FLOP INVARIANCE OF THE TOPOLOGICAL VERTEX

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Abstract. We prove transformation formulae for generating functions of Gromov–Witten invariants on general toric Calabi–Yau threefolds under flops. Our proof is based on a combinatorial identity on the topological vertex and analysis of fans of toric Calabi–Yau threefolds.

1. Introduction

Motivated by a conjecture [Mor, W] on quantum cohomology, Li and Ruan studied the transformation of Gromov–Witten (GW) invariants of projective Calabi–Yau (CY) threefolds under flops using symplectic approach [LR]. The algebro-geometric approach was pursued in [LY]. The same problem for Donaldson–Thomas invariants was studied in [HL], and this may be related since there is a conjecture that Donaldson–Thomas invariants and GW invariants are related at the level of generating functions [MNOP1, MNOP2].

In this paper, we study the behavior of GW invariants of toric Calabi–Yau (TCY) threefolds (which are noncompact) under a flop based on the method of the topological vertex. It is a formalism which expresses the partition functions of GW invariants of TCY threefolds in terms of symmetric functions [AKMV]. (In this paper, the partition function of GW invariants means the exponential of the generating function.) Although its original argument was based on the duality to the Chern-Simons theory, a mathematical theory including a definition of GW invariants for TCY threefolds has been developed later in [LLLZ] (see Remark 3.2). We remark that in [IK3], the case of some special TCY threefolds was studied (see remark 4.5).

Let us explain the results of this paper. Let X be a TCY threefold containing a torus invariant rational curve C such that its normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$. Let X^+ be another TCY threefold obtained by flopping C. Identifying the expansion parameters with respect to second homology classes, we can compare the partition function of GW invariants of X and that of X^+. We show that they are equal except for factors coming from multiples of $[C]$ and from multiples of the class $[C^+]$ of the flopped curve C^+ (Theorem 4.4). Since the difference between the two appears only at the local contributions from neighborhoods of C and C^+, showing the equality of two partition functions reduces to showing a combinatorial identity on skew Schur functions (Theorem 2.7). Then we obtain the same result as [LR, LY] on the relation between GW invariants of X and

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those of X^+. As an example, we consider the TCY threefold X containing two disjoint $\mathbb{P}^1 \times \mathbb{P}^1$'s and another related by a flop. We also show that the partition function of X reproduces Nekrasov's partition function of 4-dimensional $\mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory with a matter in the bifundamental representation $(\mathbf{2}, \bar{\mathbf{2}})$ [N] (Proposition 5.1). As another application, we consider the canonical bundle K_S of a complete smooth toric surface S and the canonical bundle $K_{\hat{S}}$ of a blown-up surface $\hat{S}$ and show that GW invariants of $K_{\hat{S}}$ with certain second homology classes are equal to those of K_S (Proposition 6.1).

The organization of this paper is as follows. In §2, we prove a key combinatorial identity. In §3, we give a definition of TCY threefolds used in this paper and review the method to write down their partition functions. In §4, we study the transformations of partition functions under a flop. In §5, we give an example and discuss the relationship with Nekrasov's partition function. In §6, we study GW invariants of the canonical bundles of smooth toric surfaces related by a blowup. Combinatorial formulae are collected in Appendix A.

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2. Topological vertex under Flops

2.1. **Definitions.** Let $\mathcal{P}$ be the set of partitions. For $\mu = (\mu_1 \geq \mu_2 \geq \cdots) \in \mathcal{P}$, we define two integers $|\mu|$ and $\kappa(\mu)$ by

$$|\mu| = \sum_{j=1}^{l(\mu)} \mu_j \ , \quad \kappa(\mu) = |\mu| + \sum_{j=1}^{l(\mu)} \mu_j(\mu_j - 2j) \ ,$$

where $l(\mu)$ is the number of nonzero components in μ.

We use the following definition of the topological vertex ([ORV]):

Definition 2.1.

$$C_{\lambda_1,\lambda_2,\lambda_3}(q) \stackrel{\text{def.}}{=} q^{\frac{1}{2}\kappa(\lambda_3)} s_{\lambda_2}(q^\rho) \sum_{\mu \in \mathcal{P}} s_{\lambda_1/\mu}(q^{\lambda_2^t+\rho}) s_{\lambda_3^t/\mu}(q^{\lambda_2+\rho}) \ , \tag{1}$$

where $s_{\mu/\nu}(q^{\mu+\rho})$ (resp. $s_\mu(q^\rho)$) is the skew Schur function with the specialization of variables:

$$s_{\mu/\nu}(x_i = q^{\mu_i - i + \frac{1}{2}}) \quad (\text{resp. } s_\mu(x_i = q^{-i+\frac{1}{2}})) \ .$$

Take four partitions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. These will be fixed throughout the rest of §2. We define

$$Z_0(q, Q_0) \overset{\text{def.}}{=} \sum_{\mu \in \mathcal{P}} (-Q_0)^{|\mu|} C_{\lambda_1,\lambda_2,\mu^t}(q) C_{\lambda_3,\lambda_4,\mu}(q) \ , \tag{2}$$

$$Z_0^+(q, Q_0^+) \overset{\text{def.}}{=} \sum_{\mu \in \mathcal{P}} (-Q_0^+)^{|\mu|} C_{\lambda_1,\mu^t,\lambda_4}(q) C_{\lambda_3,\mu,\lambda_2}(q) \ . \tag{3}$$

We also set

$$Z_{(-1,-1)}(q, Q) = \prod_{k=1}^{\infty} (1 - Qq^k)^k, \tag{4}$$

and

$$Z_0'(q, Q_0) \overset{\text{def.}}{=} \frac{Z_0(q, Q_0)}{Z_{(-1,-1)}(q, Q_0)} \ , \qquad Z_0^{+\prime}(q, Q_0^+) \overset{\text{def.}}{=} \frac{Z_0^+(q, Q_0^+)}{Z_{(-1,-1)}(q, Q_0^+)} \ .$$

The goal of this section is to show an identity relating $Z_0'(q, Q_0)$ and $Z_0^{+\prime}(q, Q_0^+)$ under the identification $Q_0^+ = Q_0^{-1}$ (Theorem 2.7). Formulae necessary for proofs can be found in Appendix A.

Remark 2.2. Let us mention the geometrical meaning of the above formal power series. $Z_{(-1,-1)}(q, Q_0)$ is the partition function of the TCY threefold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (cf. §3.2, see also [EK1, (C.18)] and [FP, Theorem 3]). $Z_0(q, Q_0)$ and $Z_0^+(q, Q_0^+)$ appear as local contributions in the partition functions of TCY threefolds related by a flop such that both a flopping curve and a flopped curve have normal bundles isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (see Figure 3).

2.2. **Individual calculations.** First, we compute $Z_0'(q, Q_0)$ and $Z_0^{+\prime}(q, Q_0^+)$ respectively.

Let us introduce the following functions:

$$f_\mu(q) = \frac{q}{q-1} \sum_{i \geq 1} (q^{\mu_i - i} - q^{-i}) \ ,$$
$$f_{\mu,\nu}(q) = (q - 2 + q^{-1}) f_\mu(q) f_\nu(q) + f_\mu(q) + f_\nu(q) \ ,$$

and let $C_k(\mu, \nu)$ be the expansion coefficients in the Laurent polynomial $f_{\mu,\nu}(q)$:

$$f_{\mu,\nu}(q) = \sum_{k \in \mathbb{Z}} C_k(\mu, \nu) q^k \ .$$

Proposition 2.3. *We have*

$$\begin{aligned} Z_0'(q, Q_0) &= q^{\frac{1}{2}\kappa(\lambda_2) + \frac{1}{2}\kappa(\lambda_4)} s_{\lambda_1}(q^\rho) s_{\lambda_3}(q^\rho) \prod_{k \in \mathbb{Z}} (1 - Q_0 q^k)^{C_k(\lambda_1^t, \lambda_3^t)} \\ &\times \sum_\tau (-Q_0)^{|\tau|} s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho}, Q_0 q^{-\lambda_3-\rho}) s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho}, Q_0 q^{-\lambda_1-\rho}) \ , \end{aligned} \tag{5}$$

$$
\begin{aligned}
{Z_0^+}'(q,Q_0^+) &= s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\prod_{k\in\mathbb{Z}}(1-Q_0^+q^k)^{C_k(\lambda_1,\lambda_3)} \\
&\qquad \times\sum_\tau(-Q_0^+)^{|\tau|}s_{\lambda_2/\tau}(q^{\lambda_3^t+\rho},Q_0^+q^{-\lambda_1^t-\rho})s_{\lambda_4/\tau^t}(q^{\lambda_1^t+\rho},Q_0^+q^{-\lambda_3^t-\rho})\ .
\end{aligned} \tag{6}
$$

Proof. By definition (1) of the topological vertex, we have

$$
\begin{aligned}
Z_0(q,Q_0) &= \sum_\mu(-Q_0)^{|\mu|}q^{\frac12\kappa(\lambda_2)}s_{\lambda_1}(q^\rho)\sum_T s_{\mu^t/T}(q^{\lambda_1^t+\rho})s_{\lambda_2^t/T}(q^{\lambda_1+\rho}) \\
&\qquad q^{\frac12\kappa(\lambda_4)}s_{\lambda_3}(q^\rho)\sum_{T'}s_{\mu/T'}(q^{\lambda_3^t+\rho})s_{\lambda_4^t/T'}(q^{\lambda_3+\rho}) \\
&= q^{\frac12\kappa(\lambda_2)+\frac12\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\sum_{T,T'}(-Q_0)^{|T|}s_{\lambda_2^t/T}(q^{\lambda_1+\rho})s_{\lambda_4^t/T'}(q^{\lambda_3+\rho}) \\
&\qquad \sum_\mu s_{\mu^t/T}(-Q_0q^{\lambda_1^t+\rho})s_{\mu/T'}(q^{\lambda_3^t+\rho})\ .
\end{aligned}
$$

We perform the sum with respect to μ by using (19):

$$
\begin{aligned}
Z_0(q,Q_0) &= q^{\frac12\kappa(\lambda_2)+\frac12\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho) \\
&\qquad \prod_{i,j\geq1}(1-Q_0q^{h_{\lambda_1^t,\lambda_3^t}(i,j)})\sum_\tau s_{T^t/\tau}(q^{\lambda_3^t+\rho})s_{(T')^t/\tau^t}(-Q_0q^{\lambda_1^t+\rho}) \\
&\qquad \sum_{T,T'}(-Q_0)^{|T|}s_{\lambda_2^t/T}(q^{\lambda_1+\rho})s_{\lambda_4^t/T'}(q^{\lambda_3+\rho}) \\
&= q^{\frac12\kappa(\lambda_2)+\frac12\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\prod_{i,j\geq1}(1-Q_0q^{h_{\lambda_1^t,\lambda_3^t}(i,j)}) \\
&\qquad \sum_\tau(-Q_0)^{|\tau|}\sum_T s_{\lambda_2^t/T}(q^{\lambda_1+\rho})s_{T/\tau^t}(Q_0q^{-\lambda_3-\rho})\sum_{T'}s_{\lambda_4^t/T'}(q^{\lambda_3+\rho})s_{T'/\tau}(Q_0q^{-\lambda_1-\rho})\ .
\end{aligned}
$$

Here for $\mu,\nu\in\mathcal{P}$,

$$
h_{\mu,\nu}(i,j)\overset{\text{def.}}{=}\mu_i-i+\nu_j-j+1\ .
$$

In passing to the second line, we have used (22). By using (20), we have

$$
\begin{aligned}
Z_0(q,Q_0) &= q^{\frac12\kappa(\lambda_2)+\frac12\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\prod_{i,j\geq1}(1-Q_0q^{h_{\lambda_1^t,\lambda_3^t}(i,j)}) \\
&\qquad \sum_\tau(-Q_0)^{|\tau|}s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho},Q_0q^{-\lambda_3-\rho})s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho},Q_0q^{-\lambda_1-\rho})\ .
\end{aligned}
$$

Applying Lemma A.1, we obtain (5). One can also compute $Z_0^+(q,Q_0^+)$ in a similar way. □

The next corollary is a consequence of Proposition 2.3.

Corollary 2.4. *${Z_0^+}'(q,Q_0^+)$ is a polynomial in Q_0^+ of degree at most $|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|$. Moreover, if $\lambda_3=\lambda_4=\emptyset$, ${Z_0^+}'(q,Q_0^+)$ is a polynomial in Q_0^+ of degree $|\lambda_1|+|\lambda_2|$.*

Similar statement also holds for $Z_0'(q,Q_0)$.

Proof. The first statement follows if we apply (16) and (21) to the expression (6). To prove the second statement, we show that the top term does not vanish. By (16), we have

$$\prod(1-Q_0^+q^k)^{C_k(\lambda_1^t,\emptyset)} = (-1)^{|\lambda|}q^{-\frac{1}{2}\kappa_{\lambda_1}}(Q_0^+)^{|\lambda_1|} + (\text{terms of lower degree in } Q_0^+).$$

Substituting this into (6) with λ_3, λ_4 set to $\emptyset$, and using (21), we obtain the claim. □

2.3. **Comparison.** Next, we compare $Z_0'(q,Q_0)$ with $Z_0^{+\prime}(q,Q_0^+)$ under the identification $Q_0^+ = Q_0^{-1}$. First we have the following

Lemma 2.5. *Under the identification $Q_0^+ = Q_0^{-1}$, we have*

$$(7)\quad \sum_\tau (-Q_0^+)^{|\tau|} s_{\lambda_2/\tau}(q^{\lambda_3^t+\rho}, Q_0^+ q^{-\lambda_1^t-\rho}) s_{\lambda_4/\tau^t}(q^{\lambda_1^t+\rho}, Q_0^+ q^{-\lambda_3^t-\rho})$$
$$= (-Q_0)^{-|\lambda_2|-|\lambda_4|}\sum_\tau (-Q_0)^{|\tau|} s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho}, Q_0 q^{-\lambda_3-\rho}) s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho}, Q_0 q^{-\lambda_1-\rho})\ .$$

Proof. Under $Q_0^+ = Q_0^{-1}$, we have

$$\begin{aligned}
(\text{LHS}) &= \sum_\tau (-Q_0^{-1})^{|\tau|} s_{\lambda_2/\tau}(q^{\lambda_3^t+\rho}, Q_0^{-1} q^{-\lambda_1^t-\rho}) s_{\lambda_4/\tau^t}(q^{\lambda_1^t+\rho}, Q_0^{-1} q^{-\lambda_3^t-\rho}) \\
&= \sum_\tau (-Q_0^{-1})^{|\tau|}(-1)^{|\lambda_2|+|\lambda_4|} s_{\lambda_2^t/\tau^t}(q^{-\lambda_3-\rho}, Q_0^{-1} q^{\lambda_1+\rho}) s_{\lambda_4^t/\tau}(q^{-\lambda_1-\rho}, Q_0^{-1} q^{\lambda_3+\rho})\ , \\
&= \sum_\tau (-Q_0)^{-|\lambda_2|+|\tau|-|\lambda_4|+|\tau|}(-Q_0^{-1})^{|\tau|} s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho}, Q_0 q^{-\lambda_3^t-\rho}) s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho}, Q_0 q^{-\lambda_1^t-\rho}) \\
&= (\text{RHS})\ .
\end{aligned}$$

Note that we have used the property (22) in the second line and the homogeneity (21) of skew Schur functions in the third line. □

The next lemma was proven in [IK3, eq.(45)].

Lemma 2.6. *The following identity holds:*

$$\prod_k (1-Q_0^{-1}q^k)^{C_k(\lambda_1,\lambda_3)} = (-Q_0)^{-|\lambda_1|-|\lambda_3|} q^{\frac{1}{2}\kappa(\lambda_1)+\frac{1}{2}\kappa(\lambda_3)} \prod_k (1-Q_0q^k)^{C_k(\lambda_1^t,\lambda_3^t)}.$$

Proof. By (17), we have

$$\begin{aligned}
\prod_k (1-Q_0^{-1}q^k)^{C_k(\lambda_1,\lambda_3)} &= \prod_k (1-Q_0^{-1}q^{-k})^{C_k(\lambda_1^t,\lambda_3^t)} \\
&= Q_0^{-\frac{1}{2}(|\lambda_1|+|\lambda_3|)} q^{-\frac{1}{4}(\kappa(\lambda_1^t)+\kappa(\lambda_3^t))} \prod_k \left(Q_0^{\frac{1}{2}} q^{\frac{k}{2}} - Q_0^{-\frac{1}{2}} q^{-\frac{k}{2}}\right)^{C_k(\lambda_1^t,\lambda_3^t)}\ .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\prod_k (1-Q_0q^k)^{C_k(\lambda_1^t,\lambda_3^t)} &= Q_0^{\frac{1}{2}(|\lambda_1|+|\lambda_3|)} q^{\frac{1}{4}(\kappa(\lambda_1^t)+\kappa(\lambda_3^t))} \prod_k \left(Q_0^{-\frac{1}{2}} q^{-\frac{k}{2}} - Q_0^{\frac{1}{2}} q^{\frac{k}{2}} \right)^{C_k(\lambda_1^t,\lambda_3^t)} \\ &= (-1)^{|\lambda_1|+|\lambda_3|} Q_0^{\frac{1}{2}(|\lambda_1|+|\lambda_3|)} q^{\frac{1}{4}(\kappa(\lambda_1^t)+\kappa(\lambda_3^t))} \prod_k \left(Q_0^{\frac{1}{2}} q^{\frac{k}{2}} - Q_0^{-\frac{1}{2}} q^{-\frac{k}{2}} \right)^{C_k(\lambda_1^t,\lambda_3^t)} .\end{aligned}$$

By comparing the above two equations and by using a symmetry (15) of a κ-factor, we get the claim. □

The following is the main result in this section. (The case of $\lambda_1 = \lambda_4 = \emptyset$ was proved in [Ka].)

Theorem 2.7. *Under the identification $Q_0^+ = Q_0^{-1}$, we have*

$$Z_0^{+\prime}(q, Q_0^+) = (-Q_0)^{-(|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|)} q^{\frac{1}{2}(\kappa(\lambda_1)-\kappa(\lambda_2)+\kappa(\lambda_3)-\kappa(\lambda_4))} Z_0'(q, Q_0) .$$

Proof. This follows from Proposition 2.3 and Lemmas 2.5 and 2.6. □

3. Toric Calabi–Yau threefolds and Partition Functions

In this section, we give definitions of toric Calabi–Yau threefolds and their partition functions. Our reference is [Ko].

3.1. Toric Calabi–Yau threefolds.

Definition 3.1. A toric Calabi–Yau (TCY) threefold is a three-dimensional smooth toric variety X over $\mathbb{C}$ associated with a fan Σ satisfying following conditions:

(i) the primitive generator $\vec{\omega}$ of every 1-cone satisfies $\vec{\omega} \cdot \vec{u} = 1$ where $\vec{u} = (0, 0, 1)$;
(ii) all maximal cones are three dimensional;
(iii) $|\Sigma| \cap \{z = 1\}$ is simply connected where $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subset \mathbb{R}^3$ is the support of Σ and z is the third coordinate of $\mathbb{R}^3$.

The condition (i) is equivalent to the condition that $\wedge^3 T^*X$ is trivial (Calabi–Yau condition) and the condition (ii) implies that $\pi_1(X) = 0$. The condition (iii) is imposed for simplicity of arguments.

We briefly describe necessary facts on (co)homology of TCY threefolds. Recall that the subset $\Sigma_n \subset \Sigma$ of n-cones is in one-to-one correspondence with the set of $(3-n)$-dimensional torus invariant subvarieties in X. Let $\Sigma_1 = \{\rho_1, \ldots, \rho_r\}$ be the set of 1-cones. Denote by $\vec{\omega}_i$ $(1 \le i \le r)$ the primitive lattice vector generating ρ_i and by $D_{\rho_i} \subset X$ $(1 \le i \le r)$ the torus invariant Weil divisor corresponding to ρ_i. The group $A_2(X)$ of all Weil divisors modulo rational equivalence is generated by $D_{\rho_1}, \ldots, D_{\rho_r}$ with rational equivalence given by $\sum_{j=1}^{r} A_{ij} D_{\rho_j} = 0$ $(i = 1, 2, 3)$ ([F], the first proposition in §3.4) where $A = (A_{ij})$ is the $3 \times r$ matrix

$$A = \big(\vec{\omega}_1, \ldots, \vec{\omega}_r\big).$$

Let Σ_2' be the set of 2-cones which lie in the interior of $|\Sigma|$:

$$\Sigma_2' = \{\tau \in \Sigma_2 |\, \tau \subset |\Sigma| \setminus \partial|\Sigma|\}.$$

It is in one-to-one correspondence with the set of torus invariant (hence rational) curves in X. Let us write $\Sigma_2' = \{\tau_1, \ldots, \tau_p\}$ and let $C_{\tau_i} \subset X$ denote the rational curve corresponding to τ_i. We define $N_1^T(X)$ to be the set of 2-cycles generated by $C_{\tau_1}, \ldots, C_{\tau_p}$ modulo numerical equivalence. Note that by the intersection pairing $A_2(X) \times N_1^T(X) \to \mathbb{Z}$, $A_2(X) \otimes \mathbb{R}$ and $N_1^T(X) \otimes \mathbb{R}$ become dual to each other.

Now let us explain the calculation of the intersection numbers and numerical equivalence. If ρ_j and τ_i spans a 3-cone, $D_{\rho_j}.C_{\tau_i} = 1$ and if ρ_j and τ_i do not span a cone in the fan, $D_{\rho_j}.C_{\tau_i} = 0$ ([F],§5,1 p.98). If two 1-cones, say ρ_1, ρ_2, are contained in τ_i, then $D_{\rho_1}.C_{\tau_i}$ and $D_{\rho_2}.C_{\tau_i}$ are obtained via rational equivalence relations of D_{ρ_j}'s. For convenience, we introduce the following injective map

$$l_X : N_1^T(X) \to \{l \in \mathbb{Z}^r \mid A.l = \vec{0}\} = L_A, \quad Z \mapsto (D_{\rho_1}.Z, \ldots, D_{\rho_r}.Z). \tag{8}$$

Then $D_{\rho_1}.C_{\tau_i}$ and $D_{\rho_2}.C_{\tau_i}$ are obtained by solving the equation $A.l_X([C_{\tau_i}]) = \vec{0}$. (Hence they satisfy the relation $D_{\rho_1}.C_{\tau_i} + D_{\rho_2}.C_{\tau_i} = -2$.) The numerical equivalence can be read from linear relations between the vectors $l_X([C_{\tau_1}]), \ldots, l_X([C_{\tau_p}])$.

By the analysis of the gluing of local coordinate systems around C_{τ_i}, we see that its normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(D_{\rho_1}.C_{\tau_i}) \oplus \mathcal{O}_{\mathbb{P}^1}(D_{\rho_2}.C_{\tau_i})$. We will use a term a $(-1,-1)$-curve for a torus invariant curve with the normal bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

3.2. **Partition functions.** Let X be a TCY threefold and Σ be its fan. We briefly review how to write down the partition function of X.

First, consider the following directed graph Γ_X (called a toric graph) with labels on edges of a certain type. The vertex set is

$$V(\Gamma_X) = V_3(\Gamma_X) \cup V_1(\Gamma_X),\ V_3(\Gamma_X) = \{v_\sigma | \sigma \in \Sigma_3(X)\},\ V_1(\Gamma_X) = \{v_\tau | \tau \in \Sigma_2(X) \setminus \Sigma_2'(X)\}.$$

The edge set is

$$E(\Gamma_X) = E_3(\Gamma_X) \cup E_1(\Gamma_X),\ E_3(\Gamma_X) = \{e_\tau | \tau \in \Sigma_2'(X)\},\ E_1(\Gamma_X) = \{e_\tau | \tau \in \Sigma_2(X) \setminus \Sigma_2'(X)\}.$$

An edge $e_\tau \in E_3(\Gamma_X)$ joins $v_\sigma, v_{\sigma'} \in V_3(\Gamma)$ iff $\tau = \sigma \cap \sigma'$ (see Figure 1) and an edge $e_\tau \in E_1(\Gamma)$ joins $v_\sigma \in V_3(\Gamma_X)$ and $v_\tau \in V_1(\Gamma_X)$ iff σ is a unique 3-cone such that τ is a face of σ. (Note that a vertex in $V_3(\Gamma_X)$ is trivalent and a vertex in $V_1(\Gamma_X)$ is univalent.)

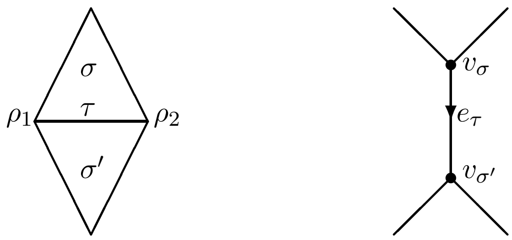

FIGURE 1. Fan (section at $z = 1$) and toric graph

$(\vec{\lambda}(e), \vec{\lambda}(e'), \vec{\lambda}(e''))$ $(\vec{\lambda}(e)^t, \vec{\lambda}(e'), \vec{\lambda}(e''))$ $(\vec{\lambda}(e)^t, \vec{\lambda}(e')^t, \vec{\lambda}(e''))$ $(\vec{\lambda}(e)^t, \vec{\lambda}(e')^t, \vec{\lambda}(e'')^t)$

FIGURE 2. $\vec{\lambda}_v$

The direction of edges can be taken arbitrarily. The label $n : E_3(\Gamma) \to \mathbb{Z}$, called the *framing*, is given as follows:

$$n(e_\tau) = \frac{D_{\rho_1}.C_\tau - D_{\rho_2}.C_\tau}{2} ,$$

where $\tau \in \Sigma_2'$ and $\rho_1, \rho_2 \in \Sigma_1$ are as shown in Figure 1. Note that Γ_X is connected by the condition (iii) in Definition 3.1

Secondly, we write down the partition function from Γ_X. Let

$$\mathcal{P}(\Gamma_X) = \{\vec{\lambda} : E_3(\Gamma) \to \mathcal{P}\}.$$

Take the set of formal variables $\vec{Q} = (Q_e)_{e \in E_3(\Gamma_X)}$ associated to $E_3(\Gamma_X)$. Then the partition function of X is a formal power series in $\vec{Q}$ given by

$$Z_X(q, \vec{Q}) = \sum_{\vec{\lambda} \in \mathcal{P}(\Gamma)} \prod_{e \in E_3(\Gamma)} (-1)^{|\vec{\lambda}(e)|(n_e+1)} q^{\frac{\kappa(\vec{\lambda}(e))}{2} n(e)} Q_e^{|\vec{\lambda}(e)|} \prod_{v \in V_3(\Gamma)} C_{\vec{\lambda}_v}(q) . \tag{9}$$

Here $C_{\vec{\lambda}_v}(q)$ is the topological vertex defined in (1) and $\vec{\lambda}_v$ ($v \in V_3(\Gamma)$, $\vec{\lambda} \in \mathcal{P}(\Gamma)$) is as in Figure 2 (for $e \in E(\Gamma_X) \setminus E_3(\Gamma_X)$, set $\vec{\lambda}(e)$ to $\emptyset$). We remark that the partition function does not depend on the directions of edges since the framing changes the sign if one gives the opposite direction to an edge $e \in E_3(\Gamma_X)$ and it is compensated by (15) and the summation.

Remark 3.2. Precisely speaking, the partition function obtained in [LLLZ] has the expression almost same as (9) except that $C_{\vec{\lambda}_v}(q)$ is replaced by $\tilde{\mathcal{W}}_{\vec{\lambda}_v}(q)$. Here $\tilde{\mathcal{W}}_{\lambda_1,\lambda_2,\lambda_3}(q)$ is a rational function in $q^{\frac{1}{2}}$ similar to $C_{\lambda_1,\lambda_2,\lambda_3}(q)$ but has a slightly different expression. It is conjectured that $\tilde{\mathcal{W}}_{\lambda_1,\lambda_2,\lambda_3}(q) = C_{\lambda_1,\lambda_2,\lambda_3}(q)$ [LLLZ, Conjecture 8.3]. Here we use $C_{\lambda_1,\lambda_2,\lambda_3}(q)$ assuming that the conjecture is true.

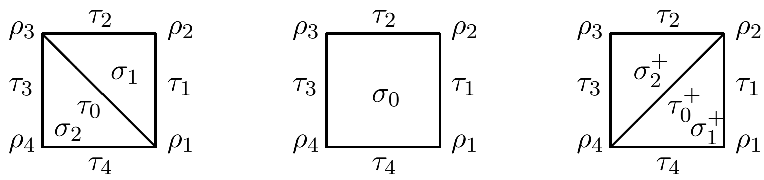

FIGURE 3. Fans (sections at $z = 1$): Σ (left), $\bar{\Sigma}$ (middle) and Σ^+ (right). The generators $\vec{\omega}_1, \ldots, \vec{\omega}_4$ of $\rho_1, \ldots, \rho_4$ satisfy the relation $\vec{\omega}_1 + \vec{\omega}_3 = \vec{\omega}_2 + \vec{\omega}_4$.

The Gromov–Witten invariant $N_{g,\beta}(X)$ of X with the genus g and the second homology class $\beta \in H_2^{cpt}(X, \mathbb{Z})$ (see [LLLZ] for a definition) is obtained as follows:

$$\sum_{g \geq 0} N_{g,\beta}(X) g_s^{2g-2} = \sum_{\substack{\vec{d}=(d_e)_{e \in E_3(\Gamma_X)}, \\ \vec{d}[\vec{C}]=[\beta]}} F_{\vec{d}}(e^{\sqrt{-1}g_s}), \tag{10}$$

where $[\vec{C}] = ([C_e])_{e \in E_3(\Gamma_X)}$ and $C_e \subset X$ is the rational curve corresponding to e. $F_{\vec{d}}(q)$ is the coefficient of $\vec{Q}^{\vec{d}} = \prod_{e \in E_3(\Gamma_X)} Q_e^{d_e}$ in $\log Z_X(q, \vec{Q})$.

4. Transformations of partition functions under flop

In this section, we study the transformation of the partition function of TCY threefolds under a flop.

Let X be a TCY threefold and let Σ be its fan. Assume that X contains at least one $(-1,-1)$-curve C_0. Denote the corresponding 2-cone by τ_0. Near τ_0, the fan looks like the left diagram in Figure 3. We set

$$\bar{\Sigma} = (\Sigma \setminus \{\tau_0, \sigma_1, \sigma_2\}) \cup \{\sigma_0\}, \quad \Sigma^+ = (\Sigma \setminus \{\tau_0, \sigma_1, \sigma_2\}) \cup \{\tau_0^+, \sigma_1^+, \sigma_2^+\}$$

where $\tau_0, \sigma_1, \sigma_2, \sigma_0, \tau_0^+ \sigma_1^+, \sigma_2^+$ are cones shown in Figure 3. Let Y be the singular toric variety associated with the fan $\bar{\Sigma}$ and X^+ be the TCY threefold associated with the fan Σ^+. Then associated to the evident maps $\Sigma \to \bar{\Sigma}$ and $\Sigma^+ \to \bar{\Sigma}^+$, there are the following birational maps:

$$\begin{array}{ccc} X & \stackrel{\phi}{\dashrightarrow} & X^+ \\ f \searrow & & \swarrow f^+ \\ & Y & \end{array}$$

The map f is a small contraction with the exceptional set C_0 and ϕ is a flop of f.

Remark 4.1. Since Σ and Σ^+ have the same set of 1-cones, there is a canonical isomorphism $A_2(X) \cong A_2(X^+)$ induced by ϕ. In turn, this induces an isomorphism $\phi_* : N_1^T(X) \otimes \mathbb{R} \to N_1^T(X^+) \otimes \mathbb{R}$ via the duality between $A_2(\cdot) \otimes \mathbb{R}$ and $N_1^T(\cdot) \otimes \mathbb{R}$ where $\cdot = X, X^+$.

From here on, we proceeds assuming that $\tau_1, \ldots, \tau_4 \in \Sigma_2'$. Other cases can be recovered by setting to zero the formal variables associated to any of $\tau_1, \ldots, \tau_4$ which are not in Σ_2'. We use the notations shown in Table 1.

	X		X^+	
2-cone	$\tau_0, \tau_1, \dots, \tau_4$	τ	$\tau_0^+, \tau_1, \dots, \tau_4$	τ
curve	$C_0, C_1, \dots, C_4$	C_τ	$C_0^+, C_1^+, \dots, C_4^+$	C_τ^+
edge	$e_0, e_1, \dots, e_4$	e_τ or just e	$e_0^+, e_1^+, \dots, e_4^+$	e_τ or just e
variable	$Q_0, Q_1, \dots, Q_4$	Q_e	$Q_0^+, Q_1^+, \dots, Q_4^+$	Q_e

TABLE 1.

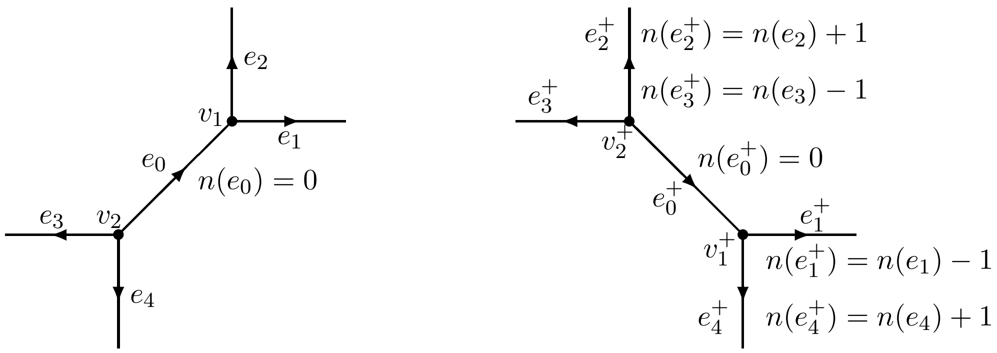

FIGURE 4. Toric graphs Γ_X (left) and Γ_{X^+} (right).

Lemma 4.2. *Under the flop $\phi : X \dashrightarrow X^+$, the curve classes transform as follows.*

$$\phi_*[C_0] = -[C_0^+], \qquad \phi_*[C_i] = [C_i^+]+[C_0^+], \qquad \phi_*[C_\tau] = [C_\tau^+] \quad (\tau \in \Sigma_2'(X) \backslash \{\tau_0, \dots, \tau_4\}).$$

Proof. The first statement follows from $l_X([C_0]) = -l_{X^+}([C_0^+])$ by remark 4.1. The proof of the other two is similar. □

Let Γ_X be a toric graph of X. Near the edge e_0, the graph looks like the left diagram in Figure 4. Under the flop ϕ, the toric diagram (and the framings) changes as follows.

Lemma 4.3. *A graph obtained from Γ_X by replacing the left diagram in Figure 4 with the right is a toric graph of X^+.*

We associate the same formal variables $\vec{Q} = (Q_e)$ to edges in $E_3(\Gamma_X) \setminus \{e_0, \dots, e_4\}$ and those in $E_3(\Gamma_{X^+}) \setminus \{e_0^+, \dots, e_4^+\}$ and write the partition functions of X and X^+ as $Z_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4)$ and $Z_{X^+}(q, \vec{Q}, Q_0^+, Q_1^+, Q_2^+, Q_3^+, Q_4^+)$ respectively. It is immediate to check that

$$\begin{aligned} Z_X(q, \vec{0}, Q_0, 0, 0, 0, 0) &= Z_{(-1,-1)}(q, Q_0), \\ Z_{X^+}(q, \vec{0}, Q_0^+, 0, 0, 0, 0) &= Z_{(-1,-1)}(q, Q_0^+). \end{aligned} \tag{11}$$

We set

$$Z'_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4) \overset{\text{def.}}{=} \frac{Z_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4)}{Z_X(q,\vec{0},Q_0,0,0,0,0)},$$
$$Z'_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+) \overset{\text{def.}}{=} \frac{Z_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+)}{Z_{X^+}(q,\vec{0},Q_0^+,0,0,0,0)}.$$

Now we will compare these. To do so, we should identify the formal variables so that the identification is compatible with Lemma 4.2:

$$Q_0=(Q_0^+)^{-1}, \qquad Q_i=Q_0^+Q_i^+.$$

Theorem 4.4. (i) *The coefficients of $\vec{Q}^{\vec{d}}Q_0^{d_0}Q_1^{d_1}\dots Q_4^{d_4}$ in $Z'_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4)$ is zero if $d_0>d_1+d_2+d_3+d_4$. A similar result holds for X^+.*

(ii) *Under the identification $Q_0=(Q_0^+)^{-1}$, $Q_i=Q_0^+Q_i^+$, we have*

$$Z'_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4)=Z'_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+).$$

(This is an equality between two formal power series in $\vec{Q},Q_0^+,\dots,Q_4^+$.)

Remark 4.5. In [IK3, §4.1], Iqbal and Kashani–Poor studied the special case such that the 2-cones $\tau_2,\tau_4\notin\Sigma'_2$ and the curves C_1,C_3 have normal bundles $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}}(-1)$ or $\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(0)$. They obtained the result of Lemma 4.2 and proved the second statement of Theorem 4.4 in that case.

Proof. (i) follows from the first statement of Corollary 2.4.

(ii) Let

$$\mathcal{P}'(\Gamma_X)=\{\vec{\nu}:E_3(\Gamma_X)\setminus\{e_0\}\to\mathcal{P}\}$$

and define $\vec{\nu}_v\in\mathcal{P}^3$ for $\vec{\nu}\in\mathcal{P}'(\Gamma_X)$ and $v\in V_3(\Gamma_X)\setminus\{v_1,v_2\}$ in the same way as $\vec{\lambda}_v$ (Figure 2). After (9), $Z_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4)$ is written as follows:

$$\begin{aligned}&Z_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4)\\&=\sum_{\vec{\nu}\in\mathcal{P}'(\Gamma_X)}\prod_{e\in E_3(\Gamma_X)\setminus\{e_0,\dots,e_4\}}(-1)^{(n(e)+1)|\vec{\nu}(e)|}Q_e^{|\vec{\nu}(e)|}\prod_{v\in V_3(\Gamma_X)\setminus\{v_1,v_2\}}C_{\vec{\nu}_v}(q)\\&\times\underbrace{\prod_{i=1}^4(-1)^{(n(e_i)+1)|\vec{\nu}(e_i)|}Q_i^{|\vec{\nu}(e_i)|}\sum_{\mu\in\mathcal{P}}C_{\vec{\nu}(e_1),\vec{\nu}(e_2),\mu^t}(q)C_{\vec{\nu}(e_3),\vec{\nu}(e_4),\mu}(q)(-Q_0)^{|\mu|}}_{(a)}.\end{aligned}$$

Similarly, $Z_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+)$ is written as follows:

$$\begin{aligned}
&Z_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+)\\
&=\sum_{\vec{\nu}\in\mathcal{P}'(\Gamma_{X^+})}\prod_{e\in E_3(\Gamma_{X^+})\setminus\{e_0^+,\dots,e_4^+\}}(-1)^{(n(e)+1)|\vec{\nu}(e)|}Q_e^{|\vec{\nu}(e)|}\prod_{v\in V_3(\Gamma_{X^+})\setminus\{v_1^+,v_2^+\}}C_{\vec{\nu}_v}(q)\\
&\times\underbrace{\prod_{i=1}^{4}(-1)^{(n(e_i^+)+1)|\vec{\nu}(e_i^+)|}(Q_i^+)^{|\vec{\nu}(e_i^+)|}\sum_{\mu\in\mathcal{P}}C_{\vec{\nu}(e_1^+),\mu^t,\vec{\nu}(e_4^+)}(q)C_{\vec{\nu}(e_3^+),\mu,\vec{\nu}(e_2^+)}(q)(-Q_0^+)^{|\mu|}}_{(b)}.
\end{aligned}$$

Here

$$\mathcal{P}'(\Gamma_{X^+})=\{\vec{\nu}:E_3(\Gamma_{X^+})\setminus\{e_0^+\}\to\mathcal{P}\}$$

and for $\vec{\nu}\in\mathcal{P}'(\Gamma_{X^+})$ and $v\in V_3(\Gamma_{X^+})\setminus\{v_1^+,v_2^+\}$, $\vec{\nu}_v\in\mathcal{P}^3$ is defined in the same way.

Since Γ_X and Γ_{X^+} are identical outside the diagrams described in Figure 4, $E_3(\Gamma_X)\setminus\{e_0,\dots,e_4\}=E_3(\Gamma_{X^+})\setminus\{e_0^+,\dots,e_4^+\}$, $V_3(\Gamma_X)\setminus\{v_1,v_2\}=V_3(\Gamma_{X^+})\setminus\{v_1^+,v_2^+\}$ and we have a natural bijection $p:\mathcal{P}'(\Gamma_X)\to\mathcal{P}'(\Gamma_{X^+})$ such that $p(\vec{\nu})=\vec{\nu}^+$ iff $\vec{\nu}(e)=\vec{\nu}^+(e)$ for all $e\in E_3(\Gamma_X)\setminus\{e_0,\dots,e_4\}$ and $\vec{\nu}(e_i)=\vec{\nu}^+(e_i^+)$ for $1\le i\le 4$. Under this identification, we could see that the two partition functions have the same expressions except for the factors (a) and (b). Taking into account the change in framings, we have

$$\left.\frac{(a)}{Z_{(-1,-1)}(q,Q_0)}\right|_{Q_0=(Q_0^+)^{-1},Q_i=Q_0^+Q_i^+}=\frac{(b)}{Z_{(-1,-1)}(q,Q_0^+)}$$

by Theorem 2.7. □

We finish this subsection by restating Theorem 4.4 in terms of GW invariants. (Compare with [LR, Corollary A.1] and [LY, Theorem 3.1.1].)

Corollary 4.6. *For $\beta\in H_2^{cpt}(X,\mathbb{Z})$ such that β is not a multiple of $[C_0]$,*

$$N_{g,\phi_*(\beta)}(X^+)=N_{g,\beta}(X).$$

Moreover,

$$N_{g,d[C_0]}(X)=N_{g,d[C_0^+]}(X^+)=N_{g,d[\mathbb{P}^1]}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)).$$

Proof. Theorem 4.4 implies that $\log Z_X(q,\vec{Q},Q_0,\dots,Q_4)$ and $\log Z_{X^+}(q,\vec{Q},Q_0^+,\dots,Q_4^+)$ are written in the following form:

$$\log Z'_X(q,\vec{Q},Q_0,\dots,Q_4)=\sum_{\vec{d}}\sum_{\substack{d_0,\dots,d_4\ge 0,\\ d_1+\cdots+d_4\ge d_0}}F_{\vec{d},d_0,d_1,d_2,d_3,d_4}(q)\vec{Q}^{\vec{d}}Q_0^{d_0}\dots Q_4^{d_4},$$

$$\log Z'_{X^+}(q,\vec{Q},Q_0^+,\dots,Q_4^+)=\sum_{\vec{d}}\sum_{\substack{d_0,\dots,d_4\ge 0,\\ d_1+\cdots+d_4\ge d_0}}F_{\vec{d},d_0,d_1,d_2,d_3,d_4}(q)\vec{Q}^{\vec{d}}(Q_0^+)^{d_1+\cdots+d_4-d_0}(Q_1^+)^{d_1}\dots(Q_4^+)^{d_4}.$$

Comparing with (10), we obtain the first statement. The second statement follows from (11). □

5. Example and geometric engineering

In this section, we first give an example of §4. Then we will discuss its relation with Nekrasov's partition function [N] along the same lines with [IK1, IK2, EK1, EK2, Z].

Let X and X^+ be the TCY threefolds associated with the left and right toric graphs in Figure 5, respectively. X contains two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ disjoint to each other and X^+ is obtained by a flop of a unique $(-1,-1)$-curve in X. In this example, formal variables should be assigned as in Figure 5: the five variables for X are independent and the nine variables for X^+ have the four relations $Q_{F_i} = Q^+_{F_i} Q^+_0$ $(i = 1,2)$ and $Q_{B_i} = Q^+_{B_i} Q^+_0$ $(i = 1,2)$. The variables of X and X^+ should be identified by $Q_0^+ = Q_0^{-1}$ and Q_{F_i}, Q_{B_i} of $X^+ = Q_{F_i}, Q_{B_i}$ of X.

Let us compute the partition function Z_X of X (we omit the variables). By Proposition 2.3, we have

$$Z_X = \sum_{\mu_1^1,\mu_2^1,\mu_1^2,\mu_2^2} \prod_{k=1}^{2} (Q_{B_k})^{|\mu_1^k|+|\mu_2^k|} s^2_{\mu_1^k}(q^\rho) s^2_{\mu_2^k}(q^\rho) \prod_{i,j\geq 1} \left(1 - Q_{F_k} q^{h_{\mu_1^k,(\mu_2^k)^t}(i,j)}\right)^{-2}$$
$$\sum_{\lambda} (-Q_0)^{|\lambda|}\ s_{\lambda^t}(q^{\mu_2^1+\rho}, Q_{F_1} q^{\mu_1^1+\rho})\ s_\lambda(q^{(\mu_2^2)^t+\rho}, Q_{F_2} q^{(\mu_1^2)^t+\rho})\ .$$

We can perform the sum in the last factor by (19):

$$\begin{aligned}
&\sum_{\lambda} (-Q_0)^{|\lambda|} s_{\lambda^t}(q^{\mu_2^1+\rho}, Q_{F_1} q^{\mu_1^1+\rho}) s_\lambda(q^{(\mu_2^2)^t+\rho}, Q_{F_2} q^{(\mu_1^2)^t+\rho}) \\
&= \prod_{i,j\geq 1} \left(1 - Q_0 q^{h_{\mu_2^1,(\mu_2^2)^t}(i,j)}\right)\left(1 - Q_0 Q_{F_1} q^{h_{\mu_1^1,(\mu_2^2)^t}(i,j)}\right) \\
&\qquad \left(1 - Q_0 Q_{F_2} q^{h_{\mu_2^1,(\mu_1^2)^t}(i,j)}\right)\left(1 - Q_0 Q_{F_1} Q_{F_2} q^{h_{\mu_1^1,(\mu_1^2)^t}(i,j)}\right) .
\end{aligned}$$

Therefore Theorem 4.4 implies that the partition function Z_{X^+} is obtained from Z_X by replacing

$$\prod_{i,j\geq 1} \left(1 - Q_0 q^{h_{\mu_2^1,(\mu_2^2)^t}(i,j)}\right) \to \prod_{k} \left(1 - (Q_0^+)^{-1} q^k\right)^{C_k(\mu_2^1,(\mu_2^2)^t)} \prod_{k\geq 1} \left(1 - Q_0^+ q^k\right)^k ,$$

and replacing Q_0 in other factors by $(Q_0^+)^{-1}$.

From the discussions in [KMV, §2.1], it seems natural to expect that the partition function of X reproduces Nekrasov's partition function for a gauge theory with a product gauge group and with a matter. We want to clarify this statement. Let us set

$$Z_X^{\text{inst}} = \frac{Z_X}{Z_X|_{Q_{B_1}=Q_{B_2}=0}} .$$

Then, by the same method with [EK1, Z], we can show the following

Proposition 5.1. *Let*

$$q = e^{-2R\hbar} , \quad Q_{F_1} = e^{-4Ra_1} , \quad Q_{F_2} = e^{-4Ra_2} , \quad Q_0 = e^{2R(a_1+a_2-m)} .$$

Then we have

$$Z_X^{\rm inst} = \sum_{\mu_1^1,\mu_2^1,\mu_1^2,\mu_2^2} \prod_{k=1}^{2} \left(\frac{Q_{B_k}}{2^4 Q_{F_k}}\right)^{|\mu_1^k|+|\mu_2^k|} \prod_{l,n=1}^{2} \prod_{i,j\geq 1} \frac{\sinh R\left(a_{ln}^{(k)} + \hbar\left(\mu_{1,i}^k - \mu_{2,j}^k + j - i\right)\right)}{\sinh R\left(a_{ln}^{(k)} + \hbar\,(j-i)\right)}$$
$$\times\, q^{\frac{1}{2}\left(\kappa(\mu_1^1)+\kappa(\mu_2^1)-\kappa(\mu_1^2)-\kappa(\mu_2^2)\right)} \,(2^2 Q_0)^{|\mu_1^1|+|\mu_2^1|+|\mu_1^2|+|\mu_2^2|}\, (Q_{F_1}^{\frac{1}{2}})^{2|\mu_1^1|+|\mu_1^2|+|\mu_2^2|}\, (Q_{F_2}^{\frac{1}{2}})^{|\mu_1^1|+|\mu_2^1|+2|\mu_1^2|}$$
$$\prod_{l,n=1}^{2} \prod_{i,j\geq 1} \frac{\sinh R\left(a_{ln}^{(1,2)} + m + \hbar\,(j-i)\right)}{\sinh R\left(a_{ln}^{(1,2)} + m + \hbar\left(\mu_{l,i}^1 - \mu_{n,j}^2 + j - i\right)\right)} ,$$

where

$$a_{11}^{(k)} = a_{22}^{(k)} = 0 \ , \quad a_{12}^{(k)} = -a_{21}^{(k)} = 2a_k \ ,$$

and

$$a_{11}^{(1,2)} = a_1 + a_2 \ , \quad a_{21}^{(1,2)} = -a_1 + a_2 \ , \quad a_{12}^{(1,2)} = a_1 - a_2 \ , \quad a_{22}^{(1,2)} = -a_1 - a_2 \ .$$

By Proposition 5.1, it is easy to see that the $R \to 0$ limit of

$$Z_X^{\rm inst}|_{q=e^{-2R\hbar},\ Q_{B_k}=2^2\Lambda_k,\ Q_{F_k}=e^{-4Ra_k},\ Q_0=e^{2R(a_1+a_2-m)}}$$

is equal to the instanton part of Nekrasov's partition function of 4-dimensional SU(2) $\times$ SU(2) gauge theory with a matter in the bifundamental representation $(\mathbf{2}, \bar{\mathbf{2}})$ [N, (66)]. (See also [FMP, HIV, MO, S] for related works.)

Remark 5.2. It is immediate to see that $Z_{X^+}^{\rm inst} = Z_{X^+}/(Z_{X^+}|_{Q_{B_1}=Q_{B_2}=0})$ also coincides with the same Nekrasov's partition function with a similar variable identification in the limit $R \to 0$. More generally, Theorem 4.4 may imply that if TCY threefolds X and X^+ are related by flops with respect to $(-1,-1)$-curves and if the partition function of X reproduces Nekrasov's partition function for a gauge theory, then the partition function of X^+ also reproduces it. (This statement itself seems to be well-known to specialists.)

6. Application to toric surface and its blowup

As an application, we compare GW invariants of the canonical bundle of a complete smooth toric surface and those of the canonical bundle of a blown-up surface. Some relevant numerical data can be found in [CKYZ].

Let S be a complete smooth toric surface (see [F, §2.5]) and $\hat{S}$ its blowup at a torus fixed point. The exceptional curve of $\psi : \hat{S} \to S$ is denoted by E. Let X be the total space of the canonical bundle K_S of S and $\hat{X} = K_{\hat{S}}$. These are TCY threefolds and E is a $(-1,-1)$-curve in $K_{\hat{S}}$.

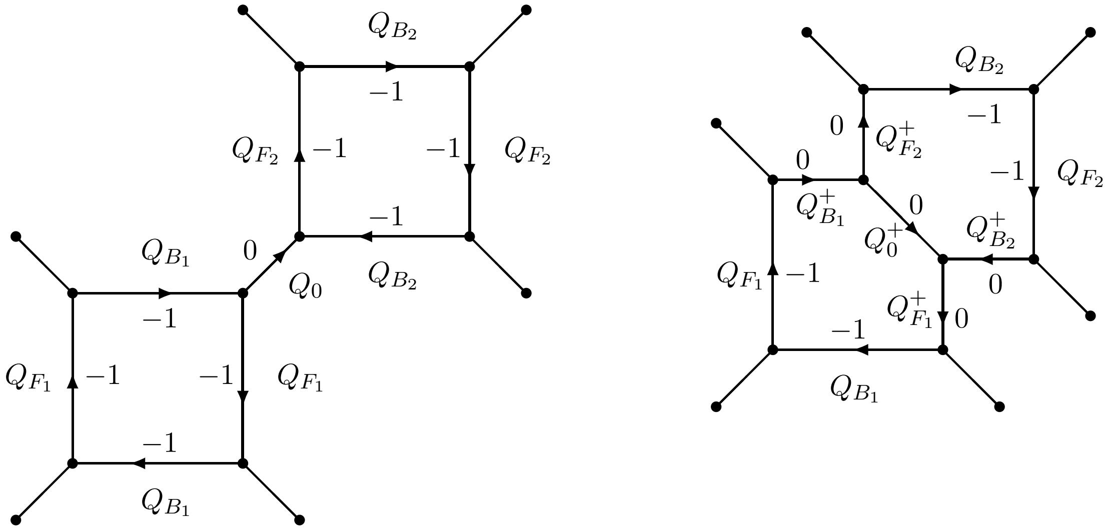

FIGURE 5. TCY threefold which contains two disjoint $\mathbb{P}^1 \times \mathbb{P}^1$ connected by a $(-1,-1)$-curve (left) and its flop (right).

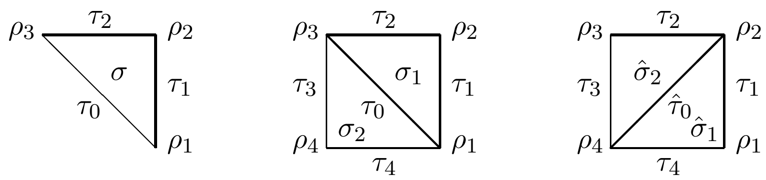

FIGURE 6. Fans (sections at $z = 1$): Σ (left), $\bar{\Sigma}$ (middle) and $\hat{\Sigma}$ (right). The generators $\vec{\omega}_1, \dots, \vec{\omega}_4$ of $\rho_1, \dots, \rho_4$ satisfy the relation $\vec{\omega}_1 + \vec{\omega}_3 = \vec{\omega}_2 + \vec{\omega}_4$.

Since all torus invariant curves in X are contained in $S \subset X$, there is a canonical map $N_1^T(X) \to H_2(S, \mathbb{Z})$. In fact, the following maps $p, \hat{p}$ are isomorphisms:

$$p : H_2(S, \mathbb{R}) \xrightarrow{\sim} N_1^T(X) \otimes \mathbb{R}, \qquad \hat{p} : H_2(\hat{S}, \mathbb{R}) \xrightarrow{\sim} N_1^T(\hat{X}) \otimes \mathbb{R}. \tag{12}$$

Proposition 6.1. (i) *For $\beta \in H_2(\hat{S}, \mathbb{Z})$ such that β is not an multiple of $[E]$ and satisfying $\beta.E < 0$,*

$$N_{g,\hat{p}(\beta)}(\hat{X}) = 0 \ .$$

(ii) *For $\beta \in H_2(\hat{S}, \mathbb{Z})$ such that $\beta.E = 0$,*

$$N_{g,\hat{p}(\beta)}(\hat{X}) = N_{g,p(\psi_*(\beta))}(X) \ .$$

(iii) *For a multiple of $[E]$,*

$$N_{g,d[E]}(\hat{X}) = N_{g,d[\mathbb{P}^1]}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Proof. Let $\bar{X}$ be the TCY threefold obtained from $\hat{X}$ by flopping the curve E. Let $\Sigma, \bar{\Sigma}, \hat{\Sigma}$ be fans of $X, \bar{X}, \hat{X}$, and let $\hat{\tau}_0$ be the 2-cone in $\hat{\Sigma}$ representing E. Then near $\hat{\tau}_0$, $\hat{\Sigma}$ looks like the right diagram in Figure 6 and $\Sigma, \hat{\Sigma}$ are like the left and the middle diagrams. $\Sigma, \bar{\Sigma}, \hat{\Sigma}$ are identical outside these parts.

A natural inclusion $\Sigma \hookrightarrow \bar{\Sigma}$ induces the isomorphism

$$\alpha : N_1^T(X) \xrightarrow{\sim} \{Z \in N_1^T(\bar{X}) \,|\, Z.D_{\rho_4} = 0\} ,$$

where ρ_4 is the 1-cone in $\bar{\Sigma}$ shown in Figure 6. Composed with the isomorphism $\phi_* : N_1^T(\bar{X}) \otimes \mathbb{R} \to N_1^T(\hat{X}) \otimes \mathbb{R}$ induced from the flop $\phi : \bar{X} \dashrightarrow \hat{X}$,

$$\phi_* \circ \alpha : N_1^T(X) \otimes \mathbb{R} \xrightarrow{\sim} \{Z \in N_1^T(\hat{X}) \otimes \mathbb{R} \,|\, Z.D_{\rho_4} = 0\} , \tag{13}$$

where ρ_4 is the 1-cone in $\hat{\Sigma}$ shown in Figure 6. By calculating intersection numbers, we see that the RHS is spanned by (recall $E = C_{\hat{\tau}_0}$)

$$[E] + [C_{\tau_1}], \qquad [E] + [C_{\tau_2}], \qquad [C_\tau] \quad (\tau \in \hat{\Sigma}_2' \setminus \{\tau_1, \tau_2\}).$$

Under the isomorphisms (12), the inverse of the isomorphism (13) becomes

$$\psi_* : \{\beta \in H_2(\hat{S}, \mathbb{R}) \,|\, \beta.E = 0\} \xrightarrow{\sim} H_2(S, \mathbb{R}). \tag{14}$$

By applying Theorem 4.4 to $\hat{X}$ and $\bar{X}$, we obtain (i). The statement (iii) follows from the second statement of Corollary 4.6. The first statement of Corollary 4.6, together with the following, implies (ii):

$$N_{g,\beta}(X) = N_{g,\alpha(\beta)}(\bar{X}) \qquad (\beta \in N_1^T(X)),$$

by the construction of partition function (9). □

Appendix A. Combinatorial formulae

We collect some combinatorial formulae which are used in this paper. Our basic references are [Mac, EK1, ORV, Z].

$\kappa(\mu)$ is always even and

$$\kappa(\mu^t) = -\kappa(\mu) , \tag{15}$$

where μ^t denotes the conjugate partition (the partition corresponding to the transposed Young diagram of μ).

$C_k(\mu, \nu)$ are nonnegative integers which are nonzero for finitely many values of k [Z, Theorem 5.1], and have the following properties ([EK1, §3.1], [Z, §5.3]):

$$\sum_k C_k(\mu, \nu) = |\mu| + |\nu| , \qquad \sum_k k C_k(\mu, \nu) = \frac{1}{2}(\kappa(\mu) + \kappa(\nu)) , \tag{16}$$

$$C_k(\mu, \nu) = C_{-k}(\mu^t, \nu^t) . \tag{17}$$

The following lemma is proved in [EK1, Lemma in §C], [Z, Proposition 6.1]:

Lemma A.1. *For $\mu, \nu \in \mathcal{P}$, the following identity holds:*

$$\prod_{i,j\geq 1}\left(1 - Qq^{h_{\mu,\nu}(i,j)}\right) = Z_{(-1,-1)}(q,Q)\prod_{k}\left(1 - Qq^{k}\right)^{C_k(\mu,\nu)} .$$

Here are some properties of skew Schur function. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be sets of variables and $(x, y) = (x_1, x_2, \dots, y_1, y_2, \dots)$. The following formulae are useful in performing the summations over partitions ([Mac, p.93,(5.10)]):

$$\sum_{\lambda\in\mathcal{P}} s_{\lambda/\lambda_1}(x)s_{\lambda/\lambda_2}(y) = \prod_{i,j\geq 1}(1 - x_iy_j)^{-1}\sum_{\mu\in\mathcal{P}} s_{\lambda_2/\mu}(x)s_{\lambda_1/\mu}(y) , \tag{18}$$

$$\sum_{\lambda\in\mathcal{P}} s_{\lambda/\lambda_1}(x)s_{\lambda^t/\lambda_2}(y) = \prod_{i,j\geq 1}(1 + x_iy_j)\sum_{\mu\in\mathcal{P}} s_{\lambda_2^t/\mu}(x)s_{\lambda_1^t/\mu^t}(y) , \tag{19}$$

$$\sum_{\xi\in\mathcal{P}} s_{\mu/\xi}(x)s_{\xi/\nu}(y) = s_{\mu/\nu}(x,y) . \tag{20}$$

Other properties are ([Z, Proposition 4.1]):

$$s_{\mu/\nu}(Qx) = Q^{|\mu|-|\nu|}s_{\mu/\nu}(x) , \tag{21}$$

where $Qx = (Qx_1, Qx_2, \dots)$.

$$s_{\lambda/\mu}(q^{\nu+\rho}) = (-1)^{|\lambda|-|\mu|}s_{\lambda^t/\mu^t}(q^{-\nu^t-\rho}) . \tag{22}$$

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