Natural G-Constellation Families

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Abstract

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. G-constellations are a scheme-theoretic generalization of orbits of G in $\mathbb{C}^n$. We study flat families of G-constellations parametrised by an arbitrary resolution of the quotient space $\mathbb{C}^n/G$. We develop a geometrical naturality criterion for such families, and show that, for an abelian G, the number of the equivalence classes of these natural families is finite.

Possible applications include the derived McKay correspondence and moduli space constructions of resolutions of quotient singularities.

0 Introduction

Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup. In this paper we study flat families of *G-constellations* parametrised by a given resolution Y of the singular quotient space $X = \mathbb{C}^n/G$.

$$Y \xrightarrow{\ \pi\ } X \xleftarrow{\ q\ } \mathbb{C}^n$$

G-constellations are a scheme theoretical generalization of a set theoretical orbit of G in $\mathbb{C}^n$. They arose in the context of trying to construct crepant resolutions of X as a moduli space construction. For further detail see the brief survey of the subject history in the latter part of this section. The definition is:

Definition 0.1 ([Cra01])**.** A G-constellation is a G-equivariant coherent sheaf $\mathcal{F}$ whose global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$ form the regular representation of G.

At the outset, we allow Y to be any normal scheme birational to the quotient space X. Let R denote the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$ of $\mathbb{C}^n$. We start by moving from working with a whole of G-constellation $\mathcal{F}$ to working with just its space of global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$. To keep track of the

G-equivariant $\mathcal{O}_{\mathbb{C}_n}$-module structure we consider $\Gamma(\mathbb{C}^n, \mathcal{F})$ as a module for the cross-product algebra $R \rtimes G$. The functor $\Gamma(\bullet)$ becomes an equivalence between the categories of G-equivariant quasi-coherent sheaves on $\mathbb{C}^n$ and of $R \rtimes G$-modules.

This is not a pure formalism - $R \rtimes G$ is one of the *non-commutative crepant resolutions* of $\mathbb{C}^n/G$, a certain class of non-commutative algebras introduced by Michel van den Bergh in [dB02] as an analogue of a commutative crepant resolution for an arbitrary non-quotient Gorenstein singularity. For three-dimensional terminal singularities, van den Bergh shows ([dB02], Theorem 6.3.1) that if a non-commutative crepant resolution Q exists, then it is possible to construct commutative crepant resolutions as moduli spaces of certain stable Q-modules.

We then define a family of G-constellations on Y as an $(R \rtimes G) \otimes \mathcal{O}_Y$-module, flat over Y, whose fiber at every point of Y is the regular representation of G. In Section 1 we develop a geometrical naturality criterion for such families: mimicking the moduli spaces M_θ of θ-stable G-constellations (see the latter half of this section) and their tautological families, we demand for a G-constellation parametrised in a family $\mathcal{F}$ by a point $p \in Y$ to be supported precisely on the G-orbit corresponding to the point $\pi(p)$ in the quotient space X. We call the families which satisfy this condition *gnat*-families (short for a *generically natural*) and demonstrate (Proposition 1.5) that they enjoy a number of other natural properties, including being equivalent (locally isomorphic) to the natural family $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ on the open set of Y which lies over the free orbits in X. In this natural family a G-constellation which lies over a free orbit is the unique G-constellation supported on that orbit - its reduced subscheme structure. Thus, in a sense, *gnat*-families can be viewed as flat deformations of free orbits of G.

Another property which characterises *gnat*-families is that it is possible to embed them into $K(\mathbb{C}^n)$, considered as a constant sheaf of $R \rtimes G \otimes \mathcal{O}_Y$-modules on Y. Thus it becomes important to study G-equivariant locally free sub-$\mathcal{O}_Y$-modules of $K(\mathbb{C}^n)$. In Section 2, we study the rank one case. A G-invariant invertible sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$ is just a Cartier divisor, and we define $G\text{-}\operatorname{Car}(Y)$, a group of G-Cartier divisors on Y, as a natural extension of the group of Cartier divisors which fits into a short exact sequence

$$1 \to \operatorname{Car}(Y) \to G\text{-}\operatorname{Car}(Y) \xrightarrow{\rho} G^\vee \to 1$$

where $G^\vee$ is the group of 1-dimensional irreducible representations of G.

We then define $\mathbb{Q}$-valued valuations of these G-Cartier divisors at prime Weil divisors of Y and define $G\text{-}\operatorname{Div} Y$, the group of G-Weil divisors of Y, as a torsion-free subgroup of $\mathbb{Q}$-Weil divisors which fits into a following exact sequence:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \operatorname{Car} Y & \longrightarrow & G\text{-}\operatorname{Car} Y & \xrightarrow{\rho} & G^\vee & \longrightarrow & 1 \\
 & & \Big\downarrow {\scriptstyle val_K} & & \Big\downarrow {\scriptstyle val_{K_G}} & & \Big\downarrow {\scriptstyle val_{G^\vee}} & & \\
0 & \longrightarrow & \operatorname{Div} Y & \longrightarrow & G\text{-}\operatorname{Div} Y & \longrightarrow & val_{G^\vee}(G^\vee) & \longrightarrow & 0
\end{array}$$

We then show that the three vertical maps in this diagram, val_K, the ordinary $\mathbb{Z}$-valued valuation of Cartier divisors, val_{K_G}, the $\mathbb{Q}$-valued valuation of G-Cartier divisors, and their quotient $val_{G^\vee}$, a $\mathbb{Q}/\mathbb{Z}$-valued valuation of $G^\vee$, are all isomorphisms when Y is smooth and proper over X.

Then, in Section 3, we observe that when our group G is abelian all its irreducible representations are of rank 1, so any *gnat*-family splits into invertible G-eigensheaves and G-Weil divisors are all that we need to classify it after an embedding into $K(\mathbb{C}^n)$. We establish a natural 1-to-1 correspondence between equivalence classes of *gnat*-families on a resolution Y and sets of G-Weil divisors $\{D_\chi\}_{\chi\in G^\vee}$, satisfying a finite set of inequalities and having $D_{\chi_0} = 0$, where χ_0 is the trivial representation of G. We further show that as any *gnat*-family $\mathcal{F}$ embedded into $K(\mathbb{C}^n)$ must be closed under the natural action of R on the latter, all the G-eigensheaves into which $\mathcal{F}$ decomposes must be, in a certain sense, close to each other inside $K(\mathbb{C}^n)$. This allows us to put a precise bound on how far from $D_{\chi_0} = 0$, numerically, all the other divisors D_χ can be in a *gnat*-family. We then show that this leaves only a finite number of possibilities for each of D_χ and thus the number of equivalence classes of *gnat*-families on Y is finite.

Our main result (Theorem 4.1) is:

Theorem (Classification of *gnat*-families)**.** *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G and Y a resolution of X. Then isomorphism classes of gnat-families on Y are in* 1*-to-*1 *correspondence with linear equivalence classes of G-divisor sets $\{D_\chi\}_{\chi\in G^\vee}$, each D_χ a χ-Weil divisor, which satisfy the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall\, \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Here $\rho(f) \in G^\vee$ is the homogeneous weight of f. Such a divisor set $\{D_\chi\}$ corresponds then to a gnat-family $\bigoplus \mathcal{L}(-D_\chi)$.

This correspondence descends to a 1*-to-*1 *correspondence between equivalence classes of gnat-families and sets $\{D_\chi\}$ as above and with $D_{\chi_0} = 0$. Furthermore, each divisor D_χ in such a set satisfies inequality*

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}$$

where $\{M_\chi\}$ is a fixed divisor set defined by

$$M_\chi = \sum_P (\min_{f\in R_\chi} v_P(f))P$$

As a consequence, the number of equivalence classes of gnat-families on Y is finite.

We now give a brief history of the subject and state motivations for studying the families of G-constellations.

The singular quotient space X is in a certain sense ([Muk03], Example 11.8) a coarse moduli space for the set-theoretical orbits of G in $\mathbb{C}^n$. A natural question to ask was whether we can refine a concept of an 'orbit of G in $\mathbb{C}^n$' and state a moduli problem for it which yields a fine moduli space Y which resolves the singularities of X.

The first step was to equip an orbit with an appropriate scheme-theoretic structure:

Definition 0.2. A G-`cluster` is a G-invariant subscheme $\mathcal{Z}$ of $\mathbb{C}^n$ of dimension 0 whose ring $\Gamma(Z, \mathcal{O}_Z)$ is a regular representation of G.

E.g. any free orbit of G supports a unique G-cluster: the reduced induced closed subscheme structure. On the other hand, we find many different G-clusters supported at the fixed point orbit at the origin of $\mathbb{C}^n$.

Following the ideas of Nakamura, Reid introduced in [Rei97] the scheme G-Hilb, the fine moduli space of all G-clusters. It comes equipped with a Hilbert-Chow morphism $G\text{-Hilb}\,\mathbb{C}^n \to X$ which sends each G-cluster to its set-theoretic support. The main irreducible component of $G\text{-Hilb}\,\mathbb{C}^n$ birational to X can be identified (e.g. [IN00], §2) with the scheme $\text{Hilb}^G\,\mathbb{C}^n$ introduced by Nakamura and Ito in [IN96]. They then proceeded to show that for G a finite subgroup of $\text{SL}_2(\mathbb{C})$, the scheme $\text{Hilb}^G\,\mathbb{C}^n$ is the unique crepant minimal resolution of $\mathbb{C}^2/G$.

Then Nakamura showed by explicit toric geometry computations [Nak00] that for G a finite abelian subgroup of $\text{SL}_3(\mathbb{C})$, the scheme $\text{Hilb}^G\,\mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$. He conjectured that the same is true for the non-abelian case.

This conjecture was settled by Bridgeland, King and Reid in [BKR01]. They use derived category methods and establish a category equivalence $D(Y) \to D^G(\mathbb{C}^n)$ between the bounded derived categories of coherent sheaves on $Y = \text{Hilb}^G\,\mathbb{C}^n$ and of G-equivariant coherent sheaves on $\mathbb{C}^n$, respectively. Under a certain assumption on the dimension of the fibers of Y, which holds automatically when $n \leq 3$, they prove that the Fourier-Mukai transform which uses the structure sheaf of the universal G-cluster $\mathcal{U}_G \subset Y \times \mathbb{C}^n$ is the requisite equivalence. In particular, this shows that Y is a crepant resolution of X, proving Nakamura's conjecture. It is then further shown ([BKR01], §8) that in the case of $n = 3$, $\text{Hilb}^G\,\mathbb{C}^3$ is the only component of $G\text{-Hilb}\,\mathbb{C}^3$, i.e. $G\text{-Hilb}\,\mathbb{C}^3$ is connected. In dimension two this was proven by Ishii in [Ish02], while in dimensions four and higher it is known to be false.

For $n \geq 3$ crepant resolutions of $\mathbb{C}^n/G$, if they exist, are not necessarily unique. The question arose whether G-clusters can be generalised further, to obtain the other crepant resolutions by a moduli space construction. Subsequent research had shown that it was not necessary to give an orbit a subscheme structure - it is sufficient to equip an orbit with a coherent sheaf that looks like what we would expect of an image of a skyscraper sheaf of a point under a derived category equivalence as above. This generalisation was a concept of a G-constellation given by Craw in his thesis [Cra01]. Note that a priori a definition of G-constellation doesn't exclude sheaves supported at more than one orbit of G. However a *gnat*-family consists only of those supported at a single orbit.

Observe that, tautologically, the structure sheaf of any G-cluster is a G-constellation. In fact on a free orbit this all we get: the concepts of a G-constellation, a G-cluster and a set-theoretic orbit coincide where G acts freely. At the origin, however, there are many G-constellations which do not arise as structure sheaves of G-clusters. Too many in fact: the moduli space of all G-constellations is non-separated at the origin, suggesting that some sort of stability conditions are needed.

These came to us courtesy of a natural 1-to-1 correspondence existing between G-constellations and representations of the McKay quiver of G into the regular representation of G. This allows for the use of an earlier result of King [Kin94] on GIT construction of moduli spaces of quiver representations to introduce the stability conditions known as θ-stability on G-constellations and to construct for any given stability condition θ a moduli space M_θ of θ-stable G-constellations together with a projective morphism to X and a universal θ-stable G-constellation $\mathcal{U}_\theta$ in $\mathbf{Coh}\, Y \times \mathbb{C}^n$. In a quiver-theoretic context, Kronheimer [Kro89] had already considered these moduli spaces and have studied the chamber structure in the space Π of stability parameters θ, where all values of θ in the same chamber yield the same M_θ. The methods of [BKR01] can be then extended to show that, under the same assumptions on the fiber dimensions of M_θ, the Fourier-Mukai transform $D(M_\theta) \to D^G(\mathbb{C}^n)$ is an equivalence of categories, which makes the main irreducible component of M_θ a crepant resolution of $\mathbb{C}^n/G$. In case of an abelian G, an explicit description of this coherent component is provided in toric terms by Craw, Maclagan and Thomas in [CMT05a], [CMT05b].

Craw in his thesis conjectured that when G is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ every crepant resolution projective over $\mathbb{C}^3/G$ can be realised as a moduli space M_θ of θ-stable G-constellations for some chamber in Π. In the case of G being abelian, this was proved by Craw and Ishii in [CI04].

Thus one motivation for the study of families of G-constellations on a fixed resolution Y is an observation that, as evident from [CI04], there exist stability parameters θ for which the GIT construction yields isomorphic moduli spaces M_θ, but equips them with different tautological families of G-constellations $\mathcal{U}_\theta$. Another is the desire to obtain for a given crepant resolution Y a direct construction of the derived McKay equivalence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ as a Fourier-Mukai functor using an appropriate

G-constellation family. Finally, the question of a moduli construction of non-projective (over X) crepant resolutions still remains open.

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1 *gnat*-Families

1.1 Families of G-Constellations

Let G be a finite abelian group and let V_{giv} be an n-dimensional faithful representation of G. We identify the symmetric algebra $S({V_{\text{giv}}}^{\vee})$ with the coordinate ring R of $\mathbb{C}^n$ via a choice of such an isomorphism that the induced action of G on $\mathbb{C}^n$ is diagonal. The (left) action of G on V_{giv} induces a (left) action of G on R, where we adopt the convention that

$$g.f(\mathbf{v}) = f(g^{-1}.\mathbf{v}) \qquad \forall\, g \in G, f \in R, \mathbf{v} \in V_{\text{giv}}, \tag{1.1}$$

When we consider the induced scheme morphisms $g : \mathbb{C}^n \to \mathbb{C}^n$ and the induced sheaf morphisms $g : \mathcal{O}_{\mathbb{C}^n} \to g_*^{-1}\mathcal{O}_{\mathbb{C}_n}$, the convention above ensures that for any point $x \in \mathbb{C}^n$ and any function f in the stalk $\mathcal{O}_{\mathbb{C}^n,x}$ at x, the function $g.f$ is, naturally, an element of the stalk $\mathcal{O}_{\mathbb{C}^n,g.x}$ at $g.x$

Corresponding to the inclusion $R^G \subset R$ of the subring of G-invariant functions we have the quotient map $q : \mathbb{C}^n \to X$, where $X = \text{Spec } R^G$ is the quotient space. This space is generally singular.

We first wish to establish a notion of a family of *G-Constellations* parametrised by an arbitrary scheme.

Definition 1.1 ([CI04]). A G-`constellation` is a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ such that $H^0(\mathcal{F})$ is isomorphic, as a $\mathbb{C}[G]$-module, to the regular representation V_{reg}.

We would like for a family of G-constellations to be a locally free sheaf on Y, whose restriction to any point of Y would give us the respective G-constellation. We'd like this restriction to be a finite-dimensional vector-space, and for this purpose, it would be better to consider, instead of the whole G-constellation $\mathcal{F}$, just its space of global sections $\Gamma(\mathcal{F})$. It is a vector space with G and R actions, satisfying

$$g.(f.\mathbf{v}) = (g.f).(g.\mathbf{v}) \tag{1.2}$$

On the other hand, for any vector space V with G and R actions satisfying (1.2), we can define maps $g : \tilde{V} \to g_*^{-1}\tilde{V}$ to give the sheaf $\tilde{V} = V \otimes_R \mathcal{O}_{\mathbb{C}^n}$

a G-equivariant structure. It is convenient to view such vector spaces as modules for the following non-commutative algebra:

Definition 1.2. A cross-product algebra $R \rtimes G$ is an algebra, which has the vector space structure of $R \otimes_{\mathbb{C}} \mathbb{C}[G]$ and the product defined by setting, for all $g_1, g_2 \in G$ and $f_1, f_2 \in R$,

$$(f_1 \otimes g_1) \times (f_2 \otimes g_2) = (f_1(g_1.f_2)) \otimes (g_1 g_2) \tag{1.3}$$

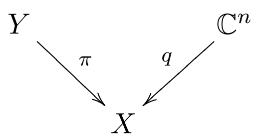
Functors $\Gamma(\bullet)$ and $\tilde{\bullet} = (\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ give an equivalence (compare to [Har77], p. 113, Corollary 5.5) between the categories of quasi-coherent G-equivariant sheaves on $\mathbb{C}^n$ and of $R \rtimes G$-modules. G-constellations then correspond to $R \rtimes G$-modules, whose underlying G-representation is V_{reg}. As an abuse of notation, we shall use the term 'G-constellation' to refer to both the equivariant sheaf and the corresponding $R \rtimes G$-module.

Definition 1.3. A `family of` G`-constellations parametrised by a scheme` S is a sheaf $\mathcal{F}$ of $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_S$-modules, locally free as an $\mathcal{O}_S$-module, and such that for any point ι_p : Spec $\mathbb{C} \hookrightarrow S$, its fiber $\mathcal{F}_{|p} = \iota_p^* \mathcal{F}$ is a G-constellation.

We shall say that two families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if they are locally isomorphic as $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_S$-modules.

1.2 *gnat*-Families

Let Y be a normal scheme and $\pi : Y \to X$ be a birational map.



We wish to refine the definition (1.3) above and develop a notion of a geometrically natural family of G-constellations parametrised by Y.

Any free G-orbit supports a unique G-cluster $Z \subset \mathbb{C}^n$: the reduced induced closed subscheme structure. Let U be an open subset of Y such that $\pi(U)$ consists of free orbits of G and consider the sheaf $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ restricted to U. It has a natural $(R \rtimes G)$-module structure induced from $\mathcal{O}_{\mathbb{C}^n}$. It is locally free as an $\mathcal{O}_U$ module, since the quotient map q is flat wherever G acts freely. Its fiber at a point $p \in Y$ is $\Gamma(\mathcal{O}_Z)$, where Z is the G-cluster corresponding to the free orbit $q^{-1}\pi(p)$. Thus $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ is a natural family of G-constellations, indeed of G-clusters, on $U \subset Y$.

Its fiber at the generic point of Y is $K(\mathbb{C}^n)$. The Normal Basis Theorem from Galois theory ([Gar86], Theorem 19.6) gives an isomorphism from $K(\mathbb{C}^n)$ to the generic fiber of any G-constellation family on Y, which we can write as $K(Y) \otimes_{\mathbb{C}} V_{\mathrm{reg}}$, but this isomorphism is only $K(Y)$ and G, but not necessarily R, equivariant.

On the other hand, for any G-constellation in a sense of G-equivariant sheaf, we can consider its support in $\mathbb{C}^n$. For instance, in the natural family $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ discussed above the support of the G-constellation parametrised by a point $p \in U$ is precisely the G-orbit $q^{-1}\pi(p)$. This turns out to be the criterion we seek and we shall show that any family satisfying it is generically equivalent to the natural one.

Definition 1.4. A *gnat*-`family` $\mathcal{F}$ (short for *generically natural* family) is a family of G-constellations parametrised by Y such that for any $p \in Y$

$$q\left(\operatorname{Supp}_{\mathbb{C}^n} \mathcal{F}_{|p}\right) = \pi(p) \tag{1.4}$$

Proposition 1.5. *Let Y be a normal scheme and $\pi : Y \to X$ be a birational map. Let $\mathcal{F}$ be a family of G-constellations on Y. Then the following are equivalent:*

1. *On any $U \subset Y$, such that πU consists of free orbits, $\mathcal{F}$ is equivalent to $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$.*

2. *There exists an $(R \rtimes G) \otimes_{\mathbb{C}} K(Y)$-module isomorphism:*

$$\mathcal{F}_{|p_Y} \xrightarrow{\sim} (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{p_Y}$$

   where p_Y is the generic point of Y.

3. *There exists an $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$-module embedding*

$$F \hookrightarrow K(\mathbb{C}^n)$$

   where $K(\mathbb{C}^n)$ is viewed as a constant sheaf on Y and given a $\mathcal{O}_Y$-module structure via the birational map $\pi : Y \to X$.

4. *$\mathcal{F}$ is a gnat-family.*

5. *The action of $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$ on $\mathcal{F}$ descends to the action of $(R \rtimes G) \otimes_{R^G} \mathcal{O}_Y$, where R^G-module structure on $\mathcal{O}_Y$ is induced by the map $\pi : Y \to X$.*

Proof. 1 $\Rightarrow$ 2 is restricting any of the local isomorphisms to the stalk at the generic point p_Y of Y. 2 $\Rightarrow$ 3: the embedding is given by the natural map $\mathcal{F} \hookrightarrow \mathcal{F} \otimes K(Y)$. As Y is irreducible and $\mathcal{F}$ is locally free, $\mathcal{F} \otimes K(Y)$ is isomorphic to $\mathcal{F}_{p_Y}$, and hence to $K(\mathbb{C}^n)$. 3 $\Rightarrow$ 5 is immediate by inspecting the natural $R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y$-module structure on $K(\mathbb{C}^n)$. 5 $\Rightarrow$ 4 is also immediate, as the descent of the action of $R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y$ to that of $R \rtimes G \otimes_{R^G} \mathcal{O}_Y$ implies that for any $p \in Y$ we have $\mathfrak{m}_{\pi(p)} \subset \operatorname{Ann}_R \mathcal{F}_{|p}$, where $\mathfrak{m}_{\pi(p)} \subset R^G$ is the maximal ideal of $\pi(p)$. Therefore $\mathfrak{m}_{\pi(p)} = (\operatorname{Ann}_R \mathcal{F}_{|p})^G$, which is equivalent to (1.4).

4 ⇒ 5: Consider the following composition of algebra morphisms:

$$R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{\alpha} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F}) \xrightarrow{\beta_p} \mathrm{End}_{\mathbb{C}}(\mathcal{F}_{|p})$$

where α is the action map of $R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y$ on $\mathcal{F}$ and β_p is restriction to the fiber at a point $p \in Y$.

To show that α filters through $R \rtimes G \otimes_{R^G} \mathcal{O}_Y$ it suffices to show that for any $f \in R^G$ we have $f \otimes 1 - 1 \otimes f \in \ker(\alpha)$. From (1.4) we have $\mathfrak{m}_{\pi(p)} = (\mathrm{Ann}_R \mathcal{F}_{|p})^G$, and therefore

$$\beta_p \alpha((f - f(p)) \otimes 1) = 0$$

Observe that $\beta_p \alpha(f(p) \otimes 1) = f(p)\, 1_{\mathrm{End}_{\mathbb{C}} \mathcal{F}_{|p}} = \beta_p \alpha(1 \otimes f)$, and therefore

$$\beta_p \alpha(f \otimes 1 - 1 \otimes f) = 0 \tag{1.5}$$

As $\mathcal{E}nd_{\mathcal{O}_Y} \mathcal{F}$ is locally free, (1.5) holding $\forall\, p \in Y$ implies $\alpha(f \otimes 1 - 1 \otimes f) = 0$, as required.

5 ⇒ 1: We have the $R \rtimes G \otimes_{R^G} \mathcal{O}_Y$-action on $\mathcal{F}$:

$$R \rtimes G \otimes_{R^G} \mathcal{O}_Y \xrightarrow{\alpha} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$$

LHS is isomorphic to $\pi^* \mathcal{E}nd_{\mathcal{O}_X}(q_* \mathcal{O}_{\mathbb{C}^n})$. Over U, since q is flat over $\pi(U)$, LHS is further isomorphic to $\mathcal{E}nd_{\mathcal{O}_U}(\pi^* q_* \mathcal{O}_{\mathbb{C}_n})$. Thus we have:

$$\mathcal{E}nd_{\mathcal{O}_U}(\pi^* q_* \mathcal{O}_{\mathbb{C}_n}) \xrightarrow{\alpha'} \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{F}) \tag{1.6}$$

This map (1.6) is an $\mathcal{O}_U$-algebra homomorphism of (split) Azumaya algebras over U of the same rank. By a general result on Azumaya algebras any such is an isomorphism (see [ACvdE05], Theorem 5.3, for full generality, but the original result in [AG60], Corollary 3.4 will also suffice here). Now Skolem-Noether theorem for Azumaya algebras ([Mil80], IV, §2, Proposition 2.3) implies that locally α' must be induced by isomorphisms $\pi^* q_* \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\sim} \mathcal{F}$. □

2 G-Cartier and G-Weil divisors

If $\mathcal{F}$ is a *gnat*-family, by Proposition 1.5 we can embed it into $K(\mathbb{C}^n)$. We need, therefore, to study G-subsheaves of $K(\mathbb{C}^n)$ which are locally free on Y. In this section we treat the rank 1 case, i.e. the invertible sheaves. Now, on an arbitrary scheme S, an invertible sheaf together with its embedding into $K(S)$ defines a unique Cartier divisor on S. But here we embed not into $K(Y)$ but into its Galois extension $K(\mathbb{C}^n)$. Recall that we identify $K(Y)$ with $K(\mathbb{C}^n)^G$ via the birational map $Y \xrightarrow{\pi} X$. We therefore seek to extend the familiar construction of Cartier divisors to accommodate for this fact.

2.1 G-Cartier divisors

We write $G^\vee$ for $\mathrm{Hom}(G, \mathbb{C}^*)$, the group of irreducible representations of G of rank 1.

Definition 2.1. We shall say that a rational function $f \in K(\mathbb{C}^n)$ is G-`homogeneous of weight` $\chi \in G^\vee$ if

$$g.f = \chi(g^{-1})f \quad \forall\, g \in G \tag{2.1}$$

We shall denote by $K_\chi(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n)$ of homogeneous elements of a specific weight χ and by $K_G(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n)$ of all the G-homogeneous elements. We shall use R_χ and R_G to mean $R \cap K_\chi(\mathbb{C}^n)$ and $R \cap K_G(\mathbb{C}^n)$ respectively.

NB: The choice of a sign is dictated by wanting $f \in R$ to be homogeneous of weight $\chi \in G^\vee$ if $f(g.v) = \chi(g)f(v)$ for all $g \in G$ and $v \in \mathbb{C}^n$.

The invertible elements of $K_G(\mathbb{C}^n)$ form a multiplicative group which we shall denote by $K^*_G(\mathbb{C}^n)$. We have a short exact sequence:

$$1 \to K^*(Y) \to K^*_G(\mathbb{C}^n) \xrightarrow{\rho} G^\vee \to 1 \tag{2.2}$$

The following replicates, almost word-for-word, the definition of a Cartier divisor in [Har77], pp. 140-141.

Definition 2.2. A group of G-`Cartier divisors` on Y, denoted by G-$\mathrm{Car}(Y)$ is the group of global sections of the sheaf of multiplicative groups $K^*_G(\mathbb{C}^n)/\mathcal{O}^*_Y$, i.e. the quotient of the constant sheaf $K^*_G(\mathbb{C}^n)$ on Y by the sheaf $\mathcal{O}^*_Y$ of invertible regular functions.

Observe that (2.2) gives a well-defined short exact sequence:

$$1 \to \mathrm{Car}(Y) \to G\text{-}\mathrm{Car}(Y) \xrightarrow{\rho} G^\vee \to 1 \tag{2.3}$$

Given a G-Cartier divisor, we call its image $\chi \in G^\vee$ under ρ `the weight` of the divisor and say, further, that the divisor is χ-`Cartier`.

A G-Cartier divisor can be specified by a choice of an open cover $\{U_i\}$ of Y and functions $\{f_i\} \subseteq K^*_G(\mathbb{C}^n)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*_Y)$. In such case, the weight of the divisor is the weight of any one of f_i.

As with ordinary Cartier divisors, we say that a G-Cartier divisor is principal if it lies in the image of the natural map $K^*_G(\mathbb{C}^n) \to K^*_G(\mathbb{C}^n)/\mathcal{O}^*_Y$ and call two divisors linearly equivalent if their difference is principal.

Consider now a χ-Cartier divisor D on Y specified by a collection $\{(U_i, f_i)\}$ where U_i form an open cover of Y and $f_i \in K^*_\chi(\mathbb{C}^n)$. We define an invertible sheaf $\mathcal{L}(D)$ on Y as the sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$ generated by f_i^{-1} on U_i. Observe that G acts on $\mathcal{L}(D)$, the action being the restriction of the one on $K(\mathbb{C}^n)$, and that it acts on every section by the character χ.

Proposition 2.3. *The map $D \to \mathcal{L}(D)$ gives an isomorphism between G-$\operatorname{Car} Y$ and the group of invertible G-subsheaves of $K(\mathbb{C}^n)$. Furthermore, it descends to an isomorphism of the group G-Cl of G-Cartier divisors up to linear equivalence and the group G-Pic of invertible G-sheaves on Y.*

Proof. A standard argument in [Har77], Proposition 6.13, shows everything claimed, apart from the fact we can embed any invertible G-sheaf $\mathcal{L}$, with G acting by some $\chi \in G^\vee$, as a sub-$\mathcal{O}_Y$-module into $K(\mathbb{C}^n)$.

Given such $\mathcal{L}$, we consider the sheaf $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y)$. On every open set U_i where $\mathcal{L}$ is trivial, it is G-equivariantly isomorphic to the constant sheaf $K_\chi(\mathbb{C}^n)$. On an irreducible scheme a sheaf constant on an open cover is constant itself, so as Y is irreducible we have $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n)$ and a particular choice of this isomorphism gives the necessary embedding as

$$\mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n) \subset K(\mathbb{C}^n)$$

□

2.2 Homogeneous valuations

We now aim to develop a matching notion of G-Weil divisors. Recall that the homomorphism from ordinary Cartier to ordinary Weil divisors is defined in terms of valuations of rational functions at prime Weil divisors of Y.

Valuations at prime divisors of Y define a unique group homomorphism val_K from $K^*(Y)$ to $\operatorname{Div} Y$, the group of Weil divisors. Looking at the short exact sequence (2.2), we see that val_K must extend uniquely to a homomorphism val_{K_G} from $K^*_G(\mathbb{C}^n)$ to $\mathbb{Q}$-$\operatorname{Div} Y$, as $G^\vee$ is finite and $\mathbb{Q}$ is injective. We further obtain a quotient homomorphism $val_{G^\vee}$ from $G^\vee$ to $\mathbb{Q}/\mathbb{Z}$-$\operatorname{Div} Y$.

Explicitly, we set:

Definition 2.4. Let P be a prime Weil divisor on Y.

For any $f \in K^*_G(\mathbb{C}^n)$, observe that $f^{|G|}$ is necessarily of trivial weight and hence lies in $K(Y)$. We define `valuation of` f `at` P to be

$$v_P(f) = \frac{1}{|G|} v_P(f^{|G|}) \in \mathbb{Q} \tag{2.4}$$

where $v_D(f^{|G|})$ is the ordinary valuation in the local ring of P.

For any $\chi \in G^\vee$, observe that for any f, f' homogeneous of weight χ their ratio f/f' is of trivial character and therefore has integer valuation. We define `valuation of` χ `at` P to be

$$v_P(\chi) = \operatorname{frac}(v_P(f)) \in \mathbb{Q}/\mathbb{Z} \tag{2.5}$$

where f is any homogeneous function of weight χ and frac(-) denotes the fractional part.

It can be readily verified that $val_{K_G} = \sum v_P(\text{-})P$ and $val_{G^\vee} = \sum v_P(\text{-})P$. Furthermore, the short exact sequence (2.3) becomes a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Car}\,Y & \longrightarrow & G\text{-}\,\mathrm{Car}\,Y & \xrightarrow{\ \rho\ } & G^\vee & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle val_K} & & \downarrow{\scriptstyle val_{K_G}} & & \downarrow{\scriptstyle val_{G^\vee}} & & \\ 0 & \longrightarrow & \mathrm{Div}\,Y & \longrightarrow & \mathbb{Q}\text{-}\,\mathrm{Div}\,Y & \longrightarrow & \mathbb{Q}/\mathbb{Z}\text{-}\,\mathrm{Div}\,Y & \longrightarrow & 0 \end{array} \tag{2.6}$$

2.3 G-Weil divisors

Aiming to have a short exact sequence similar to (2.3), we now define the group G-Div Y of G-Weil divisors to be the subgroup of $\mathbb{Q}$-Div Y, which consists of the pre-images of $val_{G^\vee}(G^\vee) \subset \mathbb{Q}/\mathbb{Z}$-Div Y.

Definition 2.5. We say that a $\mathbb{Q}$-Weil divisor $\sum q_P P$ on Y is a `G-Weil divisor` if there exists $\chi \in G^\vee$ such that

$$\mathrm{frac}(q_P) = v_P(\chi) \qquad \text{for all prime Weil } P \tag{2.7}$$

We call a G-Weil divisor principal if it is an image of a single function $f \in K_G^*(\mathbb{C}^n)$ under val_{K^G}, call two G-Weil divisors linearly equivalent if their difference is principal and call a divisor $\sum q_i D_i$ effective if all $q_i \geq 0$.

We now have a following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Car}\,Y & \longrightarrow & G\text{-}\,\mathrm{Car}\,Y & \xrightarrow{\ \rho\ } & G^\vee & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle val_K} & & \downarrow{\scriptstyle val_{K_G}} & & \downarrow{\scriptstyle val_{G^\vee}} & & \\ 0 & \longrightarrow & \mathrm{Div}\,Y & \longrightarrow & G\text{-}\,\mathrm{Div}\,Y & \longrightarrow & val_{G^\vee}(G^\vee) & \longrightarrow & 0 \end{array} \tag{2.8}$$

A warning: for general Y, even a smooth one, G-Cartier and G-Weil divisors may not be very well behaved. For an example let Y be the smooth locus of X. It can be shown, that while val_K is an isomorphism, val_{K_G} is not even injective as G-Car Y has torsion. And $val_{G^\vee}$ is the zero map, thus G-Div Y is just Div Y.

Proposition 2.6. *If Y is smooth and proper over X, then val_K, val_{K_G} and $val_{G^\vee}$ in (2.8) are isomorphisms.*

Proof. If Y is smooth, or at least locally factorial, val_K is well-known to be an isomorphism ([Har77], Proposition 6.11). It therefore suffices to show that $val_{G^\vee}$ is injective and hence an isomorphism. As diagram (2.8) commutes, val_{K_G} will then also have to be an isomorphism.

Fix $\chi \in G^\vee$. Let Y_χ denote the normalisation of $Y \times_X (\mathbb{C}^n / \ker \chi)$. It is a Galois covering of Y whose Galois group is $\chi(G)$. By Zariski-Nagata's purity of the branch locus theorem ([Zar58], Proposition 2), the ramification locus of $Y_\chi \to Y$ is either empty or of pure codimension one. As Y is smooth, $Y_\chi \to Y$ being finite and unramified would make it an étale cover. Which is impossible, since a resolution of a quotient singularity is well known to be simply-connected (see, for instance, [Ver00], Theorem 4.1).

Thus, we can assume there exists a ramification divisor $P \subset Y_\chi$. Let Q be its image in Y. Let $\mathrm{Ram}(P)$ be the subgroup of G which fixes P pointwise. Then $n_{\mathrm{ram}} = |\mathrm{Ram}(P)/\ker\chi|$ is the ramification index of P. We can take ordinary integer valuations of $K^*_\chi(\mathbb{C}^n)$ on prime divisors of Y_χ as $K^*_\chi(\mathbb{C}^n) \subset K(\mathbb{C}^n)^{\ker\chi}$. It is easy to see that for any $f \in K^*_\chi(\mathbb{C}^n)$

$$v_Q(f) = \frac{1}{n_{\mathrm{ram}}} v_P(f) \tag{2.9}$$

where LHS is a rational valuation in sense of Definition 2.4.

If $v_Q(\chi) = 0$, then $v_Q(K^*_\chi(\mathbb{C}^n)) \subset \mathbb{Z}$. Then necessarily $v_Q(K^*_\chi(\mathbb{C}^n)) = \mathbb{Z}$, as $K^*_\chi(\mathbb{C}^n)$ is a coset of $K(Y)$ in $K^*_G(\mathbb{C}^n)$. In particular, there would exist $f_\chi \in K^*_\chi(\mathbb{C}^n)$, such that $v_Q(f_\chi) = 0$, i.e. f_χ is a unit in $\mathcal{O}_{Y_\chi,P}$. Which is impossible: any $g \in \mathrm{Ram}(P)$ fixes P pointwise, in particular $f - g.f \in \mathfrak{m}_{Y,P}$ for any $f \in \mathcal{O}_{Y,P}$. As $\mathrm{Ram}(P)/\ker\chi$ is non-trivial we can choose g such that $\chi(g) \neq 1$ and then $f_\chi = \frac{1}{1-\chi(g)}(f_\chi - g.f_\chi)$ must lie in $\mathfrak{m}_{Y,P}$. This finishes the proof.

For abelian G, this all can be seen very explicitly by exploiting the toric structure of the singularity: even though we do not assume the resolution Y to be toric, it has been proven by Bouvier ([Bou98], Theorem 1.1) and by Ishii and Kollár ([KI03], Corollary 3.17, in a more general context of Nash problem) that every essential divisor over X (i.e. a divisor which must appear on every resolution) is toric. The set of essential toric divisors is well understood - it can be identified with the Hilbert basis of the positive octant of the toric lattice of weights, and then with a subset of $\mathrm{Ext}^1(G^\vee, \mathbb{Z}) = \mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$. This correspondence sends each divisor precisely to the valuation of $G^\vee$ at it, see [Log04], Section 4.3 for more detail. □

We also show that, away from a finite number of prime divisors on Y, all G-Weil divisors are ordinary Weil.

Proposition 2.7. *Unless a prime divisor $P \subset Y$ is exceptional or its image in X is a branch divisor of $\mathbb{C}^n \to X$, the valuation $v_P : G^\vee \to \mathbb{Q}/\mathbb{Z}$ is the zero-map.*

Proof. If P is not exceptional, let Q be its image in X. The valuations at P and Q are the same, so it suffices to prove the statement about v_Q. Let P' be any divisor in $\mathbb{C}^n$ which lies above Q. As in Proposition 2.6, for any $f \in K^*_G(\mathbb{C}^n)$ we have $v_Q(f) = \frac{1}{n_{\mathrm{ram}}} v_{P'}(f)$ where n_{ram} is the ramification index of P'. Unless Q is a branch divisor, $n_{\mathrm{ram}} = 1$ and $v_Q = v_{P'}$. Which makes v_Q integer-valued on $K^*_G(\mathbb{C}^n)$ and makes the quotient homomorphism $G^\vee \to \mathbb{Q}/\mathbb{Z}$ the zero map. □

3 Classification of *gnat*-families

3.1 Reductor Sets

From now on, in addition to assuming that G is a finite group acting faithfully on V_{giv}, we also assume that G is abelian. We further assume that Y is smooth and $\pi : Y \to X$ is proper.

Let $\mathcal{F}$ be a gnat-family on Y. Write the decomposition of $\mathcal{F}$ into G-eigensheaves as $\bigoplus_{\chi \in G^\vee} \mathcal{F}_\chi$. By Proposition 1.5 we can embed $\mathcal{F}$ into $K(\mathbb{C}^n)$ and, as was demonstrated in Proposition 2.3, the image of each $\mathcal{F}_\chi$ defines a χ-Cartier divisor. Hence $\mathcal{F} \simeq \bigoplus_\chi \mathcal{L}(-D_\chi)$ for some set $\{D_\chi\}_{\chi \in G^\vee}$ of G-Weil divisors.

Definition 3.1. Let $\{D_\chi\}_{\chi \in G^\vee}$ be a set of G-Weil divisors on Y. We call it a `reductor set` if each D_χ is a χ-Weil divisor and $\oplus \mathcal{L}(-D_\chi)$ is a gnat-family on Y. We call a reductor set `normalised` if $D_{\chi_0} = 0$. We say that two reductor sets $\{D_\chi\}$ and $\{D'_\chi\}$ are linearly equivalent if there exists $f \in K(Y)$ such that $D_\chi - D'_\chi = \operatorname{Div} f$ for all $\chi \in G^\vee$.

Lemma 3.2. *Let $\{D_\chi\}$ and $\{D'_\chi\}$ be two reductor sets. Any $(R \rtimes G) \otimes \mathcal{O}_Y$-module morphism $\phi : \bigoplus \mathcal{L}(-D_\chi) \to \bigoplus \mathcal{L}(-D'_\chi)$ is necessarily a multiplication inside $K(\mathbb{C}^n)$ by some $f \in K(Y)$.*

Proof. Because of G-equivariance ϕ decomposes as $\bigoplus_{\chi \in G^\vee} \phi_\chi$ with ϕ_χ a morphism $\mathcal{L}(-D_\chi) \to \mathcal{L}(-D'_\chi)$. Each ϕ_χ is a morphism of invertible sub-$\mathcal{O}_Y$-modules of $K(\mathbb{C}^n)$ and so is necessarily a multiplication by some $f_\chi \in K(Y)$: consider the induced map $\mathcal{O}_Y \to \mathcal{L}(-D_\chi + D'_\chi)$ and take f_χ to be the image of 1 under this map.

It remains to show that all f_χ are equal. Fix any $\chi \in G^\vee$ and consider any G-homogeneous $m \in R$ of weight χ. Take any $s \in \mathcal{L}(-D_{\chi_0})$. Then $ms \in \mathcal{L}(-D_\chi)$ and by R-equivariance of ϕ

$$\phi_\chi(ms) = m\phi_{\chi_0}(s) = f_{\chi_0} ms \tag{3.1}$$

and hence $f_\chi = f_{\chi_0}$ for all $\chi \in G^\vee$. □

Corollary 3.3. *Isomorphism classes of gnat-families on Y are in 1-to-1 correspondence with linear equivalence classes of reductor sets.*

Proof. If in the proof of Lemma 3.2 each ϕ_χ is an isomorphism, then f, by construction, globally generates each $\mathcal{L}(-D'_\chi + D_\chi)$. Thus $D_\chi - D'_\chi = \operatorname{Div}(f)$. □

Proposition 3.4. *Let $\{D_\chi\}$ and $\{D'_\chi\}$ be two reductor sets. Then $\bigoplus \mathcal{L}(-D_\chi)$ and $\bigoplus \mathcal{L}(-D'_\chi)$ are equivalent (locally isomorphic) if and only if there exists a Weil divisor N such that $D_\chi - D'_\chi = N$ for all $\chi \in G^\vee$.*

Proof. The 'if' direction is immediate.

Conversely, if the families are equivalent, then by applying Lemma 3.2 to each local isomorphism, we obtain the data $\{U_i, f_i\}$, where U_i are an open cover of Y and on each U_i multiplication by f_i is an isomorphism $\bigoplus \mathcal{L}(-D_\chi) \xrightarrow{\sim} \bigoplus \mathcal{L}(-D'_\chi)$. One can readily check that such $\{U_i, f_i\}$ must define a Cartier divisor and that the corresponding Weil divisor is the requisite divisor N. □

Corollary 3.5. *In each equivalence classes of gnat-families there is precisely one family whose reductor set is normalised.*

3.2 Reductor Condition

We now investigate when is a set $\{D_\chi\}$ of G-divisors a reductor set.

This issue is the issue of $\bigoplus \mathcal{L}(-D_\chi)$ actually being $(R \rtimes G)\otimes\mathcal{O}_Y$-module. By definition it is a sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$, but there is no a priori reason for it to also be closed under the natural $R \rtimes G$-action on $K(\mathbb{C}^n)$. If it is closed, it can be checked that it trivially satisfies all the other requirements in Proposition 1.5, item 3, which makes it a *gnat*-family. We further observe that $\bigoplus \mathcal{L}(-D_\chi)$ is always closed under the action of G, so it all boils down to the closure under the action of R.

Recall, that we write R_G for $R \cap K^*_G(\mathbb{C}^n)$, the G-homogeneous regular polynomials, and R_χ for $R \cap K^*_\chi(\mathbb{C}^n)$, the G-homogeneous regular polynomials of weight $\chi \in G^\vee$.

Proposition 3.6 (Reductor Condition)**.** *Let $\{D_\chi\}_{\chi\in G^\vee}$ be a set with each D_χ a χ-Weil divisor. Then it is a reductor set if and only if, for any $f \in R_G$, the divisor*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \qquad (3.2)$$

i.e. it is effective.

Remarks:

1. If we choose a G-eigenbasis of V_{giv}, then its dual basis, a set of basic monomials $x_1, \dots, x_n$, generates R_G as a semi-group. As condition (3.2) is multiplicative on f, it is sufficient to check it only for f being one of x_i. This leaves us with a finite number of inequalities to check.

2. Numerically, if we write each D_χ as $\sum q_{\chi,P} P$, inequalities (3.2) subdivide into independent sets of inequalities

$$q_{\chi,P} + v_P(f) - q_{\chi\rho(f),P} \geq 0 \qquad \forall\, \chi \in G^\vee \qquad (3.3)$$

   a set for each prime divisor P on Y. This shows that a *gnat*-family can be specified independently at each prime divisor of Y: we can construct reductor sets $\{D_\chi\}$ by independently choosing for each prime divisor P any of the sets of numbers $\{q_{\chi,P}\}_{\chi\in G^\vee}$ which satisfy (3.3).

3. There is an interesting link here with the work of Craw, Maclagan and Thomas in [CMT05a] which bears further investigation. In a toric context, they have rediscovered these inequalities as dual, in a certain sense, to the defining equations of the coherent component Y_θ of the moduli space M_θ of θ-semistable G-constellations. They then use them to compute the distinguished θ-semistable G-constellations parametrised by torus orbits of Y_θ. In particular, their Theorem 7.2 allows them to explicitly write down the tautological gnat-family on Y_θ and suggests that, up to a reflection, it is the gnat family which minimizes $\theta.\{D_\chi\}$. We shall see an example of that for the case of $Y_\theta = \mathrm{Hilb}^G$ in our Proposition 3.17.

Proof. Take an open cover U_i on which all $\mathcal{L}(-D_\chi)$ are trivialised and write $g_{\chi,i}$ for the generator of $\mathcal{L}(-D_\chi)$ on U_i. As R is a direct sum of its G-homogeneous parts, it is sufficient to check the closure under the action of just the homogeneous functions. Thus it suffices to establish that for each $f \in R_G$, each U_i and each $\chi \in G^\vee$

$$fg_{\chi,i} \in \mathcal{O}_Y(U_i)g_{\chi\rho(f),i}$$

On the other hand, with the notation above, G-Cartier divisor $D_\chi + (f) - D_{\chi\rho(f)}$ is given on U_i by $\frac{fg_{\chi,i}}{g_{\chi\rho(f),i}}$ and it being effective is equivalent to

$$\frac{fg_{\chi,i}}{g_{\chi\rho(f),i}} \in \mathcal{O}_Y(U_i)$$

for all U_i's. The result follows. ∎

3.3 Canonical family

We have not yet given any evidence of any gnat-families actually existing on an arbitrary resolution Y of X.

Proposition 3.7 (Canonical family). *Let Y be a resolution of $X = \mathbb{C}^n/G$. Define the set $\{C_\chi\}_{\chi\in G^\vee}$ of G-Weil divisors by*

$$C_\chi = \sum v(P,\chi)P$$

where P runs over all prime Weil divisors on Y and $v_P(\chi)$ are the rational numbers introduced in Definition 2.4.

Then $\{C_\chi\}_{\chi\in G^\vee}$ is a reductor set.

We call the corresponding family `the canonical gnat-family on` *Y.*

Proof. We must show that $\{C_\chi\}$ satisfies the inequalities (3.2). Choose any $\chi \in G^\vee$, any $f \in R_G$ and any prime divisor P on Y. Observe that

$0 \le v_P(\chi), v_P(\chi\rho(f)) < 1$ by definition, while $v_P(f) \ge 0$ since $f^{|G|}$ is regular on all of Y. So we must have

$$v_P(\chi) + v_P(f) - v_P(\chi\rho(f)) > -1$$

As the above expression must be integer-valued, we further have

$$v_P(\chi) + v_P(f) - v_P(\chi\rho(f)) \ge 0$$

as required. □

This family has a following geometrical description:

Proposition 3.8. *On any resolution Y, the canonical family is isomorphic to the pushdown to Y of the structure sheaf $\mathcal{N}$ of the normalisation of the reduced fiber product $Y \times_X \mathbb{C}^n$.*

Proof. First we construct a $(R \rtimes G) \otimes \mathcal{O}_Y$-module embedding of $\mathcal{N}$ into $K(\mathbb{C}^n)$. Let α be the map $\mathcal{O}_Y \otimes_{R^G} R \to K(\mathbb{C}^n)$ which sends $a \otimes b$ to ab. It is $R \rtimes G \otimes \mathcal{O}_Y$-equivariant. If we show that $\ker \alpha$ is the nilradical of $\mathcal{O}_Y \otimes_{R^G} R$, then $\mathcal{N}$ can be identified with the integral closure of the image of α in $K(\mathbb{C}^n)$. Due to G-equivariance α decomposes as $\bigoplus_{\chi \in G^\vee} \alpha_\chi$ with each α_χ a morphism $\mathcal{O}_Y \otimes_{R^G} R_\chi \to K_\chi(\mathbb{C}^n)$. Observe that $(\mathcal{O}_Y \otimes_{R^G} R_\chi)^{|G|} \subset \mathcal{O}_Y \otimes_{R^G} R_{\chi 0} = \mathcal{O}_Y$ as a product of $|G|$ homogeneous functions is invariant. Hence $(\ker \alpha_\chi)^{|G|} \subset \ker \alpha_{\chi 0} = 0$ as required.

Write $\bigoplus_{\chi \in G^\vee} \mathcal{N}_\chi$ for the decomposition of $\mathcal{N}$ into G-eigensheaves. Fix a point $p \in Y$ and observe that $f \in K_\chi(\mathbb{C}^n)$ is integral over the local ring $\mathcal{N}_p$ if and only if $f^{|G|} \in (\mathcal{N}_{\chi 0})_p = \mathcal{O}_{Y,p}$. Therefore

$$(\mathcal{N}_\chi)_p = \{ f \in K_\chi(\mathbb{C}^n) \mid G\text{-Weil divisor } \operatorname{Div}(f) \text{ is effective at } p \}$$

In particular, the generator c_χ of C_χ at p lies in $(\mathcal{N}_\chi)_p$. Observe further that for any $f \in (\mathcal{N}_\chi)_p$ the Weil divisor $\operatorname{Div}(f) - C_\chi$ is effective at p as the coefficients of C_χ are just the fractional parts of those of $\operatorname{Div}(f)$ and the latter is effective. Therefore c_χ generates $(\mathcal{N}_\chi)_p$ as $\mathcal{O}_{Y,p}$-module, giving $\mathcal{N}_\chi = \mathcal{L}(-C_\chi)$ as required. □

3.4 Symmetries

Having demonstrated that the set of equivalence classes of gnat-families is always non-empty, we now establish two types of symmetries which this set must possess.

Proposition 3.9 (Character Shift). *Let $\{D_\chi\}$ be a normalised reductor set. Then for any χ in $G^\vee$*

$$D'_{\chi\lambda} = D_\chi - D_{\lambda^{-1}} \tag{3.4}$$

is also a normalised reductor set. We call it `the χ-shift` of $\{D_\chi\}$.

Proof. Writing out the reductor condition (3.2) for the new divisor set $\{D'_\chi\}$ we get:

$$(D_\chi - D_{\lambda^{-1}}) + (m) - (D_{\chi\rho(m)} - D_{\lambda^{-1}}) \geq 0$$

Cancelling out D_λ^{-1}, we obtain precisely the reductor condition for the original set $\{D_\chi\}$. And since

$$D'_{\chi 0} = D'_{\lambda^{-1}\lambda} = D_{\lambda^{-1}} - D_{\lambda^{-1}} = 0$$

we see that the new reductor set is normalised. □

NB: Observe, that for a reductor set $\{D_\chi\}$ and for any χ-Weil divisor N, the set $\{D_\chi + N\}$ is linearly equivalent to the χ-shift of $\{D_\chi\}$.

Proposition 3.10 (Reflection)**.** *Let $\{D_\chi\}$ be a normalised reductor set. Then the set $\{-D_{\chi^{-1}}\}$ is also a normalised reductor set, which we call* `the reflection` *of $\{D_\chi\}$.*

Proof. We need to show that

$$-D_{\chi^{-1}} + (m) - (-D_{\chi^{-1}\rho(m)^{-1}}) \geq 0$$

Rearranging we get

$$D_{\chi^{-1}\rho(m)^{-1}} + (m) - D_{\chi^{-1}\rho(m)^{-1}\rho(m)} \geq 0$$

which is one of the reductor equations the original set $\{D_\chi\}$ must satisfy. As $D'_{\chi 0} = -D_{\chi 0} = 0$, the new set is normalised. □

3.5 Maximal shift family and finiteness

We now examine the individual line bundles $\mathcal{L}(-D_\chi)$ in a *gnat*-family and show that the reductor condition imposes a restriction on how far apart from each other they can be.

Lemma 3.11. *Let $\{D_\chi\}$ be a reductor set. Write each D_χ as $\sum q_{\chi,P}P$, where P ranges over all the prime Weil divisors on Y. For any $\chi_1, \chi_2 \in G^\vee$ and for any prime Weil divisor P, we necessarily have*

$$\min_{f \in R_{\chi_1/\chi_2}} v_P(f) \quad \geq \quad q_{\chi_1,P} - q_{\chi_2,P} \quad \geq \quad - \min_{f \in R_{\chi_2/\chi_1}} v_P(f) \tag{3.5}$$

Proof. Both inequalities follow directly from the reductor condition (3.2): the right inequality by setting $\chi = \chi_1 \in G^\vee$, $\rho(f) = \frac{\chi_2}{\chi_1}$ and letting f vary within $R_{\rho(f)}$; the left inequality by setting $\chi = \chi_2$ and $\rho(f) = \frac{\chi_1}{\chi_2}$. □

This suggests the following definition:

Definition 3.12. For each character $\chi \in G^\vee$, we define the `maximal shift` χ`-divisor` M_χ *to be*

$$M_\chi = \sum_P (\min_{f \in R_\chi} v_P(f))P \tag{3.6}$$

where P ranges over all prime Weil divisors on Y.

Lemma 3.13. *The G-Weil divisor set $\{M_\chi\}$ is a normalised reductor set. We call the corresponding family* `the maximal shift` *gnat*`-family` on Y.

Proof. We need to show that for any $f \in R_G$ and any prime divisor P

$$v_P(m_\chi) + v_P(f) - v_P(m_{\chi\rho(f)}) \geq 0$$

where m_χ and $m_{\chi\rho(f)}$ are chosen to achieve the minimality in (3.6).

Observe that $m_\chi f$ is also a G-homogeneous element of R, therefore by the minimality of $v_P(m_{\chi\rho(f)})$ we have

$$v_P(m_\chi f) \geq v_P(m_{\chi\rho(f)})$$

as required.

To establish that $M_{\chi_0} = 0$ we observe that for any G-homogeneous $f \in R$ we have $v_P(f) \geq 0$ on any prime Weil divisor P as $f^{|G|}$ is globally regular. Moreover for f in $R_{\chi_0} = R^G$ this lower bound is achieved by $f = 1$. □

Observe that with Lemma 3.13 we have established another gnat-family which always exists on any resolution Y. While sometimes it coincides with the canonical family, generally the two are distinct.

Proposition 3.14 (Maximal Shifts)**.** *Let $\{D_\chi\}$ be a normalised reductor set. Then for any $\chi \in G^\vee$*

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}} \tag{3.7}$$

Moreover both the bounds are achieved.

Proof. To establish that (3.7) holds set $\chi_2 = \chi_0$ in Lemma 3.11. Lemma 3.13 shows that the bounds are achieved. □

Proposition 3.15. *If the coefficient of a maximal shift divisor M_χ at a prime divisor $P \subset Y$ is non-zero, then either P is an exceptional divisor or the image of P in X is a branch divisor of $\mathbb{C}^n \xrightarrow{q} X$.*

Proof. Let P be a prime divisor on X which is not a branch divisor of q. Let $\chi \in G^\vee$. By the defining formula (3.6) it suffices to find $f \in R_\chi$ such that $v_P(f) = 0$.

As R is a PID, there exist $t_1, \ldots, t_k \in R$ such that $(t_1), \ldots, (t_k)$ are all the distinct prime divisors lying over P in $\mathbb{C}^n$. Observe that the product

$t_1 \dots t_n$ must be G-homogeneous. Since P is not a branch divisor, there exists $u \in R$ such that $t_1 \dots t_k u$ is invariant and $u \notin (t_i)$ for all i. Then $u' = u^{|G|-1}$ is a G-homogeneous function of the same weight as $t_1 \dots t_k$ and $v_P(u') = 0$. Now take any $f \in R_\chi$ and consider its factorization into irreducibles. G-homogeneity of f implies that all t_i occur with the same power k. Now replacing $(t_1 \dots t_n)^k$ in the factorization by $(u')^k$ we obtain an element of R_χ whose valuation at P is zero. □

Corollary 3.16. *The number of equivalence classes of gnat-families on Y is finite.*

Proof. Let $\{D_\chi\}$ be a normalised reductor set. Coefficients of D_χ at prime divisors P of Y have fixed fractional parts (Definition 2.5), are bound above and below (Proposition 3.14) and are zero at all but finite number of P (Proposition 3.15). This leaves only a finite number of possibilities. □

For one particular resolution Y the family provided by the maximal shift divisors has a nice geometrical description.

Proposition 3.17. *Let $Y = \operatorname{Hilb}^G \mathbb{C}^n$, the coherent component of the moduli space of G-clusters in $\mathbb{C}^n$. If Y is smooth, then $\bigoplus \mathcal{L}(-M_\chi)$ is the universal family $\mathcal{F}$ of G-clusters parametrised by Y, up to the usual equivalence of families.*

Proof. Firstly $\mathcal{F}$ is a gnat-family, as over any set $U \subset X$ such that G acts freely on $q^{-1}(U)$ we have $\mathcal{F}|_U \simeq \pi^* q_* \mathcal{O}_{\mathbb{C}^n}|_U$. Write $\mathcal{F}$ as $\oplus \mathcal{L}(-D_\chi)$ for some reductor set $\{D_\chi\}$. Take an open cover $\{U_i\}$ of Y and consider the generators $\{f_{\chi,i}\}$ of D_χ on each U_i. Working up to equivalence, we can consider $\{D_\chi\}$ to be normalised and so $f_{\chi_0,i} = 1$ for all U_i.

Now any G-cluster Z is given by some invariant ideal $I \subset R$ and so the corresponding G-constellation $H^0(\mathcal{O}_Z)$ is given by R/I. In particular note that R/I is generated by R-action on the generator of χ_0-eigenspace. Therefore any $f_{\chi,i}$ is generated from $f_{\chi_0,i} = 1$ by R-action, which means that all $f_{\chi,i}$ lie in R.

But this means that for any prime Weil divisor P on Y we have

$$v_P(f_{\chi,i}) \geq \min_{f \in R_\chi} v_P(f)$$

and therefore $D_\chi \geq M_\chi$. Now Proposition 3.14 forces the equality. □

4 Conclusion

We summarise the results achieved in the following theorem:

Theorem 4.1 (Classification of *gnat*-families)**.** *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G, Y nonsingular and $\pi : Y \to X$ a proper birational map. Then isomorphism classes of gnat-families on Y are in 1-to-1 correspondence with linear equivalence classes of G-divisor sets $\{D_\chi\}_{\chi \in G^\vee}$, each D_χ a χ-Weil divisor, which satisfy the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall\, \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Such a divisor set $\{D_\chi\}$ corresponds then to a gnat-family $\bigoplus \mathcal{L}(-D_\chi)$.

This correspondence descends to a 1-to-1 correspondence between equivalence classes of gnat-families and sets $\{D_\chi\}$ as above and with $D_{\chi_0} = 0$. Furthermore, each divisor D_χ in such a set satisfies inequality

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}$$

where $\{M_\chi\}$ is a fixed divisor set defined by

$$M_\chi = \sum_P (\min_{f \in R_\chi} v_P(f))P$$

As a consequence, the number of equivalence classes of gnat-families is finite.

Proof. Corollary 3.3 establishes the correspondence between isomorphism classes of *gnat*-families and linear equivalence classes of reductor sets. Proposition 3.6 gives description of reductor sets as the divisor sets satisfying the reductor condition inequalities.

Corollary 3.5 gives the correspondence on the level of equivalence classes of *gnat*-families and normalised reductor sets. Proposition 3.14 establishes the bounds on the set of all normalised reductor sets and Corollary 3.16 uses it to show that the set of all normalised reductor sets is finite. □

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