Virtual turning points and bifurcation of Stokes curves for higher order ordinary differential equations

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Abstract

For a higher order linear ordinary differential operator P, its Stokes curve bifurcates in general when it hits another turning point of P. This phenomenon is most neatly understandable by taking into account Stokes curves emanating from virtual turning points, together with those from ordinary turning points. This understanding of the bifurcation of a Stokes curve plays an important role in resolving a paradox recently found in the Noumi-Yamada system, a system of linear differential equations associated with the fourth Painlevé equation.

Exact WKB analysis, that is, WKB analysis based on the Borel resummation, has turned out to be an important and useful tool in mathematical physics [1]; its advantage certainly consists in its efficiency in manipulating exponentially small terms, but still more important, from the theoretical viewpoint, are the fact that the Borel transform of an ordinary differential operator $P(x, \eta^{-1}d/dx)$ with a large parameter η is a partial differential operator on the (x, y)-space with y denoting the variable dual to η, and the fact that microlocal analysis, a new and powerful machinery in mathematics [2], clarifies the structure of singularities of solutions of the Borel transformed equation, i.e., the Borel transformed WKB solutions, which are multi-valued analytic functions on (x, y)-space. An important example of the influence of microlocal analysis on WKB analysis is the introduction of the notion of a virtual turning point for differential equations of the third or higher order [3]; it is, by definition, the x-component of the self-intersection point of a bicharacteristic curve of the Borel transform of the operator $P(x, \eta^{-1}d/dx)$. Note that a bicharacteristic curve is the most "elementary" carrier of singularities of solutions of linear partial differential equations in general [2]. Note also that Voros [4] uses the corresponding result for the Tricomi-type operator in constructing his theory of exact WKB analysis for differential operators of the second order. As the so-called new Stokes curve for higher order operators [5] is nothing but an ordinary Stokes curve emanating from a virtual turning point, the importance of the notion of a virtual turning point is practically evident. Actually it plays an important role in computing the transition probabilities for the non-adiabatic transition problem of the Landau-Zener type [6]. In this paper we show how important a role a virtual turning point plays from the theoretical viewpoint. To be more concrete, we validate the following Assertion A using a concrete example we encounter in the exact WKB analysis of the Painlevé transcendent [7]:

Assertion A: *The role of a virtual turning point is commensurate with that of an ordinary turning point; theoretically speaking, there is no distinction between them.*

In validating this challenging assertion, we divide our discussion into two steps: we first show the mechanism that relates a virtual turning point with the bifurcation phenomenon of a Stokes curve that is observed when it hits a simple turning point, and then we argue how the mechanism works in understanding the true nature of a seemingly paradoxical phenomenon which has been just found [7].

The relevance of a virtual turning point and the bifurcation of a Stokes curve has been recognized since a couple of years ago [8], but it has not been published in literature.

We start with a third (or higher) order differential operator $P(x, \eta^{-1}d/dx)$ with a large parameter η. Let us consider the situation where a Stokes curve emanating from a turning point s_1 hits another simple turning point s_2. We further suppose that the Stokes curve is of type $(1, 2)$, i.e., an integral curve of the direction field

$$\text{Im}(\xi_1(x) - \xi_2(x))dx = 0, \tag{1}$$

and that $x = s_2$ satisfies

$$\xi_2(x) = \xi_3(x), \tag{2}$$

where $\xi_j(x)$ $(j = 1, 2, 3)$ are mutually distinct solutions of the characteristic equation $P(x, \xi) = 0$. Here we have used the assumption that P is of third (or higher) order; in the case of operators of the second order, this situation cannot be observed. Now, since $x = s_2$ is a simple turning point where $\xi_2(x)$ and $\xi_3(x)$ merge by the assumption (2), $\xi_2(x)$ (and $\xi_3(x)$ also) has a square-root type singularity at $x = s_2$ and the Stokes curve bifurcates there (Fig.1).

If the operator P does not contain any parameter other than η, one might be content to regard this bifurcation just as one of the pathologies which analysis of higher order equations presents. Then the reasoning would be stopped there. But, if the operator P depends on an auxiliary parameter t, it is natural to consider how the configuration of Stokes curves changes as the parameter t changes. Then it is more reasonable to take into account the Stokes curve emanating from s_2, in addition to the Stokes curve emanating from s_1. To fix the situation, let us suppose their configurations are those given in Fig.2 (resp., Fig.3) for $t = t_2$ (resp., $t = t_3$). We also suppose that $s_2(t)$ lies on the Stokes curve emanating from $s_1(t)$ when $t = t_1$. In the situation we observe in Fig.2, we know that there exists a virtual turning point $v = v(t)$ such that a Stokes curve of type $(1, 3)$ emanating from v passes through the crossing point of the Stokes curve emanating from s_1 and that from s_2 [8] (cf. Fig.2′). The configuration of these Stokes curves then becomes as described in Fig.3′ when $t = t_3$ in all cases we have examined [9]. Since the Stokes curve emanating from $v(t_1)$ is of type $(1, 3)$, it also bifurcates at $s_2(t_1)$ because of the singularity that $\xi_3(x)$ contains. The resulting configuration is given in Fig.1′. Comparing Figures 1′, 2′ and 3′, one naturally observes that the configuration of Stokes curves continuously changes as the parameter t moves, in spite of the fact that the relative location of the Stokes curve emanating from $v(t)$ and that from $s_1(t)$ is interchanged on the right of their crossing points. Thus we can understand the bifurcation of a Stokes curve to be a natural counterpart of the addition of a Stokes curve emanating from a virtual turning point; it is not an isolated pathology! We refer the reader to [9] for the concrete description of Stokes curves in the example of the Stokes geometry for the quantized Hénon map.

We now show how the mechanism described above is related to the paradoxical situation which one of us (S.S.) has recently found [7] in the computer-assisted study of the Stokes geometry of the Painlevé hierarchy of Noumi-Yamada type [10]; its first member, with which we are concerned in this paper, consists of the following symmetric form of the fourth Painlevé equation [11]

$$\eta^{-1}\frac{df_j}{dt} = f_j(f_{j+1} - f_{j+2}) + \alpha_j \quad (j = 0, 1, 2) \tag{3}$$

with $f_j = f_{j-3}$ $(j = 3, 4)$ and $\alpha_0 + \alpha_1 + \alpha_2 = \eta^{-1}$, and its underlying "Schrödinger" equation

$$-\eta^{-1}x\frac{\partial}{\partial x}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}(2\alpha_1+\alpha_2)/3 & f_1 & 1\\ x & (-\alpha_1+\alpha_2)/3 & f_2\\ xf_0 & x & -(\alpha_1+2\alpha_2)/3\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix} \tag{4}$$

together with its deformation equation

$$\eta^{-1}\frac{\partial}{\partial t}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}f_2 - \frac{t}{2} & -1 & 0\\ 0 & f_0 - \frac{t}{2} & -1\\ -x & 0 & f_1 - \frac{t}{2}\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}. \tag{5}$$

As is well-known, the equation (3) is nothing but the compatibility condition of the equations (4) and (5). As the equation (3) is equivalent to the fourth Painlevé equation, we can use more traditional pair of the "Schrödinger" equation and its deformation equation [12] so that their compatibility condition is equivalent to (3). Actually in the traditional case the "Schrödinger" equation is of the second order. Now, one of the results of [13] asserts that, if the parameter t lies on the Stokes curve of (3) (to be more precise, the appropriate linearization of (3); see [13] for the details), two turning points of the "Schrödinger" equation are connected by a Stokes curve of the "Schrödinger" equation. On the other hand, a computer-assisted study of the Stokes geometry of the equations (3) and (4) shows the following intriguing fact [7] (here we have chosen $\alpha_0 = 1.0 + 0.6i$ and $\alpha_1 = 0.2 - 0.1i$):

If the parameter t is on the Stokes curve of (3) and if it is sufficiently close to its origin, i.e., a turning point of (3), then a double turning point d and a simple turning point s_1 of equation (4) are connected by a Stokes curve of (4) (Fig.5; here we have included another simple turning point s_2 for the later reference). However, when t lies in some portion, say σ, of the same Stokes curve which is far away from the turning point of (3), no pair of turning points of equation (4) are connected by a Stokes curve of (4), unless virtual turning points are counted as turning points (Fig.6; we have also included the simple turning point s_2 in this figure).

One might be puzzled by the apparent contradiction between the above quoted result of [13] and the latter half part of the observation of [7], namely, the fact that no pair of turning points of (4) is connected by a Stokes curve of (4) for the parameter t in the portion σ of a Stokes curve of (3). As we see below, this paradox is resolved in a natural manner if a virtual turning point is counted as a turning point.

Let us first note, by comparing Fig.5 and Fig.6, that the Stokes curve connecting turning points d and s_1 should hit the turning point s_2 as the parameter t moves from the portion close to the turning point of (3) (i.e., generating Fig.5) to another portion σ of the Stokes curve of (3) that is far away from the turning point (i.e., generating Fig.6). Hence it is reasonable to surmise that bifurcation of a Stokes curve should occur in the course of the journey of t from a point, say t_5, giving rise to the configuration of Stokes curves of (4) in Fig.5, and to another point, say t_6, giving rise to Fig.6. As we know that the bifurcation phenomenon is a counterpart of the addition of Stokes curves emanating from virtual turning points, we include virtual turning points in Fig.5 to find Fig.5′. Since we have two kinds of crossing points of Stokes curves, i.e., the crossing of the Stokes curve emanating from s_1 and that from s_2, and the crossing of the Stokes curve from s_2 and that from d, we write in two virtual turning points v_1 and v_2. (We have omitted some other virtual turning points which are not of our immediate concern.) In parenthesis, an interesting fact worth mentioning is that v_1 and v_2 are connected by a Stokes curve. As t moves from t_5 to t_6, we should encounter the configuration of Stokes curves given in Fig.4. In Fig.4 the virtual turning point v_1 is connected with both the double turning point d and the virtual turning point v_2 thanks to the bifurcation of the Stokes curve emanating from v_1, and similarly the virtual turning point v_2 is connected with both the simple turning point s_1 and the virtual turning point v_1. When t moves further to reach t_6, the interchange of relative location of the Stokes curve emanating from v_1 and that from s_1 on the left of their crossing point switches the target of the Stokes curve (emanating from v_1) from v_2 to the double turning point d. In parallel with this, the target of the Stokes curve emanating from v_2 becomes s_1, not v_1. Thus we obtain Fig.6′, where s_1 (resp., d) in Fig.5 is superseded by v_1 (resp., v_2), that is, virtual turning points v_1 and v_2 are respectively connected by a Stokes curve with ordinary turning points d and s_1 but d and s_1 are not connected. Thus we clearly see that a "virtual" turning point is really a "real" object even though not "ordinary". (From the viewpoint of the topological complexity, Fig.5′ corresponds to Fig.3′ and Fig.6′ corresponds to Fig.2′. Hence it might be better, logically speaking, to arrange our argument so that we may start from $t = t_6$ and reach $t = t_5$. Here we have arranged the materials so that we may start with a "usual"situation and end up with an "unusual" situation with the change of the parameter.)

In conclusion, we emphasize that virtual turning points and (ordinary) turning points play equal roles in Fig.6′, validating **Assertion A**.

In ending this paper we note that the geometric study given here strongly indicates that connection formula for the wave function $\psi = {}^t(\psi_0, \psi_1, \psi_2)$ (i.e., a solution of (4)) across a Stokes curve emanating from a virtual turning point should be relevant to the connection formula for the Painlevé transcendents. It should be an important and interesting problem to study in general how the analytic structure of the wave function near a Stokes curve emanating from a virtual turning point is related to the connection formula for the novel transcendents that appear as solutions of a higher member in the Noumi-Yamada hierarchy, i.e., the so-called higher order fourth Painlevé equation.

Acknowledgment: The research of the authors has been supported in part by JSPS Grant-in-Aid No.14340042, No.14077213, No.15540190 and No.16540148.

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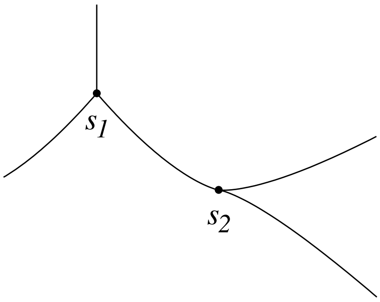

Figure 1 : Bifurcation of a Stokes curve.

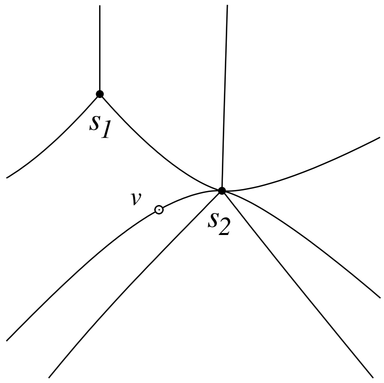

Figure 1′: Figure 1 with a virtual turning point added.

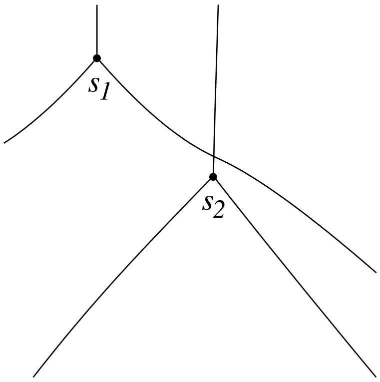

Figure 2 :Configuration of Stokes curves for $t = t_2$.

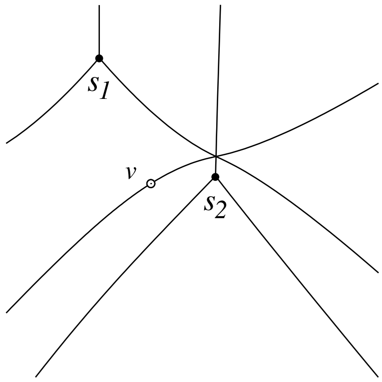

Figure 2′: Figure 2 with a virtual turning point added.

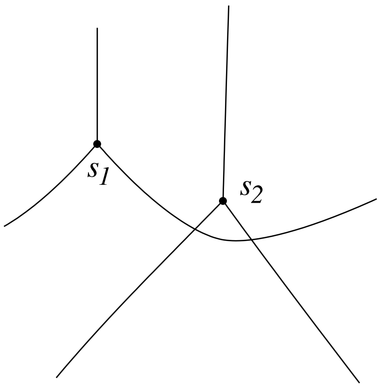

Figure 3 : Configuration of Stokes curves for $t = t_3$.

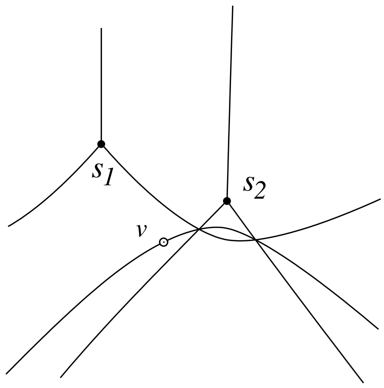

Figure 3′: Figure 3 with a virtual turning point added.

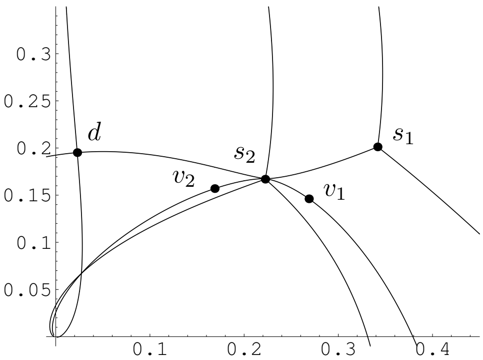

Figure 4 : Configuration of Stokes curves of (4) when s_2 meets other Stokes curves.

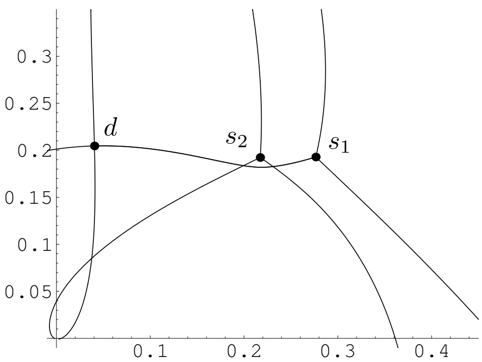

Figure 5 : Configuration of Stokes curves of (4) for $t = t_5 = -1.6104 - 0.2268i$.

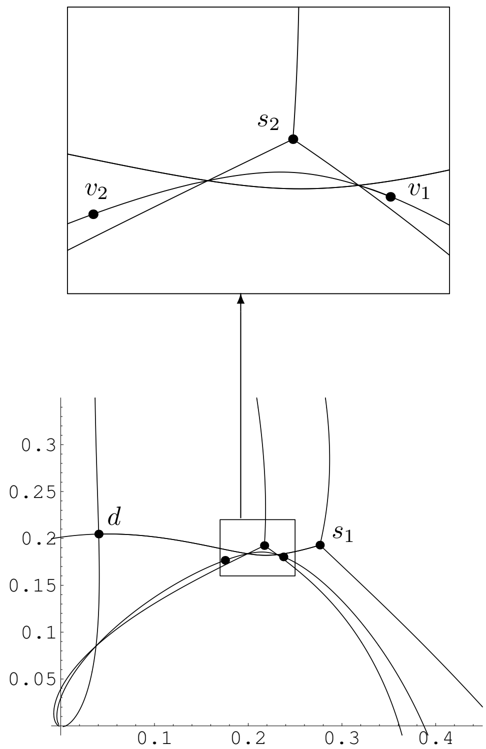

Figure 5′: Figure 5 with virtual turning points added.

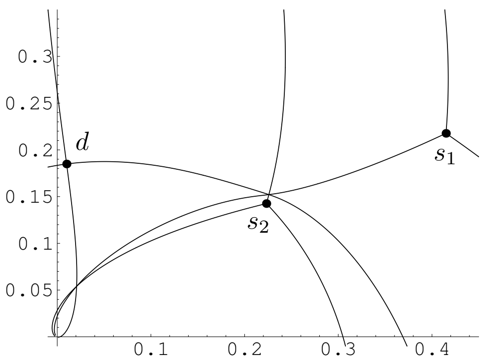

Figure 6 : Configuration of Stokes curves of (4) for $t = t_6 = -1.5783 - 0.4130i$.

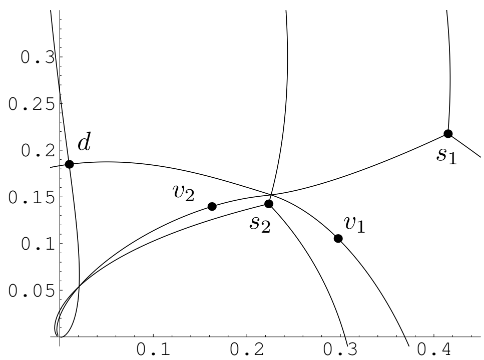

Figure 6′: Figure 6 with virtual turning points added.