldots, \gamma_r$ of H with the property that for some $s \leq r$, $\gamma_1, \ldots, \gamma_s$ form a set of free generators of F.*

Proof. First, observe that (ii) follows formally from (i) (of the present Corollary); Theorem 1.2, (i); and the elementary fact that if $\mathbb{G}_A \hookrightarrow \mathbb{G}_{B'}$ is an embedding, then any set of free generators of the fundamental group of $\mathbb{G}_A$ may be extended to a set of free generators of the fundamental group of $\mathbb{G}_{B'}$.

Thus, it suffices to prove (i). Let $\mathbb{G}_B$ be any graph whose fundamental group is equal to G. Then the subgroup $F \subseteq G$ defines an infinite graph-covering

$$\mathbb{G}_{A'} \to \mathbb{G}_B$$

of $\mathbb{G}_B$. In particular, $\mathbb{G}_{A'}$ has fundamental group equal to F. Although, in general, the graph $\mathbb{G}_{A'}$ will not necessarily be finite, it follows from the fact that its fundamental group F is *finitely generated* that there exists a *finite subgraph* $\mathbb{G}_A \subseteq \mathbb{G}_{A'}$ such that the natural injection of fundamental groups $\pi_1(\mathbb{G}_A) \hookrightarrow \pi_1(\mathbb{G}_{A'})$ is, in fact, a bijection. Moreover, the composite $\mathbb{G}_A \hookrightarrow \mathbb{G}_{A'} \to \mathbb{G}_B$ is an immersion (since it is a composite of immersions). This completes the proof of (i). ◯

Another interesting consequence of Theorem 1.2 is the following well-known result:

Corollary 1.7. (Residual Finiteness of Free Groups) *Every discrete free group F* **injects** *into its profinite completion.*

Proof. Indeed, let $\mathbb{G}$ be a *connected graph* with $\pi_1(\mathbb{G}) = F$. If $H \subseteq F$ is the *kernel* of the map from F to its profinite completion, write $\mathbb{H} \to \mathbb{G}$ for the corresponding *graph-covering*. If $\mathbb{H}$ is *not* a tree, then one verifies immediately that $\mathbb{H}$ contains a *finite connected subgraph* $\mathbb{H}'$ which is not a tree. In particular, $\mathbb{H}'$ admits *nontrivial* finite graph-coverings. Let $\mathbb{G}'$ be a finite connected subgraph of $\mathbb{G}$ which contains the image of $\mathbb{H}'$. Then if we apply Theorem 1.2 to the immersion $\mathbb{H}' \to \mathbb{G}'$, we obtain [since finite graph-coverings of a subgraph of a given graph always extend to finite graph-coverings of the given graph] that there exists a finite graph-covering $\mathbb{K}' \to \mathbb{G}'$ whose pull-back to $\mathbb{H}'$ is *nontrivial*. Thus, if we extend $\mathbb{K}' \to \mathbb{G}'$ to a finite graph-covering $\mathbb{K} \to \mathbb{G}$, we obtain a finite graph-covering of $\mathbb{G}$ whose pull-back to $\mathbb{H}$ is *nontrivial*. But this contradicts the definition of H. ◯

Remark 1.7.1. We recall in passing that there is also a *pro-l version* of this residual finiteness result — cf., e.g., [RZ], Proposition 3.3.15.

Finally, before continuing, we note the following useful result concerning *finite group actions on semi-graphs*, which is implicit in the theory of [Serre]:

Lemma 1.8. **(Finite Group Actions on Semi-graphs)** *Let* $\mathbb{G}$ *be a* **connected semi-graph***, equipped with the action of a* **finite group** G*. Then:*

(i) Every finite sub-semi-graph $\mathbb{G}'$ *of* $\mathbb{G}$ *is contained in a finite connected sub-semi-graph* $\mathbb{G}''$ *of* $\mathbb{G}$ *that is* **stabilized** *by the action of* G*.*

(ii) Suppose that $\mathbb{G}$ *is a* **tree***. Then: (a) there exists at least one vertex or edge of* $\mathbb{G}$ *that is* **fixed** *by* G*; (b) if* G *fixes* **two distinct** *vertices* w_1*,* w_2 *of* $\mathbb{G}$*, then* G *acts trivially on any* **"geodesic"** *[i.e., path of closed edges of minimal length] that joins* w_1*,* w_2*; (c) if* G *fixes* **three distinct** *vertices of* $\mathbb{G}$*, then there exists at least one* **subjoint** *of* $\mathbb{G}$ *on which* G *acts trivially.*

Proof. First, we consider *assertion (i)*. Since $\mathbb{G}$ is *connected*, we may assume without loss of generality that $\mathbb{G}'$ is *connected* and contains the G-orbit of some vertex. Then one verifies easily that if we take $\mathbb{G}''$ to be the G-orbit of $\mathbb{G}'$, then the desired properties are satisfied.

Next, we consider *assertion (ii)*. First, we verify assertion (a). This follows formally from [Serre], Chapter I, §6.5, Corollary 3 to Proposition 26, Proposition 27 — at least if one assumes, as in done in [Serre], that G fixes some orientation on the tree $\mathbb{G}$. On the other hand, by "splitting" each edge of $\mathbb{G}$ which violates this assumption into *two new edges*, corresponding to the two branches of the original edge, one sees immediately that one still obtains assertion (a), even without this assumption. This completes the proof of (ii), (a).

Next, to prove (ii), (b), recall from [Serre], Chapter I, §2.2, Proposition 8, that there is a *unique* path of minimal length from w_1 to w_2. Since G fixes w_1, w_2, it thus follows that *G fixes this path*. Thus, [since it is evident that there are no automorphisms of this path that fix w_1, w_2] we conclude that G acts trivially on this path, as desired. This completes the proof of (ii), (b). Finally, we observe that (ii), (c) follows formally from (ii), (b). ◯

Remark 1.8.1. We observe, in passing, that Lemma 1.8, (ii), (a), implies [the well-known fact — cf., e.g., [Serre], Chapter I, §3.4, Theorem 5] that a *free group* [which may be thought of as the fundamental group of some graph, hence admits a free action on some tree] *does not contain any nontrivial finite subgroups*.

Section 2: Commensurability Properties

In this §, we begin our study of the *geometry of semi-graphs of profinite groups* by considering various topics concerning *commensurators* and *slimness.*

In the following, we shall use the language of *anabelioids* [i.e., "multi-Galois categories" in the language of [SGA1]] of [Mzk4], which, in effect, amounts to working with *profinite groups up inner automorphism* [cf., e.g., [Mzk4], Proposition 1.1.4]. If $\mathcal{X}$ is a *connected anabelioid* [i.e., a Galois category], then we shall denote the *profinite fundamental group* [for some choice of basepoint] of $\mathcal{X}$ by:

$$\widehat{\pi}_1(\mathcal{X})$$

As is well-known, this profinite group is, in a natural sense, independent of the choice of basepoint, up to inner automorphism.

Definition 2.1. We shall refer to the following data $\mathcal{G}$:

(a) a *semi-graph* $\mathbb{G}$;

(b) for each vertex v of $\mathbb{G}$, a *connected anabelioid* $\mathcal{G}_v$;

(c) for each edge e of $\mathbb{G}$, a *connected anabelioid* $\mathcal{G}_e$, together with, for each branch $b \in e$ abutting to a vertex v, a morphism of anabelioids $b_* : \mathcal{G}_e \rightarrow \mathcal{G}_v$

as a *semi-graph of (connected) anabelioids* [cf. the notion of a *"semi-graph of profinite groups"* introduced in [Mzk3], Appendix]. We shall refer to the various $\mathcal{G}_v$, $\mathcal{G}_e$ as the *constituent anabelioids* of $\mathcal{G}$. Given two semi-graphs of anabelioids, there is an evident notion of *morphism between semi-graphs of anabelioids* [cf. also Remark 2.4.2 below]. If all of the b_*'s are π_1*-monomorphisms* [i.e., induce injective homomorphisms on associated fundamental groups — cf. [Mzk4], Definition 1.1.12], then we shall say that $\mathcal{G}$ is *of injective type.* When the underlying semi-graph $\mathbb{G}$ is a graph, we shall refer to a semi-graph of anabelioids $\mathcal{G}$ as a *graph of (connected) anabelioids.*

Let $\mathcal{G}$ be a *connected semi-graph of anabelioids* [i.e., the underlying semi-graph $\mathbb{G}$ is assumed to be connected]. If $\mathbb{G}$ has *at least one vertex*, then let us denote by

$$\mathcal{B}(\mathcal{G})$$

the category of *objects* given by data

$$\{S_v, \phi_e\}$$

where v (respectively, e) ranges over the vertices (respectively, edges) of $\mathbb{G}$; for each vertex v, $S_v \in \mathrm{Ob}(\mathcal{G}_v)$; for each edge e, with branches b_1, b_2 abutting to vertices

v_1, v_2, respectively, $\phi_e : b_1^* S_{v_1} \xrightarrow{\sim} b_2^* S_{v_2}$ is an isomorphism in $\mathcal{G}_e$, and *morphisms* given by morphisms [in the evident sense] between such data. If $\mathbb{G}$ has *no vertices* — and hence precisely one edge e, which is necessarily *isolated* — then we shall write $\mathcal{B}(\mathcal{G}) \stackrel{\text{def}}{=} \mathcal{G}_e$. One verifies immediately that this category $\mathcal{B}(\mathcal{G})$ is a *connected anabelioid*.

Now let $G' \in \text{Ob}(\mathcal{B}(\mathcal{G}))$; write

$$\mathcal{B}' \stackrel{\text{def}}{=} \mathcal{B}(\mathcal{G})_{G'} \to \mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}(\mathcal{G})$$

for the corresponding *finite étale covering of anabelioids* [cf. [Mzk4], Definition 1.2.2, (i)]. Then it follows from the definition of the anabelioid $\mathcal{B}(\mathcal{G})$ associated to the semi-graph of anabelioids $\mathcal{G}$ [i.e., in terms of finite étale coverings of each of the constituent anabelioids of $\mathcal{G}$, together with gluing isomorphisms] that $\mathcal{B}'$ itself arises naturally as the $\mathcal{B}(-)$ of some connected semi-graph of anabelioids $\mathcal{G}'$ equipped with a *morphism*

$$\mathcal{G}' \to \mathcal{G}$$

of semi-graphs of anabelioids which lies over some *proper morphism of semi-graphs*:

$$\mathbb{G}' \to \mathbb{G}$$

Here, if v (respectively, e) is a *vertex* (respectively, *edge*) of $\mathbb{G}$, then the vertices v' (respectively, edges e') of $\mathbb{G}'$ are the elements of the set of connected components of the [not necessarily connected!] anabelioid $\mathcal{B}' \times_{\mathcal{B}} \mathcal{G}_v$ (respectively, $\mathcal{B}' \times_{\mathcal{B}} \mathcal{G}_e$). The connected anabelioid $\mathcal{G}'_{v'}$ (respectively, $\mathcal{G}'_{e'}$) is the connected component anabelioid of $\mathcal{B}' \times_{\mathcal{B}} \mathcal{G}_v$ (respectively, $\mathcal{B}' \times_{\mathcal{B}} \mathcal{G}_e$) determined by v' (respectively, e').

Definition 2.2.

(i) We shall refer to an arrow $\mathcal{G}' \to \mathcal{G}$ that arises as in the above discussion as a *finite étale covering* [of $\mathcal{G}$].

(ii) If a morphism of semi-graphs of anabelioids satisfies the property that each of its induced morphisms between constituent anabelioids is an isomorphism (respectively, is finite étale; induces an homomorphism with open image between the respective $\widehat{\pi}_1(-)$'s), then we shall say that the morphism is *locally trivial* (respectively, *locally finite étale*; *locally open*).

Let us denote by

$$\Pi_{\mathbb{G}} \stackrel{\text{def}}{=} \widehat{\pi}_1(\mathcal{G}) \stackrel{\text{def}}{=} \widehat{\pi}_1(\mathcal{B}(\mathcal{G}))$$

the *fundamental group* of the connected anabelioid $\mathcal{B}(\mathcal{G})$ relative to some basepoint. Put another way, if we choose *basepoints* for the constituent anabelioids of $\mathcal{G}$, then $\mathcal{G}$ determines a *"semi-graph of profinite groups"* [cf. [Mzk3], Appendix, except that here the underlying semi-graph is not necessarily finite], and one may think of $\Pi_{\mathbb{G}}$ as the profinite group associated to this semi-graph of profinite groups.

For each vertex v (respectively, branch b of an edge e that abuts to the vertex v) of $\mathbb{G}$, let us write Π_v (respectively, Π_b) for the fundamental group of the anabelioid $\mathcal{G}_v$ (respectively, $\mathcal{G}_e$) for some choice of basepoint. Thus, we have natural outer homomorphisms:

$$\Pi_v \to \Pi_{\mathbb{G}}; \quad \Pi_b \to \Pi_{\mathbb{G}}$$

Moreover, the branch b determines an associated outer homomorphism:

$$\Pi_b \to \Pi_v$$

If $\mathcal{G}$ is *of injective type*, then we shall also denote the image of Π_b in Π_v, which is well-defined up to conjugation in Π_v, by Π_b.

If $\mathbb{H} \subseteq \mathbb{G}$ is a [not necessarily connected] *sub-semi-graph* of $\mathbb{G}$, then we shall write $\mathcal{G}_{\mathbb{H}}$ for the *semi-graph of anabelioids* determined by restricting $\mathcal{G}$ to $\mathbb{H}$. That is to say, the underlying semi-graph of $\mathcal{G}_{\mathbb{H}}$ is $\mathbb{H}$, and for each *component* c (i.e., either an edge or vertex) of $\mathbb{H}$, we let $(\mathcal{G}_{\mathbb{H}})_c \stackrel{\text{def}}{=} \mathcal{G}_c$; if a branch b of an edge e of $\mathbb{H}$ abuts to a vertex v of $\mathbb{H}$, then we take the associated morphism $b_* : (\mathcal{G}_{\mathbb{H}})_e \to (\mathcal{G}_{\mathbb{H}})_v$ to be the morphism associated to the corresponding branch of $\mathbb{G}$. If $\mathbb{H}$ is *connected*, then we shall write

$$\Pi_{\mathbb{H}}$$

for the fundamental group of $\mathcal{G}_{\mathbb{H}}$, for some choice of basepoint. Thus, we have a natural outer homomorphism $\Pi_{\mathbb{H}} \to \Pi_{\mathbb{G}}$.

Definition 2.3. Let $\mathcal{G}$ be a semi-graph of anabelioids of injective type, with underlying semi-graph $\mathbb{G}$.

(i) We shall say that $\mathcal{G}$ is *of bounded order* if there exists an integer $M \geq 1$ such that all of the $\widehat{\pi}_1(\mathcal{G}_v)$'s, where v ranges over the vertices of the underlying semi-graph $\mathbb{G}$, are finite groups of order dividing M.

(ii) We shall refer to a morphism of semi-graphs of anabelioids

$$\mathcal{G} \to \mathcal{G}'$$

which induces an isomorphism on underlying semi-graphs $\mathbb{G} \xrightarrow{\sim} \mathbb{G}'$ [relative to which we may identify $\mathbb{G}$, $\mathbb{G}'$] and for which $\mathcal{G}'$ is a semi-graph of anabelioids of bounded order as an *approximator* for $\mathcal{G}$. We shall say that an approximator is π_1*-epimorphic* if each of the induced morphisms between the respective constituent anabelioids is a π_1*-epimorphism* [cf. [Mzk4], Definition 1.1.12].

(iii) We shall say that $\mathcal{G}$ is *quasi-coherent* if, for every integer $M \geq 1$, and every collection of finite étale coverings $\mathcal{H}_c \to \mathcal{G}_c$ of degree $\leq M$, where c ranges over the components of $\mathbb{G}$, there exists an approximator

$$\mathcal{G} \to \mathcal{G}'$$

such that, for each component c of $\mathbb{G}$, the pull-back to $\mathcal{G}_c$ of the *"universal covering"* $\mathcal{H}'_c \to \mathcal{G}'_c$ of $\mathcal{G}'_c$ [i.e., the finite étale covering determined by the trivial subgroup of

$\widehat{\pi}_1(\mathcal{G}'_c)$] splits $\mathcal{H}_c \to \mathcal{G}_c$. In this situation, we shall say that this approximator *splits* the given collection of coverings. We shall say that a quasi-coherent $\mathcal{G}$ is *coherent* if, for each component c of $\mathbb{G}$, the profinite group $\widehat{\pi}_1(\mathcal{G}_c)$ is *topologically finitely generated* [which, as is well-known, implies that $\mathrm{Out}(\widehat{\pi}_1(\mathcal{G}_c))$ is equipped with a *natural profinite group structure*].

Remark 2.3.1. Relative to the notation of Definition 2.3, (iii), by replacing the constituent anabelioids of $\mathcal{G}'$ by the *image anabelioids* of the constituent anabelioids of $\mathcal{G}$ [cf. [Mzk4], Definition 1.1.7, (i)], one may always take the approximator of Definition 2.3, (iii), to be π_1*-epimorphic.*

Definition 2.4. Let $\mathcal{G}$ be a semi-graph of anabelioids of injective type, with underlying semi-graph $\mathbb{G}$.

(i) Let v be a vertex of $\mathbb{G}$. If, for every integer $M \geq 1$, there exists a π_1-epimorphic approximator

$$\mathcal{G} \to \mathcal{G}'$$

for $\mathcal{G}$ such that there exists a subgroup $N_M \subseteq \widehat{\pi}_1(\mathcal{G}'_v)$ of order $\geq M$ which has *trivial intersection* with all of the conjugates of all of the $\widehat{\pi}_1(\mathcal{G}'_e)$ [where e ranges over the edges abutting to v], then we shall say that v is *elevated.* If all of the vertices of $\mathbb{G}$ are elevated, then we shall say that $\mathcal{G}$ is *totally elevated.*

(ii) If, for every vertex v of $\mathbb{G}$, the anabelioid $\mathcal{G}_v$ is slim, then we shall say that $\mathcal{G}$ is *verticially slim.*

(iii) Let e be a closed edge of $\mathbb{G}$. If there exists a finite étale covering $\mathcal{G}' \to \mathcal{G}$ of $\mathcal{G}$ such that the underlying graph $\mathbb{G}'$ of $\mathcal{G}'$ contains a pair of *distinct coverticial edges* e_a, e_b, both of which map to e in $\mathbb{G}$, then we shall say that e is *sub-coverticial.* If every edge e'' of a finite étale covering $\mathcal{G}'' \to \mathcal{G}$ of $\mathcal{G}$ that maps to e is sub-coverticial, then we shall say that e is *universally sub-coverticial.* If every closed edge of $\mathbb{G}$ is sub-coverticial (respectively, universally sub-coverticial), then we shall say that $\mathcal{G}$ is *totally sub-coverticial* (respectively, *totally universally sub-coverticial*).

(iv) Let e be an edge of $\mathbb{G}$. We shall say that e is *aloof* (respectively, *estranged*) if, for every vertex v to which some branch b of e abuts and every $g \in \Pi_v$, the intersection in Π_v of Π_b with any subgroup of the form $g \cdot \Pi_{b'} \cdot g^{-1}$, where *either* $b' \neq b$ is a branch of an edge that abuts to v *or* $b' = b$ and $g \notin \Pi_b$, has *infinite index* in Π_b (respectively, *and* is, in fact, *trivial*). If every edge of $\mathbb{G}$ is aloof (respectively, estranged), then we shall say that $\mathcal{G}$ is *totally aloof* (*totally estranged).*

Remark 2.4.1. One verifies easily that if $\mathcal{G}' \to \mathcal{G}$ is a finite étale covering, and v' (respectively, e'; e'; e') is a(n) vertex (respectively, edge; edge; edge) of $\mathbb{G}'$ that maps to a(n) elevated vertex v (respectively, universally sub-coverticial edge e; aloof edge e; estranged edge e) of $\mathbb{G}$, then v' (respectively, e') is itself *elevated* (respectively, *universally sub-coverticial*; *aloof*; *estranged*).

Remark 2.4.2. Let $\phi : \mathcal{G} \to \mathcal{H}$ be a *morphism* between semi-graphs of anabelioids of injective type. Concretely speaking, this means that we are given, for each *vertex* v (respectively, *edge* e) of $\mathcal{G}$ mapping to a vertex w (respectively, edge f) of $\mathcal{H}$, a 1-morphism of anabelioids

$$\phi_v : \mathcal{G}_v \to \mathcal{H}_w \text{ (respectively, } \phi_e : \mathcal{G}_e \to \mathcal{H}_f)$$

together with the an isomorphism ϕ_b of the composite 1-morphism of anabelioids $\mathcal{G}_e \to \mathcal{G}_v \to \mathcal{H}_w$ with the composite 1-morphism of anabelioids $\mathcal{G}_e \to \mathcal{H}_f \to \mathcal{H}_w$ whenever a branch b of e abuts v. That is to say, strictly speaking, ϕ is a "1-morphism"; the 1-morphisms from $\mathcal{G}$ to $\mathcal{H}$ form a *category*, of which ϕ is an object; one can then speak of *isomorphisms* between various objects of this category.

On the other hand, observe that if we restrict our attention to *locally open morphisms* ϕ between *totally aloof* semi-graphs of anabelioids, then it follows formally from the definitions [cf. also [Mzk4], Corollary 1.1.6] that:

> The isomorphism class of ϕ is completely determined by the isomorphism class of the ϕ_v.

If, moreover, we restrict our attention to *locally open morphisms* ϕ between *totally aloof, verticially slim* semi-graphs of anabelioids, then:

> The 1-morphism ϕ has no nontrivial automorphisms.

That is to say, so long as we restrict our attention to *locally open morphisms* [e.g., *locally finite étale*] *between totally aloof, verticially slim semi-graphs of anabelioids*, we may work with such morphisms as if they are simply *"morphisms in a category"*, rather than 1-morphisms in a 2-category [cf. the situation for finite étale morphisms between slim anabelioids: [Mzk4], Proposition 1.2.5]. In the following, we shall often take this point of view without further mention.

Proposition 2.5. **(Injectivity)** *Let* $\mathcal{G}$ *be a* **connected, quasi-coherent graph of anabelioids**. *Let* $\mathbb{H} \subseteq \mathbb{G}$ *be a connected subgraph of the underlying graph* $\mathbb{G}$ *of* $\mathcal{G}$. *Then:*

(i) The natural morphisms $\Pi_b \to \Pi_{\mathbb{G}}$, $\Pi_v \to \Pi_{\mathbb{G}}$, $\Pi_{\mathbb{H}} \to \Pi_{\mathbb{G}}$ *are* **injective**. *By abuse of notation, we will denote their images, which are well-defined up to conjugation in* $\Pi_{\mathbb{G}}$, *by* Π_b, Π_v, $\Pi_{\mathbb{H}}$, *respectively.*

(ii) Suppose that $\mathcal{G}$ *is* **of bounded order**. *Then there exists a normal open subgroup* $H \subseteq \Pi_{\mathbb{G}}$, *such that all of the natural morphisms* $\Pi_b \to \Pi_{\mathbb{G}}/H$, $\Pi_v \to \Pi_{\mathbb{G}}/H$ *are* **injective**.

Proof. First, we consider assertion (i). It suffices to show that any finite étale Galois covering of $\mathcal{G}_{\mathbb{H}}$ may be *split* by a finite étale covering pulled back from $\mathcal{G}$. Note that this is immediate in the *locally trivial* case. Indeed, in this case, it suffices to extend the given finite étale covering from $\mathcal{G}_{\mathbb{H}}$ to the remainder of $\mathcal{G}$ by *gluing* [which is always possible, by the *local triviality* assumption!].

Thus, by our assumption of *quasi-coherence*, it suffices to construct, under the further assumption that $\mathcal{G}$ is *of bounded order*, a finite étale covering of $\mathcal{G}$ each of whose constituent anabelioids is *trivial*. We construct such a covering by *gluing*: Let $M \geq 1$ be an integer such all of the orders $[\widehat{\pi}_1(\mathcal{G}_v) : 1]$ divide M. Over the vertex v, we take the covering to be the union of $M/[\widehat{\pi}_1(\mathcal{G}_v) : 1]$ copies of some "universal covering" of $\mathcal{G}_v$ [i.e., the finite étale covering determined by the trivial subgroup of $\widehat{\pi}_1(\mathcal{G}_v)$]. If e is an edge that abuts to the vertex v, then the restriction of this covering to e is a union of $M/[\widehat{\pi}_1(\mathcal{G}_e) : 1]$ copies of some universal covering of $\mathcal{G}_e$ — i.e., a covering of $\mathcal{G}_e$ whose isomorphism class is *independent of* v! Thus, by choosing appropriate gluing isomorphisms, we obtain a covering of $\mathcal{G}$ having the desired properties.

As for assertion (ii), the *Galois closure* of a covering such as that constructed in the preceding paragraph determines a normal open subgroup $H \subseteq \Pi_{\mathbb{G}}$ having the properties asserted in assertion (ii). ◯

Proposition 2.6. **(Commensurability)** *Let* $\mathcal{G}$ *be a* **connected, quasi-coherent graph of anabelioids**. *Let* $\mathbb{H}, \mathbb{K} \subseteq \mathbb{G}$ *be connected subgraphs of the underlying graph* $\mathbb{G}$ *of* $\mathcal{G}$. *Suppose that there exists a component* c *of* $\mathbb{H}$ *that does* **not** *belong to* $\mathbb{K}$ *and which is either an* **elevated vertex** *or a* **sub-coverticial edge** *[i.e., relative to* $\mathcal{G}$*]. Then the intersection of* $\Pi_{\mathbb{H}}$ *with any conjugate of* $\Pi_{\mathbb{K}}$ *has* **infinite index** *in* $\Pi_{\mathbb{H}}$. *In particular, no conjugate of* $\Pi_{\mathbb{H}}$ *is commensurable to a conjugate of* $\Pi_{\mathbb{K}}$.

Proof. First, we consider the case where $c = v$ is a *vertex*. We may assume without loss of generality that $\Pi_{\mathbb{K}} \bigcap \Pi_{\mathbb{H}}$ has *finite index* in $\Pi_{\mathbb{H}}$. Then by taking "M" in Definition 2.4, (i) to be

$$[\Pi_{\mathbb{H}} : \Pi_{\mathbb{K}} \bigcap \Pi_{\mathbb{H}}] + 1$$

and applying the fact that $\mathcal{G}$ is *elevated* at v, we obtain the existence of a π_1-*epimorphic approximator*

$$\mathcal{G} \to \mathcal{G}'$$

such that, if we denote the various fundamental groups associated to $\mathcal{G}'$ by means of a "dash", then there is a subgroup $N_M \subseteq \Pi'_v$ of order $\geq M$ that has *trivial intersection* with each of the conjugates of the Π'_b, for branches b that abut to v. Since $\mathcal{G}'$ is *of bounded order*, it follows from Proposition 2.5, (ii), that there exists a normal open subgroup $K \subseteq \Pi'_{\mathbb{K}}$ such that Π'_w *injects* into $\Pi'_{\mathbb{K}}/K$, for all vertices w of $\mathbb{K}$; moreover, it follows that there exists an integer M' that is divisible both by $[\Pi'_{\mathbb{K}} : K]$ and by *twice* the orders of all of Π'_w, as w ranges over the vertices of $\mathbb{G}$.

Now we construct a [not necessarily connected!] finite étale covering

$$\mathcal{G}'' \to \mathcal{G}'$$

as follows: Over $\mathbb{K}$, we take this covering to be the union of $M'/[\Pi'_{\mathbb{K}} : K]$ copies of the covering defined by the normal open subgroup $K \subseteq \Pi'_{\mathbb{K}}$. Over vertices $w \neq v$ not contained in $\mathbb{K}$, we take this covering to be a union of $M'/[\Pi'_w : 1]$ copies of a "universal covering" of $\mathcal{G}'_w$. Over v, we take this covering to be a union of $M'/2[\Pi'_v : N_M]$ copies of the covering defined by the Π'_v-set Π'_v/N_M and of $M'/2[\Pi'_v : 1]$ copies of a "universal covering" of $\mathcal{G}'_v$. Note that the restriction of any of these coverings over w or v to an abutting edge e is isomorphic to a union of "universal coverings" of $\mathcal{G}'_e$. Thus, by choosing appropriate gluing isomorphisms, we obtain a covering $\mathcal{G}'' \to \mathcal{G}'$.

On the other hand, the existence of this covering leads to a *contradiction*, as follows: This covering determines a finite $\Pi_{\mathbb{G}}$-set S; write $H'' \stackrel{\text{def}}{=} \text{Ker}(\Pi_{\mathbb{G}} \to \text{Aut}(S))$; denote the images in $\Pi_{\mathbb{G}}/H''$ of the various fundamental groups under consideration by means of a "double dash". Thus, we have equalities $\Pi''_{\mathbb{K}} = \Pi_{\mathbb{K}}/K$, $\Pi''_v = \Pi'_v$ [because of the existence of "universal coverings" in the restriction of $\mathcal{G}''$ to v] and inequalities as follows:

$$[\Pi''_{\mathbb{H}} : \Pi''_{\mathbb{K}} \bigcap \Pi''_{\mathbb{H}}] \leq [\Pi_{\mathbb{H}} : \Pi_{\mathbb{K}} \bigcap \Pi_{\mathbb{H}}] < M$$

Moreover, since $\Pi''_{\mathbb{K}}$ acts *freely* on S, it follows that $\Pi''_{\mathbb{K}} \bigcap \Pi''_{\mathbb{H}}$ also acts *freely* on S, and hence that the isotropy subgroup $I \subseteq \Pi''_{\mathbb{H}}$ associated to an element of S has order $\leq [\Pi''_{\mathbb{H}} : \Pi''_{\mathbb{K}} \bigcap \Pi''_{\mathbb{H}}] < M$. In particular, $I_v \stackrel{\text{def}}{=} I \bigcap \Pi''_v$ has order $< M$. On the other hand, by the construction of $\mathcal{G}''$, it follows that for some such isotropy group $I_v \subseteq \Pi''_v = \Pi'_v$, we have $N_M \subseteq I_v$, where we recall that N_M has order $\geq M$, a contradiction.

Next, we consider the case where $c = e$ is an *edge*. Since e is *sub-coverticial*, it suffices [by replacing $\mathcal{G}$ by a finite étale covering of $\mathcal{G}$] to show that $\Pi_{\mathbb{K}} \bigcap \Pi_{\mathbb{H}}$ has *infinite index* in $\Pi_{\mathbb{H}}$ under the assumption that $\mathbb{H}$ contains a pair of distinct coverticial edges e_a, e_b, neither of which is contained in $\mathbb{K}$. Thus, let us suppose that $\Pi_{\mathbb{K}} \bigcap \Pi_{\mathbb{H}}$ has *finite index* in $\Pi_{\mathbb{H}}$; set $M \stackrel{\text{def}}{=} [\Pi_{\mathbb{H}} : \Pi_{\mathbb{K}} \bigcap \Pi_{\mathbb{H}}] + 1$.

Now observe that by using the *loop* $\mathbb{L}$ of $\mathbb{H}$ constituted by e_a, e_b, we may construct a *finite graph-covering of degree* M

$$\mathbb{G}' \to \mathbb{G}$$

which is *trivial* over $\mathbb{K}$, but *connected* over $\mathbb{L}$. Then considering the actions of $\Pi_{\mathbb{H}}$, $\Pi_{\mathbb{K}}$ on the corresponding finite $\Pi_{\mathbb{G}}$-set yields a contradiction. This completes the proof. ○

Corollary 2.7. **(Slimness and Commensurators)** *Let* $\mathcal{G}$ *be a* **connected, quasi-coherent graph of anabelioids**. *Let* $\mathbb{H}, \mathbb{K} \subseteq \mathbb{G}$ *be connected subgraphs of the underlying graph* $\mathbb{G}$ *of* $\mathcal{G}$*. Suppose that every vertex of* $\mathbb{H}$*,* $\mathbb{K}$ *is* **elevated** *[i.e., relative to* $\mathcal{G}$*]. Then:*

(i) We have $C_{\Pi_{\mathbb{G}}}(\Pi_{\mathbb{H}}) = \Pi_{\mathbb{H}}$*. In particular, if* v *is an* **elevated vertex** *of* $\mathbb{G}$*, then* $C_{\Pi_{\mathbb{G}}}(\Pi_v) = \Pi_v$*.*

(ii) Suppose that $\mathbb{H}$ contains a vertex v such that $\mathcal{G}_v$ is **slim***. Then the natural morphism $\mathcal{B}(\mathcal{G}_{\mathbb{H}}) \to \mathcal{B}(\mathcal{G})$ is* **relatively slim** *[cf. [Mzk4], Definition 1.2.9]. In particular, $\mathcal{B}(\mathcal{G})$ is* **slim***.*

(iii) Suppose that every edge of $\mathbb{H}$, $\mathbb{K}$ is **universally sub-coverticial** *[i.e., relative to $\mathcal{G}$] and that $\Pi_{\mathbb{H}}$ is* **commensurable** *to a conjugate of $\Pi_{\mathbb{K}}$. Then $\mathbb{H} = \mathbb{K}$.*

Proof. First, we consider assertion (i). Suppose that $g \in C_{\Pi_{\mathbb{G}}}(\Pi_{\mathbb{H}})$, but $g \notin \Pi_{\mathbb{H}}$. Then there exists a connected finite Galois étale covering

$$\mathcal{G}' \to \mathcal{G}$$

— whose restriction to $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{G}_{\mathbb{H}}$, we denote by $\mathcal{H}' \to \mathcal{H}$ — satisfying the property that there exists a connected component $\mathbb{H}''$ of $\mathbb{H}'$ such that $g \cdot \mathbb{H}'' \neq \mathbb{H}''$ in $\mathbb{G}'$. Since [as one verifies immediately] $\mathbb{H}'$ injects into $\mathbb{G}'$ as a *subgraph*, it thus follows that $g \cdot \mathbb{H}'' \bigcap \mathbb{H}'' = \emptyset$ [in $\mathbb{G}'$]. Thus, assertion (i) follows from Proposition 2.6.

Next, we consider assertion (ii). By assertion (i), we have, for any open subgroup $H \subseteq \Pi_{\mathbb{H}}$:

$$Z_{\Pi_{\mathbb{G}}}(H) \subseteq Z_{\Pi_{\mathbb{G}}}(H \bigcap \Pi_v) \subseteq C_{\Pi_{\mathbb{G}}}(\Pi_v) = \Pi_v$$

Thus, we conclude [since $\mathcal{G}_v$ is *slim*] that $Z_{\Pi_{\mathbb{G}}}(H) \subseteq Z_{\Pi_v}(H \bigcap \Pi_v) = \{1\}$.

Finally, assertion (iii) is a formal consequence of Proposition 2.6. ◯

Remark 2.7.1. There is a certain overlap between the content of Corollary 2.7 and the results obtained in [HR]. The techniques of [HR], however, are more group-theoretic in nature and somewhat different in spirit from those employed in the present exposition.

Next, we consider some concrete examples:

Example 2.8. Trivial Edge Anabelioids. One verifies immediately that every semi-graph of anabelioids $\mathcal{G}$ such that $\mathcal{G}_e$ is *trivial* for all edges e is *quasi-coherent*. In this case, a vertex v of $\mathcal{G}$ is *elevated* if and only if Π_v is *infinite*. Similarly, a closed edge abutting to vertices v, w is *sub-coverticial* (respectively, *universally sub-coverticial*) if and only if both Π_v and Π_w are *nontrivial* (respectively, *infinite*).

Definition 2.9. Let Σ be a set of *prime numbers*.

(i) We shall refer to as a Σ-*integer* any positive integer each of whose prime factors belongs to Σ.

(ii) Given a *connected anabelioid*, we shall refer to as the *pro-Σ completion* of the given anabelioid the connected anabelioid constituted by the full subcategory of the given anabelioid determined by the objects dominated by a Galois covering of the final object [of the given anabelioid] whose degree is a Σ-integer. Similarly, given a semi-graph of anabelioids, we shall refer to as the *pro-Σ completion* of the given semi-graph of anabelioids the semi-graph of anabelioids obtained by replacing each constituent anabelioid by its pro-Σ completion.

Example 2.10. Stable Curves. Let Σ be a nonempty set of prime numbers. Suppose that $\mathcal{G}$ is a semi-graph of anabelioids with the property that each Π_v is the *maximal pro-Σ quotient* of the fundamental group of a *hyperbolic Riemann surface of finite type*, and that each $\Pi_b \to \Pi_v$ is the inclusion morphism of the *inertia group of one of the points at infinity.* Then one verifies immediately [by using the well-known structure of fundamental groups of hyperbolic Riemann surfaces of finite type] that $\mathcal{G}$ is *coherent*, *totally elevated*, *totally universally sub-coverticial*, *totally estranged* [hence also *totally aloof* — cf. the proof of [Mzk3], Lemma 1.3.7], and *verticially slim* [cf. [Mzk3], Lemma 1.3.1]. In particular, the pro-Σ completion of the semi-graph of anabelioids determined by the *[semi-]graph of profinite groups [without compact structure!] associated to a stable curve over a field of characteristic* 0 [cf. [Mzk3], Appendix] satisfies these properties. Similarly, one verifies easily that if Σ contains *at least one prime $\neq p$*, then the pro-Σ completion of the semi-graph of anabelioids determined by the *[semi-]graph of profinite groups [without compact structure!] associated to a stable curve over a field of characteristic $p > 0$* is coherent, totally elevated, totally universally sub-coverticial, totally estranged [cf. the proof of [Mzk3], Lemma 1.3.12], and verticially slim [cf. [Mzk3], Lemma 1.3.10].

Remark 2.10.1. In the case of Example 2.10, it is not difficult to show, using exactly the same techniques as those used in the proofs of Proposition 2.6, Corollary 2.7, that $C_{\Pi_{\mathbb{G}}}(\Pi_b) = \Pi_b$. Since, however, we shall not need this result in the following, and, moreover, a precise description of the condition on edges in the case of a more general $\mathcal{G}$ necessary to carry out such an argument [i.e., the analogue for edges of the notion of an "elevated vertex"] would be rather technical to write out in detail, we leave the task of working out the routine details to the interested reader.

The case of Example 2.10 [cf. also Example 3.10 below] motivates the following extension of the notion of a "morphism of semi-graphs of anabelioids": First, we observe that any *semi-graph* $\mathbb{G}$ may be regarded as a *category*

$$\mathrm{Cat}(\mathbb{G})$$

as follows: The *objects* of this category are the *components* [i.e., vertices and edges] of $\mathbb{G}$. The *morphisms* of this category are the identity morphisms of the components and the branches of edges [i.e., if b is a branch of an edge e that abuts to a vertex v, then we regard b as a morphism $e \to v$] of $\mathbb{G}$. Thus, if $\mathbb{G}$ is the underlying semi-graph of a semi-graph of anabelioids $\mathcal{G}$, then for every object c of $\mathrm{Cat}(\mathbb{G})$, we have an anabelioid $\mathcal{G}_c$, and for every morphism $b : e \to v$ of $\mathrm{Cat}(\mathbb{G})$, we have a morphism of anabelioids $b_* : \mathcal{G}_e \to \mathcal{G}_v$.

Definition 2.11. Let $\mathcal{G}$, $\mathcal{H}$ be semi-graphs of anabelioids. Then a *generalized morphism of semi-graphs of anabelioids*

$$\Phi : \mathcal{G} \to \mathcal{H}$$

is defined to be a collection of data, as follows:

(a) a functor $\text{Cat}(\Phi) : \text{Cat}(\mathbb{G}) \to \text{Cat}(\mathbb{H})$;

(b) for every object c of $\text{Cat}(\mathbb{G})$ that is mapped by $\text{Cat}(\Phi)$ to an object d of $\text{Cat}(\mathbb{H})$, a morphism of anabelioids $\Phi_c : \mathcal{G}_c \to \mathcal{H}_d$;

(c) for every arrow $\phi : c \to c'$ of $\text{Cat}(\mathbb{G})$ that is mapped by $\text{Cat}(\Phi)$ to an arrow $\psi : d \to d'$ of $\text{Cat}(\mathbb{H})$, an isomorphism $\Phi_\phi : \psi_* \circ \Phi_c \xrightarrow{\sim} \Phi_{c'} \circ \phi_*$, such that Φ_ϕ is the *identity* whenever ϕ is an identity morphism.

Remark 2.11.1. It is immediate from the definitions that every [non-generalized] morphism of semi-graphs of anabelioids determines a generlized morphism of semi-graphs of anabelioids. Also, just as in the non-generalized case, it is immediate from the definitions that every generalized morphism of semi-graphs of *connected* anabelioids

$$\Phi : \mathcal{G} \to \mathcal{H}$$

determines, in a natural fashion, a *morphism* $\mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ between the associated anabelioids.

Section 3: The Tempered Fundamental Group

In this §, we define and study the basic properties of the *tempered fundamental group* of a semi-graph of anabelioids. The notion of the tempered fundamental group is introduced in [André], §4. In the present manuscript, however, we wish to study this notion from a more *categorical point of view.*

Let Π be a *topological group.* Then let us write

$$\mathcal{B}^{\text{temp}}(\Pi)$$

for the *category* whose *objects* are *countable* [i.e., of cardinality $\leq$ the cardinality of the set of natural numbers], *discrete* sets equipped with a continuous Π-action and whose *morphisms* are morphisms of Π-sets.

Definition 3.1.

(i) If Π may be written as an inverse limit of an inverse system of surjections of countable discrete topological groups, then we shall say that Π is *tempered.*

(ii) Any category equivalent to a category of the form $\mathcal{B}^{\text{temp}}(\Pi)$ for some tempered topological group Π will be referred to as a *connected temperoid.* Any category equivalent to a product [in the sense of a product of categories] of a countable [hence possibly empty!] collection of connected temperoids will be referred to as a *temperoid.*

(iii) Let $\mathcal{T}_1$, $\mathcal{T}_2$ be temperoids. Then a *morphism* $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ is defined to be a functor $\phi^* : \mathcal{T}_2 \to \mathcal{T}_1$ that preserves finite limits and countable colimits. A morphism ϕ will be called *rigid* if the functor ϕ^* is rigid [cf. §0].

(iv) A connected object T of a temperoid $\mathcal{T}$ will be called *Galois* if, for any two arrows $\psi_1, \psi_2 : S \to T$ of $\mathcal{T}$, where S is connected, there exists a [unique] automorphism $\alpha \in \text{Aut}_{\mathcal{T}}(T)$ of T such that $\psi_1 = \alpha \circ \psi_2$.

Remark 3.1.1. Observe that every *profinite group* is tempered. Moreover, just as in the case of profinite groups, if a tempered group may be written as an inverse limit of an inverse system indexed by a *countable* set of surjections of countable discrete topological groups, then the group is *countably generated* [i.e., generated as a topological group by a countable set of generators].

Remark 3.1.2. Suppose that Π is *tempered.* Then every *open subgroup* of Π is *closed* and of *countable index* in Π. Moreover, the topology of Π admits a *basis of open normal subgroups.* If $H \subseteq \Pi$ is an arbitrary subgroup, then the Π-set Π/H forms an object of $\mathcal{B}^{\text{temp}}(\Pi)$ if and only if H is *open.* If $H_1, H_2 \subseteq \Pi$ are open, then there is a *natural bijection* between the morphisms $\Pi/H_1 \to \Pi/H_2$ and the cosets $h \cdot H_2$ satisfying $h^{-1} \cdot H_1 \cdot h \subseteq H_2$. In particular, if T_1, T_2 are objects of a *temperoid* $\mathcal{T}$, and T_1 is *connected*, then the set $\text{Hom}_{\mathcal{T}}(T_1, T_2)$ is *countable.* If

Π' is also *tempered*, then any continuous homomorphism $\Pi \to \Pi'$ determines [by composing the action of Π' on a Π'-set with this homomorphism so as to obtain a Π-set] a *morphism of connected temperoids* $\mathcal{B}^{\text{temp}}(\Pi) \to \mathcal{B}^{\text{temp}}(\Pi')$.

Remark 3.1.3. Suppose that Π is *tempered*. Then an object of $\mathcal{B}^{\text{temp}}(\Pi)$ is *Galois* if and only if it is isomorphic to the object determined by a Π-set of the form Π/N, where $N \subseteq \Pi$ is an *open normal subgroup*. Alternatively, a connected object T of $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{B}^{\text{temp}}(\Pi)$ is *Galois* if and only if the product $T \times T$ is isomorphic to the *coproduct* of copies of T indexed by the elements of the [countable!] set $\text{Aut}_{\mathcal{T}}(T)$, where the restriction to the copy labeled by $\sigma \in \text{Aut}_{\mathcal{T}}(T)$ of the projection to the first (respectively, second) factor of $T \times T$ is given by the identity (respectively, σ).

Remark 3.1.4. Note that if $\mathcal{X}$ is an *anabelioid* (respectively, *connected anabelioid*), then

$$\mathcal{X}^{\top} \stackrel{\text{def}}{=} (\mathcal{X}^0)^{\top}$$

[cf. §0] is a temperoid (respectively, connected temperoid). We shall refer to $\mathcal{X}^{\top}$ as the *temperification* of $\mathcal{X}$. Just as in the case of anabelioids, the decomposition of a temperoid into a countable product of connected temperoids — each of which we shall refer to as a *connected component* of the original temperoid — may be recovered *completely category-theoretically* from the categorical structure of the original temperoid [i.e., by considering decompositions of the terminal object — cf. [Mzk4], Definitions 1.1.8, 1.1.9]. Also, just as in the case of anabelioids, to give a morphism between two temperoids is *equivalent* to giving, for each connected component of the domain temperoid, a morphism [of connected temperoids] — which we shall refer to as a *component morphism* of the morphism — from that connected component to some connected component of the codomain temperoid.

Remark 3.1.5. It is immediate from the definitions that every temperoid is a *totally epimorphic category of countably connected type* [cf. §0].

Remark 3.1.6. Observe that although every *endomorphism* of a connected object of an anabelioid is an *automorphism*, temperoids do *not*, in general, satisfy this property. Indeed, if Π is a [discrete] free group on generators e_1, e_2, and $H \subseteq \Pi$ is the subgroup generated by elements of the form $e_2^n \cdot e_1 \cdot e_2^{-n}$, where n ranges over the *positive integers*, then *conjugation by* e_2 determines an endomorphism $H \to H$ [i.e., an endomorphism of the object determined by the Π-set Π/H of $\mathcal{B}^{\text{temp}}(\Pi)$] which is *not* an automorphism. Nevertheless, it is immediate from the definitions that every endomorphism of a *Galois* connected object of a temperoid is an automorphism.

Remark 3.1.7. In some situations, instead of considering temperoids or anabelioids, it is useful to consider slightly more general versions of these notions, which we shall refer to as *"quasi-temperoids"*, *"quasi-anabelioids"*, respectively. For more on these "routine" generalizations, we refer to the Appendix.

Remark 3.1.8. For a more general treatment of "categories that behave like anabelioids and temperoids", we refer to [Dub].

Proposition 3.2. (The "Grothendieck Conjecture" for Connected Temperoids) *For $i = 1, 2$, let Π_i be a* **tempered** *[topological] group; write $\mathcal{T}_i \stackrel{\text{def}}{=} \mathcal{B}^{\text{temp}}(\Pi_i)$. Then the* **category of morphisms**

$$\mathcal{T}_1 \to \mathcal{T}_2$$

is equivalent to the category whose **objects** *are continuous group homomorphisms $\phi : \Pi_1 \to \Pi_2$ and whose* **morphisms** *$\phi \to \psi$ are elements $g \in \Pi_2$ such that $\gamma_g \circ \phi = \psi$, where we write γ_g for the automorphism of Π_2 given by conjugating by g. In particular, there is a* **natural bijective correspondence** *between the set of isomorphism classes of morphisms $\mathcal{T}_1 \to \mathcal{T}_2$ and the set of [continuous]* **outer homomorphisms** *$\Pi_1 \to \Pi_2$.*

Proof. Observe that, by thinking of a *Galois object* $T \in \text{Ob}(\mathcal{T}_2)$ as an *"$\text{Aut}_{\mathcal{T}_2}(T)$-torsor object"* [cf. Remark 3.1.3], it follows that if $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ is a morphism of temperoids, then the object $\phi^*(T)$ of $\mathcal{T}_1$ is *Galois*. In light of this observation, the proof of Proposition 3.2 is formally entirely similar to that of [Mzk4], Proposition 1.1.4 [i.e. the "Grothendieck Conjecture" for connected anabelioids]. ◯

Remark 3.2.1. In particular, Proposition 3.2 implies that, if $\mathcal{X}$ is a *connected temperoid*, then it makes sense to write $\pi_1^{\text{temp}}(\mathcal{X})$. We shall refer to this tempered group $\pi_1^{\text{temp}}(\mathcal{X})$ [which is well-defined, up to inner automorphism] as the *tempered fundamental group* of $\mathcal{X}$.

Corollary 3.3. (Rigid Morphisms of Connected Temperoids) *We maintain the notation of Proposition 3.2. Let $\phi : \Pi_1 \to \Pi_2$ be a continuous homomorphism that gives rise to a morphism of temperoids $\mathcal{B}^{\text{temp}}(\phi) : \mathcal{T}_1 \to \mathcal{T}_2$. Then $\mathcal{B}^{\text{temp}}(\phi)$ is* **rigid** *if and only if the centralizer $Z_{\Pi_2}(\text{Im}(\phi))$ is* **trivial**.

Proof. This is a formal consequence of Proposition 3.2 [cf. [Mzk4], Corollary 1.1.6]. ◯

Next, let us observe that, if $T \in \text{Ob}(\mathcal{T})$ is an *object* of a *connected temperoid* $\mathcal{T}$, then the category

$$\mathcal{T}_T$$

forms a *temperoid* [which is connected if and only if T is connected]. Moreover, forming the *product* with T yields a functor $\mathcal{T} \to \mathcal{T}_T$ which determines a *morphism of temperoids*

$$\mathcal{T}_T \to \mathcal{T}$$

Thus, in the spirit of [Mzk4], §1.2, we make the following definition:

Definition 3.4.

(i) An *étale morphism* of connected temperoids is a morphism that is abstractly equivalent to a morphism of the form $\mathcal{T}_T \to \mathcal{T}$ considered in the above discussion. An *étale morphism* of temperoids is a morphism each of whose component morphisms is étale.

(ii) A continuous homomorphism of tempered groups $\Pi_1 \to \Pi_2$ will be called *relatively temp-slim* if, for every open subgroup $H \subseteq \Pi_1$, the centralizer $Z_{\Pi_2}(\mathrm{Im}(H))$ is *trivial.* A tempered group will be called *temp-slim* if its identity morphism is relatively temp-slim. A temperoid will be called *temp-slim* if and only if each of its connected components is equivalent to the "$\mathcal{B}^{\mathrm{temp}}(-)$" of some temp-slim tempered group.

Remark 3.4.1. One verifies immediately that a temperoid is *slim* as a category [cf. §0] if and only if it is *temp-slim* as a temperoid, and that an anabelioid is *slim* as an anabelioid if and only if its temperification is *temp-slim* as a temperoid.

Remark 3.4.2. By Corollary 3.3, one may work with *relatively temp-slim morphisms of temperoids* as if they are *"morphisms in a category"* [not "1-morphisms in a 2-category"]. In particular, if one works under this convention, then the *category of étale objects* over a given *temp-slim temperoid* $\mathcal{T}$ forms a category which is *equivalent* to $\mathcal{T}$ itself [cf. [Mzk4], Proposition 1.2.5]. Moreover, just as in the case of anabelioids [cf. [Mzk4], §1.2], one may work with *"pro-temperoids"* and hence consider *universal coverings* [which are pro-temperoids] of a given temp-slim temperoid.

Now we return to *semi-graphs of anabelioids.* Let $\mathcal{G}$ be a *connected, countable* [i.e., the underlying semi-graph is countable] *semi-graph of anabelioids.* If $\mathbb{G}$ has *at least one vertex*, then let us denote by

$$\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$$

the category of *objects* given by data

$$\{S_v, \phi_e\}$$

where v (respectively, e) ranges over the vertices (respectively, edges) of $\mathbb{G}$; for each vertex v, $S_v \in \mathrm{Ob}(\mathcal{G}_v^{\top})$; for each edge e, with branches b_1, b_2 abutting to vertices v_1, v_2, respectively, $\phi_e : b_1^* S_{v_1} \xrightarrow{\sim} b_2^* S_{v_2}$ is an isomorphism in $\mathcal{G}_e^{\top}$, and *morphisms* given by morphisms [in the evident sense] between such data. If $\mathbb{G}$ has *no vertices* — and hence precisely one edge e, which is necessarily *isolated* — then we shall write $\mathcal{B}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathcal{G}_e^{\top}$.

The definition of $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$ extends immediately to *arbitrary semi-graphs of anabelioids*, each connected component of which is *countable.* Moreover, for such $\mathcal{G}$, we have a natural *full embedding*:

$$\mathcal{B}(\mathcal{G}) \hookrightarrow \mathcal{B}^{\mathrm{cov}}(\mathcal{G})$$

An object of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ that lies in the essential image of $\mathcal{B}(\mathcal{G})$ will be called *finite*.

Now observe that we may associate, in a natural way, to any object of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ a *morphism of countable semi-graphs of anabelioids*:

$$\mathcal{G}' \to \mathcal{G}$$

[cf. the discussion of §2 in the case of $\mathcal{B}(\mathcal{G})$].

Definition 3.5.

(i) We shall refer to a morphism of semi-graphs of anabelioids $\mathcal{G}' \to \mathcal{G}$ that may be constructed in this way as a *covering of semi-graphs of anabelioids* [so, in the terminology of §2, coverings of semi-graphs of anabelioids that arise from *finite* objects determine *"finite étale coverings of semi-graphs of anabelioids"*].

(ii) Suppose that $\mathcal{G}' \to \mathcal{G}$ satisfies the condition that there exists a finite étale covering $\mathcal{G}'' \to \mathcal{G}$ with the property that, for any component [i.e., vertex or edge] c of $\mathbb{G}$, the restriction of $\mathcal{G}'' \to \mathcal{G}$ to $\mathcal{G}_c$ *splits* the restriction of $\mathcal{G}' \to \mathcal{G}$ to $\mathcal{G}_c$. Then we shall say that the covering $\mathcal{G}' \to \mathcal{G}$, as well as the object of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ that gave rise to this covering, is *tempered*. Also, we shall write

$$\mathcal{B}^{\text{temp}}(\mathcal{G}) \subseteq \mathcal{B}^{\text{cov}}(\mathcal{G})$$

for the full subcategory determined by the tempered objects. Thus, we have natural *full embeddings*:

$$\mathcal{B}(\mathcal{G}) \hookrightarrow \mathcal{B}^{\text{temp}}(\mathcal{G}) \hookrightarrow \mathcal{B}^{\text{cov}}(\mathcal{G})$$

(iii) An arrow $\phi : H_1 \to H_2$ of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ with connected domain and codomain will be called *Galois* if, for any two arrows $\psi_1, \psi_2 : K \to H_1$ such that $\phi \circ \psi_1 = \phi \circ \psi_2$, there exists a [unique] automorphism $\alpha \in \text{Aut}_{H_2}(H_1)$ of H_1 over H_2 such that $\psi_1 = \alpha \circ \psi_2$.

Remark 3.5.1. It is immediate from the definitions that passing to the *underlying morphism of semi-graphs* yields an equivalence between the datum of a *locally trivial covering of semi-graph of anabelioids* of $\mathcal{G}$ and the datum of a *graph-covering with countable fibers of the semi-graph* $\mathbb{G}$.

Remark 3.5.2. It follows from Remark 2.4.2 that, if $\mathcal{G}$ is *totally aloof* and *verticially slim*, then the construction given above of a covering of semi-graphs of anabelioids associated to an object of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ determines a natural *full embedding* of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ into the *category of totally aloof, verticially slim semi-graphs of anabelioids and locally finite étale morphisms over* $\mathcal{G}$.

Now let us assume that the semi-graph of anabelioids $\mathcal{G}$ is *connected*, *quasi-coherent*, *totally elevated*, *totally aloof*, and *verticially slim*. Let

$$\{\mathcal{G}^i \to \mathcal{G}\}_{i \in I}$$

be some *cofinal* [i.e., in $\mathcal{B}(\mathcal{G})$] collection of *connected finite étale Galois coverings*, indexed by a set I. Thus, $\widehat{\pi}_1(\mathcal{G})$ may be constructed as the inverse limit

$$\varprojlim_i \ \mathrm{Gal}(\mathcal{G}^i/\mathcal{G})$$

of the resulting inverse system of finite groups $\mathrm{Gal}(\mathcal{G}^i/\mathcal{G})$. Let us write

$$\mathcal{G}^{\infty,i} \to \mathcal{G}^i$$

for the *"combinatorial universal covering"* of $\mathcal{G}^i$ [i.e., the covering of $\mathcal{G}^i$ determined by the universal graph-covering of the underlying semi-graph $\mathbb{G}^i$]. One verifies immediately that $\mathcal{G}^{\infty,i} \to \mathcal{G}$ is a *Galois tempered covering.* Then we set:

$$\pi_1^{\mathrm{temp}}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \varprojlim_i \ \mathrm{Gal}(\mathcal{G}^{\infty,i}/\mathcal{G})$$

Note that $\pi_1^{\mathrm{temp}}(\mathcal{G})$ is *independent*, up to *inner automorphism*, of the choice of the cofinal system $\{\mathcal{G}^i \to \mathcal{G}\}_{i\in I}$.

Theorem 3.6. (The Tempered Fundamental Group of a Semi-graph of Anabelioids) *Let $\mathcal{G}$, $\mathcal{G}'$ be* **connected**, **countable**, **quasi-coherent**, **totally elevated**, **totally aloof**, **verticially slim** *semi-graphs of anabelioids. Then:*

(i) The topological group $\pi_1^{\mathrm{temp}}(\mathcal{G})$ defined above is **tempered**.

(ii) There is a natural **equivalence of categories***:*

$$\mathcal{B}^{\mathrm{temp}}(\pi_1^{\mathrm{temp}}(\mathcal{G})) \xrightarrow{\sim} \mathcal{B}^{\mathrm{temp}}(\mathcal{G})$$

In particular, the category $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ is a **connected temperoid**. *We shall refer to $\pi_1^{\mathrm{temp}}(\mathcal{G})$ as the* **tempered fundamental group** *of $\mathcal{G}$.*

(iii) The full embedding $\mathcal{B}(\mathcal{G}) \hookrightarrow \mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ induces an **injection** *$\pi_1^{\mathrm{temp}}(\mathcal{G}) \hookrightarrow \widehat{\pi}_1(\mathcal{G})$ of topological groups.*

(iv) Any morphism of semi-graphs of anabelioids $\mathcal{G}' \to \mathcal{G}$ induces a morphism of temperoids

$$\mathcal{B}^{\mathrm{temp}}(\mathcal{G}') \to \mathcal{B}^{\mathrm{temp}}(\mathcal{G})$$

[by pulling back tempered coverings of $\mathcal{G}$ to tempered coverings of $\mathcal{G}'$]. Moreover, if the original morphism of semi-graphs of anabelioids is **locally finite étale**, *then this morphism of temperoids is* **relatively temp-slim**. *In particular, the temperoid $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ is* **temp-slim**.

(v) Suppose that $\mathcal{G}$ is **coherent**, *and that we are given a* **tempered covering** *$\mathcal{G}' \to \mathcal{G}$. Then the resulting morphism of temperoids*

$$\mathcal{B}^{\mathrm{temp}}(\mathcal{G}') \to \mathcal{B}^{\mathrm{temp}}(\mathcal{G})$$

is **étale**.

Proof. Assertion (i) follows from the definitions and the fact that $\mathcal{G}$ is *countable*. Next, we consider assertion (ii). First, we observe that, by the definition of a *tempered covering*, it follows that every connected tempered covering $\mathcal{H} \to \mathcal{G}$ appears as a *subcovering* of some $\mathcal{G}^{\infty,i} \to \mathcal{G}$. Moreover, unraveling the definitions [cf., especially, the definition of $\mathcal{B}^{\text{cov}}(\mathcal{G})$] reveals that the covering $\mathcal{H} \to \mathcal{G}$ may be constructed as the *quotient* of the covering $\mathcal{G}^{\infty,i} \to \mathcal{G}$ by the action of $\text{Gal}(\mathcal{G}^{\infty,i}/\mathcal{H})$; and that every morphism of coverings [of $\mathcal{G}$] from $\mathcal{G}^{\infty,i}$ to $\mathcal{H}$ may be obtained from the original morphism by composing with the action of $\text{Gal}(\mathcal{G}^{\infty,i}/\mathcal{G})$. Conversely, one verifies immediately that the quotient of $\mathcal{G}^{\infty,i} \to \mathcal{G}$ by the action of any subgroup of $\text{Gal}(\mathcal{G}^{\infty,i}/\mathcal{G})$ yields a subcovering of $\mathcal{G}^{\infty,i} \to \mathcal{G}$. This shows that there is a *natural equivalence* between $\mathcal{B}^{\text{temp}}(\text{Gal}(\mathcal{G}^{\infty,i}/\mathcal{G}))$ and the full subcategory of $\mathcal{B}^{\text{temp}}(\mathcal{G})$ determined by the subcoverings of $\mathcal{G}^{\infty,i} \to \mathcal{G}$. Passing to the limit over i then completes the proof of assertion (ii). Assertion (iii) follows from the fact that *discrete free groups* [such as $\text{Gal}(\mathcal{G}^{\infty,i}/\mathcal{G}^{i})$] are always *residually finite* [cf., e.g., Corollary 1.7]. Assertion (iv) follows from the definitions and Corollary 2.7, (ii). Finally, assertion (v) follows from the fact that the *coherence* of $\mathcal{G}$ implies that, given any finite étale covering $\mathcal{H}' \to \mathcal{G}'$, there exists a finite étale covering $\mathcal{H} \to \mathcal{G}$ whose pull-back to $\mathcal{G}'$ *splits* the restrictions of $\mathcal{H}' \to \mathcal{G}'$ to each of the $\mathcal{G}'_c$ [where c is a component of $\mathbb{G}'$]. ◯

Theorem 3.7. (Maximal Compact Subgroups of the Tempered Fundamental Group) *Let* $\mathcal{G}$ *be a* **connected**, **countable**, **quasi-coherent**, **totally elevated**, **totally estranged**, **verticially slim** *semi-graph of anabelioids. Assume that* $\mathbb{G}$ *has* **at least one vertex**. *Then:*

(i) For each vertex v of $\mathcal{G}$, there is a natural continuous, injective outer homomorphism $\widehat{\pi}_1(\mathcal{G}_v) \hookrightarrow \pi_1^{\text{temp}}(\mathcal{G})$. By abuse of notation, we shall write $\widehat{\pi}_1(\mathcal{G}_v)$ for the subgroup [well-defined up to conjugation] determined by the image of this homomorphism. We shall refer to the [necessarily **compact**!*] subgroups of $\pi_1^{\text{temp}}(\mathcal{G})$ that arise in this way as the* **verticial subgroups**.

(ii) Let us think of the verticial subgroups as being **parametrized** *by a vertex v of $\mathcal{G}$ and an element of the coset space $\pi_1^{\text{temp}}(\mathcal{G})/\widehat{\pi}_1(\mathcal{G}_v)$. Then if H_1, H_2 are verticial subgroups of $\pi_1^{\text{temp}}(\mathcal{G})$ that arise from* **distinct parametrization data**, *then $H_1 \bigcap H_2$ has* **infinite index** *in H_1. In particular, verticial subgroups that arise from distinct parametrization data are* **distinct**.

(iii) Every compact subgroup of $\pi_1^{\text{temp}}(\mathcal{G})$ is contained in **at least one** *verticial subgroup. If a nontrivial compact subgroup of $\pi_1^{\text{temp}}(\mathcal{G})$ is contained in* **more than one** *verticial subgroup, then it is contained in* **precisely two** *verticial subgroups, determined by a compatible system of pairs of vertices of the $\mathbb{G}^{\infty,i}$ [as i ranges over the elements of I] joined to one another by a* **single [closed] edge**. *In particular, in this case, this compact subgroup is contained in the image of some $\widehat{\pi}_1(\mathcal{G}_e)$, for some edge e of $\mathbb{G}$. We shall refer to such images of "$\widehat{\pi}_1(\mathcal{G}_e)$'s" as the* **edge-like subgroups** *of $\pi_1^{\text{temp}}(\mathcal{G})$.*

(iv) The **maximal compact subgroups** *of* $\pi_1^{\text{temp}}(\mathcal{G})$ *are precisely the* **verticial subgroups***. The* **nontrivial intersections of two distinct maximal compact subgroups** *of* $\pi_1^{\text{temp}}(\mathcal{G})$ *are precisely the* **edge-like subgroups***.*

Proof. Assertion (i) follows from the definitions; Theorem 3.6, (iii); Proposition 2.5. Assertion (ii) follows from Theorem 3.6, (iii); Proposition 2.6; Corollary 2.7, (i).

Now suppose that $H \subseteq \pi_1^{\text{temp}}(\mathcal{G})$ is a *nontrivial compact subgroup.* Then H acts *continuously* on the semi-graph $\mathbb{G}^{\infty,i}$, for each $i \in I$. Thus, this action factors through a *finite quotient.* In particular, by Lemma 1.8, (ii), (a), H fixes at least one edge or vertex of the semi-graph $\mathbb{G}^{\infty,i}$. Since the action of H is *over* $\mathbb{G}$, it follows that if H fixes an edge, then it *does not switch the branches* of the edge. Since $\mathbb{G}$, hence also $\mathbb{G}^{\infty,i}$, is connected and has at least one vertex, it thus follows that every edge of $\mathbb{G}^{\infty,i}$ abuts to at least one vertex. In particular, we conclude that if H fixes an edge of $\mathbb{G}^{\infty,i}$, then it fixes a vertex, i.e., that H always *fixes at least one vertex* of $\mathbb{G}^{\infty,i}$.

Now suppose that for some *cofinal* subset $J \subseteq I$, H fixes ≥ 3 *vertices* of $\mathbb{G}^{\infty,j}$, for every $j \in J$. Then by Lemma 1.8, (ii), (c), we conclude that, for every $j \in J$, H acts trivially on some *subjoint* of $\mathbb{G}^{\infty,j}$. In particular, H acts trivially on some *subjoint* of $\mathbb{G}^j$. Since the semi-graphs $\mathbb{G}^j$ are all *finite*, we thus conclude that we may choose a *compatible system of such subjoints* [i.e., on which H acts trivially] of the $\mathbb{G}^j$. But, sorting through the definitions, this implies that H is contained in *some conjugate* in the profinite fundamental group $\widehat{\pi}_1(\mathcal{G})$ of some $\widehat{\pi}_1(\mathcal{G}_v)$, and, moreover, that it is in fact contained, for two *distinct* branches b, b' abutting to v of edges e, e', respectively [where e is not necessarily distinct from e'], in the *intersection* of the images of $\widehat{\pi}_1(\mathcal{G}_e)$, $\widehat{\pi}_1(\mathcal{G}_{e'})$, via b, b'. But since $\mathcal{G}$ is assumed to be *totally estranged*, we thus conclude that H is *trivial*, in contradiction to our hypotheses.

Thus, in summary, we have shown that for some *cofinal* subset $J \subseteq I$, H fixes *at least one*, but *no more than two* vertices of $\mathbb{G}^{\infty,j}$, for every $j \in J$. Moreover, by Lemma 1.8, (b), it follows if H fixes two vertices of $\mathbb{G}^{\infty,j}$, then these two vertices are joined to one another by a *single [closed] edge.* In particular, by possibly replacing J by some smaller cofinal subset, we may assume that there exists a *compatible system* of vertices of $\mathbb{G}^{\infty,j}$, for $j \in J$, each of which is fixed by H. On the other hand, we may also conclude that *there exist at most two* such compatible systems. This completes the proof of assertion (iii). Finally, assertion (iv) follows formally from assertions (ii), (iii). ◯

Remark 3.7.1. The notion that

maximal compact subgroups correspond to points

is a recurrent theme in the geometry of group actions. Classical well-known examples of this phenomenon include the theory of symmetric spaces obtained as quotients of a real reductive group by a maximal compact subgroup or, in the p-adic case, of $\mathbb{Q}_p$-valued points of a reductive group by $\mathbb{Z}_p$-valued points. Another example of this sort of situation is the *"discrete real section conjecture"* of [Mzk5], §3.2.

Definition 3.8. We shall refer to as *quasi-geometric* any continuous homomorphism of tempered groups

$$\Pi_1 \to \Pi_2$$

that satisfies the following condition: Any maximal compact subgroup $K_1 \subseteq \Pi_1$ (respectively, nontrivial intersection $K_1 \bigcap H_1$ of two distinct maximal compact subgroups $K_1, H_1 \subseteq \Pi_1$) maps surjectively to an open subgroup of some maximal compact subgroup $K_2 \subseteq \Pi_2$ (respectively, of some nontrivial intersection $K_2 \bigcap H_2$ of two distinct maximal compact subgroups $K_2, H_2 \subseteq \Pi_2$). A *quasi-geometric morphism of temperoids* is a morphism of temperoids each of whose component morphisms arises [cf. Proposition 3.2] from a quasi-geometric continuous homomorphism of tempered groups.

Remark 3.8.1. It is immediate that any *isomorphism of temperoids* is *quasi-geometric.*

Corollary 3.9. **(Reconstruction of the Underlying Semi-graph of Anabelioids)** *Let $\mathcal{G}$, $\mathcal{H}$ be* **connected**, **countable**, **quasi-coherent**, **totally elevated**, **totally estranged**, **verticially slim** *graphs of anabelioids. Then applying "$\mathcal{B}^{\text{temp}}(-)$" determines a natural bijective correspondence between* **locally open morphisms of semi-graphs of anabelioids**

$$\mathcal{G} \to \mathcal{H}$$

and **quasi-geometric morphisms of temperoids** $\mathcal{B}^{\text{temp}}(\mathcal{G}) \to \mathcal{B}^{\text{temp}}(\mathcal{H})$.

Proof. First, we note that it is immediate from the definitions and Theorem 3.7, (iv), that any locally open morphisms of semi-graphs of anabelioids $\mathcal{G} \to \mathcal{H}$ determines a quasi-geometric morphism of temperoids $\mathcal{B}^{\text{temp}}(\mathcal{G}) \to \mathcal{B}^{\text{temp}}(\mathcal{H})$. Next, we observe that by Proposition 3.2; Definition 3.8; Theorem 3.7, (ii), (iii), any quasi-geometric $\phi : \mathcal{B}^{\text{temp}}(\mathcal{G}) \to \mathcal{B}^{\text{temp}}(\mathcal{H})$ determines a map from the vertices to $\mathbb{G}$ to the vertices of $\mathbb{H}$ — i.e., by considering the *unique* [conjugacy class of] verticial subgroup(s) of $\pi_1^{\text{temp}}(\mathcal{H})$ that contain(s) the image of a given verticial subgroup of $\pi_1^{\text{temp}}(\mathcal{G})$. Similarly, by considering nontrivial intersections of maximal compact subgroups, one obtains [in light of the fact that, since $\mathcal{H}$ is *totally elevated* and *totally aloof*, all of the edge-like subgroups of $\pi_1^{\text{temp}}(\mathcal{H})$ are *infinite*] that any quasi-geometric $\phi : \mathcal{B}^{\text{temp}}(\mathcal{G}) \to \mathcal{B}^{\text{temp}}(\mathcal{H})$ determines a map from the edges to $\mathbb{G}$ to the edges of $\mathbb{H}$ which is compatible with the map obtained above on vertices. Thus, in summary, we conclude that a quasi-geometric $\phi : \mathcal{B}^{\text{temp}}(\mathcal{G}) \to \mathcal{B}^{\text{temp}}(\mathcal{H})$ determines a *map on the underlying graphs* $\mathbb{G} \to \mathbb{H}$ that is *functorial* in ϕ.

Next, we observe that if $\phi : \mathcal{B}^{\text{temp}}(\mathcal{G}) \to \mathcal{B}^{\text{temp}}(\mathcal{H})$ is quasi-geometric, that any morphism

$$\phi' : \mathcal{B}^{\text{temp}}(\mathcal{G}') \to \mathcal{B}^{\text{temp}}(\mathcal{H}')$$

induced by ϕ between *étale coverings* of the domain and codomain of ϕ [i.e., $\mathcal{G}' \to \mathcal{G}$, $\mathcal{H}' \to \mathcal{H}$ are *tempered coverings*] is again *quasi-geometric*. Indeed, this follows immediately from the characterization of nontrivial maximal compact subgroups (respectively, nontrivial intersections of two distinct maximal compact subgroups) as the *verticial subgroups* (respectively, *edge-like subgroups*) — cf. Theorem 3.7, (iii), (iv). Thus, we obtain a morphism of graphs $\mathbb{G}' \to \mathbb{H}'$, which is *functorial* in $\mathcal{G}'$, $\mathcal{H}'$. Finally, by varying $\mathcal{G}'$, $\mathcal{H}'$, we conclude that ϕ arises from a *morphism of graphs of anabelioids* which [again by Theorem 3.7, (iii), (iv)] is manifestly *unique* and *locally open*. This completes the proof of Corollary 3.9. ◯

Remark 3.9.1. Suppose that $\mathcal{G}$ is as in Theorem 3.7. Then observe that, if $\mathbb{G}$ is a *semi-graph* which is *not* a graph, then the techniques developed here are *not* sufficient, in general, to reconstruct the *open edges* of $\mathbb{G}$ from, say, the isomorphism class of the tempered group $\pi_1^{\rm temp}(\mathcal{G})$ — cf. Remark 4.8.1.

Example 3.10. Pointed Stable Curves over *p*-adic Local Fields I. Let K be a finite extension of $\mathbb{Q}_p$; $\overline{K}$ an algebraic closure of K; $X_K^{\log}$ a *smooth pointed stable log curve* over K. Let us write $X_{\overline{K}}^{\log} \stackrel{\rm def}{=} X_K^{\log} \times_K \overline{K}$;

$$\pi_1^{\rm temp}(X_K^{\log})$$

for the *tempered fundamental group* of [André], §4. Thus, $\pi_1^{\rm temp}(X_K^{\log})$ is a *tempered topological group* [in the sense of Definition 3.1, (i)] and fits into a natural *exact sequence*:

$$1 \to \pi_1^{\rm temp}(X_{\overline{K}}^{\log}) \to \pi_1^{\rm temp}(X_K^{\log}) \to G_K \to 1$$

[where $G_K \stackrel{\rm def}{=} \mathrm{Gal}(\overline{K}/K)$; we write $\pi_1^{\rm temp}(X_{\overline{K}}^{\log})$ for the *geometric tempered fundamental group* of $X_K^{\log}$, i.e., the tempered fundamental group of $X_{\overline{K}}^{\log} \times_{\overline{K}} (\overline{K})^\wedge$; the "$\wedge$" denote the p-adic completion]. To simplify the notation, let us write:

$$\Pi \stackrel{\rm def}{=} \pi_1^{\rm temp}(X_K^{\log}); \quad \Delta \stackrel{\rm def}{=} \pi_1^{\rm temp}(X_{\overline{K}}^{\log})$$

Note that Δ is also *tempered*, so we obtain *temperoids*:

$$\mathcal{B}^{\rm temp}(\Pi); \quad \mathcal{B}^{\rm temp}(\Delta)$$

Now let us write

$$\mathcal{G} \text{ (respectively, } \mathcal{G}^{\rm c})$$

for the *graph of anabelioids* (respectively, *semi-graph of anabelioids*) determined by the semi-graph of profinite groups [without compact structure!] (respectively, with compact structure) associated to the geometric special fiber of the stable model of $X_{\overline{K}}^{\log}$ [cf. [Mzk3], Appendix] over the ring of integers $\mathcal{O}_{\overline{K}}$ of $\overline{K}$. Note that it follows from the definitions that we have a natural *equivalence* $\mathcal{B}^{\rm temp}(\mathcal{G}) \xrightarrow{\sim} \mathcal{B}^{\rm temp}(\mathcal{G}^{\rm c})$ and a natural *full embedding*:

$$\mathcal{B}^{\rm temp}(\mathcal{G}) \hookrightarrow \mathcal{B}^{\rm temp}(\Delta)$$

Now suppose that we are given an exhaustive sequence of open characteristic [hence normal] subgroups of finite index

$$\ldots \subseteq N_i \subseteq \ldots \subseteq \Delta$$

[where i ranges over the positive integers] of Δ; write $\Delta_i \stackrel{\mathrm{def}}{=} \Delta/N_i$. Thus, N_i determines a finite log étale covering of $X^{\log}_{\overline{K}}$, whose geometric special fiber gives rise to *semi-graphs of anabelioids*

$$\mathcal{G}_i; \quad \mathcal{G}_i^{\mathrm{c}}$$

on which Δ_i acts *faithfully*. Recall from Example 2.10 that $\mathcal{G}_i$, $\mathcal{G}_i^{\mathrm{c}}$ are *coherent*, *totally elevated*, *totally universally sub-coverticial*, *totally estranged*, and *verticially slim*. In particular, $\mathcal{G}_i$ satisfies the hypotheses of Theorem 3.7, Corollary 3.9.

Next, let us observe that we obtain a *natural compatible system of* **generalized** [cf. Definition 2.11] *morphisms of graphs of anabelioids*

$$\mathcal{G}_i \to \mathcal{G}_j$$

[where $i \geq j$], which are compatible with the actions of Δ_i, Δ_j, as follows: The functor

$$\mathrm{Cat}(\mathbb{G}_i) \to \mathrm{Cat}(\mathbb{G}_j)$$

is obtained by mapping a vertex v (respectively, vertex v; edge e; edge e) of $\mathbb{G}_i$ to a(n) vertex v' (respectively, edge e'; edge v'; edge e') of $\mathbb{G}_j$ whenever the map on geometric special fibers between the coverings determined by Δ_i, Δ_j maps the irreducible component corresponding to v into the irreducible component corresponding to v' in such a way that the image of the irreducible component corresponding to v is not equal to a node (respectively, the irreducible component corresponding to v into the node corresponding to e'; the node corresponding to e to a non-nodal point lying in the irreducible component corresponding to v'; the node corresponding to e to the node corresponding to e'). The remainder of the data necessary to define the generalized morphism of graphs of anabelioids $\mathcal{G}_i \to \mathcal{G}_j$ is determined naturally by considering the map on geometric special fibers between the coverings determined by Δ_i, Δ_j. For a more *group-theoretic description* of these generalized morphisms $\mathcal{G}_i \to \mathcal{G}_j$, we refer to the discussion of Example 5.6 below.

Finally, we observe that these generalized morphisms of graphs of anabelioids induce — by applying "$\mathcal{B}^{\mathrm{temp}}(-)$" [cf. Remark 2.11.1] — natural *morphisms of temperoids*

$$\ldots \to \mathcal{B}^{\mathrm{temp}}(\mathcal{G}_i) \to \ldots \to \mathcal{B}^{\mathrm{temp}}(\mathcal{G}_j) \to \ldots \to \mathcal{B}^{\mathrm{temp}}(\mathcal{G})$$

compatible with the actions of the Δ_i, hence also corresponding *surjections of tempered groups*:

$$\Delta \twoheadrightarrow \ldots \twoheadrightarrow \Delta[i] \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(\mathcal{G}_i) \rtimes \Delta_i \twoheadrightarrow \ldots \twoheadrightarrow \Delta[j] \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(\mathcal{G}_j) \rtimes \Delta_j \twoheadrightarrow \ldots \twoheadrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G})$$

Note that each $\Delta[i]$ is *temp-slim*. [Indeed, this follows from Corollary 2.7, (iii); the fact that Δ_i acts *faithfully* on $\mathcal{G}_i$; Theorem 3.6, (iv).] Since Δ is the inverse limit of the $\Delta[i]$, and G_K is *slim* [cf. e.g., [Mzk3], Theorem 1.1.1], we thus conclude that both Δ and Π are *temp-slim*.

Remark 3.10.1. We maintain the notation of Example 3.10. Write $\mathcal{B}^{\text{temp}}(X_K^{\log})$ for the category of *tempered coverings of* $X_K^{\log}$ [so $\mathcal{B}^{\text{temp}}(X_K^{\log})$ is a temperoid whose tempered fundamental group is $\pi_1^{\text{temp}}(X_K^{\log})$]. Let Σ be a *set of prime numbers.* Denote by

$$\mathcal{B}^{\text{temp}}(X_K^{\log})^{\Sigma} \subseteq \mathcal{B}^{\text{temp}}(X_K^{\log})$$

the full subcategory determined by the tempered coverings dominated by coverings which arise as a *combinatorial covering* [i.e., a covering arising from a graph-covering of the dual graph of the geometric special fiber] of a finite étale Galois covering of $X_K^{\log}$ whose degree is a Σ-*integer.* One verifies immediately that $\mathcal{B}^{\text{temp}}(X_K^{\log})^{\Sigma}$ is a *temperoid.* We shall refer to the tempered fundamental group of this temperoid as the *pro-*Σ *tempered fundamental group* of $X_K^{\log}$. Then, so long as Σ contains *at least one prime* $\neq p$, the entire discussion of Example 3.10 may be carried out for the *pro-*Σ *tempered fundamental group* of $X_K^{\log}$. [We leave the routine details to the interested reader.] Also, we observe that the analogue of "$\mathcal{G}$" (respectively, "$\mathcal{G}^{\text{c}}$") in the pro-Σ case is precisely the *pro-*Σ *completion* [in the sense of Definition 2.9, (ii)] of the semi-graph of anabelioids $\mathcal{G}$ (respectively, $\mathcal{G}^{\text{c}}$).

In the case of tempered fundamental groups of pointed stable curves, i.e., Example 3.10, we observe [cf. Remark 3.9.1] that, in the notation of Example 3.10, not only $\mathcal{G}$, but also the *semi-graph* of anabelioids $\mathcal{G}^{\text{c}}$ may be reconstructed group-theoretically from the tempered fundamental group:

Corollary 3.11. (Reconstruction of Semi-graphs of Anabelioids Associated to Pointed Stable Curves) *For* $\Box = \alpha, \beta$*, let* $K_\Box$ *be a finite extension of* $\mathbb{Q}_{p_\Box}$*;* $\overline{K}_\Box$ *an algebraic closure of* $K_\Box$*;* $(X_\Box^{\log})_{K_\Box}$ *a* **smooth pointed stable log curve** *over* $K_\Box$*. Let us write* $(X_\Box^{\log})_{\overline{K}_\Box} \stackrel{\text{def}}{=} (X_\Box^{\log})_{K_\Box} \times_{K_\Box} \overline{K}_\Box$*;* $\Delta[\Box] \stackrel{\text{def}}{=} \pi_1^{\text{temp}}((X_\Box^{\log})_{\overline{K}_\Box})$*;*

$$\mathcal{G}[\Box] \text{ (respectively, } \mathcal{G}^{\text{c}}[\Box])$$

for the **graph of anabelioids** *(respectively,* **semi-graph of anabelioids***) determined by the semi-graph of profinite groups [without compact structure!] (respectively, with compact structure) associated to the geometric special fiber of the stable model of* $(X_\Box^{\log})_{\overline{K}_\Box}$ *[cf. [Mzk3], Appendix] over the ring of integers* $\mathcal{O}_{\overline{K}_\Box}$ *of* $\overline{K}_\Box$ *[cf. Example 3.10]. Then any isomorphism of topological groups*

$$\gamma : \Delta[\alpha] \xrightarrow{\sim} \Delta[\beta]$$

determines a **compatible isomorphism of semi-graphs of anabelioids**

$$\mathcal{G}^{\text{c}}[\alpha] \xrightarrow{\sim} \mathcal{G}^{\text{c}}[\beta]$$

in a fashion that is **functorial** *with respect to* γ*. Moreover, if such a* γ *exists, then* $p_\alpha = p_\beta$*.*

Proof. Let Σ be a *set of prime numbers* such that $p_\alpha, p_\beta \notin \Sigma$. Write $\widehat{\Delta}[\Box]$ for the profinite completion of $\Delta[\Box]$; $\Delta[\Box]^{\Sigma}$ for the *pro-*Σ *tempered fundamental group* of

Remark 3.10.1; $\widehat{\Delta}[\Box]^{\Sigma}$ for the maximal pro-Σ quotient of $\widehat{\Delta}[\Box]$; $\mathcal{G}^{c}[\Box]^{\Sigma}$, $\mathcal{G}[\Box]^{\Sigma}$ for the respective pro-Σ completions [in the sense of Definition 2.9, (ii)] of $\mathcal{G}^{c}[\Box]$, $\mathcal{G}[\Box]$. Moreover, since Galois coverings of degree a Σ-integer are necessarily *admissible* [cf., e.g., [Mzk1], §3], it follows that $\Delta[\Box]^{\Sigma}$ may be identified with the tempered fundamental group $\pi_1^{\mathrm{temp}}(\mathcal{G}^{c}[\Box]^{\Sigma}) \cong \pi_1^{\mathrm{temp}}(\mathcal{G}[\Box]^{\Sigma})$.

Next, let us *observe* that the kernel

$$I_{\Sigma}[\Box] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Delta[\Box] \twoheadrightarrow \Delta[\Box]^{\Sigma})$$

may be recovered as the kernel $J_{\Sigma}[\Box]$ of the natural morphism $\Delta[\Box] \to \widehat{\Delta}[\Box]^{\Sigma}$. Indeed, since it follows from the definitions that the morphism $\Delta[\Box] \to \widehat{\Delta}[\Box]^{\Sigma}$ *factors* through $\Delta[\Box]^{\Sigma}$, we obtain that $I_{\Sigma}[\Box] \subseteq J_{\Sigma}[\Box]$. On the other hand, the fact that $J_{\Sigma}[\Box] \subseteq I_{\Sigma}[\Box]$ follows from the fact that free discrete groups *inject* into their pro-Σ completions [cf. Remark 1.7.1]. This completes the proof of the equality $I_{\Sigma}[\Box] = J_{\Sigma}[\Box]$.

In particular, we conclude that the quotients

$$\Delta[\Box] \twoheadrightarrow \Delta[\Box]^{\Sigma} \cong \pi_1^{\mathrm{temp}}(\mathcal{G}[\Box]^{\Sigma})$$

are *compatible* with γ. Thus, by Corollary 3.9, we conclude that γ induces a natural, functorial *isomorphism of graphs of anabelioids*

$$\mathcal{G}[\alpha]^{\Sigma} \xrightarrow{\sim} \mathcal{G}[\beta]^{\Sigma}$$

hence, in particular, an isomorphism of graphs $\mathbb{G}[\alpha] \xrightarrow{\sim} \mathbb{G}[\beta]$.

Next, let us observe, that:

(i) The isomorphisms obtained in the preceding paragraph may also be applied to open subgroups of finite index $\Delta'[\alpha] \subseteq \Delta[\alpha]$, $\Delta'[\beta] \subseteq \Delta[\beta]$ that correspond via γ.

Moreover, let us recall that:

(ii) The decomposition groups of cusps in $\Delta[\Box]^{\Sigma}$ are *commensurably terminal* [cf. [Mzk3], Lemma 1.3.7].

(iii) Every image via the natural morphism

$$\Delta'[\Box]^{\Sigma} \to \Delta[\Box]^{\Sigma}$$

associated to some open subgroup of finite index $\Delta'[\Box] \subseteq \Delta[\Box]$ of the decomposition group of a node in $\Delta'[\Box]^{\Sigma}$ [i.e., in the terminology of Theorem 3.7, an "edge-like subgroup"] is *either* an open subgroup of an edge-like subgroup of $\Delta[\Box]^{\Sigma}$ *or* an open subgroup of a decomposition group of a cusp in $\Delta[\Box]^{\Sigma}$.

(iv) Every decomposition group of a cusp in $\Delta[\Box]^\Sigma$ admits an open subgroup that arises as the image via *some* morphism $\Delta'[\Box]^\Sigma \to \Delta[\Box]^\Sigma$ as in (ii) of an edge-like subgroup of $\Delta'[\Box]^\Sigma$.

Indeed, (ii) follows from [Mzk3], Lemma 1.3.7; (iii) is immediate from the definitions. On the other hand, (iv) may be verified as follows: Suppose that

$$\Delta'[\Box] \subseteq \Delta[\Box]$$

is an open normal subgroup of finite index that corresponds to a covering of $(X^{\log}_\Box)_{\overline{K}_\Box}$ whose ramification indices at the cusps are *prime* to $p_\Box$, but which is *ramified* over the irreducible component $C_\Box$ of the special fiber of the stable model of $(X^{\log}_\Box)_{\overline{K}_\Box}$ that contains [the restriction to the special fiber of] the *cusp of interest*, which we shall denote by $x_\Box$. [Note that such a $\Delta'[\Box]$ always *exists*: Indeed, by passing to a Galois covering of degree a Σ-integer, one may assume that $(X^{\log}_\Box)_{\overline{K}_\Box}$ is of genus ≥ 2; then one verifies immediately that the covering arising from "multiplication by $p_\Box$ on the Jacobian" satisfies the conditions just stated.] Then the ramification over $C_\Box$ implies that this covering has $\geq p_\Box$ *distinct* cusps lying over $x_\Box$ which, nevertheless, map to the *same point* of the normalization of $C_\Box$ in the field extension of its function field determined by the covering. Thus, we conclude that [the restrictions to the special fiber of] these distinct cusps must lie on an irreducible component of the special fiber [of the stable model] of the covering that *collapses* to [the restriction to the special fiber of] $x_\Box$ [cf. also [Tama2], Theorem 0.2, for a more general result concerning the existence of coverings with collapsing irreducible components]. Now, sorting through the definitions, we see that this completes the proof of (iv).

Thus, in summary, it follows formally from (i), (ii), (iii), (iv) that the natural, functorial *isomorphism of graphs of anabelioids*

$$\mathcal{G}[\alpha]^\Sigma \xrightarrow{\sim} \mathcal{G}[\beta]^\Sigma$$

induced by γ *extends uniquely* to a natural, functorial *isomorphism of semi-graphs of anabelioids*

$$\mathcal{G}^{\mathrm{c}}[\alpha]^\Sigma \xrightarrow{\sim} \mathcal{G}^{\mathrm{c}}[\beta]^\Sigma$$

[which may also be regarded as being induced by γ], hence, in particular, an isomorphism of semi-graphs $\mathbb{G}^{\mathrm{c}}[\alpha] \xrightarrow{\sim} \mathbb{G}^{\mathrm{c}}[\beta]$.

Next, let us observe that, if $\Delta'[\alpha] \subseteq \Delta[\alpha]$, $\Delta'[\beta] \subseteq \Delta[\beta]$ are normal open subgroups of finite index that correspond via γ, the decomposition group

$$D_v \subseteq \Delta[\Box]/\Delta'[\Box]$$

determined by a vertex v of $\mathbb{G}[\Box]$ acts naturally on the anabelioid $\mathcal{G}[\Box]^\Sigma_v$. Thus, the *inertia group*

$$I_v \subseteq D_v$$

at v — i.e., the subgroup that acts trivially on this anabelioid — is necessarily a *power of* $p_\Box$. (Indeed, here we use the easily verified fact that any nontrivial automorphism of an irreducible component of the special fiber [of the stable model of the covering determined by $\Delta'[\Box]$] induces a *nontrivial outer automorphism* of the tame pro-Σ fundamental group [i.e., where "tame" means that one only allows tame ramification at the nodes and cusps] of the open subscheme of this irreducible component given by taking the complement [in this irreducible component] of the nodes and cusps.) In particular, we obtain that $p_\alpha = p_\beta$. Thus, in the following, we shall write $p \stackrel{\text{def}}{=} p_\alpha = p_\beta$.

Next, let us observe that the natural quotient

$$\Delta[\Box] \twoheadrightarrow \pi_1^{\text{temp}}(\mathcal{G}[\Box]) \cong \pi_1^{\text{temp}}(\mathcal{G}^{\text{c}}[\Box])$$

— i.e., the quotient determined by the *"admissible quotient"* of $\widehat{\Delta}[\Box]$, in the sense of [Mzk3], §2 — may be characterized as follows: A normal open subgroup of finite index $\Delta'[\Box] \subseteq \Delta[\Box]$ arises from this quotient if and only if no irreducible component of the special fiber of the stable model of the corresponding covering *collapses* in the stable model of $(X_\Box^{\log})_{\overline{K}_\Box}$, and, moreover, the decomposition groups at the nodes and cusps (respectively, inertia groups at the irreducible components) of the corresponding covering are prime to p (respectively, trivial). Indeed, this follows immediately from well-known *"purity of the branch locus"* results and the well-known *"structure of local fundamental groups of stable curves"* [cf., e.g., [Tama2], Lemma 2.1]. Moreover, observe that this characterization is equivalent to the following *"group-theoretic"* condition [i.e., condition compatible with γ]:

> The natural map $\Delta'[\Box] \to \Delta[\Box]$ is *quasi-geometric*; the stabilizer $\subseteq \Delta[\Box]/\Delta'[\Box]$ of any edge of the semi-graph $\mathbb{G}^{\text{c}}[\Box]$ has order prime to p; the stabilizer $\subseteq \Delta[\Box]/\Delta'[\Box]$ of any vertex v of the semi-graph $\mathbb{G}^{\text{c}}[\Box]$ acts faithfully on the anabelioid $\mathcal{G}^{\text{c}}[\Box]_v^\Sigma$.

Thus, we conclude that γ induces an isomorphism

$$\pi_1^{\text{temp}}(\mathcal{G}[\alpha]) \cong \pi_1^{\text{temp}}(\mathcal{G}[\beta])$$

hence, by Corollary 3.9, we conclude that γ induces an isomorphism of graphs of anabelioids

$$\mathcal{G}[\alpha] \xrightarrow{\sim} \mathcal{G}[\beta]$$

which — by applying the functorial isomorphisms "$\mathbb{G}^{\text{c}}[\alpha] \xrightarrow{\sim} \mathbb{G}^{\text{c}}[\beta]$" obtained above to arbitrary normal open subgroups of finite index $\Delta'[\Box] \subseteq \Delta[\Box]$ that arise from the "admissible quotient" — induces a uniquely determined *isomorphism of semi-graphs of anabelioids*

$$\mathcal{G}^{\text{c}}[\alpha] \xrightarrow{\sim} \mathcal{G}^{\text{c}}[\beta]$$

[which is, of course, functorial in γ], as desired. ○

Remark 3.11.1. Note that for any set of primes Σ of *cardinality* ≥ 3 [i.e., so that Σ contains at least one prime $\neq p_\alpha, p_\beta$], the argument given above also yields a *"pro-Σ version"* of Corollary 3.11, i.e., where one replaces the isomorphism

$$\gamma : \Delta[\alpha] \xrightarrow{\sim} \Delta[\beta]$$

in the statement of Corollary 3.11 by an isomorphism

$$\Delta[\alpha]^{\Sigma} \xrightarrow{\sim} \Delta[\beta]^{\Sigma}$$

between the respective pro-Σ tempered fundamental groups [in the sense of Remark 3.10.1].

Remark 3.11.2. Once one recovers the *"admissible quotients"* $\Delta[\Box] \twoheadrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G}^{\mathrm{c}}[\Box])$, one may apply the results of [Tama1] to the various *verticial subgroups* of $\pi_1^{\mathrm{temp}}(\mathcal{G}^{\mathrm{c}}[\Box])$ to recover, in certain cases, the *isomorphism class* of the curve determined by the complement of the nodes and cusps in the irreducible component of the special fiber corresponding to this verticial subgroup.

Section 4: Categories of Localizations

In this §, we consider *categories of localizations* of a semi-graph of anabelioids satisfying certain conditions.

Let $\mathcal{G}$ be a *totally aloof, verticially slim semi-graph of anabelioids* [so that we may apply Remark 2.4.2]. Also, we assume that we have been given a *finite group* Γ *of automorphisms of* $\mathcal{G}$ [i.e., Γ acts *faithfully* on $\mathcal{G}$].

If $\mathbb{H} \to \mathbb{G}$ is a *morphism of semi-graphs*, then we shall write $\mathcal{G}_{\mathbb{H}}$ for the semi-graph of anabelioids obtained by pulling back [in the evident sense] the semi-graph of anabelioids structure of $\mathcal{G}$ via $\mathbb{H} \to \mathbb{G}$. If v (respectively, e; b) is a(n) vertex (respectively, edge; branch of an edge) of $\mathbb{G}$, then we shall write

$$\mathcal{G}[v] \stackrel{\text{def}}{=} \mathcal{G}_{\mathbb{G}[v]}; \quad \mathcal{G}[e] \stackrel{\text{def}}{=} \mathcal{G}_{\mathbb{G}[e]}; \quad \mathcal{G}[b] \stackrel{\text{def}}{=} \mathcal{G}_{\mathbb{G}[b]}$$

[i.e., relative to the natural morphisms $\mathbb{G}[v] \to \mathbb{G}$, $\mathbb{G}[e] \to \mathbb{G}$, $\mathbb{G}[b] \to \mathbb{G}$ of §1].

Definition 4.1.

(i) We shall say that Γ acts *piecewise faithfully* on $\mathcal{G}$ if every element $\gamma \in \Gamma$ satisfies the following condition: If there exists a vertex v of $\mathbb{G}$ such that γ fixes v as well as all of the branches of closed edges of $\mathbb{G}$ that abut to v, then γ is the identity.

(ii) Any locally trivial morphism of totally aloof, verticially slim semi-graphs of anabelioids $\mathcal{G}' \to \mathcal{G}$ whose underlying morphism of semi-graphs is an immersion (respectively, excision; embedding) will also be referred to as an *immersion* (respectively, *excision*; *embedding*), or, alternatively, as an *immersive* (respectively, *excisive*; *embedding*) morphism.

(iii) Let $\mathcal{H}$ be a totally aloof, verticially slim semi-graph of anabelioids. Then we shall refer to as a $(\mathcal{G}, \Gamma)$*-structure on* $\mathcal{H}$ any Γ-orbit [relative to the action of Γ on $\mathcal{G}$] of locally finite étale morphisms of totally aloof, verticially slim semi-graphs of anabelioids $\mathcal{H} \to \mathcal{G}$. We shall refer to any of the morphisms in this orbit as a *structure morphism* [relative to this particular $(\mathcal{G}, \Gamma)$-structure on $\mathcal{H}$]. We shall say that a $(\mathcal{G}, \Gamma)$-structure on $\mathcal{H}$ is *iso-immersive* (respectively, *iso-excisive*) if some [or, equivalently, every] structure morphism $\mathcal{H} \to \mathcal{G}$ factors as the composite of an immersion (respectively, excision) $\mathcal{H} \to \mathcal{G}'$ with a finite étale morphism $\mathcal{G}' \to \mathcal{G}$ such that $\mathbb{G}'$ is *untangled* [so the composites of the excision $\mathbb{H} \to \mathbb{G}'$ with the various $\mathbb{H}[v] \to \mathbb{H}$, $\mathbb{H}[e] \to \mathbb{H}$, $\mathbb{H}[b] \to \mathbb{H}$ are all *embeddings*]. If $\mathcal{H} \to \mathcal{H}'$ is a locally finite étale morphism between totally aloof, verticially slim semi-graphs of anabelioids that are equipped with $(\mathcal{G}, \Gamma)$-structures, then we shall say that this morphism is *compatible* with the $(\mathcal{G}, \Gamma)$-structures if the composite of this morphism with a structure morphism of $\mathcal{H}'$ yields a structure morphism of $\mathcal{H}$.

(iv) Let $\mathcal{H}$ be a totally aloof, verticially slim semi-graph of anabelioids. Then we shall refer to as a *local* $(\mathcal{G}, \Gamma)$*-structure on* $\mathcal{H}$ the datum of a $(\mathcal{G}, \Gamma)$-structure for each $\mathcal{H}[c]$, where c varies among the components [i.e., vertices and edges] of the underlying semi-graph $\mathbb{H}$ satisfying the property that if a branch b of an edge e of $\mathbb{H}$ abuts to a vertex v of $\mathbb{H}$, then the given $(\mathcal{G}, \Gamma)$-structures on $\mathcal{H}[v]$, $\mathcal{H}[e]$ *coincide* on $\mathcal{H}[b]$. We shall say that a local $(\mathcal{G}, \Gamma)$-structure is *iso-immersive* (respectively, *iso-excisive*) if each of its constituent $(\mathcal{G}, \Gamma)$-structures is iso-immersive (respectively, iso-excisive). We shall say that a local $(\mathcal{G}, \Gamma)$-structure is *verticially iso-excisive* if each of its constituent $(\mathcal{G}, \Gamma)$-structures at a *vertex* is iso-excisive. If $\mathcal{H} \to \mathcal{H}'$ is a locally finite étale morphism between totally aloof, verticially slim semi-graphs of anabelioids that are equipped with local $(\mathcal{G}, \Gamma)$-structures, then we shall say that this morphism is *compatible* with the local $(\mathcal{G}, \Gamma)$-structures if each of the induced morphisms $\mathcal{H}[c] \to \mathcal{H}[c']$ (where c is a component of $\mathbb{H}$ mapping to a component c' of $\mathbb{H}'$) is compatible with the given $(\mathcal{G}, \Gamma)$-structures.

Remark 4.1.1. Let $\mathcal{H}$ be a totally aloof, verticially slim semi-graph of anabelioids, equipped with a local $(\mathcal{G}, \Gamma)$-structure. Then each component c of $\mathbb{H}$ determines a *well-defined* Γ*-orbit of components of* $\mathbb{G}$ [by mapping c to $\mathbb{G}$ via a structure morphism]. In particular, if c is an *edge*, then it makes to say that c lies over an *open* (respectively, *closed*) edge of $\mathbb{G}$. In this case, we shall say that c is $\mathbb{G}$-*open* (respectively, $\mathbb{G}$-*closed*). One verifies immediately that any locally finite étale morphism *compatible* with given local $(\mathcal{G}, \Gamma)$-structures maps $\mathbb{G}$-open (respectively, $\mathbb{G}$-closed) edges to $\mathbb{G}$-open (respectively, $\mathbb{G}$-closed) edges.

Next, we assume further that Γ acts *piecewise faithfully* on $\mathcal{G}$, and that $\mathcal{G}$ is *finite*, *connected*, *coherent*, *totally elevated*, and *totally universally sub-coverticial.* Then we may define a *category of localizations* of $\mathcal{G}$

$$\mathfrak{Loc}(\mathcal{G}, \Gamma)$$

associated to the pair $(\mathcal{G}, \Gamma)$ as follows: The *(finite) closed objects* are the connected finite étale coverings $\mathcal{G}'$ of $\mathcal{G}$, which we regard as being equipped with the resulting $\mathcal{G}$- [i.e., $(\mathcal{G}, \{1\})$-)] *structure.* The *infinite open objects* are the semi-graphs of anabelioids $\mathcal{G}''$ that appear as *connected tempered coverings* of $\mathcal{G}$ of *infinite degree.* We regard infinite open objects as being equipped with the resulting $\mathcal{G}$*-structure.* An object that is either closed or infinite open will be called *tempered.* A *finite open object* $\mathcal{H}$ is a finite, connected, quasi-coherent, totally elevated, totally universally sub-coverticial, totally aloof, verticially slim semi-graph of anabelioids, equipped with an iso-immersive, verticially iso-excisive local $(\mathcal{G}, \Gamma)$-structure, such that $\mathbb{H}$ contains at least one *non-isolated open edge* which is, however, $\mathbb{G}$*-closed.* The *morphisms between tempered objects* (respectively, *morphisms from a finite open object to an arbitrary object*) are the locally finite étale morphisms of semi-graphs of anabelioids compatible with the $\mathcal{G}$-structure (respectively, local $(\mathcal{G}, \Gamma)$-structure). There are no morphisms from a tempered to a finite open object. This completes the definition of the category $\mathfrak{Loc}(\mathcal{G}, \Gamma)$.

Definition 4.2.

(i) The (possibly infinite) *verticial length* (respectively, *edge-wise length*) of an object of $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is defined to be the cardinality of the set of vertices (respectively, closed edges) of the underlying semi-graph. An open object of verticial length 1 (respectively, 2) and edge-wise length 0 (respectively, 1) will be referred to as a *nuclear object* (respectively, *link*). A locally trivial morphism from a nuclear object to a link (respectively, an arbitrary object) will be referred to as an *NL-morphism* (respectively, a *verticial morphism*). The *verticial degree* of an arrow $\mathcal{H} \to \mathcal{K}$ in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ at a vertex v of $\mathbb{H}$ mapping to a vertex w of $\mathbb{K}$ is defined to be the [necessarily finite] degree of the finite étale morphism of anabelioids $\mathcal{H}_v \to \mathcal{K}_w$ induced by the arrow.

(ii) A *graph-localization morphism* in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is defined to be a locally trivial morphism which satisfies the condition that it is an isomorphism whenever its domain is closed. A *strict graph-localization morphism* is a graph-localization morphism for which the induced arrow on underlying semi-graphs is *injective on vertices*.

Remark 4.2.1. Note that every *locally trivial* morphism in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is *excisive*. Moreover, every *embedding* in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is a *strict graph-localization morphism*). The converse to both of these statements, however, may easily be seen to be false in general.

Proposition 4.3. (Basic Properties of the Category of Localizations)

(i) The underlying semi-graph of anabelioids of an object of $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *is* **connected**, **coherent**, **totally elevated**, **totally universally sub-coverticial**, **totally aloof**, **verticially slim**, **of injective type**, *and* **of positive verticial length**. *The underlying morphism of semi-graphs of a morphism of* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *is* **locally finite étale**.

(ii) If $\mathcal{H}$ *is a finite open object of* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$, *then any* **excision** $\mathbb{H}' \to \mathbb{H}$ *of finite connected graphs of positive verticial length determines an excision* $\mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H}_{\mathbb{H}'} \to \mathcal{H}$ *of* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$.

(iii) Let $\phi : \mathcal{H} \to \mathcal{K}$ *be a morphism in* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *from a finite open object* $\mathcal{H}$ *to a tempered object* $\mathcal{K}$. *Then* ϕ *is* **not** *an isomorphism of semi-graphs of anabelioids.*

(iv) Let $\mathcal{H}$ *be a finite open object of* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *such that* $\mathbb{H}$ *is a* **tree**. *Then the local* $(\mathcal{G}, \Gamma)$*-structure on* $\mathcal{H}$ *arises from a* **unique** $(\mathcal{G}, \Gamma)$**-structure** *on* $\mathcal{H}$ *[hence, in particular, from a [not necessarily unique]* $\mathcal{G}$*-structure].*

(v) Every morphism in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *is an* **epimorphism**. *In particular, the category* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *is* **totally epimorphic**.

(vi) Every **endomorphism** *of a finite [open or closed] object of* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *is an* **automorphism**. *Moreover, the automorphism group of any finite object of* $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ *is* **finite**.

Proof. Assertions (i) (respectively, (ii)) is immediate from the definitions [cf. also Remark 2.4.1] (respectively, [cf. also the fact that the domain of any nonproper excision admits an edge that maps to an edge of *strictly greater* verticial cardinality]). As for assertion (iii), we may assume without loss of generality that $\mathcal{K}$ is *finite*. Then assertion (iii) follows from the fact that ϕ necessarily maps an open, $\mathbb{G}$-closed edge of $\mathbb{H}$ [which always exists, by the definition of an open object] to an open, $\mathbb{G}$-closed edge of $\mathbb{K}$. But this contradicts the easily verified fact that every $\mathbb{G}$-closed edge of $\mathbb{K}$ is closed in $\mathbb{K}$.

Next, we consider assertion (iv). The desired morphism $\mathcal{H} \to \mathcal{G}$ may be constructed by *gluing* together local structure morphisms to $\mathcal{G}$; the fact that such a gluing operation may be performed — despite the "Γ-ambiguities" involved — follows from our assumption that $\mathbb{H}$ is a *tree*. Finally, the *uniqueness* of the $(\mathcal{G}, \Gamma)$-structure follows by reducing to the case of *nuclear* $\mathcal{H}$ by assertion (v) below [one checks immediately that there are no vicious circles in the argument], in which case the desired uniqueness is immediate from the definitions.

Next, we consider assertion (v). Let $\phi, \psi : \mathcal{H} \to \mathcal{K}$, $\xi : \mathcal{H}' \to \mathcal{H}$ be morphisms in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ such that $\phi \circ \xi = \psi \circ \xi$. By localizing on $\mathbb{H}$ [and applying the fact that any finite étale morphism of slim anabelioids is an *epimorphism* in the category of finite étale morphisms of slim anabelioids], one verifies immediately that it suffices to treat the case where ξ is an *NL-morphism* [so, in particular, $\mathcal{H}'$ is *nuclear*; $\mathcal{H}$ is a *link*]. Write e_H for the *unique closed edge* of $\mathbb{H}$; b_H for the *branch* of e_H that abuts to the unique vertex v of $\mathbb{H}'$. Since ϕ, ψ then coincide on the edges of $\mathbb{H}$ that abut to v, we conclude that ϕ, ψ *coincide* on $\mathbb{H}[e_H]$. In particular, we may also assume that $\mathcal{K}$ is a *link*. Write e_K for the *unique closed edge* of $\mathbb{K}$.

Now I *claim* that it suffices to show that ϕ, ψ *coincide* on $\mathcal{H}[e_H]$. Indeed, since $\mathcal{H}$ is a *link*, $\mathcal{H}[e_H]$ may be obtained from $\mathcal{H}$ by simply omitting the *open edges*. Thus, it suffices to check that ϕ, ψ map each open edge e' of $\mathbb{H}$ to the same open edge e'' of $\mathbb{K}$ and induce the same morphism $\mathcal{H}_{e'} \to \mathcal{K}_{e''}$. On the other hand, since ϕ, ψ coincide on $\mathcal{H}[e_H]$, both of these assertions follow from the fact that $\mathcal{K}$ is *totally aloof* [cf. also Remark 2.4.1]. This completes the proof of the claim.

To show that ϕ, ψ coincide on $\mathcal{H}[e_H]$, we reason as follows: First, we observe that [by our definition of the category $\mathfrak{Loc}(\mathcal{G}, \Gamma)$; the *piecewise faithfulness* of Γ] $\mathcal{H}[b_H]$; $\mathcal{H}[e_H]$; $\mathcal{K}[e_K]$ may be equipped with $\mathcal{G}$*-structures* that are *compatible* with ξ, ϕ, and ψ. Thus, these $\mathcal{G}$-structures induce *injective* [by Proposition 2.5, (i)] *outer homomorphisms*

$$\widehat{\pi}_1(\mathcal{H}') = \widehat{\pi}_1(\mathcal{H}[b_H]) \hookrightarrow \widehat{\pi}_1(\mathcal{G});$$

$$\widehat{\pi}_1(\mathcal{H}) = \widehat{\pi}_1(\mathcal{H}[e_H]) \hookrightarrow \widehat{\pi}_1(\mathcal{G}); \quad \widehat{\pi}_1(\mathcal{K}) = \widehat{\pi}_1(\mathcal{K}[e_K]) \hookrightarrow \widehat{\pi}_1(\mathcal{G})$$

which are *compatible* with the *outer homomorphisms*

$$\widehat{\pi}_1(\xi) : \widehat{\pi}_1(\mathcal{H}') \to \widehat{\pi}_1(\mathcal{H}); \quad \widehat{\pi}_1(\phi), \widehat{\pi}_1(\psi) : \widehat{\pi}_1(\mathcal{H}) \to \widehat{\pi}_1(\mathcal{K})$$

induced by ξ, ϕ, ψ. This compatibility implies that $\widehat{\pi}_1(\phi)$, $\widehat{\pi}_1(\psi)$ differ by conjugation by some element $g \in \widehat{\pi}_1(\mathcal{G})$. By the coincidence of ϕ, ψ on $\mathcal{H}'$, however, we may assume that g *centralizes* $\widehat{\pi}_1(\mathcal{H}')$. Thus, by Corollary 2.7, (i), we conclude that g is the identity, and hence that $\widehat{\pi}_1(\phi)$, $\widehat{\pi}_1(\psi)$ *coincide*. On the other hand, by Corollary 2.7, (i), and the fact that $\mathcal{K}$ is *totally aloof*, this implies that ϕ, ψ *coincide* on $\mathcal{H}[e_H]$, as desired.

Finally, we consider assertion (vi). Let ϕ be an endomorphism of a *finite* [open or closed] object $\mathcal{H}$. By *finiteness*, it is immediate that, for some integer $M \geq 1$, ϕ^M fixes some vertex v of $\mathcal{H}$ and induces the identity on the anabelioid $\mathcal{H}_v$. Thus, by assertion (v), we conclude that ϕ^M is the *identity*, so ϕ is an automorphism, as desired. The finiteness of the automorphism group of $\mathcal{H}$ is immediate from the finiteness of $\mathcal{H}$ itself. ◯

Proposition 4.4. **(Associated Anabelioids)** *Let $\mathcal{H} \to \mathcal{K}$, $\mathcal{L} \to \mathcal{K}$ be morphisms between* **finite** *[open or closed] objects in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$. Then:*

(i) There exists a finite étale covering $\mathcal{K}' \to \mathcal{K}$ of $\mathcal{K}$ such that the induced morphism $\mathcal{H}' \to \mathcal{K}'$ from any connected component $\mathcal{H}'$ of the pull-back of this finite étale covering to $\mathcal{H}$ is an **embedding***.*

(ii) There exists a finite étale covering $\mathcal{K}' \to \mathcal{K}$ of $\mathcal{K}$ such that $\mathcal{K}'$ **embeds** *into a finite étale covering $\mathcal{G}' \to \mathcal{G}$ of $\overline{\mathcal{G}}$.*

(iii) The morphism $\mathcal{H} \to \mathcal{K}$ in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ induces a **relatively slim π_1-monomorphism** *of* **slim** *anabelioids*

$$\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$$

which **completely determines** *the original morphism $\mathcal{H} \to \mathcal{K}$ [among all morphisms in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ from $\mathcal{H}$ to $\mathcal{K}$].*

(iv) Suppose that $\mathcal{H} \to \mathcal{K}$, $\mathcal{L} \to \mathcal{K}$ are **finite étale***. Then every morphism $\mathcal{H} \to \mathcal{L}$ in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ lying over $\mathcal{K}$ is* **finite étale** *and induces a finite étale morphism on associated anabelioids [cf. assertion (iii)]. Moreover, the full subcategory of such finite étale objects over $\mathcal{K}$ determines a* **full embedding***:*

$$\mathcal{B}(\mathcal{K})^0 \hookrightarrow \mathfrak{Loc}(\mathcal{G}, \Gamma)_{\mathcal{K}}$$

When $\mathcal{K} = \mathcal{G}$, we have $\mathfrak{Loc}(\mathcal{G}, \Gamma)_{\mathcal{K}} = \mathfrak{Loc}(\mathcal{G}, \Gamma)$; the essential image of this embedding is the full subcategory of closed objects.

Proof. First, we consider assertion (i). Since $\mathcal{K}$ is *quasi-coherent*, it follows from Proposition 2.5, (i), that we may reduce immediately to the case where the given morphism $\mathcal{H} \to \mathcal{K}$ is *locally trivial*. Thus, we are reduced to a problem in *graph theory* — a problem solved in §1 — cf. Theorem 1.2 [i.e., *"Zariski's main theorem for semi-graphs"*].

Assertion (ii) may be shown as follows: Let $\mathcal{K}'' \to \mathcal{K}$ be a *combinatorial universal covering*. Although $\mathcal{K}''$ [i.e., $\mathcal{K}'' \to \mathcal{K}$] will not, in general, determine an object of $\mathfrak{Loc}(\mathcal{G}, \Gamma)_{\mathcal{K}}$, it can, nevertheless, be thought of as as *inductive limit* of

$\mathcal{K}_\alpha \in \mathrm{Ob}(\mathfrak{Loc}(\mathcal{G}, \Gamma)_{\mathcal{K}})$ associated to connected finite sub-semi-graphs $\mathbb{K}_\alpha$ of $\mathbb{K}''$. By Proposition 4.3, (iv), these $\mathcal{K}_\alpha$ admit compatible [i.e., as α varies] morphisms [in $\mathfrak{Loc}(\mathcal{G}, \Gamma)_{\mathcal{K}}$] to $\mathcal{G}$. Moreover, the *uniqueness*, up to finitely many possibilities, [cf. Proposition 4.3, (iv)] of such a compatible system implies that some finite index subgroup of $\mathrm{Gal}(\mathcal{K}''/\mathcal{K})$ *fixes* such a compatible system. In particular, we conclude that, for some finite subcovering $\mathcal{K}''' \to \mathcal{K}$ of $\mathcal{K}''/\mathcal{K}$, we obtain a morphism [in $\mathfrak{Loc}(\mathcal{G}, \Gamma)_{\mathcal{K}}$] $\mathcal{K}''' \to \mathcal{G}$. Thus, the existence of a $\mathcal{K}' \to \mathcal{K}$ as asserted follows from assertion (ii).

Next, we consider of assertion (iii). The fact that $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{K})$ are *slim* follows from Corollary 2.7, (ii). By assertion (i), slimness, and the injectivity of Proposition 2.5, (i), to show that $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a π_1-*monomorphism*, it suffices to show, under the assumption that $\mathcal{H} \to \mathcal{K}$ is an *embedding*, that any *locally trivial* finite étale covering $\mathcal{H}' \to \mathcal{H}$ of $\mathcal{H}$ may be split by the pull-back of a *locally trivial* finite étale covering $\mathcal{K}' \to \mathcal{K}$ of $\mathcal{K}$. But this is immediate [cf. the proof of Proposition 2.5, (i)]. This completes the proof of the fact that $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a π_1-*monomorphism.*

Next, we show that $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is *relatively slim* and *determines* $\mathcal{H} \to \mathcal{K}$. By Proposition 4.3, (v), we reduce immediately to the case where $\mathcal{H}$ is *nuclear.* Then our conclusion follows from Corollary 2.7, (i), (ii), (iii). This completes the proof of assertion (iii).

Finally, assertion (iv) is a formal consequence of assertion (iii); Proposition 4.3, (iii); and the definitions. ○

Proposition 4.5. (The Subcategory of Tempered Objects) *Let $\mathcal{H}$, $\mathcal{K}$ be* **tempered** *objects of $\mathfrak{Loc}(\mathcal{G}, \Gamma)$. Then every morphism $\mathcal{H} \to \mathcal{K}$ in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is a* **tempered covering**. *In particular, we have a natural* **full embedding**

$$\mathcal{B}^{\mathrm{temp}}(\mathcal{G})^0 \hookrightarrow \mathfrak{Loc}(\mathcal{G}, \Gamma)$$

whose essential image is the full subcategory of tempered objects.

Proof. Indeed, by Theorem 3.6, (iv), (v), we obtain a *morphism of temperoids*

$$\mathcal{B}^{\mathrm{temp}}(\mathcal{H}) \to \mathcal{B}^{\mathrm{temp}}(\mathcal{K})$$

that is compatible with the *étale morphisms of temperoids* $\mathcal{B}^{\mathrm{temp}}(\mathcal{H}) \to \mathcal{B}^{\mathrm{temp}}(\mathcal{G})$, $\mathcal{B}^{\mathrm{temp}}(\mathcal{K}) \to \mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ induced by the $\mathcal{G}$-structures. It thus follows formally that the morphism $\mathcal{B}^{\mathrm{temp}}(\mathcal{H}) \to \mathcal{B}^{\mathrm{temp}}(\mathcal{K})$ is *étale*, hence corresponds to some *tempered covering* $\mathcal{H} \to \mathcal{K}$. But this tempered covering must *coincide* with the original morphism $\mathcal{H} \to \mathcal{K}$. Indeed, both morphisms induce the same arrow $\mathcal{B}^{\mathrm{temp}}(\mathcal{H}) \to \mathcal{B}^{\mathrm{temp}}(\mathcal{K})$. Thus, if v is a vertex of $\mathbb{H}$, and $\mathcal{K}' \to \mathcal{K}$ is an excision with finite open domain such that the restrictions to $\mathcal{H}[v]$ of the two morphisms in question both factor through $\mathcal{K}'$, then we conclude that these two morphisms both induce the same arrow $\mathcal{B}(\mathcal{H}_v) \to \mathcal{B}(\mathcal{K}')$ [cf. Proposition 2.5, (i); Corollary 2.7, (i)], so we conclude by Proposition 4.3, (v); Proposition 4.4, (iii). ○

Proposition 4.6. (Valuative Criterion for Finite Étale Morphisms) *Let $\phi : \mathcal{H} \to \mathcal{K}$ be a morphism between finite objects in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$. Then ϕ is a* **finite étale morphism** *if and only if the following condition is satisfied: for every* **NL-morphism** $\mathcal{H}_0 \to \mathcal{H}_1$ *and every commutative diagram*

$$\begin{array}{ccc} \mathcal{H}_0 & \longrightarrow & \mathcal{H}_1 \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \mathcal{K} \end{array}$$

in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$, there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_0 & \longrightarrow & \mathcal{H}_2 \\ \downarrow \text{id} & & \downarrow \\ \mathcal{H}_0 & \longrightarrow & \mathcal{H}_1 \end{array}$$

— where the horizontal arrows are NL-morphisms — such that the composite commutative diagram

$$\begin{array}{ccc} \mathcal{H}_0 & \longrightarrow & \mathcal{H}_2 \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \mathcal{K} \end{array}$$

admits a morphism $\mathcal{H}_2 \to \mathcal{H}$ in $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ that makes the resulting triangles in this diagram **commute**.

Proof. First, we consider *necessity*. By pulling back the finite étale morphism $\mathcal{H} \to \mathcal{K}$ to $\mathcal{H}_1$, we reduce immediately to the case where $\mathcal{H}_1 = \mathcal{K}$. But then the fact that the condition in question is satisfied follows immediately from the definition of a finite étale morphism.

Next, we consider *sufficiency*. By base-changing $\mathcal{H} \to \mathcal{K}$ by some finite étale morphism as in Proposition 4.4, (i), we reduce immediately to the case where $\mathcal{H} \to \mathcal{K}$ is an *embedding*. But then the condition in question implies that $\mathbb{H} \to \mathbb{K}$ is *surjective*, which implies that $\mathcal{H} \to \mathcal{K}$ is an *isomorphism* [hence, in particular, finite étale], as desired. ◯

Proposition 4.7. (Domination of Links) *For $i = 1, 2$, let $\mathcal{H}_0 \to \mathcal{H}_i$ be an* **NL-morphism**. *Suppose that the unique closed edge of $\mathbb{H}_i$ is the image of the* **same** *[i.e., for $i = 1, 2$] edge of $\mathbb{H}_0$. Then there exists an* **NL-morphism** $\mathcal{H}_0 \to \mathcal{H}_3$ *which fits into commutative diagrams*

$$\begin{array}{ccc} \mathcal{H}_0 & \longrightarrow & \mathcal{H}_3 \\ \downarrow \text{id} & & \downarrow \\ \mathcal{H}_0 & \longrightarrow & \mathcal{H}_i \end{array}$$

for $i = 1, 2$.

Proof. First, we observe that we may choose $\mathcal{G}$-structures on $\mathcal{H}_1$, $\mathcal{H}_2$ that coincide when restricted to $\mathcal{H}_0$ [cf. Proposition 4.3, (iv)]. In the following, "i" will always range over the elements of the set $\{1, 2\}$. Now by the definition of $\mathfrak{Loc}(\mathcal{G}, \Gamma)$, there exists a finite Galois étale covering

$$\mathcal{G}' \to \mathcal{G}$$

together with finite étale subcoverings $\mathcal{G}' \to \mathcal{G}_i \to \mathcal{G}$ [for $i = 1, 2$] such that $\mathbb{G}'$, $\mathbb{G}_i$ are *untangled*; there exists an *embedding* $\mathcal{H}_i \to \mathcal{G}_i$ compatible with the $\mathcal{G}$-structures. Then [by *conjugating* $\mathcal{G}_i$ appropriately] we may assume that there exists a vertex v' (respectively, edge e') of $\mathcal{G}'$ whose image in $\mathcal{G}_i$ is equal to the image of the unique vertex of $\mathcal{H}_0$ (respectively, the unique closed edge of $\mathcal{H}_i$) via the composite morphism $\mathcal{H}_0 \to \mathcal{H}_i \to \mathcal{G}_i$ (respectively, $\mathcal{H}_i \to \mathcal{G}_i$), and, moreover, that $I \bigcap H_1 = I \bigcap H_2$, where we write

$$H_i \subseteq \mathrm{Gal}(\mathcal{G}'/\mathcal{G})$$

for the subgroup determined by the subcovering $\mathcal{G}_i \to \mathcal{G}$ and

$$I \subseteq \mathrm{Gal}(\mathcal{G}'/\mathcal{G})$$

for the *isotropy subgroup* associated to v'. Set $H_3 \stackrel{\mathrm{def}}{=} I \bigcap H_1 = I \bigcap H_2$; write $\mathcal{G}_3 \to \mathcal{G}$ for the subcovering determined by H_3. Then if we take $\mathcal{H}_3$ to be the *link* contained in $\mathcal{G}_3$ which is determined by the images v_3, e_3 of v', e', respectively, then the natural morphism $\mathcal{G}_3 \to \mathcal{G}_i$ restricts to a morphism

$$\mathcal{H}_3 \to \mathcal{H}_i$$

with the desired properties. ○

Remark 4.7.1. Note that by applying Proposition 4.7 in an iterative fashion, one may construct an *NL-morphism* with domain $\mathcal{H}_0$ that dominates an arbitrary given finite collection of NL-morphisms $\mathcal{H}_0 \to \mathcal{H}_i$, for $i = 1, \ldots, n$. Moreover, given any *pair* of such dominating NL-morphisms $\mathcal{H}_0 \to \mathcal{H}_a$, $\mathcal{H}_0 \to \mathcal{H}_b$, there exists [again by Propsition 4.7] an NL-morphism $\mathcal{H}_0 \to \mathcal{H}_c$ that dominates $\mathcal{H}_0 \to \mathcal{H}_a$, $\mathcal{H}_0 \to \mathcal{H}_b$. Observe that in this situation, although *a priori*, we obtain two arrows $\mathcal{H}_c \to \mathcal{H}_i$ [i.e., one passing through $\mathcal{H}_a$, the other passing through $\mathcal{H}_b$], these two arrows necessarily *coincide* [by Proposition 4.3, (v)]. Thus, in summary, for each *edge* e of $\mathcal{H}_0$, we obtain a *natural system of dominating NL-morphisms with domain* $\mathcal{H}_0$, each of whose codomains is a link with closed edge given by the image [via the NL-morphism under consideration] of e.

Theorem 4.8. **(Category-Theoreticity of Categories of Localizations)** *For $i = 1, 2$, let $\mathcal{G}_i$ be a* **finite, connected, coherent, totally elevated, totally universally sub-coverticial, totally aloof, verticially slim graph of anabelioids**, *with underlying graph $\mathbb{G}_i$; let Γ_i be a finite group that acts* **piecewise faithfully** *on $\mathcal{G}_i$. Suppose that $\mathbb{G}_i$ has* **at least one edge**. *Write*

$$\mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)^{\text{fin}} \subseteq \mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)$$

for the full subcategory determined by the **finite objects**. *Then the categories $\mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)^{\text{fin}}$ (respectively, $\mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)$) are* **slim**; *every* **equivalence of categories**

$$\Phi : \mathfrak{Loc}(\mathcal{G}_1, \Gamma_1)^{\text{fin}} \xrightarrow{\sim} \mathfrak{Loc}(\mathcal{G}_2, \Gamma_2)^{\text{fin}} \ \textit{(respectively, } \Phi : \mathfrak{Loc}(\mathcal{G}_1, \Gamma_1) \xrightarrow{\sim} \mathfrak{Loc}(\mathcal{G}_2, \Gamma_2)\textit{)}$$

arises, up to unique isomorphism, from a **unique isomorphism of graphs of anabelioids**

$$\mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$$

together with a compatible isomorphism $\Gamma_1 \xrightarrow{\sim} \Gamma_2$.

Proof. First, we reconstruct the *underlying semi-graph of anabelioids* of an object $\mathcal{H}$ of $\mathcal{C} \stackrel{\text{def}}{=} \mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)^{\text{fin}}$ (respectively, $\mathcal{C} \stackrel{\text{def}}{=} \mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)$) [where $i = 1, 2$] *category-theoretically* as follows: The *objects of verticial length* 1 are precisely the *strongly indissectible* [cf. §0] objects. An object $\mathcal{H}$ of verticial length 1 is *nuclear* if and only if the domain of every morphism with codomain $\mathcal{H}$ is of verticial length 1. If $\mathcal{H}$ is nuclear, then the result of applying "$\perp$" to the category $\mathcal{C}_{\mathcal{H}}$ of objects and morphisms over $\mathcal{H}$ is a *Galois category* isomorphic to the anabelioid $\mathcal{B}(\mathcal{H})$. The *verticial morphisms* are precisely the morphisms with nuclear domain which are, moreover, *minimal-adjoint* [cf. §0] to the morphisms with nuclear codomain. The *vertices* of the underlying semi-graph of an object $\mathcal{H}$ are precisely the isomorphism classes, over $\mathcal{H}$, of verticial morphisms $\mathcal{K} \to \mathcal{H}$. Thus, in particular, we conclude that Φ induces a bijection between the sets of *vertices* of the underlying semi-graphs of corresponding objects, together with compatible isomorphisms of the various *constituent anabelioids* at the vertices; moreover, these bijections and isomorphisms are *compatible* with arrows in $\mathcal{C}$. In particular, Φ preserves *locally trivial morphisms*.

An object $\mathcal{H}$ is a *link* if and only if $\mathcal{H}$ is of verticial length 2, and, moreover, any locally trivial morphism $\mathcal{K} \to \mathcal{H}$, where $\mathcal{K}$ is also of verticial length 2, is an isomorphism. The *closed edges* of the underlying semi-graph of an object $\mathcal{H}$ are precisely the isomorphism classes, over $\mathcal{H}$, of locally trivial morphisms $\mathcal{K} \to \mathcal{H}$, where $\mathcal{K}$ is a link. An *[open] edge* of a nuclear object $\mathcal{H}$ is a *system* of compatible closed edges of NL-morphisms $\mathcal{H} \to \mathcal{K}$, as we vary the NL-morphism as described in Remark 4.7.1. Thus, we conclude that Φ induces an isomorphism between the *underlying semi-graphs of corresponding objects*; moreover, these isomorphisms are *compatible* with arrows in $\mathcal{C}$. Note that this implies, for instance, that Φ preserves the *embeddings*.

In particular, Φ preserves the *finite objects* [i.e., objects of finite verticial length], as well as the *NL-morphisms*. Moreover, by Proposition 4.6, Φ preserves the *finite étale morphisms* between finite objects. Thus, by considering the *isotropy subgroups* associated to the various edges in the Galois group of a finite étale Galois covering, one sees that Φ induces an isomorphism between the *underlying semi-graphs of anabelioids of corresponding finite objects* in a fashion that is *compatible* with arrows in $\mathcal{C}$. Moreover, these induced isomorphisms may be extended immediately to the case of *infinite objects* [i.e., when such objects exist in $\mathcal{C}$] by representing such objects as inductive limits of inductive systems consisting of finite objects and embeddings.

Next, we observe that the *closed objects* $\mathcal{H}$ of $\mathcal{C}$ are precisely the finite objects whose underlying semi-graph $\mathbb{H}$ is a *graph*; $\mathcal{G}_i$ is the *unique closed object* of $\mathcal{C}$, up to isomorphism, to which every other closed object maps.

Next, we observe that one may recover the various *local* $(\mathcal{G},\Gamma)$-*structures* on open objects as follows: First, we note that we may reconstruct the automorphisms $\mathcal{G}_i \to \mathcal{G}_i$ in Γ_i by *localizing on* $\mathcal{G}_i$. That is to say, if $\phi : \mathcal{H} \to \mathcal{G}_i$ is a morphism with *finite open* domain [where we note that such a ϕ always exists, for instance, if $\mathbb{H}$ is a *tree* — cf. Proposition 4.3, (iv)], then by Proposition 4.3, (v), the Γ_i-span of ϕ is precisely the set of all morphisms $\mathcal{H} \to \mathcal{G}_i$ in $\mathcal{C}$. Moreover, since Γ_i acts *piecewise faithfully*, it follows that the cardinality of this set is always equal to the order of Γ_i. Thus, by taking $\mathcal{H}$ to be various localizations of $\mathcal{G}_i$ and then gluing, we recover first the set of morphisms of semi-graphs of anabelioids $\mathcal{G}_i \to \mathcal{G}_i$ arising from elements of Γ_i and then the group structure [by composing morphisms of semi-graphs of anabelioids].

Thus, in summary, we have shown that Φ induces an *isomorphism of semi-graphs of anabelioids*

$$\mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$$

together with a compatible isomorphism $\Gamma_1 \xrightarrow{\sim} \Gamma_2$. Moreover, we have shown that Φ induces an isomorphism between the *underlying semi-graph of anabelioids of corresponding objects* in a fashion that is *compatible* with arrows in $\mathcal{C}$, as well as with the given *local* $(\mathcal{G},\Gamma)$-*structures*. Thus, it is an easily verified tautology that the equivalence Φ is isomorphic to the equivalence induced by the isomorphisms $\mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$, $\Gamma_1 \xrightarrow{\sim} \Gamma_2$.

Finally, it remains to verify that $\mathcal{C}$ is *slim*. Let $A \in \mathrm{Ob}(\mathcal{C})$; suppose that ψ is an *automorphism* of the natural functor $\mathcal{C}_A \to \mathcal{C}$. Concretely speaking, this means that for every object $\beta : B \to A$ of $\mathcal{C}_A$, we are given an automorphism $\psi_\beta \in \mathrm{Aut}_{\mathcal{C}}(B)$ in such a way that the assignment $\beta \mapsto \psi_\beta$ is compatible with the image in $\mathcal{C}$ of arrows of $\mathcal{C}_A$. Since A is *arbitrary*, it suffices [by considering, for given $\beta : B \to A$, the automorphism of the resulting composite functor $\mathcal{C}_B \to \mathcal{C}_A \to \mathcal{C}$ induced by ψ] to show that ψ_A [i.e., the automorphism assigned to the identity $A \to A$] is the identity. By considering β with *nuclear* domain, we conclude immediately that ψ_β *fixes* the set of vertices of A. Thus, since $\mathcal{G}_i$ is *totally aloof*, we reduce to the case where A is *nuclear*.

Now since A is nuclear, to show that ψ is trivial, it suffices to show the following: If v be a *vertex* of $\mathbb{G}_i$; $\Gamma_v \subseteq \Gamma_i$ is the corresponding isotropy subgroup; and H_v is the *extension* of Γ_v by $\widehat{\pi}_1((\mathcal{G}_i)_v)$ arising from the action of Γ_v on $(\mathbb{G}_i)_v$, then the profinite group H_v is *slim*. But this is an immediate formal consequence of the fact that $\mathcal{G}_i$ is *verticially slim* and *totally aloof*, together with our assumption that the action of Γ_i on $\mathcal{G}_i$ is *piecewise faithful*. ○

Remark 4.8.1. Returning to the notation used in the discussion preceding Theorem 4.8, suppose that the semi-graph $\mathbb{G}$ has *at least one vertex*. Write $\mathbb{H} \subseteq \mathbb{G}$ for the *maximal subgraph* [cf. §1] of $\mathbb{G}$. Then observe that [whenever $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is defined] if we set $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{G}_{\mathbb{H}}$, then Γ acts naturally and piecewise faithfully on $\mathcal{H}$; $\mathfrak{Loc}(\mathcal{H}, \Gamma)$ is defined; and we have *natural equivalences*

$$\mathfrak{Loc}(\mathcal{G}, \Gamma)^{\text{fin}} \xrightarrow{\sim} \mathfrak{Loc}(\mathcal{H}, \Gamma)^{\text{fin}}; \quad \mathfrak{Loc}(\mathcal{G}, \Gamma) \xrightarrow{\sim} \mathfrak{Loc}(\mathcal{H}, \Gamma)$$

[defined by simply omitting all $\mathbb{G}$-open edges]. Thus, there is no essential loss of generality in restricting Theorem 4.8 to the case where $\mathbb{G}$ is a *graph*.

Remark 4.8.2. One verifies easily that [whenever $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is defined] the following five conditions are *equivalent*:

(i) $\mathbb{G}$ has *no closed edges*.

(ii) $\mathbb{G}$ is a *tree* [cf. §1] with *at most one vertex*.

(iii) $\mathfrak{Loc}(\mathcal{G}, \Gamma)^{\perp}$ is a *Galois category*.

(iv) Every monomorphism of $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is an isomorphism.

(v) Every object of $\mathfrak{Loc}(\mathcal{G}, \Gamma)$ is closed.

Moreover, if any of these conditions is satisfied, then there is a *natural equivalence*

$$\mathfrak{Loc}(\mathcal{G}, \Gamma)^{\perp} \xrightarrow{\sim} \mathcal{B}(\mathcal{G})$$

— so, in particular, $\mathfrak{Loc}(\mathcal{G}, \Gamma)^{\perp}$ does not depend on the action of Γ [which is, at any rate, trivial, if $\mathbb{G}$ admits at least one vertex]. Thus, since equivalences between connected anabelioids are "well-understood", there is no essential loss of generality in excluding from Theorem 4.8 the case in which these conditions are satisfied.

Remark 4.8.3. In the resp'd case of Theorem 4.8 [i.e., where one includes the *infinite objects*], if one assumes further that $\mathcal{G}_i$ is *totally estranged*, then the proof of Theorem 4.8 may be simplified somewhat, by applying Proposition 4.5, Corollary 3.9.

Remark 4.8.4. In the notation of the proof of Theorem 4.8, if an object A of $\mathcal{C}$ is *nuclear*, then the category $\mathcal{C}_A^{\perp}$ is easily verified to be a *connected anabelioid*; the category $\mathcal{C}[A]^{\perp}$ [cf. §0] is easily verified to be a *"connected quasi-anabelioid"* [cf. Remark A.4.2 of the Appendix].

Section 5: Arithmetic Semi-graphs of Anabelioids

In this §, we consider semi-graphs of anabelioids equipped with a *continuous action of a profinite group*, which we think of as an *"arithmetic structure"* on the semi-graph of anabelioids. We then proceed to study a certain *"arithmetic analogue"* of the theory of *maximal compact subgroups* of §3.

Definition 5.1.

(i) Let $\mathcal{G}$ be a *connected, coherent, totally aloof, verticially slim semi-graph of anabelioids.* Let $\mathcal{A}$ be a *slim connected anabelioid*, equipped with a *basepoint*, so we may speak of $\widehat{\pi}_1(\mathcal{A})$. We shall refer to as an *action of* $\widehat{\pi}_1(\mathcal{A})$ *on* $\mathcal{G}$ the datum of a homomorphism

$$\rho_{\mathcal{G}} : \widehat{\pi}_1(\mathcal{A}) \to \mathrm{Aut}(\mathcal{G})$$

[where $\mathrm{Aut}(\mathcal{G})$ denotes the group of automorphisms of $\mathcal{G}$ as a totally aloof, verticially slim semi-graph of anabelioids]. Note that any such pair $(\mathcal{G}, \rho_{\mathcal{G}})$ admits an *"inner action"* by $\widehat{\pi}_1(\mathcal{A})$ — i.e., by letting $\widehat{\pi}_1(\mathcal{A})$ act on $\widehat{\pi}_1(\mathcal{A})$ by conjugation and on $\mathcal{G}$ via $\rho_{\mathcal{G}}$. We shall say that an action of $\widehat{\pi}_1(\mathcal{A})$ on $\mathcal{G}$ is *continuous* if, for some open subgroup $H \subseteq \widehat{\pi}_1(\mathcal{A})$, the following conditions are satisfied:

(a) $\widehat{\pi}_1(\mathcal{A})$ is *topologically finitely generated.*

(b) The semi-graph $\mathbb{G}$ is *locally finite.*

(c) The action of H on $\mathbb{G}$ is trivial; the resulting outer homomorphism $H \to \mathrm{Out}(\widehat{\pi}_1(\mathcal{G}_v))$, where v ranges over the vertices of $\mathbb{G}$, is *continuous* [i.e., relative to the natural profinite group topology on $\mathrm{Out}(\widehat{\pi}_1(\mathcal{G}_v))$].

(d) There is a *finite* set V of vertices of $\mathbb{G}$ such that for every vertex w of $\mathbb{G}$, there exists a $v \in V$ and an *isomorphism* of semi-graphs of anabelioids $\mathcal{G}[v] \xrightarrow{\sim} \mathcal{G}[w]$ that is *compatible* with the action of H on both sides.

(ii) A triple $\overline{\mathcal{G}} = (\mathcal{G}, \mathcal{A}, \rho_{\mathcal{G}})$ as in (i), where $\rho_{\mathcal{G}}$ is a continuous action of $\widehat{\pi}_1(\mathcal{A})$ on $\mathcal{G}$, will be referred to as a *connected arithmetic semi-graph of anabelioids (over $\mathcal{A}$).* Suppose that $\overline{\mathcal{G}} = (\mathcal{G}, \mathcal{A}, \rho_{\mathcal{G}})$ is a connected arithmetic semi-graph of anabelioids. Then we shall refer to $\mathcal{G}$ (respectively, $\mathcal{A}$; $\rho_{\mathcal{G}}$) as the *geometric component* (respectively, *arithmetic component*; *arithmetic action*) of $\overline{\mathcal{G}}$. The arithmetic action of an arithmetic semi-graph of groups *induces* [what, by abuse of terminology, we shall also refer to as] *"arithmetic actions"* on various objects [e.g., the underlying semi-graph, etc.] associated to this arithmetic semi-graph of groups.

(iii) A [not necessarily connected] *arithmetic semi-graph of anabelioids* $\overline{\mathcal{G}}$ is defined to be a formal collection of connected arithmetic semi-graphs of anabelioids; each object in this collection will be referred to as a *connected component* of $\overline{\mathcal{G}}$. Note that the geometric components of each of the connected components of $\overline{\mathcal{G}}$ together determine a natural *geometric component* [i.e., a (not necessarily connected) semi-graph of anabelioids] of $\overline{\mathcal{G}}$. We shall say that an arithmetic semi-graph of anabelioids is *finite* (respectively, *elevated at a vertex*; *totally elevated*; *sub-coverticial at a closed edge*; *universally sub-coverticial at a closed edge*; *totally sub-coverticial*; *totally universally sub-coverticial*; *aloof at an edge*; *totally aloof*; *estranged at an edge*; *totally estranged*) if its geometric component is so.

(iv) Given two connected arithmetic semi-graphs of anabelioids $\overline{\mathcal{G}}' = (\mathcal{G}', \mathcal{A}', \rho_{\mathcal{G}'})$, $\overline{\mathcal{G}} = (\mathcal{G}, \mathcal{A}, \rho_{\mathcal{G}})$, a *morphism of connected arithmetic semi-graphs of anabelioids*

$$\overline{\mathcal{G}}' \to \overline{\mathcal{G}}$$

consists of a pair

$$\widehat{\pi}_1(\mathcal{A}') \to \widehat{\pi}_1(\mathcal{A}); \quad \mathcal{G}' \to \mathcal{G}$$

[i.e., a continuous homomorphism of profinite groups and a morphism of semi-graphs of anabelioids] which is *compatible* with $\rho_{\mathcal{G}'}$, $\rho_{\mathcal{G}}$, and which we regard *up to composition with the inner action* of $\widehat{\pi}_1(\mathcal{A})$ on $(\mathcal{G}, \rho_{\mathcal{G}})$. We shall refer to the morphism $\mathcal{G}' \to \mathcal{G}$ (respectively, the induced morphism of anabelioids $\mathcal{A}' \to \mathcal{A}$) as the *geometric* (respectively, *arithmetic*) *component* of the morphism. [In particular, if we restrict our attention to $\overline{\mathcal{G}}' \to \overline{\mathcal{G}}$ whose geometric component is *locally open*, then we may work with such morphisms as if they are "morphisms in a category" — cf. Remark 2.4.2.] A *morphism of* [not necessarily connected] *arithmetic semi-graphs of anabelioids*

$$\overline{\mathcal{G}}' \to \overline{\mathcal{G}}$$

is defined to be a collection of morphisms, one from each connected component of $\overline{\mathcal{G}}'$ to some connected component of $\overline{\mathcal{G}}$. We shall say that $\overline{\mathcal{G}}' \to \overline{\mathcal{G}}$ is *finite étale* (respectively, *tempered*; *locally trivial*; *locally open*; *locally finite étale*; *immersive*; *excisive*; an *embedding*; *BC-finite étale*) if each of its *geometric components* is finite étale (respectively, a tempered covering; locally trivial; locally open; locally finite étale; immersive; excisive; an embedding; an isomorphism); each of its *arithmetic components* is finite étale (respectively, finite étale; an isomorphism; a composite of a π_1-epimorphism with a finite étale morphism; finite étale; an isomorphism; an isomorphism; an isomorphism; finite étale); and its induced map on *connected components* has finite [but possibly empty] fibers (respectively, has countable fibers; is arbitrary; is arbitrary; is arbitrary; is arbitrary; is arbitrary; is injective; has finite [but possibly empty] fibers). [Here, the abbreviation "BC" is to be understood to stand for the phrase *"base of constants"*.]

Proposition 5.2. **(Arithmetic Tempered Coverings)** *Let $\overline{\mathcal{G}} = (\mathcal{G}, \mathcal{A}, \rho_{\mathcal{G}})$ be a* **connected arithmetic semi-graph of anabelioids**. *Then:*

(i) Every tempered covering of $\mathcal{G}$ appears as the geometric component of a tempered covering of $\overline{\mathcal{G}}$.

(ii) Suppose that $\overline{\mathcal{H}} \to \overline{\mathcal{G}}$, $\overline{\mathcal{K}} \to \overline{\mathcal{G}}$ are tempered coverings with isomorphic geometric components. Then there exist BC-finite étale coverings $\overline{\mathcal{H}}' \to \overline{\mathcal{H}}$, $\overline{\mathcal{K}}' \to \overline{\mathcal{K}}$ such that $\overline{\mathcal{H}}'$, $\overline{\mathcal{K}}'$ are isomorphic as tempered coverings over $\overline{\mathcal{G}}$.

(iii) Let us denote by

$$\mathcal{B}^{\mathrm{temp}}(\overline{\mathcal{G}})$$

the category whose **objects** *are* **tempered morphisms** $\overline{\mathcal{G}}' \to \overline{\mathcal{G}}$ *and whose* **morphisms** *are tempered morphisms over* $\overline{\mathcal{G}}$*. Then* $\mathcal{B}^{\mathrm{temp}}(\overline{\mathcal{G}})$ *is a* **connected temperoid***. Similarly, the full subcategory*

$$\mathcal{B}(\overline{\mathcal{G}}) \subseteq \mathcal{B}^{\mathrm{temp}}(\overline{\mathcal{G}})$$

determined by the finite étale coverings forms a **connected anabelioid***. If* $\overline{\mathcal{G}}$ *is* **totally elevated***, then* $\mathcal{B}^{\mathrm{temp}}(\overline{\mathcal{G}})$ **temp-slim***, and* $\mathcal{B}(\overline{\mathcal{G}})$ *is* **slim***.*

(iv) Write:

$$\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(\overline{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(\mathcal{B}^{\mathrm{temp}}(\overline{\mathcal{G}})); \quad \Pi^{\mathrm{temp}}_{\mathcal{G}} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(\mathcal{B}^{\mathrm{temp}}(\mathcal{G})) = \pi_1^{\mathrm{temp}}(\mathcal{G})$$
$$\Pi_{\overline{\mathcal{G}}} \stackrel{\mathrm{def}}{=} \widehat{\pi}_1(\overline{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \widehat{\pi}_1(\mathcal{B}(\overline{\mathcal{G}})); \quad \Pi_{\mathcal{G}} \stackrel{\mathrm{def}}{=} \widehat{\pi}_1(\mathcal{B}(\mathcal{G})) = \widehat{\pi}_1(\mathcal{G})$$

Then there are natural morphisms

$$\mathcal{B}^{\mathrm{temp}}(\mathcal{G}) \to \mathcal{B}^{\mathrm{temp}}(\overline{\mathcal{G}}) \to \mathcal{A}^{\top}; \quad \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\overline{\mathcal{G}}) \to \mathcal{A}$$

which induce **natural exact sequences***:*

$$1 \to \Pi^{\mathrm{temp}}_{\mathcal{G}} \to \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}} \to \Pi_{\mathcal{A}} \to 1$$
$$1 \to \Pi_{\mathcal{G}} \to \Pi_{\overline{\mathcal{G}}} \to \Pi_{\mathcal{A}} \to 1$$

We shall refer to $\Pi_{\mathcal{G}}$ *(respectively,* $\Pi^{\mathrm{temp}}_{\mathcal{G}}$*;* $\Pi_{\overline{\mathcal{G}}}$*;* $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$*;* $\Pi_{\mathcal{A}} \stackrel{\mathrm{def}}{=} \widehat{\pi}_1(\mathcal{A})$*) as the* **geometric** *(respectively,* **geometric tempered***;* **arithmetic***;* **arithmetic tempered***;* **BC-***)* **fundamental group** *of* $\overline{\mathcal{G}}$*.*

Proof. Assertions (i), (ii) follow from the various *finiteness* assumptions in our definition of a *"continuous action"* [cf. Definition 5.1, (i); the fact that $\mathcal{G}$ is *coherent*]. [Note, in particular, that one must make use of the assumption of Definition 5.1, (i), (d), in order for assertion (ii) to hold for arbitrary *infinite* $\overline{\mathcal{G}}$.] Assertions (iii), (iv) follow immediately from the definitions; Corollary 2.7, (ii); and assertions (i), (ii). ○

Remark 5.2.1. At this point, one could proceed to develop a theory of *"categories of arithmetic localizations"* of arithmetic semi-graphs of anabelioids, in the style of §4. Although this is quite possible [we leave the details to the enthusiastic reader!], it is rather *cumbersome*, so instead we restrict ourselves [cf. Remark 4.8.3] to considering the categorical representation of arithmetic semi-graphs of anabelioids afforded by the *"arithmetically maximal compact subgroups"* [cf. Definition 5.3 below] of the tempered fundamental groups of Proposition 5.2, in the style of Corollary 3.9.

Let $\overline{\mathcal{G}}$ be a *connected, countable, totally elevated, totally estranged arithmetic semi-graph of anabelioids*, with underlying semi-graph $\mathbb{G}$. In the notation of Proposition 5.2, we would like to consider *compact subgroups* of the arithmetic tempered fundamental group $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$. Note that for every *vertex* v of $\mathbb{G}$, we obtain an associated *decomposition group*

$$\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},v} \subseteq \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$$

[well-defined up to conjugation in $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$], which [by Corollary 2.7, (i), (iii)] may be thought of as the *commensurator* in $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ of $\Pi^{\mathrm{temp}}_{\mathcal{G},v} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},v} \bigcap \Pi^{\mathrm{temp}}_{\mathcal{G}}$. Similarly, if b is a branch of an edge e of $\mathbb{G}$ that abuts to v, then we obtain a *decomposition group*

$$\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},b} \subseteq \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},v} \subseteq \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$$

[well-defined up to conjugation in $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$], which [since $\mathcal{G}$ is totally estranged, hence, in particular *totally aloof*] may be thought of as the *commensurator* in $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},v}$ of $\Pi^{\mathrm{temp}}_{\mathcal{G},b} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},b} \bigcap \Pi^{\mathrm{temp}}_{\mathcal{G}}$.

Definition 5.3.

(i) A closed subgroup of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ will be called *arithmetically ample* if it surjects onto an open subgroup of $\Pi_{\mathcal{A}}$. A compact subgroup of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ will be called *arithmetically maximal* if it is maximal among arithmetically ample compact subgroups of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$.

(ii) Let e be an edge of $\mathbb{G}$. We shall say that e is *arithmetically estranged* if, for every vertex v to which some branch b of e abuts and every $g \in \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},v}$, the intersection in $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},v}$ of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},b}$ with any subgroup of the form $g \cdot \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},b'} \cdot g^{-1}$, where *either* $b' \neq b$ is a branch of an edge that abuts to v *or* $b' = b$ and $g \notin \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},b}$, fails to be arithmetically ample. If every edge of $\mathbb{G}$ is arithmetically estranged, then we shall say that $\mathcal{G}$ is *totally arithmetically estranged*.

(iii) We shall refer to subgroups of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ of the form "$\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},v}$" (respectively, "$\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}},b}$") as *verticial* (respectively, *edge-like*).

Remark 5.3.1. Note that all verticial and edge-like subgroups of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ are *compact* and *arithmetically ample*. Also, the intersection with $\Pi^{\mathrm{temp}}_{\mathcal{G}}$ of a(n) verticial (respectively, edge-like) subgroup of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ is a(n) verticial (respectively, edge-like) subgroup of $\Pi^{\mathrm{temp}}_{\mathcal{G}}$ in the sense of Theorem 3.7.

The main result of the present §5 is the following *"arithmetic analogue"* of Theorem 3.7, Corollary 3.9:

Theorem 5.4. **(Arithmetically Maximal Compact Subgroups)** *Let* $\overline{\mathcal{G}}$, $\overline{\mathcal{H}}$ *be* **connected**, **countable**, **totally elevated**, **totally arithmetically estranged** *arithmetic graphs of anabelioids, with the same* **arithmetic component** $\mathcal{A}$. *Suppose, moreover, that the arithmetic actions on the underlying graphs* $\mathbb{G}$, $\mathbb{H}$ *do* **not** *switch the branches of any edge. Then:*

(i) Every arithmetically ample compact subgroup of $\pi_1^{\text{temp}}(\overline{\mathcal{G}})$ *is contained in* **at least one** *verticial subgroup. If an arithmetically ample compact subgroup of* $\pi_1^{\text{temp}}(\overline{\mathcal{G}})$ *is contained in* **more than one** *verticial subgroup, then it is contained in* **precisely two** *verticial subgroups, whose intersection forms an* **edge-like subgroup**.

(ii) The **arithmetically maximal compact subgroups** *of* $\pi_1^{\text{temp}}(\overline{\mathcal{G}})$ *are precisely the* **verticial subgroups**. *The* **arithmetically ample intersections** *of two distinct arithmetically maximal compact subgroups of* $\pi_1^{\text{temp}}(\overline{\mathcal{G}})$ *are precisely the* **edge-like subgroups**.

(iii) Applying "$\mathcal{B}^{\text{temp}}(-)$*" determines a natural bijective correspondence between* **locally open morphisms of arithmetic semi-graphs of anabelioids**

$$\overline{\mathcal{G}} \to \overline{\mathcal{H}}$$

over $\mathcal{A}$ *and* **"arithmetically quasi-geometric" morphisms of temperoids** $\mathcal{B}^{\text{temp}}(\overline{\mathcal{G}}) \to \mathcal{B}^{\text{temp}}(\overline{\mathcal{H}})$ *over* $\mathcal{A}^{\top}$, *i.e., morphisms that arise from a continuous morphism* $\Pi_{\overline{\mathcal{G}}}^{\text{temp}} \to \Pi_{\overline{\mathcal{H}}}^{\text{temp}}$ *that maps any arithmetically maximal compact subgroup* $K_1 \subseteq \Pi_{\overline{\mathcal{G}}}^{\text{temp}}$ *(respectively, arithmetically ample intersection* $K_1 \bigcap H_1$ *of two distinct arithmetically maximal compact subgroups* $K_1, H_1 \subseteq \Pi_{\overline{\mathcal{G}}}^{\text{temp}}$*) to an open subgroup of some arithmetically maximal compact subgroup* $K_2 \subseteq \Pi_{\overline{\mathcal{H}}}^{\text{temp}}$ *(respectively, of some arithmetically ample intersection* $K_2 \bigcap H_2$ *of two distinct arithmetically maximal compact subgroups* $K_2, H_2 \subseteq \Pi_{\overline{\mathcal{H}}}^{\text{temp}}$*).*

Proof. Modulo the evident *"arithmetic translation"* — e.g., "nontrivial" is to be replaced by "arithmetically ample" and "estranged" by "arithmetically estranged" — the proofs are entirely parallel to those of Theorem 3.7, Corollary 3.9. ◯

Before proceeding, we review the following well-known result:

Lemma 5.5. **(Decomposition Groups of Proper Hyperbolic Curves over Finite Fields)** *Let* X *be a* **proper hyperbolic curve** *over a* **finite field** k. *Write* Π_X *for the étale fundamental group of* X*;* $\Pi_X \twoheadrightarrow G_k$ *for the natural augmentation to the absolute Galois group of* k. *Then a* k*-valued point* $x \in X(k)$ *is* **determined** *by the outer homomorphism* $\sigma_x : G_k \to \Pi_X$ *that it induces.*

Proof. Write J for the *Jacobian* of X; assume for simplicity that there exists a point $x_0 \in X(k)$. Then x_0 determines a closed embedding $X \hookrightarrow J$ whose induced morphism on étale fundamental groups $\Pi_X \twoheadrightarrow \Pi_J$ may be identified with the quotient of Π_X by the commutator subgroup of the kernel of the surjection $\Pi_X \twoheadrightarrow G_k$. Thus, it suffices to show that a point $a \in J(k)$ is determined by the outer homomorphism $\sigma_a : G_k \to \Pi_J$ that it induces.

Write $\sigma_0 : G_k \to \Pi_J$ for the outer homomorphism induced by the identity element of $J(k)$. Then the difference between σ_a and σ_0 may be thought of as an element of $\eta_a \in H^1(k, T)$, where we define T to be the kernel of the natural surjection $\Pi_J \twoheadrightarrow G_k$. Note, moreover, that we have a natural isomorphism $T \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, J(\overline{k}))$, where $\overline{k}$ is the algebraic closure of k determined by the basepoint of k implicit in the discussion. On the other hand, by well-known general nonsense [cf., e.g., [Naka], Claim (2.2); [NTs], Lemma (4.14); [Mzk2], the Remark preceding Definition 6.2], there is a natural isomorphism $H^1(k, T) \xrightarrow{\sim} J(k)$, which maps η_a to a. In particular, η_a, hence also σ_a, is sufficient to determine a itself. ○

Example 5.6. Pointed Stable Curves over p-adic Local Fields II. We work in the notation of Example 3.10. Also, Suppose that we are given an exhaustive sequence of open characteristic [hence normal] subgroups of finite index

$$\ldots \subseteq M_i \subseteq \ldots \subseteq \Pi$$

[where i ranges over the positive integers] of Π such that $N_i = M_i \bigcap \Delta$; write $\Pi_i \stackrel{\mathrm{def}}{=} \Pi/M_i$. Thus, M_i determines a finite log étale covering of $X_K^{\log}$; we assume that M_i has been chosen so that this covering has *stable reduction* over the ring of integers of the finite extension of K that it determines. Then the outer action of M_i on N_i determines an *arithmetic action* on the semi-graphs of anabelioids $\mathcal{G}_i$, $\mathcal{G}_i^{\mathrm{c}}$ of Example 3.10; that is to say, we obtain *arithmetic semi-graphs of anabelioids*

$$\overline{\mathcal{G}}_i; \quad \overline{\mathcal{G}}_i^{\mathrm{c}}$$

with underlying semi-graphs of anabelioids $\mathcal{G}_i$, $\mathcal{G}_i^{\mathrm{c}}$, respectively, equipped with natural actions by Π_i. Moreover, $\overline{\mathcal{G}}_i$, $\overline{\mathcal{G}}_i^{\mathrm{c}}$ are *connected*, *finite*, *totally elevated*, and *totally universally sub-coverti-cial* [cf. Example 3.10]. Also, it follows immediately from Lemma 5.5 that $\overline{\mathcal{G}}_i$, $\overline{\mathcal{G}}_i^{\mathrm{c}}$ are *totally arithmetically estranged*. In particular, [at least for i sufficiently large] $\overline{\mathcal{G}}_i$ satisfies the hypotheses of Theorem 5.4.

Now I *claim* that the *generalized morphisms of arithmetic graphs of anabelioids*

$$\overline{\mathcal{G}}_i \to \overline{\mathcal{G}}_j$$

[where $i \geq j$] — i.e., the generalized morphisms of graphs of anabelioids of Example 3.10 considered together with the natural arithmetic actions on the domain and codomain — may be recovered *group-theoretically* from the associated *morphisms of tempered fundamental groups*

$$\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}_i} \to \Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}_j}$$

as follows: First, we consider the functor $\mathrm{Cat}(\mathbb{G}_i) \to \mathrm{Cat}(\mathbb{G}_j)$. Now observe that if v (respectively, e) is a(n) vertex (respectively, edge) of $\mathbb{G}_i$ such that the image in $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}_j}$

of the verticial (respectively, edge-like) subgroup determined by v (respectively, e) is contained in a [necessarily unique] edge-like subgroup H of $\Pi^{\text{temp}}_{\overline{\mathcal{G}}_j}$, then this functor maps v (respectively, e) to the *edge* of $\mathbb{G}_j$ determined by H. On the other hand, if v (respectively, e) is a(n) vertex (respectively, edge) of $\mathbb{G}_i$ such that the image in $\Pi^{\text{temp}}_{\overline{\mathcal{G}}_j}$ of the verticial (respectively, edge-like) subgroup determined by v (respectively, e) is *not* contained in an edge-like subgroup of $\Pi^{\text{temp}}_{\overline{\mathcal{G}}_j}$, but *is* contained in a verticial subgroup H of $\Pi^{\text{temp}}_{\overline{\mathcal{G}}_j}$, then this functor maps v (respectively, e) to the *vertex* of $\mathbb{G}_j$ determined by H [cf. Lemma 5.5 in the case where this image *fails* to be an open subgroup of H]. That these characterizations make sense and, moreover, do indeed yield the map on objects determined by the functor in question follows from Theorem 5.4, (i), (ii); Lemma 5.5. The remainder of the data necessary to define the generalized morphism of arithmetic graphs of anabelioids $\overline{\mathcal{G}}_i \to \overline{\mathcal{G}}_j$ is determined naturally by considering the maps between the various verticial and edge-like subgroups of $\Pi^{\text{temp}}_{\overline{\mathcal{G}}_i}$, $\Pi^{\text{temp}}_{\overline{\mathcal{G}}_j}$. This completes the proof of the claim.

Moreover, by a similar argument, together with the technique of Corollary 3.11, one may reconstruct the *generalized morphisms of arithmetic semi-graphs of anabelioids*

$$\overline{\mathcal{G}}^{\text{c}}_i \to \overline{\mathcal{G}}^{\text{c}}_j$$

[where $i \geq j$] *group-theoretically* from the corresponding *morphisms of tempered groups* $M_i \to M_j$.

Remark 5.6.1. There is an immediate *profinite* generalization of the *group-theoretic reconstruction* in Example 5.6 of the *generalized morphism of arithmetic graphs of anabelioids*

$$\overline{\mathcal{G}}^{\text{c}}_i \to \overline{\mathcal{G}}^{\text{c}}_j$$

[where $i \geq j$] from the corresponding *morphism of profinite groups*:

$$M_i^{\wedge} \to M_j^{\wedge}$$

[where the "$\wedge$" denotes profinite completion]. Indeed, this follows by applying [in place of Theorem 5.4, the technique of Corollary 3.11] the fact that the "dual semi-graph with compact structure of the geometric special fiber" may be recovered even in the profinite case, from the Galois action on the geometric profinite fundamental group [cf. [Mzk3], Lemma 2.3].

Section 6: Tempered Anabelian Geometry

In this §, we observe that the theory of *Galois sections in absolute anabelian geometry* [cf. [Mzk8]] admits a fairly straightforward generalization to the case of *tempered fundamental groups.*

Let K be a finite extension of $\mathbb{Q}_p$; $\overline{K}$ an algebraic closure of K; X_K a *hyperbolic curve* over K. Let us write $X_{\overline{K}} \stackrel{\mathrm{def}}{=} X_K \times_K \overline{K}$;

$$\pi_1^{\mathrm{temp}}(X_K)$$

for the *tempered fundamental group* of [André], §4 [cf. also the group "$\pi_1^{\mathrm{temp}}(X_K^{\log})$" of Examples 3.10, 5.6]. Thus, $\pi_1^{\mathrm{temp}}(X_K)$ is a *tempered topological group* [in the sense of Definition 3.1, (i)] and fits into a natural *exact sequence*:

$$1 \to \pi_1^{\mathrm{temp}}(X_{\overline{K}}) \to \pi_1^{\mathrm{temp}}(X_K) \to G_K \to 1$$

[where $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$; we write $\pi_1^{\mathrm{temp}}(X_{\overline{K}})$ for the *geometric tempered fundamental group* of X_K, i.e., the tempered fundamental group of $X_{\overline{K}} \times_{\overline{K}} \widehat{\overline{K}}$; the "$\wedge$" denotes the p-adic completion]. To simplify the notation, let us write:

$$\Pi_{X_K}^{\mathrm{temp}} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(X_K); \quad \Delta_X^{\mathrm{temp}} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(X_{\overline{K}})$$

In the following discussion, we shall denote the *profinite completion* of a group by means of a "$\wedge$". Also, we shall write $\Pi_{X_K} \stackrel{\mathrm{def}}{=} \widehat{\Pi}_{X_K}^{\mathrm{temp}}$; $\Delta_X \stackrel{\mathrm{def}}{=} \widehat{\Delta}_X^{\mathrm{temp}}$. It follows from the well-known *residual finiteness* of discrete free groups [cf., e.g., Corollary 1.7] that we have natural injections $\Pi_{X_K}^{\mathrm{temp}} \hookrightarrow \Pi_{X_K}$, $\Delta_X^{\mathrm{temp}} \hookrightarrow \Delta_X$.

Lemma 6.1. (Profinite Normalizers)

(i) Let F be a **finitely generated** *[discrete]* **free group** *of rank > 1. Then $N_{\widehat{F}}(F) = F$.*

(ii) We have: $N_{\Delta_X}(\Delta_X^{\mathrm{temp}}) = \Delta_X^{\mathrm{temp}}$.

(iii) We have: $N_{\Pi_{X_K}}(\Pi_{X_K}^{\mathrm{temp}}) = \Pi_{X_K}^{\mathrm{temp}}$.

Proof. Assertion (i) (respectively, (ii)) is the content of [André], Lemma 3.2.1 (respectively, [André], Corollary 6.2.2). Assertion (iii) follows immediately from assertion (ii). ○

Definition 6.2. Let F, F_1, F_2 be *tempered groups* [cf. Definition 3.1, (i)]. Then:

(i) We shall refer to a [not necessarily closed] subgroup $H \subseteq F$ as being *of DFG-type* [i.e., "dense, finitely generated type"] if it is *dense* in some open subgroup of the profinite completion $\widehat{F}$, and, moreover, for any open normal subgroup $J \subseteq F$, the image of H in F/J is *finitely generated*.

(ii) We shall refer to a [not necessarily closed] subgroup $H \subseteq F$ as being *of DOF-type* [i.e., "dense in an open subgroup of finite index type"] if it is dense in some open subgroup of F of finite index.

(iii) A continuous homomorphism $F_1 \to F_2$ will be said to be *of DFG-type* (respectively, *of DOF-type*) if its image is a subgroup of F_2 of DFG-type (respectively, of DOF-type).

Lemma 6.3. (Dense Subgroups)

(i) Let F be a **finitely generated** *[discrete]* **free group** *of rank > 1. Suppose that $H \subseteq F$ is a* **finitely generated** *subgroup which is* **dense** *in $\widehat{F}$. Then $H = F$.*

(ii) Let F be either $\Pi^{\mathrm{temp}}_{X_K}$ or Δ^{temp}_X; write $\widehat{F}$ for the profinite completion of F. Then a subgroup $H \subseteq F$ is **of DFG-type** *if and only if it is of* **DOF-type**.

(iii) Let F, $\widehat{F}$ be as in (ii). Suppose that $F_1, F_2 \subseteq F$ are subgroups of **DOF-type** *which are dense in $\widehat{F}$. Then, for any $f \in \widehat{F}$ such that $f \cdot F_1 \cdot f^{-1} = F_2$, it follows that $f \in F$.*

Proof. Assertion (i) follows immediately from the "structure theory of finitely generated subgroups of free groups of finite rank" [cf., e.g., Corollary 1.6]. As for assertion (ii), let us first observe that by replacing F by an open subgroup of F of finite index containing H, we may assume that H is dense in $\widehat{F}$. Now *sufficiency* is immediate. To prove *necessity*, we note that it follows from assertion (i), together with the assumption that H is dense in $\widehat{F}$, that the image of H in each F/J is equal to F/J, i.e., that H is dense in F, as desired. Finally, assertion (iii) follows from Lemma 6.1, (i) [cf. the proofs of Lemma 6.1, (ii), (iii)]. ◯

Now suppose that $L \subseteq \overline{K}$ is also a *finite extension* of $\mathbb{Q}_p$; Y_L is a *hyperbolic curve* over L. We shall use similar notation for the various fundamental groups [i.e., tempered, profinite étale, etc.] associated to Y_L to the notation used thus far for X_K. Now we have the following result [cf. [Mzk8], Theorem 1.2]:

Theorem 6.4. (Tempered Anabelian Theorem for Hyperbolic Curves over Local Fields) *The tempered fundamental group functor determines a bijection between the set of* **dominant morphisms of schemes**

$$X_K \to Y_L$$

and the set of **outer homomorphisms of DOF-type** *$\phi : \Pi^{\mathrm{temp}}_{X_K} \to \Pi^{\mathrm{temp}}_{Y_L}$ that fit into a commutative diagram*

$$\begin{array}{ccc} \Pi^{\mathrm{temp}}_{X_K} & \xrightarrow{\phi} & \Pi^{\mathrm{temp}}_{Y_L} \\ \downarrow & & \downarrow \\ G_K & \longrightarrow & G_L \end{array}$$

for which the induced morphism $G_K \to G_L$ is an open immersion [i.e., an isomorphism onto an open subgroup of G_L] which arises from an embedding of fields $L \hookrightarrow K$.

Proof. One verifies immediately from the definition of the "tempered fundamental group" that any $\Pi^{\text{temp}}_{X_K} \to \Pi^{\text{temp}}_{Y_L}$ that arises geometrically is *of DFG-type*, hence, by Lemma 6.3, (ii), *of DOF-type*. On the other hand, given a homomorphism $\phi : \Pi^{\text{temp}}_{X_K} \to \Pi^{\text{temp}}_{Y_L}$ of DOF-type, profinite completion yields an open homomorphism $\widehat{\phi} : \Pi_{X_K} \to \Pi_{Y_L}$, so by [Mzk8], Theorem 1.2 [i.e., in essence, [Mzk2], Theorem A], we obtain that $\widehat{\phi}$ arises, up to inner automorphism, from a dominant morphism of schemes $X_K \to Y_L$. In particular, this dominant morphism of schemes induces a homomorphism $\psi : \Pi^{\text{temp}}_{X_K} \to \Pi^{\text{temp}}_{Y_L}$ of DOF-type, whose profinite completion $\widehat{\psi} : \Pi_{X_K} \to \Pi_{Y_L}$ differs from $\widehat{\phi}$ by composition with an inner automorphism of Π_{Y_L}. On the other hand, by Lemma 6.3, (iii), we thus conclude that ϕ differs from ψ by composition with an inner automorphism of $\Pi^{\text{temp}}_{Y_L}$, as desired. ◯

Next, let us write $X_K \hookrightarrow \overline{X}_K$ for the *compactification* [cf. §0] of X_K. Let

$$x \in \overline{X}_K$$

be a *closed point*. Thus, x determines, up to conjugation by an element of $\Pi^{\text{temp}}_{X_K}$, a *decomposition group*:

$$D_x \subseteq \Pi^{\text{temp}}_{X_K}$$

We shall refer to a closed subgroup of $\Pi^{\text{temp}}_{X_K}$ which arises in this way as a *decomposition group* of $\Pi^{\text{temp}}_{X_K}$. If x is a *cusp*, then we shall refer to the decomposition group D_x as *cuspidal*. Note that D_x always *surjects* onto an open subgroup of G_K. Moreover, the subgroup

$$I_x \stackrel{\text{def}}{=} D_x \bigcap \Delta^{\text{temp}}_X$$

is isomorphic to $\widehat{\mathbb{Z}}(1)$ [i.e., the profinite completion of $\mathbb{Z}$, Tate twisted once] (respectively, $\{1\}$) if x is (respectively, is not) a *cusp*. We shall refer to a closed subgroup of $\Pi^{\text{temp}}_{X_K}$ which is equal to "I_x" for some *cusp* x as a *cuspidal geometric decomposition group.*

Theorem 6.5. (Tempered Decomposition Groups)

(i) **(Determination of the Point)** *The closed point x is completely determined by the conjugacy class of the closed subgroup $D_x \subseteq \Pi^{\text{temp}}_{X_K}$. If x is a* **cusp**, *then x is completely determined by the conjugacy class of the closed subgroup $I_x \subseteq \Pi^{\text{temp}}_{X_K}$.*

(ii) **(Commensurable Terminality)** *The subgroup D_x is commensurably terminal in $\Pi^{\text{temp}}_{X_K}$. If x is a* **cusp**, *then $D_x = C_{\Pi^{\text{temp}}_{X_K}}(H)$ for any open subgroup $H \subseteq I_x$.*

(iii) **(Absoluteness of Cuspidal Decomposition Groups)** *Every isomorphism of tempered groups*

$$\alpha : \Pi^{\text{temp}}_{X_K} \xrightarrow{\sim} \Pi^{\text{temp}}_{Y_L}$$

preserves cuspidal decomposition groups and cuspidal geometric decomposition groups.

(iv) **(Cuspidal and Noncuspidal Decomposition Groups)** *No noncuspidal decomposition group of* $\Pi^{\text{temp}}_{X_K}$ *is contained in a cuspidal decomposition group of* $\Pi^{\text{temp}}_{X_K}$*.*

Proof. Assertions (i), (ii), (iv) follow formally from [Mzk8], Theorem 1.3, (i), (ii), (iv), respectively. Assertion (iii) follows from Corollary 3.11. ◯

To a large expect, the *absolute anabelian geometry* of *tempered fundamental groups* is *essentially equivalent* to the absolute anabelian geometry of *profinite fundamental groups*. Indeed, we have the following result:

Theorem 6.6. **(Tempered and Profinite Outer Isomorphisms)** *Every outer isomorphism*

$$\Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

of **profinite** *groups arises from a* **unique** *outer isomorphism*

$$\Pi^{\text{temp}}_{X_K} \xrightarrow{\sim} \Pi^{\text{temp}}_{Y_L}$$

of **tempered** *groups.*

Proof. Indeed, it is immediate that every outer isomorphism $\Pi^{\text{temp}}_{X_K} \xrightarrow{\sim} \Pi^{\text{temp}}_{Y_L}$ determines an outer isomorphism $\Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$. Now let

$$\widehat{\alpha} : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

be an arbitrary outer isomorphism. Let $H_X \subseteq \Pi_{X_K}$, $H_Y \subseteq \Pi_{Y_L}$ be open normal subgroups [of finite index] that correspond via $\widehat{\alpha}$. Then by [Mzk3], Lemma 2.3, $\widehat{\alpha}$ determines a natural isomorphism

$$\widehat{\alpha}_H : \mathcal{G}_{H_X} \xrightarrow{\sim} \mathcal{G}_{H_Y}$$

between the *"semi-graphs of anabelioids with compact data"* $\mathcal{G}_{H_X}, \mathcal{G}_{H_Y}$ [cf. Example 2.10; [Mzk3], Appendix] associated to the geometric special fibers of the coverings corresponding to H_X, H_Y. Moreover, $\widehat{\alpha}_H$ is compatible with the *natural actions* of $H'_X \stackrel{\text{def}}{=} \Pi_{X_K}/(H_X \bigcap \Delta_X)$, $H'_Y \stackrel{\text{def}}{=} \Pi_{Y_L}/(H_Y \bigcap \Delta_Y)$, relative to the isomorphism $\widehat{\alpha}_{H'} : H'_X \xrightarrow{\sim} H'_Y$ induced by $\widehat{\alpha}$.

In particular, we conclude that the closed subgroups $J_X \subseteq H_X \bigcap \Delta^{\text{temp}}_X \subseteq \Delta^{\text{temp}}_X$, $J_Y \subseteq H_Y \bigcap \Delta^{\text{temp}}_Y \subseteq \Delta^{\text{temp}}_Y$ determined by considering the pro-tempered coverings of $X_{\overline{K}}$, $Y_{\overline{K}}$ arising from the various *tempered coverings* [cf. §3] of $\mathcal{G}_{H_X}$, $\mathcal{G}_{H_Y}$ satisfy $\widehat{\alpha}(J_X^{\wedge}) = J_Y^{\wedge}$ [where the "$\wedge$" denotes profinite completion, or, equivalently, closure in Π_{X_K}, Π_{Y_L}]. Also, we note that the natural outer actions of H'_X, H'_Y on $\pi_1^{\text{temp}}(\mathcal{G}_{H_X})$, $\pi_1^{\text{temp}}(\mathcal{G}_{H_Y})$, respectively, determine *natural isomorphisms*

$$\Pi_{X_K}^{\text{temp}}/J_X \xrightarrow{\sim} \text{Aut}(\pi_1^{\text{temp}}(\mathcal{G}_{H_X})) \times_{\text{Out}(\pi_1^{\text{temp}}(\mathcal{G}_{H_X}))} H'_X$$
$$\Pi_{Y_L}^{\text{temp}}/J_Y \xrightarrow{\sim} \text{Aut}(\pi_1^{\text{temp}}(\mathcal{G}_{H_Y})) \times_{\text{Out}(\pi_1^{\text{temp}}(\mathcal{G}_{H_Y}))} H'_Y$$

which we shall use in the following discussion to identify the quotients on the left with the fibered products on the right.

Now let us write

$$\beta_H : \text{Aut}(\pi_1^{\text{temp}}(\mathcal{G}_{H_X})) \times_{\text{Out}(\pi_1^{\text{temp}}(\mathcal{G}_{H_X}))} H'_X \xrightarrow{\sim} \text{Aut}(\pi_1^{\text{temp}}(\mathcal{G}_{H_Y})) \times_{\text{Out}(\pi_1^{\text{temp}}(\mathcal{G}_{H_Y}))} H'_Y$$

for the *isomorphism induced by* $\widehat{\alpha}_H$, $\widehat{\alpha}_{H'}$. The *functoriality* of the construction of $\widehat{\alpha}_H$ in [Mzk3], Lemma 2.3, implies that the profinite completion $\widehat{\beta}_H$ differs from the isomorphism $\Pi_{X_K}/J_X^{\wedge} \xrightarrow{\sim} \Pi_{Y_L}/J_Y^{\wedge}$ induced by $\widehat{\alpha}$ by composition with an *inner automorphism*. Also, we observe that, as one varies H_X, H_Y, consideration of the resulting *"generalized morphisms of arithmetic graphs of anabelioids"* [cf. Example 5.6, Remark 5.6.1] shows that the resulting β_H's are *compatible* [up to inner automorphism]. Thus, by passing to the corresponding *inverse limit*, we conclude that the various β_H determine an *isomorphism of tempered fundamental groups*

$$\beta : \Pi_{X_K}^{\text{temp}} \xrightarrow{\sim} \Pi_{Y_L}^{\text{temp}}$$

whose profinite completion $\widehat{\beta}$ differs from $\widehat{\alpha}$ by an *inner automorphism*, as desired. That such a β is *unique*, up to inner automorphism, follows from Lemma 6.1, (iii). ○

Remark 6.6.1. One verifies easily that the technique used in the proof of Theorem 6.6 may also be applied to give another proof of Theorem 6.5, (iii) [i.e., without resorting to the theory of §3].

Now that we have the *tempered versions* — i.e., Theorems 6.4, 6.5 — of [Mzk8], Theorems 1.2, 1.3, the theory of [Mzk8], §2, concerning the *category of dominant localizations* $\text{DLoc}_K(X_K)$ [cf. *loc. cit.*] generalizes in a fairly straightforward fashion to the tempered case:

First, we define the category

$$\text{DLoc}_{G_K}(\Pi_{X_K}^{\text{temp}})$$

as follows: An *object* of this category is a surjection of tempered groups

$$H \twoheadrightarrow J$$

where $H \subseteq \Pi_{X_K}^{\text{temp}}$ is an open subgroup of finite index; J is the quotient of H by the closed normal subgroup generated by some collection of *cuspidal geometric decomposition groups*; and we assume that J is *"hyperbolic"*, in the sense that the image of $\Delta_X^{\text{temp}} \bigcap H$ in J is *nonabelian.* Given two objects $H_i \twoheadrightarrow J_i$, where $i = 1, 2$, of this category, a morphism in this category is defined to be a diagram of the form

$$\begin{array}{ccc} H_1 & & H_2 \\ \downarrow & & \downarrow \\ J_1 & \longrightarrow & J_2 \end{array}$$

where the vertical morphisms are the given morphisms, and the horizontal morphism is an *outer homomorphism of DOF-type* that is *compatible* with the various natural [open] outer homomorphisms from the H_i, J_i to G_K.

Next, let

$$D_x \subseteq \Pi_{X_K}^{\text{temp}}$$

be a *decomposition group* associated to some closed point $x \in \overline{X}_K$.

Definition 6.7. We shall say that x or D_x is *of tempered* DLoc*-type* if D_x admits an open subgroup that arises as the image via a morphism $Z \to X_K$ of $\text{DLoc}_K(X_K)$ of some *cuspidal* decomposition group of Π_Z^{temp}.

Theorem 6.8. (Tempered Group-theoreticity of the Category of Dominant Localizations) *Let K, L be finite extensions of $\mathbb{Q}_p$; X_K (respectively, Y_L) a* **hyperbolic curve** *over K (respectively, L). Then:*

(i) The tempered fundamental group functor determines **equivalences of categories**

$$\text{DLoc}_K(X_K) \xrightarrow{\sim} \text{DLoc}_{G_K}(\Pi_{X_K}^{\text{temp}}); \quad \text{DLoc}_L(Y_L) \xrightarrow{\sim} \text{DLoc}_{G_L}(\Pi_{Y_L}^{\text{temp}})$$

(ii) Every isomorphism of tempered groups

$$\alpha : \Pi_{X_K}^{\text{temp}} \xrightarrow{\sim} \Pi_{Y_L}^{\text{temp}}$$

induces an **equivalence of categories**

$$\text{DLoc}_{G_K}(\Pi_{X_K}^{\text{temp}}) \xrightarrow{\sim} \text{DLoc}_{G_L}(\Pi_{Y_L}^{\text{temp}})$$

hence also [by applying the equivalences of (i)] an **equivalence of categories**

$$\text{DLoc}_K(X_K) \xrightarrow{\sim} \text{DLoc}_L(Y_L)$$

in a fashion that is **functorial***, up to unique isomorphisms of equivalences of categories, with respect to α. Moreover, α preserves the* **decomposition groups of tempered** DLoc**-type***.*

(iii) In the situation of (ii) above, suppose further that X_K, Y_L are **once-punctured elliptic curves**. *Then α preserves the decomposition groups of the "torsion closed points" — i.e., the closed points that arise from* **torsion points** *of the underlying elliptic curve. Moreover, the resulting bijection between torsion closed points of X_K, Y_L is* **compatible** *with the isomorphism on abelianizations of geometric fundamental groups $\Delta_X^{\text{ab}} \xrightarrow{\sim} \Delta_Y^{\text{ab}}$ — i.e., "Tate modules" — induced by α.*

(iv) In the situation of (ii) above, suppose further that X_K, Y_L are **isogenous** *[cf. §0] to hyperbolic curves of* **genus zero**. *Then the isomorphism α preserves the decomposition groups of the* **algebraic** *closed points. In particular, X_K is* **defined over a number field** *if and only if Y_L is.*

Proof. In light of Theorems 6.4, 6.5, the present Theorem 6.8 follows by exactly the same arguments as those applied in [Mzk8] to prove [Mzk8], Theorem 2.3; Corollaries 2.5, 2.6, 2.8. ○

Remark 6.8.1. Just as in the case of [Mzk8], Corollaries 2.6, 2.8, the proofs of Theorem 6.8, (iii), (iv), only require the *isomorphism version* of Theorem 6.4 [cf. [Mzk8], Remark 2.8.1].

Corollary 6.9. (Tempered Absoluteness of Decomposition Groups for Genus Zero) *In the situation of Theorem 6.8, (iv), suppose further both X_K and Y_L are* **defined over a number field**. *Then the isomorphism α preserves the decomposition groups of* **all** *the closed points.*

Proof. Corollary 6.9 follows from Theorem 6.8, (iv), by applying a similar argument to the argument used in the proof of [Mzk8], Corollary 3.2. In the present tempered case, one must therefore verify the tempered analogue of [Mzk8], Lemma 3.1. We do this as follows: First, we choose a sequence of *characteristic open subgroups* [cf., e.g., [André], Lemma 6.1.2, (i)]

$$\ldots \subseteq \Pi_{X_K}^{\text{temp}}[j+1] \subseteq \Pi_{X_K}^{\text{temp}}[j] \subseteq \ldots \subseteq \Pi_{X_K}^{\text{temp}}$$

[where j ranges over the positive integers] of $\Pi_{X_K}^{\text{temp}}$ such that the $\Pi_{X_K}^{\text{temp}}[j]$ form a *base of the topology* of $\Pi_{X_K}^{\text{temp}}$; write $\Delta_X^{\text{temp}}[j] \stackrel{\text{def}}{=} \Delta_X^{\text{temp}} \bigcap \Pi_{X_K}^{\text{temp}}$. We may assume, without loss of generality, that the *dual graph* $\mathbb{G}_j$ of the [geometric] special fiber of each of the tempered coverings of $X_{\overline{K}}$ corresponding to the $\Delta_X^{\text{temp}}[j]$ is a *tree.* In particular, given any *section*

$$\sigma : G_K \to \Pi_{X_K}^{\text{temp}}$$

we obtain open subgroups

$$\Pi_{X_K[j,\sigma]}^{\text{temp}} \stackrel{\text{def}}{=} \text{Im}(\sigma) \cdot \Delta_X^{\text{temp}}[j] \subseteq \Pi_{X_K}^{\text{temp}}$$

[where $\mathrm{Im}(\sigma)$ denotes the image of σ in $\Pi^{\mathrm{temp}}_{X_K}$] corresponding to a tower of *tempered coverings* of X_K:

$$\ldots \to X_K[j+1,\sigma] \to X_K[j,\sigma] \to \ldots \to X_K$$

Also, we observe that the natural action of $\Pi^{\mathrm{temp}}_{X_K}$ on the tree $\mathbb{G}_j$ *factors through* $\Delta^{\mathrm{temp}}_{X[j]}$.

Now suppose that $\mathrm{Im}(\sigma)$ is *not* contained in any cuspidal decomposition group of $\Pi^{\mathrm{temp}}_{X_K}$. Then the following conditions on σ are *equivalent*:

(i) σ arises from a point $x \in X_K(K)$ [i.e., "$\mathrm{Im}(\sigma) = D_x$"].

(ii) For every integer $j \geq 1$, $X_K[j,\sigma](K) \neq \emptyset$.

(iii) For every integer $j \geq 1$, $X_K[j,\sigma](K)^{\mathrm{alg}} \neq \emptyset$ [where the superscript "alg" denotes the subset of algebraic K-rational points, i.e., K-rational points that map to algebraic points of $X_K(K)$].

(iv) For every integer $j \geq 1$, $\Pi^{\mathrm{temp}}_{X_K[j,\sigma]}$ contains a decomposition group [i.e., relative to $\Pi^{\mathrm{temp}}_{X_K}$] of an *algebraic* closed point of X_K that *surjects* onto G_K.

Indeed, the implications (i) $\Longrightarrow$ (ii); (iii) $\Longrightarrow$ (ii), (iv); and (iv) $\Longrightarrow$ (iii) follow formally as in the proof of [Mzk8], Lemma 3.1. Moreover, the implication (ii) $\Longrightarrow$ (iii) — i.e., *"approximation via Krasner's lemma"* [cf. the proof of [Mzk8], Lemma 3.1] — follows as in *loc. cit.*, since given any point $x_j \in X_K[j,\sigma](K)$ with image $x \in X_K(K)$, the completion at [the $\mathcal{O}_K$-valued point determined by] x_j of the *normalization* in $X_K[j,\sigma]$ of some proper model of X_K over $\mathcal{O}_K$ is *finite* over the completion of this proper model at [the $\mathcal{O}_K$-valued point determined by] x.

Finally, we consider the implication (ii) $\Longrightarrow$ (i). In the case of *loc. cit.*, this implication followed formally from the fact that the topological space

$$\prod_{j \geq 1} \overline{X}_K[j,\sigma](K)$$

was [in the case of *loc. cit.*] *manifestly compact*. In the present tempered case, although this compactness is not immediate, we may nevertheless conclude, at least for some *cofinal set* of j, the compactness of $\overline{X}_K[j,\sigma](K)$ by observing that the points of $\overline{X}_K[j,\sigma](K)$ always determine *components* [i.e., vertices or edges] of $\mathbb{G}_j$ that are *fixed* by the natural action of the image $\mathrm{Im}(\Pi^{\mathrm{temp}}_{X_K[j,\sigma]}) \subseteq \Pi^{\mathrm{temp}}_{X_K}/\Delta^{\mathrm{temp}}_{X[j]}$, i.e., by the natural action of G_K on $\mathbb{G}_j$ via σ. On the other hand, it follows from our assumption that the $\mathbb{G}_j$ are *trees* [cf. Theorem 3.7, Theorem 5.4, and their proofs; Lemma 1.8, (ii)] that, at least for some *cofinal set* of j, this set of fixed components of $\mathbb{G}_j$ is *finite*, thus implying the desired *compactness* of $\overline{X}_K[j,\sigma](K)$. ○

Remark 6.9.1. As observed in Remark 3.7.1, the argument used in the final portion of the proof of Corollary 6.9 is reminiscent of the argument used in the *"discrete real section conjecture"* of [Mzk5], §3.2. This is interesting since Corollary 6.9 itself may be regarded as a weak form of the "section conjecture" for $\Pi_{X_K}^{\text{temp}} \twoheadrightarrow G_K$ [i.e., roughly speaking, the assertion that all sections of this surjection arise geometrically].

Next, let

$$D_x \subseteq \Pi_{X_K}^{\text{temp}}$$

be a *decomposition group* associated to some *closed point* $x \in \overline{X}_K(K)$. Then one has *tempered analogues* of the various notions of *"absoluteness"* given in [Mzk8], Definition 4.1, (iii); [Mzk8], Definition 4.8 — which we denote by means of a prefix *"temp-"*. Observe that D_x also forms a *"profinite $D_x \subseteq \Pi_{X_K}$"* in the sense of [Mzk8]. Moreover, since finitely generated free groups are well-known to be *"good"* [i.e., the natural map from the cohomology of the profinite completion of this group with coefficients in a finite [i.e., as a set] module to the cohomology of the original group in the same module is an *isomorphism*], it is immediate that $\Pi_{X_K}^{\text{temp}}$ is also *good*. We thus conclude [cf. Theorem 6.6] the following:

Corollary 6.10. (Tempered Absoluteness)

(i) The point x is a **discretely absolute cusp** *(respectively, a* **integrally absolute cusp***) if and only if it is a* **discretely temp-absolute cusp** *(respectively, an* **integrally temp-absolute cusp***).*

(ii) The hyperbolic curve X_K is **unitwise absolute** *if and only if it is* **unitwise temp-absolute***.*

Corollary 6.11. (Unitwise and Integral Temp-absoluteness for Genus Zero) *Let X_K be a* **hyperbolic curve** *over K, with* **stable reduction** *over $\mathcal{O}_K$, which is* **isogenous** *to a hyperbolic curve of* **genus zero***. Then X_K is* **unitwise temp-absolute***, and every cusp of X_K is* **integrally temp-absolute***.*

Proof. Indeed, this follows formally from Corollary 6.10, (i), (ii), and [Mzk8], Corollary 4.11. ○

Finally, we also observe that it is immediate that the tempered analogue of [Mzk8], Theorem 4.3, holds:

Theorem 6.12. (Rigidity of Cuspidal Geometric Decomposition Groups) *In the notation of Theorem 6.8, (ii), suppose that α induces isomorphisms*

$$I_x \xrightarrow{\sim} I_y; \quad \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{L})$$

where $x \in \overline{X}_K(K)$ (respectively, $y \in \overline{Y}_L(L)$) is a cusp; $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(-)$ is as in [Mzk8], Theorem 4.3. Then these isomorphisms are **compatible** *with the* **natural isomorphisms** $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}) \xrightarrow{\sim} I_x$; $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{L}) \xrightarrow{\sim} I_y$.

Remark 6.12.1. Note that unlike Corollaries 6.10, 6.11; Theorem 6.12, results such as Theorem 6.8, (iii), (iv); Corollary 6.9 do *not* follow formally from their profinite analogues, since, in the latter case, it is by no means clear that any Π_{X_K}-conjugate of the decomposition group of a closed point that happens to be contained in $\Pi_{X_K}^{\text{temp}} \subseteq \Pi_{X_K}$ is necessarily a $\Pi_{X_K}^{\text{temp}}$-conjugate of the decomposition group of a closed point.

Appendix: Quasi-temperoids

In this Appendix, we discuss a certain *minor generalization* of the notion of a *temperoid* introduced in §3.

Let $\mathcal{T}$ be a *temperoid*. If A is an object of $\mathcal{T}$, write $\pi_0(A)$ for the set of connected components of A and

$$\mathcal{T}[A] \subseteq \mathcal{T}$$

for the *full subcategory* determined by the objects of $\mathcal{T}$ that admit a morphism to A [cf §0].

Definition A.1.

(i) Any category equivalent to a category of the form

$$\mathcal{T}[A]$$

— where A is a connected object, and $\mathcal{T}$ is a connected temperoid — will be referred to as a *connected quasi-temperoid*.

(ii) A category equivalent to a product [in the sense of a product of categories] of a countable [hence possibly empty!] collection of connected quasi-temperoids will be referred to as a *quasi-temperoid*. An object A of a quasi-temperoid $\mathcal{Q}$ will be called *nondegenerate* if, for every connected object B of $\mathcal{Q}$, there exist arrows $C \to B$, $C \to A$, for some connected object C of $\mathcal{Q}$.

(iii) Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quasi-temperoids. Then a *quasi-morphism* $\phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ is defined to be a functor $\phi^* : \mathcal{Q}_2 \to \mathcal{Q}_1$ that preserves finite limits and countable colimits. A quasi-morphism ϕ will be called *rigid* (respectively, a *morphism*) if the functor ϕ^* is rigid [cf. §0] (respectively, preserves nondegenerate objects).

Remark A.1.1. One verifies immediately that a quasi-temperoid is an *almost totally epimorphic category countably connected type* [cf. §0].

Remark A.1.2. Unlike the situation with temperoids, a quasi-temperoid does *not*, in general, admit a *terminal object*. Thus, it is *not* in general the case that a *quasi-morphism* of quasi-temperoids is necessarily a *morphism*. Indeed, if $\mathcal{Q}$ is a connected quasi-temperoid that does not admit a terminal object, then one verifies immediately that the functor $\mathcal{Q} \to \mathcal{Q}$ that maps all objects of $\mathcal{Q}$ to some empty object of $\mathcal{Q}$ preserves finite limits and countable colimits, but fails to preserve nondegenerate objects.

Proposition A.2. (Connected Components of Quasi-temperoids) *Let E, E' be countable sets; for each $e \in E$ (respectively, $e' \in E'$), let $\mathcal{Q}_e$ (respectively, $\mathcal{Q}'_{e'}$) be a* **connected quasi-temperoid***; set:*

$$\mathcal{Q} \stackrel{\text{def}}{=} \prod_{e \in E} \mathcal{Q}_e; \quad \mathcal{Q}' \stackrel{\text{def}}{=} \prod_{e' \in E'} \mathcal{Q}'_{e'}$$

Also, let $\phi : \mathcal{Q} \to \mathcal{Q}'$ *be a* **quasi-morphism of quasi-temperoids***. Then:*

(i) For each $e \in E$*, the natural* **projection functor**

$$\pi_e^* : \mathcal{Q} \to \mathcal{Q}_e$$

determines a **morphism of quasi-temperoids** $\mathcal{Q}_e \to \mathcal{Q}$*.*

(ii) For each $e \in E$*, write*

$$\iota_e : \mathcal{Q}_e \to \mathcal{Q}$$

for the natural **inclusion functor** *[i.e., the functor whose composite with* π_f^**, where* $f \in E$*, maps all objects of* $\mathcal{Q}_e$ *to empty [i.e., initial] objects of* $\mathcal{Q}_f$ *if* $f \neq e$*, and is the identity if* $f = e$*]. If* ϵ *is a connected component of the full subcategory of connected objects* $\mathcal{Q}^0 \subseteq \mathcal{Q}$ *[cf. §0], then write*

$$\mathcal{Q}_\epsilon \subseteq \mathcal{Q}$$

for the full subcategory determined by the objects A *of* $\mathcal{Q}$ *such that all of the connected components of* A *belong to* ϵ*. Then the* **essential image** *of* ι_e *is equal to* $\mathcal{Q}_\epsilon$ *for a* **unique** ϵ*, and, moreover, the resulting correspondence*

$$e \mapsto \epsilon$$

determines a **bijection** *between* E *and the set of connected components of* $\mathcal{Q}^0$*.*

(iii) The **nondegenerate objects** *of* $\mathcal{Q}$ *are precisely the objects each of whose component objects* $\in \mathrm{Ob}(\mathcal{Q}_e)$ *is* **nonempty***.*

(iv) If ϕ *is a* **morphism of quasi-temperoids***, and both* E *and* E' *are* **of cardinality one***, then the functor* ϕ^* *is* **faithful***.*

(v) The **quasi-morphism** *of quasi-temperoids* ϕ *induces a* **map** $\psi : E \to E'$*, and, for each* $e \in E$*, a* **quasi-morphism** *of quasi-temperoids* $\phi_e : \mathcal{Q}_e \to \mathcal{Q}'_{\psi(e)}$ *such that* ϕ *coincides with the quasi-morphism of quasi-temperoids formed by "taking the product" [in the evident sense] of the* ϕ_e*. Moreover,* ϕ *is a morphism of quasi-temperoids if and only if every* ϕ_e *is a morphism of quasi-temperoids.*

Proof. Assertion (i) (respectively, (iii)) follows immediately from the definitions (respectively, and assertion (ii)). To prove assertion (ii), we argue as follows: By unraveling the definitions, one verifies immediately that every connected object of $\mathcal{Q}$ lies in the essential image of a unique ι_e, and that two connected objects of $\mathcal{Q}$ lie in the essential image of the same ι_e if and only if they belong to the same connected component of $\mathcal{Q}^0$. Now assertion (ii) follows formally from these observations.

Next, we verify assertion (iv). First, in light of Remark A.1.1, it suffices to check faithfulness on arrows $A \to B$ between *connected* objects A, B of $\mathcal{Q}'$. But since A, B are connected, it follows immediately that there exists a *connected* object C of $\mathcal{Q}'$ such that the products $A \times C$, $B \times C$ *split* as coproducts of copies of C. Since A, B, C are connected [hence nonempty], they are *nondegenerate* [cf. assertion (iii)]. Moreover, again by Remark A.1.1, arrows $A \to B$ are represented *faithfully* by the maps $\pi_0(A \times C) \to \pi_0(B \times C)$ they induce. But, since $\phi^*(A)$, $\phi^*(B)$, $\phi^*(C)$ are *nondegenerate*, hence, in particular, *nonempty*, this implies that ϕ^* itself is faithful, as desired.

Finally, we verify assertion (v). Let us consider the various composite functors

$$\kappa[e', e] \stackrel{\text{def}}{=} \pi_e^* \circ \phi^* \circ \iota_{e'} : \mathcal{Q}'_{e'} \to \mathcal{Q}_e$$

where $e \in E$, $e' \in E'$. Now observe that, if we *fix* e and allow e' to *vary*, then since the product of objects belonging to distinct $\mathcal{Q}'_{\epsilon'}$'s will always be an *empty object* of $\mathcal{Q}'$, it follows from the fact that π_e, ϕ are *quasi-morphisms of quasi-temperoids* [together with the easily verified observation that any product of nonempty objects of a connected quasi-temperoid will always be *nonempty*] that there is *at most one* $e' \in E'$ such that the essential image of $\kappa[e', e]$ *contains nonempty objects*. Moreover, since ϕ^* preserves *nondegenerate objects*, it follows [cf. assertion (iii)] that there exists *at least one* $e' \in E'$ such that the essential image of $\kappa[e', e]$ *contains nonempty objects*. Thus, in summary, for each $e \in E$, there *exists a unique* $e' \in E'$ such that the essential image of $\kappa[e', e]$ *contains nonempty objects*; set $\psi(e) \stackrel{\text{def}}{=} e'$. Then it follows immediately from the definitions that the functor $\kappa[\psi(e), e]$ determines a *quasi-morphism of quasi-temperoids* $\mathcal{Q}_e \to \mathcal{Q}'_{\psi(e)}$. Thus, unraveling the definitions, we see that the remainder of assertion (v) follows formally from what we have done thus far. ○

Remark A.2.1. Thus, Proposition A.2 serves, in effect, to *reduce* the theory of *arbitrary quasi-temperoids* to the theory of *connected quasi-temperoids*.

Definition A.3. Let $\mathcal{Q}$ be a *connected quasi-temperoid*. Then:

(i) Any pair (A, Γ_A), where A is a object of $\mathcal{Q}$, and $\Gamma_A \subseteq \text{Aut}_{\mathcal{Q}}(A)$ is a subgroup, will be referred to as a *QD-* [or *quotient data*] *pair* [of $\mathcal{Q}$]. If Γ_A acts *transitively* on $\pi_0(A)$, then we shall say that this QD-pair is *weakly connected*; if A is connected, then we shall say that this QD-pair is *strongly connected*. [Thus, every strongly connected QD-pair is weakly connected.]

(ii) A *morphism of QD-pairs of* $\mathcal{Q}$

$$(A, \Gamma_A) \to (B, \Gamma_B)$$

is defined to be a morphism $\phi : A \to B$ such that, for every $\gamma_A \in \Gamma_A$, there exists a $\gamma_B \in \Gamma_B$ such that $\gamma_B \circ \phi = \phi \circ \gamma_A$. If, moreover, ϕ induces a *surjection* $\pi_0(A) \twoheadrightarrow \pi_0(B)$, then we shall say that this morphism is 0-*proper*. Thus, in the 0-proper case, it follows from the fact that the category $\mathcal{Q}$ is *almost totally epimorphic* that γ_B is *unique*, hence that the correspondence $\gamma_A \mapsto \gamma_B$ determines an *associated group homomorphism* $\Gamma_A \to \Gamma_B$.

(iii) Let (A, Γ_A) be a QD-pair. Then we shall say that an arrow $\phi : A \to B$ of $\mathcal{Q}$ *forms a quotient of this QD-pair* — and write $B \cong A/\Gamma_A$ — if the following two properties are satisfied: (a) $\phi \circ \gamma_A = \phi$, $\forall\, \gamma_A \in \Gamma_A$; (b) for every arrow $\psi_A : A \to C$ satisfying $\psi_A \circ \gamma_A = \psi_A$, $\forall\, \gamma_A \in \Gamma_A$, there exists a unique arrow $\psi_B : B \to C$ such that $\psi_B \circ \phi = \psi_A$. [Thus, one verifies immediately that the quotient of a QD-pair is *unique*, up to unique isomorphism, if it exists, and that the quotient of a QD-pair is connected if and only if the QD-pair is *weakly connected*.]

(iv) A 0-proper morphism of QD-pairs of $\mathcal{Q}$

$$(A, \Gamma_A) \to (B, \Gamma_B)$$

will be called 1-*proper* if the associated homomorphism $\Gamma_A \to \Gamma_B$ is *surjective*, and, moreover, the arrow $A \to B$ *forms a quotient* of the QD-pair $(A, \mathrm{Ker}(\Gamma_A \twoheadrightarrow \Gamma_B))$. [Thus, under the 1-properness assumption, one verifies immediately that if $B \to C$ forms a quotient of (B, Γ_B), then the composite arrow $A \to B \to C$ forms a quotient of (A, Γ_A).]

Remark A.3.1. One verifies immediately that the following conditions on a morphism of QD-pairs $(A, \Gamma_A) \to (B, \Gamma_B)$ are *equivalent*:

(a) $(A, \Gamma_A) \to (B, \Gamma_B)$ is 0-*proper*.

(b) $A \to B$ is an *epimorphism* in $\mathcal{Q}$ [cf. Remark A.1.1].

(c) B is the *colimit* in $\mathcal{Q}$ of the diagram formed by the two projections $A \times_B A \to A$.

Moreover, the notion of a *"quotient"* given in Definition A.3, (iii), may also be stated in terms of *colimits*. In particular, the condition that $(A, \Gamma_A) \to (B, \Gamma_B)$ be 1-*proper* may be stated entirely in terms of *colimits*.

Proposition A.4. **(Connected Quasi-temperoids)** *For $i = 1, 2$, let $\mathcal{T}_i$ be a* **connected temperoid***; let A_i be a* **connected object** *of $\mathcal{T}_i$; write*

$$\lambda_i : \mathcal{Q}_i \stackrel{\mathrm{def}}{=} \mathcal{T}_i[A_i] \to \mathcal{T}_i$$

for the natural functor. Then any **morphism of quasi-temperoids** $\phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ *fits into a* **1-commutative diagram**

$$\begin{array}{ccc} \mathcal{Q}_1 & \stackrel{\phi}{\longrightarrow} & \mathcal{Q}_2 \\ \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_2} \\ \mathcal{T}_1 & \stackrel{\psi}{\longrightarrow} & \mathcal{T}_2 \end{array}$$

— where the morphism of [quasi-]temperoids $\psi : \mathcal{T}_1 \to \mathcal{T}_2$ that makes this diagram 1*-commute is* **unique***, up to unique isomorphism.*

Proof. Write

$$\mathcal{D}_i$$

for the category whose *objects* are QD-pairs of $\mathcal{Q}_i$ and whose *morphisms* are morphisms of QD-pairs. Note that it follows immediately from the definitions that if (B, Γ_B) is a QD-pair of $\mathcal{Q}_i$, then the QD-pair $\lambda_i(B, \Gamma_B)$ of $\mathcal{T}_i$ *admits a quotient* in $\mathcal{T}_i$. Thus, we obtain a *natural functor*

$$q_i : \mathcal{D}_i \to \mathcal{T}_i$$

given by the assignment $(B, \Gamma_B) \mapsto B/\Gamma_B$. Note that this functor maps 1-*proper morphisms* of $\mathcal{D}_i$ to *isomorphisms* of $\mathcal{T}_i$. Moreover, one verifies immediately that this functor q_i is *essentially surjective*; that $q_i((B, \Gamma_B))$ is connected if and only if (B, Γ_B) is *weakly connected*; and that every connected object of $\mathcal{T}_i$ is isomorphic to the image via q_i of a *strongly connected* QD-pair.

Thus, to reconstruct $\mathcal{T}_i$ from $\mathcal{D}_i$, it suffices to reconstruct the *morphisms*

$$B/\Gamma_B \to C/\Gamma_C$$

in $\mathcal{T}_i$ between the images via q_i of two objects (B, Γ_B), (C, Γ_C) of $\mathcal{D}_i$. To this end, we define

$$\underline{\mathrm{Hom}}((B, \Gamma_B), (C, \Gamma_C)) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{D}_i}((B, \Gamma_B), (C, \Gamma_C))/\Gamma_C$$

[i.e., where Γ_C acts by composition from the right], so that the functor q_i induces a *natural map*:

$$\underline{\mathrm{Hom}}((B, \Gamma_B), (C, \Gamma_C)) \to \mathrm{Hom}_{\mathcal{T}_i}(B/\Gamma_B, C/\Gamma_C)$$

Since $\mathcal{T}_i$ is an *almost totally epimorphic category countably connected type*, it suffices to describe the morphisms between *connected objects* of $\mathcal{T}_i$.

Next, suppose that (B, Γ_B), (C, Γ_C) are *strongly connected QD-pairs*. Then observe that the functor q_i induces an *injection*:

$$\underline{\mathrm{Hom}}((B, \Gamma_B), (C, \Gamma_C)) \hookrightarrow \mathrm{Hom}_{\mathcal{T}_i}(B/\Gamma_B, C/\Gamma_C)$$

Indeed, this follows immediately by considering the natural splitting [in $\mathcal{Q}_i$ or $\mathcal{T}_i$] of $C \times_{C/\Gamma_C} C$ into a coproduct of copies of C indexed by Γ_C [together with the fact that (B, Γ_B), (C, Γ_C) are *strongly connected QD-pairs*]. Moreover, one verifies immediately that every element of $\mathrm{Hom}_{\mathcal{T}_i}(B/\Gamma_B, C/\Gamma_C)$ arises from some morphism of strongly connected QD-pairs

$$(B', \Gamma_{B'}) \to (C, \Gamma_C)$$

where $(B', \Gamma_{B'}) \to (B, \Gamma_B)$ is a 1-*proper* morphism of strongly connected QD-pairs [so we obtain a morphism $B/\Gamma_B \to C/\Gamma_C$ by composing the induced morphism $B'/\Gamma_{B'} \to C/\Gamma_C$ with the inverse of the induced isomorphism $B'/\Gamma_{B'} \stackrel{\sim}{\to} B/\Gamma_B$].

Now one verifies easily that any two 1-proper morphisms of strongly connected QD-pairs $(B', \Gamma_{B'}) \to (B, \Gamma_B)$, $(B'', \Gamma_{B''}) \to (B, \Gamma_B)$ fit into a commutative diagram of 1-proper morphisms of strongly connected QD-pairs of $\mathcal{D}_i$:

$$\begin{array}{ccc} (B''', \Gamma_{B'''}) & \longrightarrow & (B'', \Gamma_{B''}) \\ \big\downarrow & & \big\downarrow \\ (B', \Gamma_{B'}) & \longrightarrow & (B, \Gamma_B) \end{array}$$

Thus, we conclude that $\mathrm{Hom}_{\mathcal{T}_i}(B/\Gamma_B, C/\Gamma_C)$ may be reconstructed as the following *filtered inductive limit*:

$$\underline{\underline{\mathrm{Hom}}}((B, \Gamma_B), (C, \Gamma_C)) \stackrel{\mathrm{def}}{=} \varinjlim \ \underline{\mathrm{Hom}}((B', \Gamma_{B'}), (C, \Gamma_C))$$

[i.e., over 1-proper morphisms of strongly connected QD-pairs $(B', \Gamma_{B'}) \to (B, \Gamma_B)$ and transition morphisms $(B'', \Gamma_{B''}) \to (B', \Gamma_{B'})$ over (B, Γ_B)]. Moreover, one verifies immediately that this reconstruction is compatible with *composition of arrows* [i.e., composition of arrows in $\mathcal{D}_i$ induces composition of "$\underline{\underline{\mathrm{Hom}}}$-arrows"].

Next, suppose that (B, Γ_B), (C, Γ_C) are *weakly connected QD-pairs*. Then observe that each connected component B' of B determines a *strongly connected* QD-pair $(B', \Gamma_{B'})$ [i.e., where we take $\Gamma_{B'} \subseteq \Gamma_B$ to be the subgroup of automorphisms that fix the element $[B'] \in \pi_0(B)$ determined by B'] such that $q_i((B', \Gamma_{B'})) \cong q_i((B, \Gamma_B))$; a similar statement holds for (C, Γ_C). Moreover, if B', B'' (respectively, C', C'') are connected components of B (respectively, C), then one verifies immediately that any choice of elements $\gamma_B \in \Gamma_B$, $\gamma_C \in \Gamma_C$ such that $\gamma_B(B') = B''$, $\gamma_C(C') = C''$ determines a *bijection*

$$\underline{\underline{\mathrm{Hom}}}((B', \Gamma_{B'}), (C', \Gamma_{C'})) \xrightarrow{\sim} \underline{\underline{\mathrm{Hom}}}((B'', \Gamma_{B''}), (C'', \Gamma_{C''}))$$

which is, in fact, *independent* of the choice of γ_B, γ_C. Thus, if we define

$$\underline{\underline{\mathrm{Hom}}}((B, \Gamma_B), (C, \Gamma_C)) \subseteq \prod_{B', C'} \underline{\underline{\mathrm{Hom}}}((B', \Gamma_{B'}), (C', \Gamma_{C'}))$$

[where the product ranges over all choices of connected components B', C' of B, C, respectively] to be the subset of collections of elements that correspond via these bijections, then the natural projections of this direct product determine *bijections* as follows:

$$\begin{aligned} \underline{\underline{\mathrm{Hom}}}((B, \Gamma_B), (C, \Gamma_C)) &\xrightarrow{\sim} \underline{\underline{\mathrm{Hom}}}((B', \Gamma_{B'}), (C', \Gamma_{C'})) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}_i}(q_i((B', \Gamma_{B'})), q_i((C', \Gamma_{C'}))) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}_i}(q_i((B, \Gamma_B)), q_i((C, \Gamma_C))) \end{aligned}$$

Moreover, if follows immediately from the definitions that we have a *natural map*:

$$\underline{\mathrm{Hom}}((B, \Gamma_B), (C, \Gamma_C)) \to \underline{\underline{\mathrm{Hom}}}((B, \Gamma_B), (C, \Gamma_C))$$

Finally, as observed above, since $\mathcal{T}_i$ is an *almost totally epimorphic category countably connected type*, the *definition* of "$\underline{\underline{\underline{\text{Hom}}}}$", as well as the resulting *bijection* of "$\underline{\underline{\text{Hom}}}$" with "$\text{Hom}_{\mathcal{T}_i}$" and the *natural map* from "$\underline{\text{Hom}}$" to "$\underline{\underline{\underline{\text{Hom}}}}$" extend immediately to pairs of objects of $\mathcal{D}_i$ that are *not necessarily* weakly connected.

Thus, in summary, if we write

$$\mathcal{P}_i$$

for the category whose *objects* are the objects of $\mathcal{D}_i$ and whose *morphisms* are given by the "$\underline{\underline{\underline{\text{Hom}}}}$'s", then we obtain *natural functors*

$$\mathcal{Q}_i \to \mathcal{D}_i \to \mathcal{P}_i \xrightarrow{\sim} \mathcal{T}_i$$

— i.e., the first functor maps an object B of $\mathcal{Q}_i$ to the QD-pair $(B, \{1\})$; the second functor arises from the construction of $\mathcal{P}_i$; the third functor is the *equivalence* induced by the natural functor $\mathcal{D}_i \to \mathcal{T}_i$ considered above.

Now observe that the functor $\phi^* : \mathcal{Q}_2 \to \mathcal{Q}_1$ induces a 1-commutative diagram

$$\begin{array}{ccccc}
\mathcal{Q}_2 & \longrightarrow & \mathcal{D}_2 & \longrightarrow & \mathcal{P}_2 \\
\Big\downarrow \phi^* & & \Big\downarrow & & \Big\downarrow \\
\mathcal{Q}_1 & \longrightarrow & \mathcal{D}_1 & \longrightarrow & \mathcal{P}_1
\end{array}$$

Indeed, the construction of the second vertical arrow is immediate from the definitions. The construction of the third vertical arrow follows by observing that since ϕ^* preserves countable *colimits*, it follows that the functor $\mathcal{D}_2 \to \mathcal{D}_1$ *preserves* 0- and 1-proper morphisms [cf. Remark A.3.1]. Thus, by combining this diagram with the equivalences $\mathcal{P}_i \xrightarrow{\sim} \mathcal{T}_i$, we obtain a diagram as in the statement of Proposition A.4.

The fact that the resulting functor $\psi^* : \mathcal{T}_2 \to \mathcal{T}_1$ preserves countable colimits (respectively, fibered products) follows by a routine argument from the fact that ϕ^* preserves countable colimits (respectively, countable colimits and finite limits). Thus, to show that ψ is a *morphism of temperoids* [i.e., that ψ^* preserves finite limits], it suffices to show that ψ^* preserves *terminal objects*. Now let B be a connected object of $\mathcal{Q}_2$ such that $B \times B$ *splits* as a coproduct of copies of B [i.e., in other words, B is a connected Galois object of $\mathcal{T}_2$ that admits a morphism in $\mathcal{T}_2$ to A_2]. Note that such a B always exists. Then if we let $\text{Aut}(B)$ act on, say, the second factor of $B \times B$, then the resulting QD-pair $(B \times B, \text{Aut}(B))$ maps via ϕ^* to a QD-pair $(\phi^*(B) \times \phi^*(B), \phi^*(\text{Aut}(B)))$ of $\mathcal{Q}_1$. Moreover, since the first projection $B \times B \to B$ forms a *quotient* of the former QD-pair, it follows that the first projection $\phi^*(B) \times \phi^*(B) \to \phi^*(B)$ forms a *quotient* of the latter QD-pair. It thus follows immediately that $\phi^*(\text{Aut}(B))$ acts *transitively* on $\pi_0(\phi^*(B))$. Next, observe that $B \times B$ is a *coproduct* of copies of B. Thus, we conclude that $\phi^*(B) \times \phi^*(B)$ is a coproduct of copies of $\phi^*(B)$. But this implies that the connected components of $\phi^*(B)$, all of which are isomorphic to one another [by the *transitivity* observed above], form *Galois* objects of $\mathcal{T}_1$. Thus, the preceding observation concerning the quotient of the QD-pair in $\mathcal{Q}_1$ implies that the stabilizer in $\phi^*(\text{Aut}(B))$ of any connected component $\phi^*(B)$ is necessarily equal to the *entire automorphism group* of this connected component. Moreover, since ϕ^* preserves *nondegenerate objects*, it follows that $\phi^*(B)$ is *nonempty* [i.e., $\phi^*(B)$ has *at least one* connected component]. Thus, in summary, ϕ^* maps the QD-pair $(B, \mathrm{Aut}(B))$ [any quotient of which forms a terminal object in $\mathcal{T}_2$] to a QD-pair $(\phi^*(B), \phi^*(\mathrm{Aut}(B)))$ any quotient of which forms a terminal object in $\mathcal{T}_1$. That is to say, we have shown that ψ^* preserves terminal objects, as desired.

Finally, the asserted *uniqueness* of ψ follows immediately from the fact that arbitrary objects of $\mathcal{T}_i$ may be obtained as quotients of QD-pairs of $\mathcal{Q}_i$. ◯

Remark A.4.1. Thus, Proposition A.4 serves, in effect, to *reduce* the theory of *arbitrary connected quasi-temperoids* to the theory of *connected temperoids.*

Remark A.4.2. One verifies immediately that, by replacing the term "temperoid" by the term "anabelioid" [and the terms "countable/countably" by the terms "finite/finitely"], one obtains an entirely analogous [but, in fact, slightly easier] theory of *"quasi-anabelioids"* to the theory of quasi-temperoids developed above. We leave the routine details to the reader.

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