A note on quantization operators on Nichols algebra model for Schubert calculus on Weyl groups

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Dedicated to Kyoji Saito on the occasion of his sixtieth birthday

Abstract

We give a description of the (small) quantum cohomology ring of the flag variety as a certain commutative subalgebra in the tensor product of the Nichols algebras. Our main result can be considered as a quantum analog of a result by Y. Bazlov.

Introduction

In this paper, we give a description of the (small) quantum cohomology rings of the flag varieties in terms of the braided differential calculus. Here, we give some remarks on the preceding works on this subject. In [5], Fomin and one of the authors gave a combinatorial description of the Schubert calculus of the flag variety Fl_n of type A_{n-1}. They introduced a noncommutative quadratic algebra $\mathcal{E}_n$ determined by the root system, which contains the cohomology ring of the flag variety Fl_n as a commutative subalgebra. One of remarkable properties of the algebra $\mathcal{E}_n$ is that it admits the quantum deformation, and the deformed algebra $\tilde{\mathcal{E}}_n$ also contains the quantum cohomology ring of the flag variety Fl_n as its commutative subalgebra. A generalization of the algebras $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ was introduced by the authors in [9]. On the other hand, Fomin, Gelfand and Postnikov introduced the quantization operator on the polynomial ring to obtain the quantum deformation of the Schubert polynomials. Their approach was generalized for arbitrary root systems by Maré [12]. Our main idea is to lift their quantization operators onto the level of the Nichols algebras.

The term “Nichols algebra” was introduced by Andruskiewitsch and Schneider [1]. The similar object was also discovered by Woronowicz [15] and Majid [10] in the context of the braided differential calculus. The relationship between the quadratic algebra $\mathcal{E}_n$ and the Nichols algebra $\mathcal{B}(V_W)$ associated to a certain Yetter-Drinfeld module V_W over the Weyl group W was pointed out by Milinski and Schneider [13]. Majid [11] showed that it relates to a noncommutative differential structure on the permutation group S_n. In fact, the higher order differential structure on S_n gives a “super-analogue” of

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the algebra $\mathcal{E}_n$. Recently, Bazlov [2] showed that the Nichols algebra $\mathcal{B}(V_W)$ contains the coinvariant algebra $\mathbf{S}_W$ of the finite Coxeter group W. His method is based on the correspondence between braided derivations on $\mathcal{B}(V_W)$ and divided difference operators on the polynomial ring. Conjecturally, the algebra $\mathcal{E}_n$ is isomorphic to the Nichols algebra $\mathcal{B}(V_W)$ for $W = S_n$. Our aim is to quantize his model for the coinvariant algebra in case W is the Weyl group.

Fix B a Borel subgroup of a semisimple Lie group G. Let $\mathfrak{h}$ be the Cartan subalgebra in the Lie algebra of G. We regard $\mathfrak{h}$ as the reflection representation of the Weyl group W. We have a set of positive roots Δ_+ in the set of all roots $\Delta \subset \mathfrak{h}^*$. Denote by Σ the set of simple roots. We need symbols $q^{\alpha^\vee}$ corresponding to the simple roots α as the parameters for the quantum deformation. Let R be the polynomial ring $\mathbf{C}[q^{\alpha^\vee} | \alpha \in \Sigma]$. We also consider the algebra $\tilde{\mathcal{B}}(V)$ with a modified multiplication, see Section 1. Then our main result is:

Theorem. The algebra $(\mathcal{B}(V_W) \otimes \tilde{\mathcal{B}}(V_W)) \otimes R$ contains the quantum cohomology ring of the flag variety G/B as a subalgebra.

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1 Preliminaries

Let us consider the Nichols algebra $\mathcal{B}(V)$ associated to the Yetter-Drinfeld module $V = \oplus_{\alpha \in \Delta_+} \mathbf{C}[\alpha]$ over the Weyl groups W. The symbols $[\alpha]$ are subject to the condition $[-\alpha] = -[\alpha]$, and the W-action on V is defined by $w.[\alpha] = [w(\alpha)]$. The W-degree of $[\alpha]$ is a reflection $s_\alpha \in W$. The Yetter-Drinfeld module V is a naturally braided vector space with a braiding $\psi_{V,V}$. We can identify $\mathcal{B}(V)$ with its dual algebra $\mathcal{B}(V^*)$ via the W-invariant pairing $\langle [\alpha], [\beta] \rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta_+$. Denote by $\tilde{\mathcal{B}}(V)$ the algebra $\mathcal{B}(V)$ with a modified multiplication $a * b = m(\psi^{-1}_{\mathcal{B}(V),\mathcal{B}(V)}(a \otimes b))$, where m is the multiplication map in the Nichols algebra $\mathcal{B}(V)$.

Definition 1 *For each positive root α, the twisted derivation $\bar{D}_\alpha$ acting on $\mathcal{B}(V)$ from the left is defined by the rule*

$$\bar{D}_\alpha([\beta]) = \delta_{\alpha,\beta}, \quad \beta \in \Delta_+,$$

$$(\dagger) \quad \bar{D}_\alpha(xy) = \bar{D}_\alpha(x)y + s_\alpha(x)\bar{D}_\alpha(y).$$

The algebra $\tilde{\mathcal{B}}(V)$ acts on $\mathcal{B}(V^*)$ as an algebra generated by twisted derivations, and the twisted Leibniz rule $(\dagger)$ determines the algebra structure on $\mathcal{B}(V^*) \otimes \tilde{\mathcal{B}}(V)$:

$$(x \otimes [\alpha]) \cdot (u \otimes v) = x\bar{D}_\alpha(u) \otimes v + xs_\alpha(u) \otimes [\alpha] * v.$$

Lemma 1 *The representation of the algebra $\mathcal{B}(V^*) \otimes \tilde{\mathcal{B}}(V)$ on $\mathcal{B}(V^*)$ given by*

$$([\alpha_1] \cdots [\alpha_i] \otimes [\beta_1] * \cdots * [\beta_j])(x) := [\alpha_1] \cdots [\alpha_i]\bar{D}_{\beta_1} \cdots \bar{D}_{\beta_j}(x), \quad x \in \mathcal{B}(V),$$

is faithful.

Proof. This follows from the non-degeneracy of the duality pairing between $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$, *cf.* [2]. ■

Since the twisted derivations $\bar{D}_\alpha$ satisfy the Coxeter relations, one can define operators $\bar{D}_w$ for any element $w \in W$ by $\bar{D}_w = \bar{D}_{\alpha_1} \cdots \bar{D}_{\alpha_l}$ for a reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_l}$. Let $R = \mathbf{C}[q^{\alpha^\vee} | \alpha \in \Delta_+]$, where the parameters q^a satisfy the condition $q^{a+b} = q^a q^b$. We denote by $\mathcal{B}_R(V)$ the scalar extension $R \otimes \mathcal{B}(V)$. Here, we define the quantization of the element $[\alpha] \in \mathcal{B}(V)$. Let $\tilde{\Delta}_+$ be the set of positive roots α satisfying the condition $l(s_\alpha) = 2\mathrm{ht}(\alpha^\vee) - 1$, where the height $\mathrm{ht}(\alpha^\vee)$ is defined by $\mathrm{ht}(\alpha^\vee) = m_1 + \cdots + m_n$ if $\alpha^\vee = m_1 \alpha_1^\vee + \cdots + m_n \alpha_n^\vee$, $\alpha_i \in \Sigma$.

Definition 2 *Let $(c_\alpha)_{\alpha \in \Delta}$ be a set of nonzero constants with the condition $c_\alpha = c_{w\alpha}$, $w \in W$. For each root $\alpha \in \Delta_+$, we define an element $\widetilde{[\alpha]} \in \mathcal{B}_R(V^*) \otimes_R \tilde{\mathcal{B}}_R(V)$ by*

$$\widetilde{[\alpha]} := \begin{cases} c_\alpha[\alpha] \otimes 1 + d_\alpha q^{\alpha^\vee} \otimes [\alpha_1] * \cdots * [\alpha_l], & \text{if } \alpha \in \tilde{\Delta}_+, \\ c_\alpha[\alpha] \otimes 1, & \text{otherwise,} \end{cases}$$

where $\alpha_1, \ldots, \alpha_l$ are simple roots appearing in a reduced decompositon $s_\alpha = s_{\alpha_1} \cdots s_{\alpha_l}$, and $d_\alpha = (c_{\alpha_1} \cdots c_{\alpha_l})^{-1}$. We identify $\widetilde{[\alpha]}$ with an operator $c_\alpha[\alpha] + d_\alpha q^{\alpha^\vee} \bar{D}_{s_\alpha}$ or a multiplication operator $c_\alpha[\alpha]$ acting on $\mathcal{B}_R(V^*)$ by Lemma 1.

We define an R-linear map $\tilde{\mu} : \mathfrak{h}_R \to V_R \otimes_R \mathcal{B}_R(V^*)$ in similar way to Bazlov [2], i.e.,

$$\tilde{\mu}(x) = \sum_{\alpha \in \Delta_+} (x, \alpha) \widetilde{[\alpha]}.$$

Proposition 1 *The subalgebra of $\mathcal{B}_R(V^*) \otimes_R \tilde{\mathcal{B}}_R(V)$ generated by $\mathrm{Im}(\tilde{\mu})$ is commutative.*

Proof. We have to show $\tilde{\mu}(x)\tilde{\mu}(y) = \tilde{\mu}(y)\tilde{\mu}(x)$ for arbitrary $x, y \in \mathfrak{h}$. The left hand side is expanded as

$$(*) \quad \sum_{\alpha,\beta \in \Delta_+} (x,\alpha)(y,\beta) c_\alpha c_\beta [\alpha][\beta]$$

$$+ \sum_{\alpha \in \tilde{\Delta}_+, \beta \in \Delta_+} (x,\alpha)(y,\beta) d_\alpha c_\beta q^{\alpha^\vee} \bar{D}_{s_\alpha} \cdot [\beta] + \sum_{\alpha \in \Delta_+, \beta \in \tilde{\Delta}_+} (x,\alpha)(y,\beta) c_\alpha d_\beta q^{\beta^\vee} [\alpha] \cdot \bar{D}_{s_\beta}$$

$$+ \sum_{\alpha \in \tilde{\Delta}_+, \beta \in \tilde{\Delta}_+} (x,\alpha)(y,\beta) d_\alpha d_\beta q^{\alpha^\vee + \beta^\vee} \bar{D}_{s_\alpha} \bar{D}_{s_\beta}.$$

We have already known the commutativity of the classical part ([2], [9]), so we can ignore the first term. We also have

$$\bar{D}_{s_\alpha} \bar{D}_{s_\beta} = \begin{cases} \bar{D}_{s_\alpha s_\beta} & \text{if } l(s_\alpha s_\beta) = l(s_\alpha) + l(s_\beta), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{D}_{s_\alpha} \cdot [\beta] - s_\alpha([\beta]) \bar{D}_{s_\alpha} = \begin{cases} \bar{D}_{s_\alpha s_\beta} & \text{if } l(s_\alpha s_\beta) = l(s_\alpha) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$A = \{(\alpha, \beta) \in \tilde{\Delta}_+ \times \Delta_+ |\ l(s_\alpha s_\beta) = l(s_\alpha) - 1\}$$

and

$$B = \{(\alpha, \beta) \in \tilde{\Delta}_+^2 |\ l(s_\alpha s_\beta) = l(s_\alpha) + l(s_\beta)\}.$$

Then, we have

$$\sum_{\alpha \in \tilde{\Delta}_+, \beta \in \Delta_+} (x, \alpha)(y, \beta) d_\alpha c_\beta q^{\alpha^\vee} \bar{D}_{s_\alpha} \cdot [\beta] + \sum_{\alpha \in \Delta_+, \beta \in \tilde{\Delta}_+} (x, \alpha)(y, \beta) c_\alpha d_\beta q^{\beta^\vee} [\alpha] \cdot \bar{D}_{s_\beta}$$

$$= \sum_{\alpha \in \Delta_+, \beta \in \tilde{\Delta}_+} c_\alpha d_\beta \left((x, \alpha)(y, \beta) + (x, \beta)(y, \alpha) - 2(\alpha, \beta)(x, \beta)(y, \beta)\right) q^{\beta^\vee} [\alpha] \cdot \bar{D}_{s_\beta}$$

$$+ \sum_{(\alpha, \beta) \in A} d_\alpha c_\beta (x, \alpha)(y, \beta) q^{\alpha^\vee} \bar{D}_{s_\alpha s_\beta},$$

and

$$\sum_{\alpha, \beta \in \tilde{\Delta}_+} d_\alpha d_\beta (x, \alpha)(y, \beta) q^{\alpha^\vee + \beta^\vee} \bar{D}_{s_\alpha} \bar{D}_{s_\beta} = \sum_{(\alpha, \beta) \in B} d_\alpha d_\beta (x, \alpha)(y, \beta) q^{\alpha^\vee + \beta^\vee} \bar{D}_{s_\alpha s_\beta}.$$

For each element $(\alpha, \beta) \in A$ with $\alpha \neq \beta$, we can find an element $(\gamma, \delta) \in B$ such that $\alpha^\vee = \gamma^\vee + \delta^\vee$ and $s_\alpha s_\beta = s_\gamma s_\delta$ from the argument in [12, Section 3]. This correspondence gives a bijection between the sets $A' = A \setminus \{(\alpha, \beta) | \alpha = \beta\}$ and $B' = B \setminus \{(\gamma, \delta) | s_\gamma s_\delta = s_\delta s_\gamma\}$, and $(x, \alpha)(y, \beta) + (x, \gamma)(y, \delta)$ is symmetric in x and y under the correspondence between $(\alpha, \beta) \in A'$ and $(\gamma, \delta) \in B'$. Hence, $(*)$ is symmetric in x and y. ■

Remark. We can use the opposite algebra $\mathcal{B}(V)^{op}$ and the twisted derivation $\overleftarrow{D}_\alpha$ acting from the right, instead of $\tilde{\mathcal{B}}(V)$ and $\bar{D}_\alpha$. The algebra $\mathcal{B}(V)^{op}$ is the opposite algebra of $\mathcal{B}(V)$, whose multiplication $\star$ is obtained by reversing the order of the multiplication in $\mathcal{B}(V)$, i.e.,

$$a_1 \star \cdots \star a_m = a_m \cdots a_1.$$

The twisted derivation $\overleftarrow{D}_\alpha$, $\alpha \in \Delta_+$, is determined by the conditions:

$$[\beta] \overleftarrow{D}_\alpha = \delta_{\alpha, \beta}, \quad \beta \in \Delta_+,$$

$$(fg) \overleftarrow{D}_\alpha = f(g \overleftarrow{D}_\alpha) + (f \overleftarrow{D}_\alpha) s_\alpha(g).$$

Then, the algebra $\mathcal{B}(V^*) \otimes \mathcal{B}(V)^{op}$ faithfully acts on the algebra $\mathcal{B}(V^*)$ from the left via $1 \otimes [\alpha] \mapsto \overleftarrow{D}_\alpha$ and $[\beta] \otimes 1 \mapsto$ (left multiplication by $[\beta]$). We can also define the quantized element $\widetilde{[\alpha]}$ as an element in $\mathcal{B}_R(V^*) \otimes_R \mathcal{B}_R(V)^{op}$ in a similar way to Definition 2:

$$\widetilde{[\alpha]} := \begin{cases} c_\alpha [\alpha] \otimes 1 + d_\alpha q^{\alpha^\vee} \otimes [\alpha_1] \star \cdots \star [\alpha_l], & \text{if} \quad \alpha \in \tilde{\Delta}_+, \\ c_\alpha [\alpha] \otimes 1, & \text{otherwise.} \end{cases}$$

The arguments in this section work well for this definition, in particular, the subalgebra generated by $\mathrm{Im}(\tilde{\mu})$ is again commutative. This construction of the quantized elements $\widetilde{[\alpha]}$ by using $\mathcal{B}_R(V)^{op}$ and the twisted derivations from the right was suggested by Bazlov.

2 Main result

Now we can extend $\tilde{\mu}$ as an R-algebra homomorphism $\mathrm{Sym_R}(\mathfrak{h}_R) \to \mathcal{B}_R(V^*) \otimes_R \tilde{\mathcal{B}}_R(V)$. Let $\mu : \mathrm{Sym}_R(\mathfrak{h}_R) \to \mathcal{B}_R(V^*)$ be the scalar extension of the homomorphism introduced in [2], i.e.,

$$\mu(x) = \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)[\alpha].$$

The Demazure operator ∂_α, $\alpha \in \Delta_+$, acting on the polynomial ring $\mathrm{Sym}(\mathfrak{h})$ is defined by $\partial_\alpha(f) = (f - s_\alpha(f))/\alpha$. For each element $w \in W$, the operator ∂_w can be defined as $\partial_w = \partial_{\alpha_1} \cdots \partial_{\alpha_l}$ for a reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_l}$, $\alpha_1, \dots, \alpha_l \in \Sigma$. This is well-defined since the Demazure operators satisfy $\partial_\alpha^2 = 0$ and the Coxeter relations.

Lemma 2 ([2]) *For $f \in \mathrm{Sym}(\mathfrak{h})$, we have*

$$\bar{D}_\alpha \mu(f) = c_\alpha \mu(\partial_\alpha f).$$

Proposition 2 *Let I_i^q, $1 \le i \le n = \mathrm{rk}\mathfrak{h}$, be the quantum fundamental W-invariants given by* [7] *and* [8]*. Then, $\tilde{\mu}(I_i^q)\mu(f) = 0$, $\forall f \in \mathrm{Sym}_R(\mathfrak{h}_R)$.*

Proof. For each simple root $\alpha \in \Sigma$, we define

$$\eta_\alpha := \sum_{\gamma \in \Delta_+} \langle \omega_\alpha, \gamma^\vee \rangle \widetilde{[\gamma]} = \sum_{\gamma \in \Delta_+} \langle \omega_\alpha, \gamma^\vee \rangle c_\gamma [\gamma] + \sum_{\gamma \in \tilde{\Delta}_+} \langle \omega_\alpha, \gamma^\vee \rangle d_\gamma q^{\gamma^\vee} \bar{D}_{s_\gamma},$$

where ω_α is a fundamental dominant weight corresponding to α. Then, Lemma 2 shows that

$$\eta_\alpha \mu(f) = \mu(Y_\alpha f),$$

where

$$Y_\alpha = \omega_\alpha + \sum_{\gamma \in \tilde{\Delta}_+} \langle \omega_\alpha, \gamma^\vee \rangle q^{\gamma^\vee} \partial_{s_\gamma}.$$

Hence, $\tilde{\mu}(\varphi)\mu(f) = \mu(\varphi((Y_\alpha)_\alpha)(f))$ for any polynomial $\varphi \in \mathrm{Sym}_R(\mathfrak{h}_R)$. From the quantum Pieri or Chevalley formula ([4], [6], [14]), we have $\tilde{\mu}(I_i^q)(1) = 0$. For any $f \in \mathrm{Sym}_R(\mathfrak{h}_R)$, there exists a polynomial $\tilde{f} \in \mathrm{Sym}_R(\mathfrak{h}_R)$ such that $\tilde{f}((Y_\alpha)_\alpha)(1) = f$. Then, we have

$$\tilde{\mu}(I_i^q)\mu(f) = \tilde{\mu}(I_i^q)\tilde{\mu}(\tilde{f})(1) = \tilde{\mu}(\tilde{f})\tilde{\mu}(I_i^q)(1) = 0. \quad \blacksquare$$

Theorem 1 $\mathrm{Im}(\tilde{\mu})$ *generates a subalgebra in $\mathcal{B}_R(V^*) \otimes_R \tilde{\mathcal{B}}_R(V)$ isomorphic to the quantum cohomology ring of the corresponding flag variety G/B.*

Proof. We assign the degree 1 to the elements $[\alpha]$ and -1 to $\bar{D}_\alpha$. Define the filter $F_\bullet$ on the algebra $\mathrm{Im}(\tilde{\mu})$ by $F_i(\mathrm{Im}(\tilde{\mu})) = \{x | \deg(x) \le i\}$. Then, $Gr_F(\mathrm{Im}\tilde{\mu}) \cong \mathrm{Im}(\mu)$. The faithfulness of the representation of the subalgebra $\mathrm{Im}(\mu)$ in $\mathcal{B}_R(V)$ on itself implies that of the representation of the algebra generated by $\mathrm{Im}(\tilde{\mu})$ on $\mathrm{Im}(\mu)$. Hence, we have $\tilde{\mu}(I_i^q) = 0$ from Proposition 2. Since $Gr_F(\mathrm{Im}\tilde{\mu}) \cong \mathrm{Im}(\mu)$, we conclude that $\mathrm{Im}(\tilde{\mu}) \cong \mathrm{Sym}_R(\mathfrak{h}_R)/(I_1^q, \dots, I_n^q)$. $\blacksquare$

Corollary 1 (1) *In the case of root systems of type A_n, denote by $\mathfrak{S}_w$ and $\mathfrak{S}_w^q$ the Schubert polynomial and its quantization corresponding to $w \in S_{n+1}$. Then, $\tilde{\mu}(\mathfrak{S}_w^q)(1) = \mu(\mathfrak{S}_w)$.*
(2) *For general crystallographic root systems, let X_w and X_w^q be the Bernstein-Gelfand-Gelfand polynomial* ([3]) *and its quantization coresponding to $w \in W$* ([9],[12]). *Then, $\tilde{\mu}(X_w^q)(1) = \mu(X_w)$.*

Remark. In A_n-cases, the operators η_α induce the operators on the algebra $\mathrm{Sym}_R(\mathfrak{h}_R)$ introduced by Fomin, Gelfand and Postnikov [4]. For other cases, they induce Maré's operators [12]. The above corollary is a restatement of their results and [9, Proposition 8.1].

Proposition 3 *The identity*

$$\widetilde{[\alpha]}^2 = \begin{cases} c_\alpha d_\alpha q^{\alpha^\vee}, & \text{if} \quad \alpha: \text{ simple}, \\ 0, & \text{otherwise} \end{cases}$$

holds in $\mathcal{B}_R(V^*) \otimes_R \tilde{\mathcal{B}}_R(V)$.

Proof. This follows from $[\alpha]^2 = 0$, $\bar{D}_{s_\alpha}^2 = 0$ and

$$\bar{D}_{s_\alpha} \cdot [\alpha] = \begin{cases} 1 - [\alpha]\bar{D}_{s_\alpha}, & \text{if} \quad \alpha: \text{ simple}, \\ -[\alpha]\bar{D}_{s_\alpha}, & \text{otherwise.} \end{cases}$$

Example. In B_n-case, the algebra $\mathcal{B}(V)$ is generated by the symbols $[i,j]$, $\overline{[i,j]}$ and $[i]$ with $1 \le i, j \le n$ and $i \ne j$. After normalizing $c_\alpha = 1$ for all $\alpha \in \Delta$, the quantized operators are given by

$$\widetilde{[i,j]} = [i,j] + Q_{ij}\bar{D}_{(ij)}, \quad (i < j),$$
$$\widetilde{\overline{[i,j]}} = \overline{[i,j]} + Q_{\overline{ij}}\bar{D}_{\overline{(ij)}},$$
$$\widetilde{[i]} = [i], \quad (i < n),$$
$$\widetilde{[n]} = [n] + Q_n\bar{D}_{(n)},$$

where $Q_{ij} = q_i q_j^{-1}$ $(i < j)$, $Q_{\overline{ij}} = q_i q_j$ and $Q_n = q_n^2$ are elements in the Laurent polynomial ring $\mathbf{C}[q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$. We put $\widetilde{[j,i]} = -\widetilde{[i,j]}$. We can check that $\widetilde{[i,j]}$, $\widetilde{\overline{[i,j]}}$ and $\widetilde{[i]}$ satisfy the relations of the quantum B_n-bracket algebra introduced by the authors [9]:

(1) $\widetilde{[i,i+1]}^2 = Q_{i\,i+1}$, $\widetilde{[n]}^2 = Q_n$,
$\widetilde{[i,j]}^2 = 0$, if $|i-j| \ne 1$; $\widetilde{[i]}^2 = 0$, if $i < n$; $\widetilde{\overline{[i,j]}}^2 = 0$, if $i \ne j$,

(2) $\widetilde{[i,j]}\widetilde{[k,l]} = \widetilde{[k,l]}\widetilde{[i,j]}$, $\widetilde{\overline{[i,j]}}\widetilde{[k,l]} = \widetilde{[k,l]}\widetilde{\overline{[i,j]}}$, $\widetilde{\overline{[i,j]}}\widetilde{\overline{[k,l]}} = \widetilde{\overline{[k,l]}}\widetilde{\overline{[i,j]}}$,
if $\{i,j\} \cap \{k,l\} = \emptyset$,

(3) $\widetilde{[i]}\widetilde{[j]} = \widetilde{[j]}\widetilde{[i]}$, $\widetilde{[i,j]}\widetilde{\overline{[i,j]}} = \widetilde{\overline{[i,j]}}\widetilde{[i,j]}$, $\widetilde{[i,j]}\widetilde{[k]} = \widetilde{[k]}\widetilde{[i,j]}$, $\widetilde{\overline{[i,j]}}\widetilde{[k]} = \widetilde{[k]}\widetilde{\overline{[i,j]}}$, if $k \ne i, j$,

(4) $\widetilde{[i,j]}\widetilde{[j,k]} + \widetilde{[j,k]}\widetilde{[k,i]} + \widetilde{[k,i]}\widetilde{[i,j]} = 0$,
$\widetilde{\overline{[i,k]}}\widetilde{[i,j]} + \widetilde{[j,i]}\widetilde{\overline{[j,k]}} + \widetilde{\overline{[k,j]}}\widetilde{[i,k]} = 0$,

$\widetilde{[i,j]}\widetilde{[i]} + \widetilde{[j]}\widetilde{[j,i]} + \widetilde{[i]}\widetilde{\overline{[i,j]}} + \widetilde{\overline{[i,j]}}\widetilde{[j]} = 0,$
if all i, j and k are distinct,

(5) $\widetilde{[i,j]}\widetilde{[i]}\widetilde{\overline{[i,j]}}\widetilde{[i]} + \widetilde{\overline{[i,j]}}\widetilde{[i]}\widetilde{[i,j]}\widetilde{[i]} + \widetilde{[i]}\widetilde{[i,j]}\widetilde{[i]}\widetilde{\overline{[i,j]}} + \widetilde{[i]}\widetilde{\overline{[i,j]}}\widetilde{[i]}\widetilde{[i,j]} = 0$, if $i < j$.

Remark. As in the remark at the end of Section 1, we also have another construction of the quantized elements by using $\mathcal{B}(V)^{op}$ and $\overleftarrow{D}_\alpha$. Since

$$\mu(f)\overleftarrow{D}_\alpha = c_\alpha \mu(\partial_\alpha f)$$

is also correct, we can show that the algebra $\mathcal{B}_R(V^*) \otimes_R \mathcal{B}_R(V)^{op}$ contains the quantum cohomology ring of G/B as a commutative subalgebra.

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