Irreducible representations of the Cuntz algebras arising from polynomial embeddings

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For an embedding φ of the Cuntz algebra $\mathcal{O}_M$ into $\mathcal{O}_N$, $\mathcal{O}_M$ is identified with a subalgebra $\varphi(\mathcal{O}_M)$ of $\mathcal{O}_N$. We construct irreducible representations of $\mathcal{O}_N$ with continuous parameters by extending irreducible representations of $\mathcal{O}_M$. They are not unitarily equivalent to any generalized permutative representation, especially not to any permutative representation by Bratteli-Jorgensen and Davidson-Pitts. We show their unitary equivalence by parameters and give another characterization for them by states or eigenequations of cyclic vectors without the information of the embedding.

1. Main theorem

In general, representations of C*-algebras do not have unique decomposition(up to unitary equivalence) into sums or integrals of irreducibles. However, the permutative representations of the Cuntz algebra $\mathcal{O}_N$ do ([**2**, **4**, **5**]). We generalized the permutative representations in [**6**, **7**, **8**] by keeping the uniqueness of decomposition. In this paper. we show an essentially new class of representations of $\mathcal{O}_N$ by using generalized permutative(=GP) representations of $\mathcal{O}_M$ and an embedding of $\mathcal{O}_M$ into $\mathcal{O}_N$. In order to introduce new representations, we start to review GP representations.

Let $S(\mathbf{C}^N) \equiv \{z \in \mathbf{C}^N : \|z\| = 1\}$, $S(\mathbf{C}^N)^{\otimes k} \equiv \{z^{(1)} \otimes \cdots \otimes z^{(k)} : z^{(i)} \in S(\mathbf{C}^N),\, i = 1, \ldots, k\}$ for $k \geq 1$ and $S(\mathbf{C}^N)^{\infty} \equiv \{(z^{(n)})_{n\in\mathbf{N}} : z^{(n)} \in S(\mathbf{C}^N),\, n \in \mathbf{N}\}$. Let $s_1, \ldots, s_N$ be canonical generators of $\mathcal{O}_N$. For $z = (z_i)_{i=1}^N \in \mathbf{C}^N$, define $s(z) \equiv z_1 s_1 + \cdots + z_N s_N$. Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$. For $z = z^{(1)} \otimes \cdots \otimes z^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$, $(\mathcal{H}, \pi)$ is $GP(z)$ of $\mathcal{O}_N$ if there is a cyclic vector $\Omega \in \mathcal{H}$ such that $\pi(s(z))\Omega = \Omega$ where $s(z) \equiv s(z^{(1)}) \cdots s(z^{(k)})$. For $z = (z^{(n)})_{n\in\mathbf{N}} \in S(\mathbf{C}^N)^{\infty}$, $(\mathcal{H}, \pi)$ is $GP(z)$ of $\mathcal{O}_N$ if there is a unit cyclic vector $\Omega \in \mathcal{H}$ such that $\{\pi(s(z^{(n)})^* \cdots s(z^{(1)})^*)\Omega : n \in \mathbf{N}\}$ is an orthonormal family in $\mathcal{H}$. For both cases, we call Ω by the *GP vector* of $(\mathcal{H}, \pi)$. We call $GP(z)$ by the *GP representation of* $\mathcal{O}_N$ *by* z. If $z \in S(\mathbf{C}^N)^{\otimes k}$ is *non periodic*,

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that is, there is no $y \in S(\mathbf{C}^N)^{\otimes l}$ such that z equals to the tensor power $y^{\otimes p}$ of y for some $p \geq 2$, then $GP(z)$ exists uniquely up to unitary equivalence. $GP(z)$ is irreducible if and only if z is non periodic. If both $z \in S(\mathbf{C}^N)^{\otimes k}$ and $y \in S(\mathbf{C}^N)^{\otimes l}$ are non periodic, then $GP(z) \sim GP(y)$ if and only if $l = k$ and $z = y^{(\sigma(1))} \otimes \cdots \otimes y^{(\sigma(k))}$ for some $\sigma \in \mathbf{Z}_k$ where $\sim$ means unitary equivalence. For each $z \in S(\mathbf{C}^N)^\infty$, $GP(z)$ exists uniquely up to unitary equivalence. Any cyclic permutative representation is a GP representation.

Abe([**1**]) constructed a new representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_2$ with a cyclic vector $\Omega \in \mathcal{H}$ which satisfies

$$\frac{1}{\sqrt{2}}\pi(s_2 s_1 + s_1)\Omega = \Omega. \tag{1.1}$$

By generalizing Abe's example, we obtain a large class of new representations of $\mathcal{O}_N$ and show their properties from GP representations of $\mathcal{O}_M$ when $M = (N-1)k + 1$ for $k \geq 2$.

Let $s_1, \ldots, s_N$ and $t_1, \ldots, t_M$ be canonical generators of $\mathcal{O}_N$ and $\mathcal{O}_M$, respectively. Define an embedding φ of $\mathcal{O}_M$ into $\mathcal{O}_N$ by

$$\begin{cases} \varphi(t_{(N-1)(l-1)+i}) & \equiv s_N^{l-1} s_i \qquad (i = 1, \ldots, N-1,\, l = 1, \ldots, k), \\ \varphi(t_M) & \equiv s_N^k \end{cases} \tag{1.2}$$

where we denote $s_N^0 \equiv I$ for convenience. We identify $\varphi(t_i)$ and t_i.

Theorem 1.1. *For $z = (z_i)_{i=1}^M \in S(\mathbf{C}^M)$ with $|z_M| < 1$, assume that $(\mathcal{H}, \pi_0)$ is $GP(z)$ of $\mathcal{O}_M$. Then the following holds:*

(i) *There exists unique representation π of $\mathcal{O}_N$ on $\mathcal{H}$ such that $\pi \circ \varphi = \pi_0$ with respect to φ in (1.2).*

(ii) *Any representation $(\mathcal{H}', \pi')$ of $\mathcal{O}_N$ with a cyclic vector Ω' which satisfies*

$$\pi'(s(\hat{z}))\Omega' = \Omega', \quad \text{where } s(\hat{z}) \equiv z_1 t_1 + \cdots + z_M t_M \in \mathcal{O}_N,$$

is unitarily equivalent to $(\mathcal{H}, \pi)$ in (i) We denote such representation by $GP(\hat{z})$ and Ω' is called the GP vector of $(\mathcal{H}', \pi')$.

(iii) *$GP(\hat{z})$ is irreducible.*

(iv) *For $w = (w_i)_{i=1}^M \in S(\mathbf{C}^M)$ with $|w_M| < 1$, $GP(\hat{z}) \sim GP(\hat{w})$ if and only if $z = w$.*

By Theorem 1.1, the symbol $GP(\hat{z})$ makes sense as an equivalence class of representations of $\mathcal{O}_N$. The following theorem shows that $GP(\hat{z})$ is essentially new as an equivalence class.

Theorem 1.2. *Let $\tilde{\varphi} : S(\mathbf{C}^N) \hookrightarrow S(\mathbf{C}^M)$; $\tilde{\varphi}(y) \equiv (\tilde{\varphi}_i(y))_{i=1}^M$ by*

$$\tilde{\varphi}_{(N-1)(l-1)+j}(y) \equiv y_N^{l-1} y_j, \quad \tilde{\varphi}_M(y) \equiv y_N^k \quad (y = (y_i)_{i=1}^N \in S(\mathbf{C}^N)) \tag{1.3}$$

for $j=1,\ldots,N-1$ *and* $l=1,\ldots,k$*. If* $z\in S(\mathbf{C}^M)$ *satisfies* $|z_M|<1$*, then the following holds:*

(i) *If* $z\in\tilde{\varphi}(S(\mathbf{C}^N))$*, then* $GP(\hat{z})\sim GP(\tilde{\varphi}^{-1}(z))$*.*

(ii) *If* $\|z(l)\|=1$ *for some* $l\in\{1,\ldots,k\}$*, then* $GP(\hat{z})$ *is the GP representation by* $(0,\ldots,0,1)^{\otimes(l-1)}\otimes z(l)\in S(\mathbf{C}^N)^{\otimes l}$ *where* $\{z(l)\}_{l=1}^k\subset\mathbf{C}^N$ *is defined by* $z(l)\equiv(z_{(N-1)(l-1)+1},\ldots,z_{(N-1)l},0)$ *for* $l=1,\ldots,k-1$ *and* $z(k)\equiv(z_{(N-1)(k-1)+1},\ldots,z_{(N-1)k+1})$*.*

(iii) *If* $z\notin\tilde{\varphi}(S(\mathbf{C}^N))$ *and* $\|z(l)\|<1$ *for each* $l=1,\ldots,k$*, then* $GP(\hat{z})\not\sim GP(y)$ *for any* $y\in S(\mathbf{C}^N)^\infty\cup\bigcup_{k\ge1}S(\mathbf{C}^N)^{\otimes k}$*. Furthermore* $GP(\hat{z})$ *is not equivalent to any permutative representation.*

Consider the case $(N,M,k)=(2,3,2)$ for Theorem 1.1. Then the embedding in (1.2) is as follows: $\varphi:\mathcal{O}_3\hookrightarrow\mathcal{O}_2;\quad \varphi(t_1)\equiv s_1,\quad \varphi(t_2)\equiv s_2s_1,\quad \varphi(t_3)\equiv s_2s_2$. Abe's example in (1.1) is just $GP(\hat{z})$ when $z=(2^{-1/2},2^{-1/2},0)\in S(\mathbf{C}^3)$. Therefore it is irreducible and unique up to unitary equivalence. Because $(2^{-1/2},2^{-1/2},0)\notin\{(y_1,y_2y_1,y_2^2)\in S(\mathbf{C}^3):(y_1,y_2)\in S(\mathbf{C}^2)\}$, Abe's example is not unitarily equivalent to any permutative representation.

In § 2, we prove Theorem 1.1 and Theorem 1.2. In § 3, we show another characterization of new representations by extensions of representations, states and basis. In § 4, we show examples.

2. Proof of the main theorem

In this paper, any representation and embedding are unital and $*$-preserving. For $N\ge2$, let $\mathcal{O}_N$ be the Cuntz algebra([**3**]), that is, it is a C*-algebra which is universally generated by generators $s_1,\ldots,s_N$ satisfying $s_i^*s_j=\delta_{ij}I$ for $i,j=1,\ldots,N$ and $s_1s_1^*+\cdots+s_Ns_N^*=I$. In this section, we assume that $M=(N-1)k+1$ for $k\ge2$ and $t_1,\ldots,t_M$ are canonical generators of $\mathcal{O}_M$. For φ in (1.2), we identify $\varphi(t_i)$ and t_i.

Proof of Theorem 1.1.(i) We identify $\pi_0(t_i)$ and t_i. Define operators $s_1,\ldots,s_N$ on $\mathcal{H}$ by

$$(2.1)\qquad \begin{cases} s_i\equiv t_i,\quad s_Nt_{(N-1)(k-1)+i}v\equiv t_Mt_iv\quad (i=1,\ldots,N-1),\\[2ex] s_Nt_jv\equiv t_{j+N-1}v\quad (j=1,\ldots,M-N),\\[2ex] s_Nt_Mv\equiv t_Ms_Nv,\quad s_N\Omega\equiv(I-z_Mt_M)^{-1}Y\Omega \end{cases}$$

for $v\in\mathcal{H}$ where $Y\equiv\sum_{j=1}^{M-N}z_jt_{j+N-1}+\sum_{j=1}^{N-1}z_{(N-1)(k-1)+j}t_Mt_j$. Then $t_{(N-1)(l-1)+i}=s_N^{l-1}s_i$ for $i=1,\ldots,N-1$, $l=1,\ldots,k$ and $t_M=s_N^k$. From these, we can verify that $s_1,\ldots,s_N$ satisfy the relations of canonical generators of $\mathcal{O}_N$. This gives just the representation π of $\mathcal{O}_N$ in the statement. Hence the existence is shown. If $\pi^{'}$ is a representation of $\mathcal{O}_N$ on $\mathcal{H}$ which

satisfies $\pi^{'}\circ\varphi=\pi_0$, then $t_i^{'}\equiv\pi^{'}(t_i)$ satisfies (2.1). This implies $\pi=\pi^{'}$. Hence the uniqueness is shown.
(ii) If $(\mathcal{H}^{'},\pi^{'})$ and $\Omega^{'}$ are in the assumption, then $(\mathcal{H}^{'},\pi^{'}|_{\mathcal{O}_M})$ is $GP(z)$ of $\mathcal{O}_M$ with the GP vector $\Omega^{'}$ because $s(\hat{z})=\sum_{j=1}^M z_jt_j=t(z)$. Hence $(\mathcal{H}^{'},\pi^{'}|_{\mathcal{O}_M})$ is $GP(z)$ of $\mathcal{O}_M$. Therefore $(\mathcal{H}^{'},\pi^{'}|_{\mathcal{O}_M})\sim(\mathcal{H},\pi_0)$. By (i), $(\mathcal{H}^{'},\pi^{'})$ satisfies (2.1) with respect to $\pi^{'}(t_i)$, $\Omega^{'}$ and $v\in\mathcal{H}^{'}$. The unitary which gives the unitary equivalence among $(\mathcal{H}^{'},\pi^{'}|_{\mathcal{O}_M})$ and $(\mathcal{H},\pi_0)$ implies that among $(\mathcal{H}^{'},\pi^{'})$ and $(\mathcal{H},\pi)$. Hence the statement holds.
(iii) It is sufficient to show only the irreducibility of $(\mathcal{H},\pi)$. Because $(\mathcal{H},\pi|_{\mathcal{O}_M})$ is $GP(z)$ and $GP(z)$ is irreducible, $(\mathcal{H},\pi|_{\mathcal{O}_M})$ is irreducible. Since $\mathcal{O}_M\subset\mathcal{O}_N$, $(\mathcal{H},\pi)$ is also irreducible.
(iv) If $GP(\hat{z})\sim GP(\hat{w})$, then $GP(z)=GP(\hat{z})|_{\mathcal{O}_M}\sim GP(\hat{w})|_{\mathcal{O}_M}=GP(w)$. This implies that $z=w$. The inverse direction holds by (i). □

In order to show Theorem 1.2, we prepare some lemmata.

Lemma 2.1. *For $z=(z_i)_{i=1}^M\in S(\mathbf{C}^M)$ and $y=y^{(1)}\otimes\cdots\otimes y^{(L)}\in S(\mathbf{C}^N)^{\otimes L}$, assume that $(\mathcal{H},\pi)$ is a representation of $\mathcal{O}_N$ and there are $\Omega,\Omega^{'}\in\mathcal{H}$ such that $\pi(s(\hat{z}))\Omega=\Omega$ and $\pi(s(y))\Omega^{'}=\Omega^{'}$. Let $\rho_i\equiv<\Omega|v_i>$ for $v_i\equiv\pi(s(y^{(i)})\cdots s(y^{(L)}))\Omega^{'}$ for $i=1,\ldots,L$. We denote $\rho_{Lm+i}\equiv\rho_i$ for each $m\geq 1$. Then the following holds for each $i=1,\ldots,L$:*

- (i) $|\rho_i|^2\leq\sum_{l=1}^k|\rho_{i+l}|^2\|z(l)\|^2$ *for $i=1,\ldots,L$. If $\rho_{i+k}\neq 0$ and the equality holds, then $z(k)$ and $y^{(i+k-1)}$ are linearly dependent.*
- (ii) *If $L\geq 2$, $z\in S_0(\mathbf{C}^M)\equiv\{z\in S(\mathbf{C}^M):\forall l,\ \|z(l)\|<1\}$ and y is non periodic, then $|\rho_1|=\cdots=|\rho_L|=0$.*
- (iii) *If $L=1$ and $z\in S_0(\mathbf{C}^M)$, then there is $\tau(z,y)\in\mathbf{C}$ such that $t(z)^*\Omega^{'}=\tau(z,y)\cdot\Omega^{'}$ and $|\tau(z,y)|\leq 1$. $\tau(z,y)=1$ if and only if $z=\tilde{\varphi}(y)$ for $\tilde{\varphi}$ in (1.3).*

Proof. For $z\in S(\mathbf{C}^M)$, define $t(z;l)\equiv\sum_{j=(N-1)(l-1)+1}^{(N-1)l}z_jt_j$ for $l=1,\ldots,k-1$ and $t(z;k)\equiv\sum_{j=(N-1)(k-1)+1}^{M}z_jt_j$. Then $t(z)=\sum_{l=1}^k t(z;l)$.
(i) Define $Y_{i,1}\equiv 1$ and $Y_{i,l}\equiv y_N^{(i)}\cdots y_N^{(i+l-2)}$ for $l=2,\ldots,k$. Then $t_i^*v_j=\overline{z_i}y_i^{(j)}v_{j+1}$, $t^*_{(N-1)(l-1)+i}v_j=Y_{j,l}\cdot y_i^{(j+l-1)}\ \overline{z_{(N-1)(l-1)+i}}v_{j+l}$. Define $Q_{i,l}:\mathbf{C}\to\mathbf{C}$; $Q_{i,l}(c)\equiv\rho_{i+l}\cdot<z(l)|y^{(i+l-1)}>+c\cdot y_N^{(i+l-1)}$, $R_{i,l}\equiv(Q_{i,l}\circ\cdots\circ Q_{i,k-1})(\rho_{i+k}<z(k)|y^{(i+k-1)}>)$ for $l=1,\ldots,k-1$ and $R_{i,k}\equiv\rho_{i+k}<z(k)|y^{(i+k-1)}>$. Then $R_{i,1}=\sum_{l=1}^k Y_{i+1,l}\cdot<z(l)|y^{(i+l-1)}>\cdot\rho_{i+l}$. Because $<t(z;l)\Omega|v_i>=Y_{i,l}<z(l)|y^{(i+l-1)}>\rho_{i+l}$ for each $l=1,\ldots,k$, $\rho_i=R_{i,1}$. Since $|R_{i,l}|^2\leq|\rho_{i+l}|^2\|z(l)\|^2+|R_{i,l+1}|^2$ for $l=1,\ldots,k-1$,

$$|\rho_i|^2=|R_{i,1}|^2\leq\sum_{l=1}^{k-1}|\rho_{i+l}|^2\|z(l)\|^2+|R_{i,k}|^2\leq\sum_{l=1}^k|\rho_{i+l}|^2\|z(l)\|^2.$$

From this, the first statement holds. If the equality holds, then the statement follows by the Schwarz inequality of the term $|R_{i,k}|^2$.
(ii) Because (i) and $\|z(l)\| < 1$ for each $l = 1, \ldots, k$, $|\rho_1| = \cdots = |\rho_L|$. Put $\alpha \equiv |\rho_1|^2 = \cdots = |\rho_L|^2$. If $\alpha \neq 0$, then there are $c_1, \ldots, c_L \in \mathbf{C} \setminus \{0\}$ such that $y^{(i+k-1)} = c_i \cdot z(k)$ for each $i = 1, \ldots, L$ by (i). Because $y = c \cdot (z(k))^{\otimes L}$ for some $c \in \mathbf{C}$, y is periodic. This contradicts with the assumption of y. Therefore $\alpha = 0$.
(iii) Define $C_l : \mathbf{C} \to \mathbf{C};\ \ C_l(c) \equiv < z(l)|y > + cy_N$ and $D_l \equiv (C_l \circ \cdots \circ C_{k-1})(< z(k)|y >)$ for $l = 1, \ldots, k-1$ and $D_k \equiv < z(k)|y >$. For each $l \in \{1, \ldots, k\}$, $t(z;l)^* s(y)^l = y_N^{l-1} < z(l)|y > I$. Hence $t(z)^*\Omega' = \sum_{l=1}^k t(z;l)^* s(y)^l \Omega'$ Hence $\tau(z,y) = \sum_{l=1}^k y_N^{l-1} < z(l)|y >$. Furthermore we see that $\tau(z,y) = D_1$. Because $|D_l|^2 \leq \|z(l)\|^2 + |D_{l+1}|^2$ for each $l = 1, \ldots, k-1$, $|\tau(z,y)|^2 = |D_1|^2 \leq \|z(1)\|^2 + \cdots + \|z(k-1)\|^2 + |D_k|^2 \leq 1$. If $\tau(z,y) = 1$, then $|D_l|^2 = \|z(l)\|^2 + |D_{l+1}|^2$. Hence there are $c_2, \ldots, c_k \in \mathbf{C}$ such that $c_l y = z(l) + (0, \ldots, 0, \overline{D_{l+1}})$ for $l = 2, \ldots, k-1$ and $c_k y = z(k)$. From these, we have $c_l = y_N^{l-1}$ for $l = 2, \ldots, k$. Hence $z = \tilde{\varphi}(y)$. On the other hand, if $z = \tilde{\varphi}(y)$, then we see that $\tau(z,y) = 1$. □

Lemma 2.2. *Let $z = (z_i)_{i=1}^M \in S(\mathbf{C}^M)$ and $y = (y^{(n)})_{n \in \mathbf{N}} \in S(\mathbf{C}^N)^\infty$ and let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$ with unit vectors Ω and Ω' such that $t(z)\Omega = \Omega$ and $\{v_n\}_{n \in \mathbf{N}}$ is an orthonormal family in $\mathcal{H}$ where $v_n \equiv s(y^{(n)})^* \cdots s(y^{(1)})^* \Omega'$ for $n \in \mathbf{N}$. Then $< \Omega | v_n >= 0$ for each $n \in \mathbf{N}$.*

Proof. Let $\rho_n \equiv < \Omega | v_n >$. By using the notation in the poof of Lemma 2.1, we see that $(Q_{n,1} \circ \cdots \circ Q_{n,k-1})(< z(k)|y^{(n+k-1)} > \rho_{n+k})$
$= < z(1)|y^{(n+1)} > \rho_{n+1} + \sum_{l=2}^k y_N^{(n+1)} \cdots y_N^{(n+l-1)} \cdot < z(l)|y^{(n+l-1)} > \cdot \rho_{n+l}$
and $\rho_n = < s(z(1))\Omega | v_n > + \sum_{l=2}^k y_N^{(n+1)} \cdots y_N^{(n+l-1)} \cdot < s(z(l))\Omega | v_{n+l-1} >$. By comparing each term, we have $\rho_n = (Q_{n,1} \circ \cdots \circ Q_{n,k-1})(< z(k)|y^{(n+k-1)} > \rho_{n+k})$. By this and the proof of Lemma 2.1 (i), we obtain

$$|\rho_n|^2 \leq \sum_{l=1}^k \|z(l)\|^2 |\rho_{n+l}|^2 \leq \max\{|\rho_{n+l}|^2 : l = 1, \ldots, k\} \leq 1.$$

From this, for any $n \in \mathbf{N}$, there is $m_n \geq n+1$ such that $|\rho_n| \leq |\rho_{m_n}|$. Hence for any $n \in \mathbf{N}$, there is a sequence $\{m_{n,i}\}_{i \in \mathbf{N}}$ such that $n = m_{n,1}$ and $\{|\rho_{m_{n,i}}|\}_{i \in \mathbf{N}}$ is monotone increasing. Because $\Omega = \sum_n < v_n | \Omega > v_n + w$ for a vector $w \in \mathcal{H}$ which satisfies $< v_n | w >= 0$ for each $n \in \mathbf{N}$, $1 = \|\Omega\|^2 = \sum_n |\rho_n|^2 + < \Omega | w >$. Hence $|\rho_n| \leq \lim_{i \to \infty} |\rho_{m_{n,i}}| = 0$. This implies the statement. □

Define $\{1, \ldots, N\}^k \equiv \{(j_i)_{i=1}^k : j_1, \ldots, j_k \in \{1, \ldots, N\}\}$ for $k \geq 1$, $\{1, \ldots, N\}^0 \equiv \{0\}$ and $\{1, \ldots, N\}^* \equiv \bigcup_{k \geq 0} \{1, \ldots, N\}^k$. For $J \in \{1, \ldots, N\}^*$, $|J| \equiv k$ if $J \in \{1, \ldots, N\}^k$. Denote $s_J \equiv s_{j_1} \cdots s_{j_k}$, $s_J^* \equiv (s_J)^*$ for $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$.

Proof of Theorem 1.2.We denote $\pi(s_i)$ by s_i simply.
(i) Let $(\mathcal{H},\pi)$ be $GP(y)$ of $\mathcal{O}_N$ with the GP vector Ω for $y \equiv \tilde{\varphi}^{-1}(z)$. Then $s(\hat{z})\Omega = t(z)\Omega = \Omega$. Therefore Ω is the GP vector of $GP(\hat{z})$ and Ω is cyclic for $(\mathcal{H},\pi)$. This implies that $(\mathcal{H},\pi)$ is $GP(\hat{z})$. Hence the statement holds.
(ii) This follows by definition of $z(l)$ and GP representation.
(iii) For $y \in S(\mathbf{C}^N)^{\otimes L}$ with $L \geq 1$, assume that $GP(y) \sim GP(\hat{z})$. Then there is a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ with two cyclic vectors $\Omega,\Omega^{'}$ which satisfy $s(\hat{z})\Omega = \Omega$ and $s(y)\Omega^{'} = \Omega^{'}$. By Theorem 1.1 (iii), $GP(\hat{z})$ is irreducible. Hence $GP(y)$ must be also irreducible. From this, y must be non periodic. If $L \geq 2$, then $<\Omega|v_j>= 0$ by Lemma 2.1 (ii). Because $s_K^* s_J v_j \in \mathbf{C}v_{j+|K|-|J|}$ when $|K| \geq |J|$ and the lowest degree of $(t(z)^*)^n$ with respect to $s_1,\ldots,s_N$ is n at least, $(t(z)^*)^{|J|}s_J v_i = \sum_{j=1}^{L} c_{J,i,j}v_j$. Then $<\Omega|s_J v_i> = <\Omega|(t(z)^*)^{|J|}s_J v_i> = \sum_{j=1}^{L} c_{J,i,j} <\Omega|v_j> = 0$ for each $J \in \{1,\ldots,N\}^*$ and $i = 1,\ldots,L$. By the condition of $\Omega^{'}$, we see that $\mathrm{Lin}<\{s_J v_i : J \in \{1,\ldots,N\}^*, i = 1,\ldots,L\}>$ is dense in $\mathcal{H}$. Therefore $\Omega = 0$. This is contradiction. Hence $GP(y) \not\sim GP(\hat{z})$. If $L = 1$, then $<\Omega|\Omega^{'}> = <t(z)\Omega|\Omega^{'}> = \tau(z,y) <\Omega|\Omega^{'}>$ by Lemma 2.1 (iii). Because $z \notin \tilde{\varphi}(S(\mathbf{C}^N))$ and Lemma 2.1 (iii), $<\Omega|\Omega^{'}> = 0$. In the same way as the case $L \geq 2$, we have $\Omega = 0$. This is contradiction. Hence $GP(y) \not\sim GP(\hat{z})$.

Assume that $y \in S(\mathbf{C}^N)^{\infty}$ and $GP(y) \sim GP(\hat{z})$. Then there is a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ with cyclic vectors Ω and $\Omega^{'}$ such that $t(z)\Omega = \Omega$ and $\{s(y^{(n)})^* \cdots s(y^{(1)})^*\Omega^{'}\}_{n\in\mathbf{N}}$ is an orthonormal family of $\mathcal{H}$. By Lemma 2.2 and the same way as the proof of the case $y \in S(\mathbf{C}^N)^{\otimes L}$, we obtain $\Omega = 0$. This is contradiction. Hence $GP(y) \not\sim GP(\hat{z})$. In consequence, the statement holds. $\square$

3. Another characterization of $GP(\hat{z})$

3.1. Extension of representation. We introduce a notion of extension of representation with respect to a homomorphism among C*-algebras. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a unital $*$-homomorphism among two unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, and $(\mathcal{H}_0,\pi_0)$ be a representation of $\mathcal{A}$. $(\mathcal{H},\pi)$ is an *extension of* $(\mathcal{H}_0,\pi_0)$ *with respect to* φ if $(\mathcal{H},\pi)$ is a representation of $\mathcal{B}$ such that $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$ and $\pi \circ \varphi = \pi_0$. We denote the set of all extensions of $(\mathcal{H}_0,\pi_0)$ with respect to φ by $\mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$. $(\mathcal{H},\pi) \in \mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$ is *irreducible(resp. cyclic)* if $(\mathcal{H},\pi)$ is irreducible(resp. cyclic). $(\mathcal{H},\pi) \in \mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$ is *minimal* if $(\mathcal{H},\pi)$ is a subrepresentation of any element in $\mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$. $(\mathcal{H},\pi) \in \mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$ is *nonincreasing* if $\mathcal{H} = \mathcal{H}_0$. If $(\mathcal{H},\pi) \in \mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$ is irreducible, then $(\mathcal{H},\pi)$ is cyclic. If $(\mathcal{H}_0,\pi_0)$ is irreducible, $(\mathcal{H},\pi) \in \mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$ is nonincreasing and φ is injective, then $(\mathcal{H},\pi)$ is irreducible.

Proposition 3.1. *Assume that* $\mathcal{O}_M$ *is embedded into* $\mathcal{O}_N$ *by* φ *in* (1.2)*. Let* $(\mathcal{H},\pi)$ *be a representation of* $\mathcal{O}_N$ *and* $z=(z_i)_{i=1}^M\in S(\mathbf{C}^M)$ *with* $|z_M|<1$*. Then the following are equivalent:*

(i) *There is a cyclic vector* $\Omega\in\mathcal{H}$ *such that* $\pi(s(\hat{z}))\Omega=\Omega$*.*
(ii) $(\mathcal{H},\pi)$ *is an irreducible extension of* $GP(z)$ *of* $\mathcal{O}_M$ *with respect to* φ*.*
(iii) $(\mathcal{H},\pi)$ *is a cyclic extension of* $GP(z)$ *of* $\mathcal{O}_M$ *with respect to* φ *with a common cyclic vector for* $GP(z)$*.*
(iv) $(\mathcal{H},\pi)$ *is the minimal extension of* $GP(z)$ *of* $\mathcal{O}_M$ *with respect to* φ*.*
(v) $(\mathcal{H},\pi)$ *is a nonincreasing extension of* $GP(z)$ *of* $\mathcal{O}_M$ *with respect to* φ*.*

Proof. If there is a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ such that $(\mathcal{H},\pi\circ\varphi)=GP(z)$, then there is a subrepresentation of $(\mathcal{H},\pi)$ which is $GP(\hat{z})$. By Theorem 1.1 (iii), (i) implies (ii),…, (v).
(ii)⇒ (i): If (ii) holds, then $(\mathcal{H},\pi)$ has a subrepresentation which is equivalent to $GP(\hat{z})$. Because $(\mathcal{H},\pi)$ is irreducible, $(\mathcal{H},\pi)\sim GP(\hat{z})$. Hence (i) holds.
(iii)⇒ (i): If (iii) is assumed, then there is a representation $(\mathcal{H}_0,\pi_0)$ of $\mathcal{O}_M$ such that $(\mathcal{H},\pi)\in\mathcal{E}_\varphi(\mathcal{H}_0,\pi_0)$ and there is a cyclic vector $\Omega\in\mathcal{H}_0$ such that $\overline{\pi(\mathcal{O}_N)\Omega}=\mathcal{H}$ and $\overline{\pi_0(\mathcal{O}_M)\Omega}=\mathcal{H}_0$. If $\Omega_0\in\mathcal{H}_0$ is the GP vector of $GP(z)$ of $\mathcal{O}_M$, then $\Omega_0\in\mathcal{H}$. Hence $\Omega\in\mathcal{H}_0=\overline{\pi(\mathcal{O}_N)\Omega_0}\subset\overline{\pi(\mathcal{O}_N)\Omega_0}$. $\mathcal{H}=\overline{\pi(\mathcal{O}_N)\Omega}\subset\overline{\pi(\mathcal{O}_N)\Omega_0}\subset\mathcal{H}$. Therefore Ω_0 is a cyclic vector of $\mathcal{H}$ and $\pi(s(\hat{z}))\Omega_0=\Omega_0$. Hence (i) follows.
(iv)⇒ (i): If (iv) is assumed, then $(\mathcal{H},\pi)$ has a subrepresentation $(\mathcal{H}_0,\pi_0)$ of $\mathcal{O}_N$ which is $GP(\hat{z})$. By assumption, $(\mathcal{H},\pi)$ is a subrepresentation of $(\mathcal{H}_0,\pi_0)$. Therefore $(\mathcal{H},\pi)=(\mathcal{H}_0,\pi_0)=GP(\hat{z})$. Hence (i) holds.
(v)⇒ (ii): If (v) is assumed, then $(\mathcal{H},\pi)$ has a subrepresentation which is equivalent to $GP(\hat{z})$. Because $(\mathcal{H},\pi_0)$ is irreducible, $(\mathcal{H},\pi)$ is also irreducible. Hence (ii) holds. □

By Proposition 3.1, the following holds:

Corollary 3.2. *Assume that* $(\mathcal{H},\pi)$ *is* $GP(z)$ *of* $\mathcal{O}_M$ *for* $z\in S(\mathbf{C}^M)$ *and* $|z_M|<1$*. Then the following holds:*

(i) *Both the nonincreasing extension and the irreducible extension of* $(\mathcal{H},\pi)$ *with respect to* φ *are unique.*
(ii) *There is the minimal extension of* $(\mathcal{H},\pi)$ *with respect to* φ*.*
(iii) *There is unique cyclic extension of* $(\mathcal{H},\pi)$ *with respect to* φ *which has a common cyclic vector for* $(\mathcal{H},\pi)$*.*

By Corollary 3.2, we call $GP(\hat{z})$ by the *canonical extension of* $GP(z)$ *of* $\mathcal{O}_M$ *with respect to* φ.

3.2. States and GNS representations associated with $GP(\hat{z})$**.** Operator algebraists prefer *state* than representation. We realize $GP(\hat{z})$ as the GNS representation of a state of $\mathcal{O}_N$.

Proposition 3.3. *Let $s_1, \ldots, s_N$ and $t_1, \ldots, t_M$ be canonical generators of $\mathcal{O}_N$ and $\mathcal{O}_M$, respectively and let $\varphi : \mathcal{O}_M \hookrightarrow \mathcal{O}_N$ be the embedding in (1.2). We identify $\mathcal{O}_M$ and $\varphi(\mathcal{O}_M)$.*

(i) *If $z = (z_j)_{j=1}^M \in S(\mathbf{C}^M)$ satisfies $|z_M| < 1$, then $GP(\hat{z})$ of $\mathcal{O}_N$ is equivalent to the GNS representation by a state ω of $\mathcal{O}_N$ which satisfies the following equations:*

$$\omega(t_J t_K^*) = \overline{z_J} \cdot z_K, \quad \omega(t_K^*) = z_K \quad (J, K \in \{1, \ldots, M\}^*) \tag{3.1}$$

where $|J|, |K| \geq 1$ and $z_J \equiv z_{j_1} \cdots z_{j_m}$ for $J = (j_l)_{l=1}^m \in \{1, \ldots, M\}^m$.

(ii) *ω in (i) is pure.*

Proof. Let $(\mathcal{H}, \pi, \Omega)$ and $(\mathcal{H}_0, \pi_0, \Omega_0)$ be GNS representations of $\mathcal{O}_N$ and $\mathcal{O}_M$ by ω and $\omega_0 \equiv \omega|_{\mathcal{O}_M}$, respectively.
(i) $(\mathcal{H}_0, \pi_0)$ is $GP(z)$ and $t(z)\Omega_0 = \Omega_0$ by § 6.1 in [**6**]. By assumption, $\omega_0 = \omega|_{\mathcal{O}_M} =< \Omega|\pi|_{\mathcal{O}_M}(\cdot)\Omega >$. Put $\mathcal{K} \equiv \overline{\pi(\mathcal{O}_M)\Omega}$. By the uniqueness of the GNS representation, there is a unitary $U : \mathcal{H}_0 \to \mathcal{K}$ such that $U\pi_0(\cdot)U^* = \pi|_{\mathcal{O}_M}$ and $U\Omega_0 = \Omega$. This implies that $\pi|_{\mathcal{O}_M}(t(z))\Omega = \Omega$. Hence $(\mathcal{K}, \pi|_{\mathcal{O}_M})$ is $GP(z)$ of $\mathcal{O}_M$ with the GP vector Ω. By Corollary 3.2 (iii), $(\mathcal{H}, \pi)$ is $GP(\hat{z})$ of $\mathcal{O}_N$.
(ii) By (i) and Theorem 1.1 (iii), $(\mathcal{H}, \pi)$ is irreducible. Hence the statement holds. □

3.3. The canonical basis and the action of canonical generators. We show a complete orthonormal basis of $GP(\hat{z})$ which is canonically given up to freedom of the choice of parameters. Further we do the action of canonical generators of $\mathcal{O}_N$ on it.

Proposition 3.4. *Assume that $\mathcal{O}_M$ is embedded into $\mathcal{O}_N$ by φ in (1.2). For $z = (z_i)_{i=1}^M \in S(\mathbf{C}^M)$ with $|z_M| < 1$, choose $\{z^{(n)}\}_{n=2}^M \subset \mathbf{C}^M$ so that $\{z, z^{(2)}, \ldots, z^{(M)}\}$ is an orthonormal basis of $\mathbf{C}^M$. Then the following holds:*

(i) *If $(\mathcal{H}, \pi)$ is $GP(\hat{z})$ of $\mathcal{O}_N$ with the normalized GP vector Ω, then the following is a complete orthonormal basis of $\mathcal{H}$:*

$$\{\Omega, t_J t(z^{(n)})\Omega : n = 2, \ldots, M, \ J \in \{1, \ldots, M\}^*\}. \tag{3.2}$$

The action of $\mathcal{O}_N$ on (3.2) is given by (2.1).

(ii) *If $(\mathcal{H}, \pi_0)$ is $GP(z)$ of $\mathcal{O}_M$, then operators $s_1, \ldots, s_N$ on $\mathcal{H}$ which satisfy (2.1) define a representation of $\mathcal{O}_N$ on $\mathcal{H}$ and it is $GP(\hat{z})$.*

Proof. (i) In general, if $(\mathcal{H}, \pi')$ is $GP(z)$ of $\mathcal{O}_M$ with the normalized GP vector Ω, then a complete orthonormal basis of $\mathcal{H}$ is canonically given up to the freedom of the choice of parameters $\{z^{(n)}\}_{n=2}^M$ as (3.2) by § 4.3 in [**6**]. Because $(\mathcal{H}, \pi|_{\mathcal{O}_M})$ is $GP(z)$ of $\mathcal{O}_M$, (3.2) is a complete orthonormal basis of $\mathcal{H}$. By Theorem 1.1 (i), the action of $\mathcal{O}_N$ on (3.2) coincides with (2.1).

(ii) This follows from the proof of Theorem 1.1 (i). □

4. Examples

Example 4.1. A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ with a cyclic vector Ω which satisfies any one of the following eigenequations is irreducible:

(i) $N = 2$: For $z = (z_1, z_2, z_3) \in S(\mathbf{C}^3)$ with $|z_3| < 1$,

$$\pi(z_1 s_{12} + z_2 s_2 + z_3 s_{11})\Omega = \Omega.$$

(ii) $N = 2$: For $z = (z_i)_{i=1}^5 \in S(\mathbf{C}^5)$ with $|z_5| < 1$,

$$\pi(z_1 s_1 + z_2 s_{21} + z_3 s_{221} + z_4 s_{2221} + z_5 s_{2222})\Omega = \Omega.$$

(iii) $N = 3$: For $z = (z_i)_{i=1}^5 \in S(\mathbf{C}^5)$ with $|z_5| < 1$,

$$\pi(z_1 s_1 + z_2 s_2 + z_3 s_{31} + z_3 s_{32} + z_5 s_{33})\Omega = \Omega.$$

Example 4.2. A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_2$ with a cyclic vector Ω which satisfies any one of the following eigenequations is irreducible:
(i) $\pi(s_1 + s_2)\Omega = \sqrt{2}\Omega$. (ii) $\pi(\sqrt{2}s_1 + s_2 s_1 + s_2^2)\Omega = 2\Omega$.
Representations in (i) and (ii) are equivalent by Theorem 1.2 (i).

Example 4.3. Assume that $\mathcal{O}_M$ is embedded into $\mathcal{O}_N$ by φ in (1.2). If $(\mathcal{H}, \pi)$ is a representation of $\mathcal{O}_N$ with a cyclic vector Ω which satisfies

$$\pi((I - z_M t_M)^{-1}(z_1 t_1 + \cdots + z_{M-1} t_{M-1}))\Omega = \Omega$$

for some $z = (z_i)_{i=1}^M \in S(\mathbf{C}^M)$ with $|z_M| < 1$, then $(\mathcal{H}, \pi)$ is $GP(\hat{z})$.
Proof. Define $B \equiv z_M t_M$ and $X \equiv z_1 t_1 + \cdots + z_{M-1} t_{M-1}$. Then $s(\hat{z}) = B + X$. $\pi(s(\hat{z}))\Omega = \Omega$ if and only if $\pi(B + X)\Omega = \Omega$ if and only if $\pi((I - B)^{-1}X)\Omega = \Omega$. Hence the two eigenequations are equivalent. By definition of $GP(\hat{z})$, the statement holds. □

Example 4.4. We show an example which is a little different from the main theorem without proof. Let s_1, s_2 and r_1, r_2, r_3, r_4, r_5 be canonical generators of $\mathcal{O}_2$ and $\mathcal{O}_5$, respectively. If $z = (z_i)_{i=1}^5 \in S(\mathbf{C}^5)$ satisfies $(1 - |z_2|)(1 - |z_5|) > 0$, then there exists a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_2$ with a cyclic vector Ω which satisfies

$$\pi(z_1 s_{21} + z_2 s_{22} + z_3 s_{121} + z_4 s_{122} + z_5 s_{11})\Omega = \Omega$$

uniquely up to unitary equivalence. We denote such representation by $GP(\hat{z})$. Furthermore the following holds:

(i) $GP(\hat{z})$ is irreducible.
(ii) Under identification of $\mathcal{O}_5$ with a subalgebra of $\mathcal{O}_2$ by $\varphi : \mathcal{O}_5 \hookrightarrow \mathcal{O}_2; (r_i)_{i=1}^5 \mapsto (s_{21}, s_{22}, s_{121}, s_{122}, s_{11})$, $GP(\hat{z})|_{\mathcal{O}_5}$ is $GP(z)$ of $\mathcal{O}_5$.

(iii) If $z \in \tilde{\varphi}(S(\mathbf{C}^2))$, then $GP(\hat{z}) \sim GP(\tilde{\varphi}^{-1}(z))$ where $\tilde{\varphi} : S(\mathbf{C}^2) \hookrightarrow S(\mathbf{C}^5); \tilde{\varphi}(y_1, y_2) \equiv (y_2 y_1, y_2^2, y_2 y_1^2, y_1 y_2^2, y_1^2)$.

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