Finite generation of the Nagata invariant rings in A-D-E cases

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Let $\mathbf{G}_a^n \curvearrowright V = \bigoplus_{i=1}^n V_i$ be the direct sum of n copies $V_1, \ldots, V_n$ of the 2-dimensional standard unipotent action of the 1-dimensional additive group $\mathbf{G}_a$. The induced action on the polynomial ring $S_{2n} = \mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is as follows:

$$(t_1, \ldots, t_n) \in \mathbf{C}^n \curvearrowright S_{2n} = \mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n], \quad \begin{cases} x_i \longmapsto x_i \\ y_i \longmapsto t_i x_i + y_i, \end{cases}$$

The restriction of this action to a general linear subspace $G \subset \mathbf{C}^n$ is called *an action of Nagata type*. In [M], generalizing the result of Nagata [N1] ($r = 3$ and $n = 16$), we proved the infinite generation of the invariant ring S^G in the case where the inequality $1/2 + 1/(n-r) + 1/r \leq 1$ holds, where r is the codimension of G. In this article, we shall show the converse:

Theorem *The invariant ring S^G of Nagata type is finitely generated if* $1/2 + 1/(n-r) + 1/r > 1$.

This inequality is equivalent to the finiteness of the Weyl group of the Dynkin diagram $T_{2,r,n-r}$ with three legs of length 2, r and $n-r$. There are four infinite series [1]–[4] and five exceptional cases [5]–[9] for which this holds:

	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
Cartan's symbol			BDII	DIII	EIII	EVII	EVI	EIX	EVIII
r	1		2		3	3	4	3	5
$n-r$		1		2	3	4	3	5	3
diagram	A_n	A_n	D_n	D_n	E_6	E_7	E_7	E_8	E_8

In the cases [1] and [3], the invariant ring is very explicit and the proof is immediate ([M, §1]). The case [2] is classical and the invariant ring S^G is

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the homogeneous coordinate ring of the Grassmannian variety $G(2, n+1)$. We assume $s := \dim G \geq 2$ in the sequel.

In the rest of cases, we start the proof with the following key fact on the Nagata invariant ring: S^G is isomorphic to the total coordinate ring

$$\mathcal{TC}(X) := \bigoplus_{a,b_1,\dots,b_n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(ah - b_1e_1 - \cdots - b_ne_n)) \simeq \bigoplus_{L \in \operatorname{Pic} X} H^0(X, L)$$

of the variety $X = Bl_{n\,\mathrm{pts}}\,\mathbf{P}^{r-1}$ ([M, §1], [N1, §3] in the case $r = 3$). More precisely, X is the blow-up of the $(r-1)$-dimensional projective space $\mathbf{P}_*(\mathbf{C}^n/G)$ with center the n points $p_1, \dots, p_n$ corresponding to the standard basis of $\mathbf{C}^n$. In the case $r = 3$, X is a del Pezzo surface and the theorem follows from [BP].

We make use of the fact that X is the moduli spaces of certain vector bundles in the case $s = 2$ and 3. Note that $G \subset \mathbf{C}^n$ and the standard basis determine the n points $q_1, \dots, q_n$ on the projective space $\mathbf{P}_*G \simeq \mathbf{P}^{s-1}$ also. We reduce the finite generation of $\mathcal{TC}(X)$ to a geometry of the n-pointed projective space $(\mathbf{P}^{s-1}; q_1, \dots, q_s)$, which is the *Gale transform* of $(\mathbf{P}^{r-1}; p_1, \dots, p_s)$ ([DO, III], [EP]). Let $I_{q_1,\dots,q_n} \subset \mathcal{O}_{\mathbf{P}}$ be the ideal sheaf of the set of n points $\{q_1, \cdots, q_n\} \subset \mathbf{P}^{s-1}$. Then we obtain a family of exact sequences of coherent sheaves of $\mathcal{O}_{\mathbf{P}}$-modules

$$\mathbf{E}_x : 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(1) \otimes I_{q_1,\dots,q_n} \longrightarrow E_x \xrightarrow{\pi} \mathcal{O}_{\mathbf{P}} \longrightarrow 0 \qquad (1)$$

on $\mathbf{P}^{s-1}$ parameterized by $x \in \mathbf{P}_*H^1(\mathcal{O}_{\mathbf{P}}(1) \otimes I_{q_1,\dots,q_n}) = \mathbf{P}^{r-1}$. By the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbf{P}}(1)) \longrightarrow H^0(\bigoplus_{i=1}^{n} \mathbf{C}(p_i)) = \mathbf{C}^n \longrightarrow H^1(\mathcal{O}_{\mathbf{P}}(1) \otimes I_{q_1,\dots,q_n}) \longrightarrow 0,$$

$H^1(\mathbf{P}^{s-1}, \mathcal{O}_{\mathbf{P}}(1) \otimes I_{q_1,\dots,q_n})$ is isomorphic to the vector space $\mathbf{C}^n/G$ including the assignment of bases. The exact sequence $\mathbf{E}_{p_i}$ splits outside q_i for every $1 \leq i \leq n$, that is, E_{p_i} contains a subsheaf $\simeq I_{q_i}$ on which π is nonzero.

In the case $s = 2$, $\mathbf{E}_x$ is regarded as a quasi-parabolic rank 2 vector bundle on the n-pointed projective line $(\mathbf{P}^1; q_1, \dots, q_n)$. By the correspondence $x \mapsto \mathbf{E}_x$, the moduli space $\mathcal{U}(\alpha)$ of parabolic 2-bundles with a certain weight α is isomorphic to $\mathbf{P}^{r-1}$ (§1). The moduli space $\mathcal{U}(\alpha')$ is isomorphic to the blow up X_G for another weight α'. We apply the result of Bauer[B] on the variation of the moduli spaces $\mathcal{U}(\alpha)$ to determine the movable cone of them. Then the finite generation follows from the GIT construction of such moduli spaces by Mehta-Seshadri[MS] and a result of Zariski.

In the case $s \geq 3$, the sheaf E_x is not locally free at $q_1, \ldots, q_n$ but determines uniquely a vector bundle $\tilde{E}_x$ on the blow-up $S = Bl_{q_1,\ldots,q_n}\mathbf{P}^{s-1}$. Especially, In the cases [9] and [7], the correspondence $x \mapsto \tilde{E}_x \otimes \mathcal{O}_S(1)$ gives rise to an isomorphism

$$\mathbf{P}^{r-1} \xrightarrow{\sim} M_{S,L}(2, -K_S, c_2 = 2) \tag{2}$$

of the $(r-1)$-dimensional projective space to the moduli space of 2-bundles with the above described invariants on a del Pezzo surface S (of degree 1 and 2) which are stable with respect to a certain ample divisor L. The blow-up X_G is isomorphic to $M_{S,L'}(2, -K_S, c_2 = 2)$ for another ample divisor L'. The finite generation essentially follows from the ampleness of $-K_S$ (§2).

§1 Moduli of parabolic 2-bundles on $\mathbf{P}^1$

Let C be a complete algebraic curve. A pair $(E' \subset E)$ of an (algebraic) vector bundle E of rank 2 on C and its subsheaf E' of rank 2 is called a *quasi-parabolic 2-bundle*. The inclusion $\det E' \subset \det E$ determines an effective divisor on C, which we denote by Δ. E' coincides with E outside the support of D. Let $q_1, \ldots, q_n$ be a set of distinct n points on C. $(E' \subset E)$ with $\Delta = q_1 + \cdots + q_n$ is called a quasi-parabolic 2-bundle on the n-pointed curve $(C; q_1, \ldots, q_n)$. A pair $(E' \subset E; \alpha)$ of a quasi-parabolic 2-bundle and an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of real numbers in the closed interval $[0, 1]$ is called a *parabolic 2-bundle*.

Definition 1 A parabolic 2-bundle $(E' \subset E; \alpha)$ is *semi-stable* if

$$\deg L - \sum_{i=1}^{n} \alpha_i \mathrm{length}_{p_i} L/(L \cap E') \leq \frac{1}{2}(\deg E - \sum_{i=1}^{n} \alpha_i)$$

holds for every line subbundle $L \subset E$. It is *stable* if the strict inequality holds for every line subbundle $L \subset E$.

We only need the case $C = \mathbf{P}^1$. Let $q_1, \ldots, q_n \in \mathbf{P}^1$ and $p_1, \ldots, p_n \in \mathbf{P}^{n-3}$ be as in the introduction. We denote by $\mathcal{U}(\alpha)$ the moduli space of semi-stable parabolic 2-bundles $(E' \subset E; \alpha)$ on the n-pointed projective line $(\mathbf{P}^1 : q_1, \ldots, q_n)$ with $\det E \simeq \mathcal{O}_{\mathbf{P}}(1)$. Since the 2-bundle E_x in (1) is a subsheaf of the direct sum $\mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}}$, we obtain a quasi-parabolic 2-bundle $(E_x \subset \mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}})$ for each $x \in \mathbf{P}^{n-3}$. First we consider the case where the weight α is diagonal, that is, $\alpha = (a, \dots, a)$, for $a \in [0,1]$. By [B], we have the following:

Proposition 1 (1) *If $1/n < a < 1/(n-2)$, then $(E_x \subset \mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}})$ is stable for every $x \in \mathbf{P}^{n-3}$ and the classification morphism*

$$\mathbf{P}_* H^1(\mathcal{O}_{\mathbf{P}}(1) \otimes I_{q_1,\dots,q_n}) \simeq \mathbf{P}^{n-3} \longrightarrow \mathcal{U}(a, \dots, a), \quad x \mapsto (E_x \subset \mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}})$$

is an isomorphism. (The moduli space is empty if $0 \le a < 1/n$ and consists of one point if $a = 1/n$.)

(2) *$\mathcal{U}(a, \dots, a)$ is isomorphic to the blow-up $X_G = Bl_{p_1,\dots,p_n}\mathbf{P}^{n-3}$ if $n \ge 5$ and $1/(n-2) < a < 1/(n-4)$.*

In order to describe the moduli space $\mathcal{U}(\alpha)$ for a general weight α, we need the family of hyperplanes

$$H_{I,k} : \sum_{j \notin I} \alpha_j + \sum_{i \in I} (1 - \alpha_i) = k$$

in the hypercube $[0,1]^n$, where I is a subset of $\{1, \dots, n\}$ and k is an integer with $|I| \equiv k + 1 \bmod 2$. A connected component of the complement of the union of all these hyperplanes is called a *chamber*. The hyperplane $H_{I,k}$ coincides with $H_{I^c, n-k}$, where I^c is the complement of I. Hence we assume $k \le n/2$ in the sequel. We recall some results of [B, §2] for our proof.

Proposition 2 (1) *Let $\mathcal{C}$ be a chamber. Then the moduli space $\mathcal{U}(\beta)$ with $\beta \in \mathcal{C}$ is smooth of dimension $n-3$. Moreover, their isomorphism classes do not depend on β. We denote the isomorphism class by $\mathcal{U}_{\mathcal{C}}$.*

(2) *For each $\alpha \in \overline{\mathcal{C}}$, there exists a (contraction) morphism $f_{\mathcal{C},\alpha} : \mathcal{U}_{\mathcal{C}} \longrightarrow \mathcal{U}(\alpha)$.*

(3) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two adjacent chambers separated by the hyperplane $H_{I,k}$ Assume that $\sum_{j \notin I} \alpha_j + \sum_{i \in I}(1 - \alpha_i) - k$ non-positive on $\mathcal{C}$ and non-negative on $\mathcal{C}'$. Then the two moduli spaces $\mathcal{U}_{\mathcal{C}}$ and $\mathcal{U}_{\mathcal{C}'}$ are related in the following way.*

i) If $k = 2$, then $\mathcal{U}_{\mathcal{C}'}$ is the blow-up of $\mathcal{U}_{\mathcal{C}}$ at a point.

ii) If $3 \le k (\le n/2)$, then $\mathcal{U}_{\mathcal{C}'}$ is a flop of $\mathcal{U}_{\mathcal{C}}$. Let α_0 be a general point of $\overline{\mathcal{C}} \cap \overline{\mathcal{C}'}$. The morphism $f_{\mathcal{C},\alpha_0} : \mathcal{U}_{\mathcal{C}} \longrightarrow \mathcal{U}(\alpha_0)$ contracts a subvariety isomorphic to $\mathbf{P}^{k-2}$ to a singular point and $f_{\mathcal{C}',\alpha_0}$ contracts a subvariety $\simeq \mathbf{P}^{n-k-2}$ to the same point. Both $f_{\mathcal{C},\alpha_0}$ and $f_{\mathcal{C}',\alpha_0}$ are isomorphisms outside the subvarieties.

We also need the behavior of $\mathcal{U}(\alpha)$ in the neighborhood of the facets of $[0,1]^n$, which is described by the neglect of the parabolic structure at a (parabolic) point. Let $(E' \subset E)$ be a parabolic 2-bundle on $(\mathbf{P}^1 : q_1, \dots, q_n)$ and E_i the subsheaf of E which is E' outside q_i and E itself in the neighborhood of q_i. Then $(E_i \subset E)$ is a parabolic 2-bundle on the $(n-1)$-pointed projective line $(\mathbf{P}^1 : q_1, \dots, \check{q}_i, \dots, q_n)$. Similarly, let E^i be the subsheaf of E which is E outside q_i and E' in the neighborhood of q_i. Then $(E' \subset E^i)$ is also a parabolic 2-bundle.

Proposition 3 *Let C be a chamber with $\alpha_i = 0$ as its supporting hyperplane. Then the neglect $(E' \subset E) \mapsto (E_i \subset E)$ defines a morphism $\mathcal{U}_C \longrightarrow \mathcal{U}'$ onto a moduli spaces of parabolic 2-bundles on $(\mathbf{P}^1 : q_1, \dots, \check{q}_i, \dots, q_n)$. A general fiber is isomorphic to $\mathbf{P}^1$. Similarly if C has $\alpha_i = 1$ as its supporting hyperplane, then $(E' \subset E) \mapsto (E' \subset E^i)$ defines a morphism $\mathcal{U}_C \longrightarrow \mathcal{U}''$ whose general fiber is also $\mathbf{P}^1$.*

This is a moduli theoretic interpretation of the following birational geometry in the case $s = 2$:

Example 1 The projection $\mathbf{P}^{r-1} \cdots \rightarrow \mathbf{P}^{r-2}$ with center p_n induces a rational map $X_G = Bl_n\mathbf{P}^{r-1} \cdots \rightarrow Bl_{n-1}\mathbf{P}^{r-2}$ to the blow-up of $\mathbf{P}^{r-2}$ at the image of $(n-1)$ points $p_1, \dots, p_{n-1}$. This image is the Gale transform of $q_1, \dots, q_{n-1} \in \mathbf{P}^{s-1}$. The indeterminacy of this rational map is resolved by the flop with center the strict transforms of the $n-1$ lines joining p_n and p_i, $1 \le i \le n-1$. The resulting morphism is a $\mathbf{P}^1$-bundle.

Let $\overline{\Pi}$ be the polytope in $[0,1]^n$ defined by the system of 2^{n-1} inequalities $\sum_{j \notin I} \alpha_j + \sum_{i \in I}(1-\alpha_i) \ge 2$ for the subsets $I \subset \{1, \dots, n\}$ with $|I|$ odd. Let Π be its interior. By virtue of (3) of Proposition 2, $\mathcal{U}(\beta)$'s with $\beta \in \Pi$ are isomorphic to each other in codimension one. So they have the common Picard group and the common total coordinate ring.

The polytope Π is empty if $n = 3$ and consists of one point $(1/2, \cdots, 1/2)$ if $n = 4$. So we assume $n \ge 5$. The diagonal weight $(a, \dots, a)$ with $1/(n-2) < a < 1/(n-4)$ is contained in Π. Hence, by Proposition 1, $\mathcal{U}(\beta)$ is isomorphic to X_G in codimension one for every interior point β of Π.

For our proof we need a fact from the construction in [MS] also. The moduli space $\mathcal{U}_{(C:q_1,\dots,q_n)}(\alpha)$ is a GIT quotient of the product of a suitable Quot scheme and Grassmannians by suitable linearization. Since $\mathcal{U}(\alpha)$ is the projective spectrum Proj R of a graded ring R, it carries a natural ample (Cartier) divisor, which we regard as a divisor on X_G by Proposition 2 and denote by D_α. The choice of linearization in [MS] is linear with respect to the weight α. Hence we have

Lemma 1 *If weights $\alpha, \alpha', \alpha'' \in \Pi$ are colinear, then the divisors $D_\alpha, D_{\alpha'}, D_{\alpha''} \in \operatorname{Pic} X_G$ are linearly dependent.*

Proof of Theorem. Let $\tilde{\Pi}$ be the cone generated by D_α with $\alpha \in \overline{\Pi}$ in $\operatorname{Pic} X_G \otimes \mathbf{R}$. For a chamber C, we denote the subcone generated by D_α with $\alpha \in \overline{C}$ by $\tilde{C}$. Then D_α is semi-ample on the moduli space $\mathcal{U}_C$ by (2) of Proposition 2. Since C is finitely generated, so is $\tilde{C} \cap \operatorname{Pic} X_G$ by Lemma 1. Therefore, by a lemma of Zariski ([HK, Lemma 2.8]), the $\tilde{C}$-part $\bigoplus_{L \in \tilde{C} \cap \operatorname{Pic} X_G} H^0(L)$ of the total coordinate ring $\mathcal{TC}(X_G)$ is finitely generated (over $\mathbf{C}$). Since $\overline{\Pi}$ is the union of finitely many $\overline{C}$, the $\tilde{\Pi}$-part of $\mathcal{TC}(X_G)$ is also finitely generated.

The supporting hyperplanes of the polytope $\overline{\Pi}$ are $H_{I,2}$'s and $\alpha_i = 0, 1$ for $1 \le i \le n$. Let $C \subset \Pi$ be a chamber with $H_{I,2}$ as its supporting hyperplane. Let β_I be a general point of the intersection $\overline{C} \cap H_{I,2}$. Then $\mathcal{U}_C \to \mathcal{U}(\beta_I)$ is a one-point blow-up by Proposition 2. Let e_I be the exceptional divisor and Z_I the line in it. Then $(D_\alpha . Z_I)$ is positive for every $\alpha \in C$ and zero for $\alpha \in \overline{C} \cap H_{I,2}$ by (3) of Proposition 2. Therefore, by Lemma 1, the intersection number $(D.Z_I)$ is non-negative for every $D \in \tilde{\Pi}$ and $(D.Z_I) = 0$ is a supporting hyperplane of $\tilde{\Pi}$.

Let $C \subset \Pi$ be as in Proposition 3 and let F_i be a general fiber of the morphism $\mathcal{U}_C \longrightarrow \mathcal{U}'$. The intersection number $(D_\alpha . F)$ is positive for every $\alpha \in C$ and zero for $\alpha \in \overline{C} \cap \{\alpha_i = 0\}$. Therefore, by Lemma 1, the intersection number $(D.F_i)$ is non-negative for every $D \in \tilde{\Pi}$ and $(D_\alpha . F_i) = 0$ is a supporting hyperplane of $\tilde{\Pi}$.

Now let D be a divisor of X_G. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ does not belong to Π, then either $\sum_{j \notin I} \alpha_j + \sum_{i \in I} (1 - \alpha_i) < 2$ holds for a subset I of $\{1, \ldots, n\}$ or $\alpha_i < 0$ or $\alpha_i > 1$ holds for $1 \le i \le n$. By Lemma 1, if D does not belong to $\tilde{\Pi}$, then either $(D.Z_I) < 0$ holds for some I or $(D.F_i) < 0$ or $(D.F'_i) < 0$ holds for $1 \le i \le n$, where F'_i is a general fiber of the morphism $\mathcal{U}_C \longrightarrow \mathcal{U}''$ in Proposition 3. Assume that D is effective. Then the latter is impossible. Hence an effective divisor $D \notin \tilde{\Pi}$ contains the exceptional divisor e_I as irreducible component for some I. Therefore,

$TC(X_G)$ is generated as ring by its $\tilde{\Pi}$-part and the canonical global sections $1 \in H^0(\mathcal{O}_X(e_I))$ of the 2^{n-1} exceptional divisors e_I's.

§2 Moduli of certain 2-bundles on a del Pezzo surface

Let $p_1, \dots, p_n \in \mathbf{P}^{r-1}$ and $q_1, \dots, q_n \in \mathbf{P}^{s-1}$, $r + s = n$, be as in the introduction. They are the Gale transform of each other. Let $X = X_G$ and $S = S_G$ be their blow-ups. We need a certain linear isomorphism between $\operatorname{Pic} X \otimes \mathbf{Q}$ and $\operatorname{Pic} S \otimes \mathbf{Q}$ for our proof.

Generally the correspondence $e_i - e_{i+1} \mapsto e_{n+1-i} - e_{n-i}$ for $1 \leq i \leq n$ and $h - \sum_1^r e_i \mapsto h - \sum_1^s e_i$ gives an isomorphism from the Dynkyn diagram $T_{2,r,n-r}$ of X to $T_{2,s,n-s}$ of S, and hence an isometry φ_0 between two lattices $(-K_X)^\perp \subset \operatorname{Pic} X$ and $(-K_S)^\perp \subset \operatorname{Pic} S$ with respect to the inner product defined in [M, §3]. We identify the two Weyl groups $W(T_{2,s,n-s})$ and $W(T_{2,r,n-r})$ by this correspondence. The following is easily verified:

Proposition 4 *Let Ψ be the standard Cremona transformation of $\mathbf{P}^{s-1}$ with center the s points $q_1, \dots, q_s$ and Ψ' that of $\mathbf{P}^{r-1}$ with center the r points $p_{s+1}, \dots, p_n$. Then*

$$q_1, \dots, q_s, \Psi(q_{s+1}), \dots, \Psi(q_n) \in \mathbf{P}^{s-1}$$

and

$$\Psi'(p_1), \dots, \Psi'(p_s), p_{s+1}, \dots, p_n \in \mathbf{P}^{r-1}$$

are the Gale transform of each other.

Now we assume that $s = 3$ and extend the isometry φ_0 to a linear isomorphism $\varphi : \operatorname{Pic} X \otimes \mathbf{Q} \longrightarrow \operatorname{Pic} S \otimes \mathbf{Q}$ by setting $\varphi(K_X) = 2K_S$. The following is easily calculated:

$$\varphi(e_i) = h - e_i, \quad \varphi(h) = (n-2)h - e. \tag{3}$$

Remark Though φ is not an isometry, $(\varphi(D)^2) = (D^2) - (K_S.D)^2/4$ holds for every $D \in \operatorname{Pic} S$.

The main tool of our proof is vector bundle as in previous section. More precisely we consider torsion free sheaves E on S with

$$r(E) = 2, c_1(E) = -K_S \quad \text{and} \quad c_2(E) = 2. \tag{4}$$

For an ample divisor L on S, we denote by $\overline{M}_{S,L}$ the moduli space of such torsion free sheaves E which are semi-stable with respect to L in the sense of Gieseker [G]. It contains the moduli space $M_{S,L}$ of stable bundles as an open set. $M_{S,L}$ is smooth of dimension $n-4$ by the general theory. We study the variation of $M_{S,L}$ as L moves. See [EG], [FQ] and [MW] for the general theory.

We further assume that $n = (6,)7,8$. Then S is a del Pezzo surface, that is, a surface with ample $-K_S$. The degree (K_S^2) is equal to $9-n$.

Lemma 2 *Every member of $E \in \overline{M}_{S,L}$ has a nonzero global section.*

Proof. By the Riemann-Roch formula, we have $\chi(E) = 9 - n \geq 1$. Since $H^2(E) \simeq \mathrm{Hom}\,(E, \mathcal{O}_S(K_S))^\vee = 0$, we have $H^0(E) \neq 0$. $\square$

Let l be a *line*, *i.e.*, a smooth rational curve $l \subset S$ with $(l. -K_S) = 1$. When L crosses crosses the hyperplane $H_{l,1} : (2l + K_S.L) = 0$ from the positive side to the negative, the non-trivial extensions

$$0 \longrightarrow \mathcal{O}_S(-K_S - l) \longrightarrow E \longrightarrow \mathcal{O}_S(l) \longrightarrow 0,$$

which are parameterized by $\mathbf{P}^{n-6}$, are replaced by the opposite non-trivial extensions

$$0 \longrightarrow \mathcal{O}_S(l) \longrightarrow E' \longrightarrow \mathcal{O}_S(-K_S - l) \longrightarrow 0,$$

which are parameterized by $\mathbf{P}^1$, in the moduli spaces. We denote this $\mathbf{P}^1$ by Z_l. In the case $n = 8$, $-K_S$ belongs to the positive side and the moduli space is flipped when L crosses the hyperplane $H_{l,1}$.

Similarly, let C be a *conic*, *i.e.*, a smooth rational curve C with $(C. - K_S) = 2$. When L crosses the hyperplane $H_{C,1} : (2C + K_S.L) = 0$ from the positive side, the family of non-trivial extensions E of $\mathcal{O}_S(C)$ by $\mathcal{O}_S(-K_S - C)$ parameterized by $\mathbf{P}^{n-5}$ is replaced by the unique non-trivial opposite extension E_C. In fact, the moduli space is blow down to the point $[E_C]$. We denote the exceptional divisor $\simeq \mathbf{P}^{n-5}$ parameterizing E's in the moduli space by e_C.

Let $\Pi \subset \mathrm{Pic}\, S \otimes \mathbf{R}$ be the cone of ample divisor classes L on S such that $(L.2C + K_S) > 0$ for every conic $C \subset S$.

Lemma 3 *If $E \in \overline{M}_{S,L}$ is strictly μ-semi-stable with respect to an ample divisor $L \in \overline{\Pi}$, then we have either $(2l + K_S.L) = 0$ for a line l or $(2C + K_S.L) = 0$ for a conic C.*

Proof. E is an extension of a line bundle by another line bundle of the same degree outside a finite set of points. By Lemma 2, one of these two line bundles has a nonzero global section and is isomorphic to $\mathcal{O}_S(D)$ for an effective divisor D. By the strict μ-semi-stability, we have $(2D+K_S.L)=0$. Assume that $h^0(\mathcal{O}_S(D))=1$. Then D is supported by a disjoint union of lines $l_1,\dots,l_n$. Since $2=(l_1.-K_S-l_1)\leq(D.-K_S-D)\leq c_2(E)=2$, we have $D=l_1$. Assume that $h^0(\mathcal{O}_S(D))\geq 2$. Then either $|D+K_S|\neq\emptyset$ or $|D-C|\neq\emptyset$ for a conic C. But the former contradicts to $(2D+K_S.L)=0$. The latter implies $D-C=0$ since $L\in\overline{\Pi}$. □

Let $\mathcal{C}$ be a *chamber* of Π, that is, a connected component of the complement of $\bigcup_{l:\,\mathrm{line}} H_{l,1}$ in Π. For every $L\in\mathcal{C}$, every member $E\in\overline{M}_{S,L}$ is stable. Hence all $M_{S,L}$ $(=\overline{M}_{S,L})$, $L\in\mathcal{C}$, are isomorphic to each other. We denote this isomorphism class by $M_{S,\mathcal{C}}$. In particular, $M_{S,L}$'s, $L\in\Pi$, are isomorphic to each other in codimension one.

We relate $M_{S,L}$ with the blow-up X_G. By the Riemann-Roch formula, we have $\chi(\mathcal{H}om\,(E,\mathcal{O}_S(h)))=1$. Since $H^2(S,\mathcal{H}om\,(E,\mathcal{O}_S(h)))\simeq \mathrm{Hom}\,(\mathcal{O}_S(h),E(K_S))^\vee=0$, we have $\dim\mathrm{Hom}\,(E,\mathcal{O}_S(h))\geq 1$ for every semi-stable bundle $E\in\overline{M}_{S,L}$. In particular, if $(L.-K_S)/2>(L.h)$, then the moduli space $\overline{M}_{S,L}$ is empty. For example, this applies if $L=ah-K_S$ and if $a>n-3$. In the range $n-5<a<n-3$, a nonzero homomorphism $f:E\longrightarrow\mathcal{O}_S(h)$ is surjective and unique up to constant multiplication. Hence $M_{S,L}$ is isomorphic to the $(n-4)$-dimensional projective space $\mathbf{P}_*\mathrm{Ext}^1(\mathcal{O}_S(h),\mathcal{O}_S(2h-e))\simeq\mathbf{P}_*H^1(\mathbf{P}^2,I_{q_1,\dots,q_n}(1))$, where we put $e=\sum_1^n e_i$. This identification is nothing but (2) in the introduction.

Among these extensions E of $\mathcal{O}_S(h)$ by $\mathcal{O}_S(2h-e)$, there is a unique E_i which contains $\mathcal{O}_S(h-e_i)$ as its subsheaf for each $1\leq i\leq n$. E_i is nothing but $\tilde{E}_{p_i}\otimes\mathcal{O}_S(h)$ in the introduction. Hence $M_{S,L}$ is the blow-up X_G of the $\mathbf{P}^{n-4}$ at the n points $p_1,\dots,p_n$ between $a=n-5$ and the next critical value $(=n-7)$. Since $ah-K_S$ belongs to Π for $n-7<a<n-5$, $M_{S,\mathcal{C}}$ is isomorphic to X_G in codimension one for every chamber $\mathcal{C}\subset\Pi$. When $a=n-7$, we have $(2l+K_S.ah-K_S)=0$ for every $l=h-e_i-e_j$, $1\leq i<j\leq n$. In fact, at $a=n-7$ the moduli space $M_{S,ah-K_S}$ is flopped with center the strict transforms of lines joining p_i and p_j.

A line l yields another 1-cycle other than Z_l. Let $\pi:S\longrightarrow S'$ be the blow-down of $l\subset S$ to a point q on a smooth surface S' and assume that an ample divisor L is sufficiently near to the pull-back of an ample divisor

L' on S'. The direct image $\pi_* E$ of a member E of $M_{S,L}$, is not locally free at $q \in S'$. But its double dual belongs to $\overline{M}_{S',L'}$ and we get a morphism

$$M_{S,L} \longrightarrow \overline{M}_{S',L'}, \quad E \mapsto (\pi_* E)^{\vee\vee}. \tag{5}$$

This morphism is a $\mathbf{P}^1$-bundle ove the open set $M_{S',L'}$ and interprets Example 1 moduli theoretically in the case $s = 3$. We denote by F_l a general fiber of this morphism.

The following is a substitute for Lemma 1 in the cases [7] and [9].

Lemma 4 *Let l be a line. Then*

$$2(Z_l.D) = -(2l + K_S.\varphi(D)) \quad \text{and} \quad (F_l.D) = (l.\varphi(D))$$

hold for every divisor D on X.

Proof. We prove the case $n = 8$. Other cases are similar and easier. The isomorphism φ is $W(E_8)$-equivariant and the Weyl group $W(E_8)$ acts transitively on the set of 240 classes of all lines. Hence, by Proposition 4, it suffices to verify the assertion for one line l. For the first formula, we take $h - e_1 - e_2$ as l. As we saw above, Z_l is the strict transform of the line passing through p_1 and p_2. Hence we have $(Z_l.e_1) = (Z_l.e_2) = 1$, $(Z_l.e_i) = 0$ for $3 \le i \le 8$ and $(Z_l. - K_X) = -1$. On the other hand we have $(l.h - e_1) = (l.h - e_2) = 0$, $(l.h - e_i) = 1$ for $3 \le i \le 8$ and $(l. - 2K_S) = 1$. Hence, we have the equality $(Z_l.D) = -(\frac{1}{2}K_S + l.\varphi(D))$ for $D = e_1, \dots, e_8, -K_X$ by (3). Since $e_1, \dots, e_8$ and $-K_X$ generate $\operatorname{Pic} X \otimes \mathbf{Q}$, the equality holds for every D.

For the second formula, we take e_8 as l. By Example 1, F_l is the strict transform of a general line passing through p_8. Hence we have $(F_l.e_i) = 0$ for $1 \le i \le 7$, $(F_l.e_8) = 1$ and $(F_l. - K_X) = 2$. These intersection numbers on X are equal to $(e_8.h - e_i)$ and $(e_8. - 2K_S)$, respectively. □

By the lemma, the hyperplanes $H_{l,1}$ and $H_{l,0}$ are mapped to those in $\operatorname{Pic} X \otimes \mathbf{R}$ defined by the 1-cycles Z_l and F_l by φ^{-1} respectively. A similar computation shows that $H_{C,1}$ is mapped to the hyperplane defined by Z_C for every conic C.

Proof of Theorem. We prove the theorem by the induction on $n = (6,)7$ and 8. First we show the finite generation of $\mathcal{TC}(X_G)$ over $\varphi^{-1}\overline{\Pi} \subset \operatorname{Pic} X \otimes \mathbf{R}$. This is equivalent to the following:

Claim. The $\varphi^{-1}\overline{C}$-part of $\mathcal{TC}(X_G)$ is finitely generated for every chamber $\mathcal{C}$ in Π.

Every facet $\overline{\Pi}$ corresponds to either the blow-down of $e_C \simeq \mathbf{P}^{n-5}$ or a generic $\mathbf{P}^1$-bundle over $\overline{M}_{S',L'}$, where S' is the blow-down of a line from S. The blow-down of e_C is isomorphic in codimension one to $Bl_{n-1}\mathbf{P}^{n-4}$. Hence, by induction and by the result of §1, $\varphi^{-1}\mathcal{F}$-part of $\mathcal{TC}(X_G)$ is finitely generated for every facet $\mathcal{F}$ of Π. Let $R_1, \dots, R_n$ be the edges of $\overline{\mathcal{C}}$ contained in Π. We choose an ample divisor L_i on S from each R_i. By the GIT construction, $\overline{M}_{S,L_i}$ carries a natural ample (Cartier) divisor, which we denote by D_i. Then D_i is semi-ample on $\overline{M}_{S,\mathcal{C}}$. By the first formula of Lemma 4, D_i belongs to the ray $\varphi^{-1}R_i$. Therefore, by a lemma of Zariski ([HK, Lemma 2.8]), $\varphi^{-1}\mathcal{C}$-part of $\mathcal{TC}(X_G)$ is finitely generated. Thus the claim is proved.

The cone $\varphi^{-1}\overline{\Pi}$ is defined by two kinds of supporting hyperplanes, $\varphi^{-1}H_{C,1}$'s of divisorial (contraction) type and $\varphi^{-1}H_{l,0}$'s of fiber type. By the same argument as the case [4] in §1, $\mathcal{TC}(X_G)$ is generated by its $\varphi^{-1}\overline{\Pi}$-part and $\bigoplus_{C\,:\,\mathrm{conic}} H^0(\mathcal{O}_X(e_C))$. □

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