ETA-PRODUCT $\eta(7\tau)^7/\eta(\tau)$

KYOJI SAITO

Abstract. Let $L_{\Phi_7}(s)$ be the Dirichlet series associated to the eta-product $\eta(7\tau)^7/\eta(\tau) \in M_3(\Gamma_0(7), \varepsilon)$ (here $\varepsilon(n) := \left(\frac{n}{7}\right) = \left(\frac{-7}{n}\right)$ is the Dirichlet character defined by the residue symbol). We show that $L_{\Phi_7}(s)$ decomposes into the difference of two L-functions:

$$L_{\Phi_7}(s) = \frac{1}{8}\big(L(s,\varepsilon)L(s-2,1) - L(s-1,\xi)\big),$$

where i) $L(s,\varepsilon)$ and $L(s,1)$ are Dirichlet L-functions for the characters ε and 1 modulo 7, respectively, and ii) $L(s,\xi)$ is the L-function for a Hecke character ξ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$.

This expression of $L_{\Phi_7}(s)$ gives a new proof of the non-negativity of the Fourier coefficients of the product $\eta(7\tau)^7/\eta(\tau)$, conjectured in [S3] and proven by Ibukiyama [I]. We also prove the uniqueness of the above decomposition of $L_{\Phi_7}(s)$ in a suitable sense.

1. introduction

Let $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^{\eta})$, $q = \exp(2\pi\sqrt{-1}\tau)$ be the Dedekind eta-function (e.g. [R]). A product $\Pi_{i\in I}\eta(i\tau)^{e(i)}$, where I is a finite set of positive integers and $e : I \to \mathbb{Z}$ is any map, is called an eta-product. The eta-product can be developed in a Laurent series in powers of q, whose coefficients are called the *Fourier coefficients*.

Ibukiyama [I] has shown the following result, which answers to a part of a conjecture given by the author [S3] (see the next paragraph).

Theorem 1.1. *Let p be a rational prime number. Then the Fourier coefficients of the eta product $\eta_{\Phi_p} := \eta(p\tau)^p/\eta(\tau)$ are non-negative.*

The proof in [I] is given by expressing the eta-product as a difference of two generating functions of two arithmetically constructed lattices.

More in general than the theorem, for any positive integer h which may not be prime, we have the following non-negativity conjecture.

Conjecture ([S3]). Define the sequence $\Phi_h(\lambda)$ ($h \in \mathbb{Z}_{>0}$) of polynomials in λ by the recursive relation: $\frac{(1-\lambda^h)^h}{1-\lambda} = \prod_{d|h}\Phi_d(\lambda^{h/d})$. Explicitly, $\Phi_h(\lambda) = \frac{(1-\lambda^h)^{\phi(h)}}{\prod_{d|h}(1-\lambda^d)^{\mu(d)}}$ where ϕ and μ are the Euler function and the Möbius function. Then the Fourier coefficients of the eta-product $\eta_{\Phi_h}(\tau) := \frac{\eta(h\tau)^{\phi(h)}}{\prod_{d|h}\eta(d\tau)^{\mu(d)}}$ are non-negative integers.

This was proven for $h=2,3,4,5,6$ [S1,2,3] by a use of the Dirichlet series $L_{\Phi_h}(s)$ associated to the eta-products η_{Φ_h}[1]. Precisely, we show that $L_{\Phi_h}(s)$ admits either an Euler product for $h=2,3,5$ or a decomposition into the difference of two Euler products for $h=4,6$, and that these expressions lead to a direct proof of the positivity of the coefficients.

In the present note, we prove in section 2 that the Dirichlet series $L_{\Phi_7}(s)$ decomposes into a difference of two L-functions, which admit Euler products, as stated in the abstract. In section 3, we show that this expression implies the non-negativity of the Dirichlet coefficients of $L_{\Phi_7}(s)$. In section 4, we prove a general lemma on the uniqueness of the decomposition of Dirichlet series into a difference of two Euler products, and apply it to $L_{\Phi_7}(s)$ (and also to $L_{\Phi_4}(s)$ and $L_{\Phi_6}(s)$). Finally, we remark in section 5 that such difference decomposition of $L_{\Phi_p}(s)$ for any prime $p \geq 11$ does not exist. If h is a composite number, we do not know when $L_{\Phi_h}(s)$ admits such a difference decomposition.

Remark 1. The interest on the positivity of Fourier coefficients appeared, first, in the study of elliptic root systems [S1]. Namely a simply laced elliptic root system admits the flat (Frobenius manifold) structure on its invariant space if and only if its associated eta product has non-negative Fourier coefficients, and this happens exactly for the 4 exceptional types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ of elliptic root systems. The proof uses the associated Dirichlet series as explained above.

In [S2,Conjecture 13.5], we construct, by a use of regular weight system, a wide class of eta-products whose Fourier coefficients are conjecturally non-negative and are of interest.

2. L-function $L(s,\xi)$ for a Hecke character ξ of $\mathbb{Q}(\sqrt{-7})$

We recall Hecke's L-function for a character ξ on the imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$, and, then, decompose $L_{\Phi_7}(s)$ by a use of it. For a back ground on analytic number theory, one is referred to [M] and [R].

Since the class number of $\mathbb{Q}(\sqrt{-7})$ is equal to 1, we can introduce the Hecke character ξ for the non-zero ideals of $K := \mathbb{Q}(\sqrt{-7})$ by

$$(1) \qquad \xi((a)) := \Big(\frac{a}{|a|}\Big)^2 \qquad (a \in K \setminus \{0\}).$$

Then, the L-function for ξ is defined by the following Dirichlet series, which, as a result of definition, has the Euler product:

$$(2) \qquad L(s,\xi) := \sum_{\mathfrak{a}\ \subset \mathcal{O}_K} \xi(\mathfrak{a}) N_K(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}\ :\ \text{prime}} (1-\xi(\mathfrak{p})N_K(\mathfrak{p})^{-s})^{-1}.$$

[1]The Dirichlet series $\sum_{n>0} c(n)n^{-s}$ is associated to a Fourier series $\sum_{n\geq 0} c(n)q^n$.

Here, $\mathfrak{a}$ (resp. $\mathfrak{p}$) runs over all non-zero integral (resp. prime) ideals of $\mathcal{O}_K$, and $N_K(\mathfrak{a})$ is the absolute norm of $\mathfrak{a}$ (i.e. $N_K(\mathfrak{a}) = |\mathcal{O}_K/\mathfrak{a}|$).

The first main result of the present note is the following.

Lemma 2.1. *The Dirichlet series $L_{\Phi_7}(s)$ associated to the eta-product $\eta(7\tau)^7/\eta(\tau)$ decomposes into a difference of two L-functions as follows:*

$$L_{\Phi_7}(s) = \frac{1}{8}\big(L(s,\varepsilon)L(s-2,1) - L(s-1,\xi)\big), \tag{3}$$

where we recall that $\varepsilon := \left(\frac{*}{7}\right) = \left(\frac{-7}{*}\right)$ is the residue symbol modulo 7.

Proof. Recall that $\eta_{\Phi_7}(\tau)=\eta(7\tau)^7/\eta(\tau)$ belongs to the space $M_3(\Gamma_0(7),\varepsilon)$ of automorphic forms of weight 3, charcter ε on the group $\Gamma_0(7)$ (e.g. [S2,13.3]). The L-function $L(s-1,\xi)$ is associated to a Fourier series

$$f(\tau) := \sum_{\mathfrak{a}} \xi(\mathfrak{a}) N_K(\mathfrak{a}) e^{2\pi\sqrt{-1}N_K(\mathfrak{a})\tau}. \tag{4}$$

According to Hecke [H1][H2], $f(\tau)$ is a cusp form belonging to $S_3(\Gamma_0(7),\varepsilon)$ (see [M, Th.4.8.2]). Similarly, $L(s,\varepsilon)L(s-2,1)$ and $L(s-2,\varepsilon)L(s,1)$ are associated to Eisenstein series, say $E(\tau)$ and $E'(\tau)$, in $M_3(\Gamma_0(7),\varepsilon)$. Since $\Gamma_0(7)\backslash\mathcal{H}$ has two cusps and $\dim_{\mathbb{C}} S_3(\Gamma_0(7),\varepsilon)=1$, $M_3(\Gamma_0(7),\varepsilon)$ is spanned by E, E' and f. Since their Fourier coefficients until degree 3 are already linearly independent, to show the equality: $\eta_{\Phi_7}(\tau) = \frac{1}{8}\big(E(\tau) - f(\tau)\big)$, it suffices to show that nth Fourier coefficients $c(n)$ of $\eta_{\Phi_7}(\tau)$ coincide with nth Dirichlet coefficients of $\frac{1}{8}(L(s,\varepsilon)L(s-2,1) - L(s-1,\xi))$ for $1 \le n \le 3$. Let us give an explicit integral description (which we shall use in the next section) of the coefficients of $L(s-1,\xi)$. For this end, *we factorize $L(s-1,\xi)$ w.r.t. rational primes p, q in $\mathbb{Z}_{>0}$:*

$$L(s-1,\xi) := \frac{1}{1+7^{-s+1}} \cdot \prod_{\varepsilon(q)=-1} \frac{1}{1-q^{-2s+2}} \cdot \prod_{\varepsilon(p)=1} \frac{1}{P_p(p^{-s})}, \tag{5}$$

where $P_p(\lambda)\in\mathbb{Z}[\lambda]$ for a prime p with $\varepsilon(p)=1$ is defined in (6) below.

A proof of (5). We recall a well-known (e.g. [T]) list of all prime ideals in $\mathbb{Q}(\sqrt{-7})$ (where we note that all ideals are principla).

i) for any rational prime q with $\varepsilon(q) = -1$, (q) is a prime ideal,

ii) for any odd rational prime number p with $\varepsilon(p)=1$, one has the decomposition: $p = x_p^2 + 7\cdot y_p^2 = (x_p + y_p\sqrt{-7})(x_p - y_p\sqrt{-7})$ $((x_p,y_p)\in\mathbb{Z}^2_{>0})$,

iii) $2 = \frac{7\cdot 1+1}{4} = \frac{1+\sqrt{-7}}{2}\cdot\frac{1-\sqrt{-7}}{2}$ and $7 = -(\sqrt{-7})^2$.

Put $\pi_2 := \frac{1+\sqrt{-7}}{2}$ and $\pi_p := x_p + y_p\sqrt{-7}$ for an odd rational prime number p with $\varepsilon(p) = 1$ and, define the quadratic polynomials

$$\begin{aligned} P_2(X) &:= (1-\pi_2^2 X)(1-\overline{\pi}_2^2 X) = 1 + 3X + 2^2X^2 \text{ and} \\ P_p(X) &:= (1-\pi_p^2 X)(1-\overline{\pi}_p^2 X) = 1 - 2(x_p^2 - 7y_p^2)X + p^2X^2. \end{aligned} \tag{6}$$

Then (5) follows from the Euler product in (2) and
i) $\xi((\pi_p)) = \pi_p^2/p$ and $N_K((\pi_p)) = p$ for $\varepsilon(p) = 1$,
ii) $\xi((q)) = 1$ and $N_K((q)) = q^2$ for $\varepsilon(q) = -1$,
iii) $\xi((\sqrt{-7})) = -1$ and $N_K((\sqrt{-7})) = 7$.

Put $L(s,\varepsilon)L(s-2,1) = \sum_{n=1}^{\infty} a(n)n^{-s}$ and $L(s-1,\xi) = \sum_{n=1}^{\infty} b(n)n^{-s}$, and we give explicite expressions of the coefficients $a(n)$ and $b(n)$. Let

$$n = 7^k \prod_{i\in I} p_i^{l_i} \prod_{j\in J} q_j^{m_j}$$

be the prime decompostion of $n \in \mathbb{Z}_{>0}$ where $\{p_i \mid i \in I\}$ and $\{q_j \mid j \in J\}$ are finite sets of distinct prime numbers with $\varepsilon(p_i) = 1$ and $\varepsilon(q_j) = -1$.

Then, by a use of (5) together with (6), one obtains the formulae:

$$(7) \qquad a(n) = 7^{2k} \prod_{i\in I} \frac{p^{2(l_i+1)} - 1}{p_i^2 - 1} \prod_{j\in J} \frac{q_j^{2(m_j+1)} - (-1)^{m_j+1}}{q_j^2 + 1}$$

$$(8) \qquad b(n) = (-7)^k \prod_{i\in I} \Big(\sum_{t=0}^{l_i} \pi_{p_i}^{2t}\overline{\pi}_{p_i}^{2(l_i-t)}\Big) \prod_{j\in J} \frac{1-(-1)^{m_j+1}}{2} q_j^{m_j}$$

Finally, we give the Fourier expansion of η_{Φ_7} up to degree 50.

$$\begin{aligned}\eta_{\Phi_7} = {} & q^2 + q^3 + 2q^4 + 3q^5 + 5q^6 + 7q^7 + 11q^8 + 8q^9 + 15q^{10} + 16q^{11} + 21q^{12} + 21q^{13} \\ & +28q^{14} + 24q^{15} + 44q^{16} + 36q^{17} + 49q^{18} + 45q^{19} + 63q^{20} + 49q^{21} + 74q^{22} + 64q^{23} \\ & +85q^{24} + 72q^{25} + 105q^{26} + 82q^{27} + 133q^{28} + 112q^{29} + 120q^{30} + 120q^{31} + 165q^{32} \\ & +122q^{33} + 180q^{34} + 147q^{35} + 186q^{36} + 176q^{37} + 225q^{38} + 168q^{39} + 255q^{40} + 210q^{41} \\ & +245q^{42} + 224q^{43} + 324q^{44} + 219q^{45} + 338q^{46} + 276q^{47} + 341q^{48} + 294q^{49} + 385q^{50} + \cdots\end{aligned}$$

By inspection, we check the equality $c(n) = \frac{1}{8}(a(n) - b(n))$ for n with $1 \leq n \leq 3$. This completes the proof of Lemma 2.1. □

Remark 2. As we see in the above proof, once one guesses a correct formula (3), then its proof is straightforward. However, we do not know yet what is a "correct formula" for $L_{\Phi_h}(s)$ for $h > 7$ (see §5).

3. Positivity of Fourier coefficients of $\eta(7\tau)^7/\eta(\tau)$

As an immediate consequence of Lemma 2.1. together with the explicit formulae (6) and (7), we obtain the following positivity.

Corollary. *All Fourier coefficients of $\eta(7\tau)^7/\eta(\tau)$ are positive.*

Proof. Lemma 2.1. says $c(n) = \frac{1}{8}(a(n) - b(n))$ for all $n \in \mathbb{Z}_{\geq 1}$. To show $a(n) > b(n)$ for all $n \in \mathbb{Z}_{\geq 1}$, it is sufficient to show $a(p^k) > |b(p^k)|$ for any primary number p^k (i.e. p is a prime number and $k \in \mathbb{Z}_{>0}$) because of the multiplicativity of $a(n)$ and $b(n)$. We separate cases:
Case $p = 7$. $a(7^k) = 7^{2k} > 7^k = |b(7^k)|$.
Case $\varepsilon(p) = 1$. $a(p^k) > p^{2k} \geq (k+1)p^k = \sum_{i=0}^{k} |\pi_p^{2i}\overline{\pi}_p^{2(k-i)}| \geq |b(p^k)|$.
Case $\varepsilon(q) = -1$. $a(q^k) - |b(q^k)| \geq \frac{q^{2(k+1)}-1}{q^2+1} - q^k = \frac{(q^{k+2}-1)(q^k-1)-2}{q^2+1} > 0$. □

4. Uniqueness of decomposition of Dirichlet series

We show the second main result of the present note:
Under a mild assumption on a Dirichlet series $L(s) = \sum_{n\in\mathbb{Z}_{\geq 1}} c(n)n^{-s}$, we show *the uniqueness of the decomposition of $L(s)$ into the form:*

$$L(s) = aM(s) + bN(s) \tag{9}$$

where $M(s)$ and $N(s)$ are Dirichlet series which admit Euler product and a, b are constants. For our applications, we assume that $c(1) = 0$ so that one automatically has $a+b = 0$ (since the first Dirichlet coefficients of $M(s)$ and $N(s)$ are automatically equal to 1) and

(9)' $$L(s) = c(M(s) - N(s)) \quad (c := a = -b).$$

Lemma 4.1. *Let $L(s) = \sum_{n\in\mathbb{Z}_{\geq 1}} c(n)n^{-s}$ be a Dirichlet series such that* i) $c(1) = 0$ *and* ii) *there are five relatively prime integers $l, m, n, u, v \in \mathbb{Z}_{\geq 1}$ such that $c(l)c(m)c(n)c(u)c(v) \neq 0$. If there exists a decomposition* (9)*, where $M(s)$ and $N(s)$ are Dirichlet series having Euler products, then it is unique up to the transposition of $M(s)$ and $N(s)$.*

Proof. Put $M(s) = \sum_{n\in\mathbb{Z}_{\geq 1}} a(n)n^{-s}$, $N(s) = \sum_{n\in\mathbb{Z}_{\geq 1}} b(n)n^{-s}$ and $c := a = -b$ so that one has the relation among the Dirichlet coefficients:

$$c(n) = c(a(n) - b(n)) \qquad (n \in \mathbb{Z}_{\geq 1}). \tag{10}$$

Clearly $c \neq 0$, else $L(s) = 0$ contradicting to the assumption on $L(s)$.

We first remark that one sees from (10) that if $c(n) = c(m) = 0$ for relatively prime positive integers n and m then $c(nm)=0$. Consequently, if $c(n)\neq 0$, then there exists a primary factor p^k of n (i.e. p is a prime number and k is a positive integer s.t. $p^k|n$) such that $c(p^k) \neq 0$.

Suppose there exist another decomposition $L(s) = c'(M'(s)-N'(s))$. Using Dirichlet coefficients $a'(n), b'(n)$ of $M'(s), N'(s)$, this means

$$c(n) = c'(a'(n) - b'(n)) \qquad (n \in \mathbb{Z}_{\geq 1}) \tag{11}$$

Let $n, m \in \mathbb{Z}_{\geq 1}$ be relatively prime to each other, then the multiplicativities of the Dirichlet coefficients a, b, a' and b' implies

$$c(mn) = c(a(n)a(m) - b(n)b(m)) = c'(a'(n)a'(m) - b'(n)b'(m))$$

Substituting $b(n) = a(n) - c(n)/c$, $b'(n) = a'(n) - c(n)/c'$ and $b(m) = a(m) - c(m)/c$, $b'(m) = a'(m) - c(m)/c'$ in this equality, we obtain

$$E(m,n): \ c(n)(a(m)-a'(m)) + c(m)(a(n)-a'(n)) = (\frac{1}{c} - \frac{1}{c'})c(n)c(m).$$

Let $k, m, n \in \mathbb{Z}_{\geq 1}$ be relatively prime to each other and $c(m)c(n)\neq 0$, then $(c(k)E(m,n)–c(m)E(n,k)–c(n)E(k,m))/c(m)c(n)$ is the equality

$$* \qquad a(k) - a'(k) = \frac{1}{2}(\frac{1}{c} - \frac{1}{c'})c(k).$$

This, together with (10) and (11), can be rewritten as the linear relations among $a(k), b(k)$ and $a'(k), b'(k)$ for all k prime to mn:

$$a'(k) = (1-\lambda)a(k) + \lambda b(k) \quad \text{and} \quad b'(k) = \lambda a(k) + (1-\lambda)b(k),$$

where $\lambda := \frac{c}{2}(\frac{1}{c} - \frac{1}{c'})$ so that $\lambda = 0$ or 1 if and only if $c = c'$ or $c = -c'$, respectively. Summing two relations, we also obtain the relation:

$$** \qquad a(k) + b(k) = a'(k) + b'(k).$$

If $c = c'$ (i.e. $\lambda = 0$), then the proof of Lemma 4.1. is already achieved as follows: by substituting $c = c'$ in $*$ and using $**$, one has

$$*** \qquad a(k) = a'(k) \quad \text{and} \quad b(k) = b'(k)$$

for any $k \in \mathbb{Z}_{\geq 1}$ prime to m, n. By replacing the role of m, n by u, v, the equalities $***$ hold for any primary numbers k. The $***$ extends, further, for any positive integers k due to the multiplicativity of a, a', b and b'. This means $M(s) = M'(s)$ and $N(s) = N'(s)$.

Suppose $c \neq c'$ (i.e. $\lambda \neq 0$). Then, $*$ means another decomposition:

$$(11)' \qquad c(k) = \frac{c}{\lambda}(a(k) - a'(k))$$

for any $k \in \mathbb{Z}_{\geq 1}$ prime to m, n. Replacing (11) by (11)', we can repeat the previous discussions to induce $*$ and $**$, where we replace the role of m, n by u, v, and consider integers k which is prime to m, n and also to u, v. Then, in addition to $*$ and $**$, we obtain: $*' : \quad 0 = a(k) - a(k) = \frac{1-\lambda}{2c}c(k)$ and $**' : \quad a(k)+b(k) = a(k)+a'(k)$ for all k prime to m, n, u, v. Taking $k = l$ with $c(l) \neq 0$, which exists by the assumption of Lemma, we obtain $\lambda = 1$, i.e. $c = -c'$. By the similar argument for the case $c = c'$, we obtain: $***' : \quad a(k) = b'(k),\ b(k) = a'(k)$ for all $k \in \mathbb{Z}_{\geq 1}$ and, therefore, $M(s) = N'(s)$ and $N(s) = M'(s)$. □

Corollary. *The Dirichlet series* $L_{\Phi_7}(s)$ *satisfies the assumptions* i) *and* ii) *so that the decomposition* (3) *is unique in the sense of Lemma* 4.1.

Remark 3. Lemma 4.1. can be formulated more precisely according to the # of relatively prime n's with $c(n) \neq 0$. The case #=5 of Lemma 4.1. is the strongest case. Since the other cases for $\# < 5$ are involved but not used in the present note, they are omitted.

Remark 4. There are a few more known Dirichlet series associated to eta-products, which decompose as (9) ((9)') and satisfy the assumption of Lemma 4.1, namely, $\eta(48\tau)^3/\eta(24\tau)$, $\eta_{\Phi_4}(8\tau) = \eta(32\tau)^2\eta(16\tau)/\eta(8\tau)$ and $\eta_{\Phi_6}(12\tau) = \eta(72\tau)\eta(36\tau)\eta(24\tau)/\eta(12\tau)$. They have an origin in a study of elliptic root systems (see [S1]).

5. NON-DECOMPOSABILITY OF $L_{\Phi_p}(s)$ FOR $p \geq 11$

We finally give the following remark, which can be shown easily.

Fact. *The Dirichlet series $L_{\Phi_p}(s)$ associated to the eta-product $\eta(p\tau)^p/\eta(\tau)$ for a prime number p with $p \geq 11$ does not admit a decomposition* (9).

Proof. Suppose a decomposition (9)' exists, i.e. there is a Dirichlet series $M(s)$ and a constant $c \neq 0$ such that $M(s) - \frac{1}{c}L_{\Phi_p}(s)$ is a Dirichlet series admitting an Euler product. Let $c(n)$, $a(n)$ and $b(n)$ be the Dirichlet coefficients of $L_{\Phi_p}(s)$, $M(s)$ and $M(s) - \frac{1}{c}L_{\Phi_p}(s)$. The following fact follows from the explicit expression of the eta product $\eta(p\tau)^p/\eta(\tau)$:

i) $c(n) = 0$ for $1 \leq n < (p^2-1)/24$ (≥ 5),

ii) $c(n) \neq 0$ for $(p^2-1)/24 \leq n < (p^2-1)/24 + p$.

Thus, we can find an odd integer m such that $1 < m < (p^2-1)/24$ and $(p^2-1)/24 \leq 2m < (p^2-1)/24+p$. Then, $a(2)a(m) = b(2)b(m) = b(2m) = a(2m) - \frac{1}{c}c(2m) = a(2)a(m) - \frac{1}{c}c(2m)$ should imply $\frac{1}{c}c(2m) = 0$. Since $c(2m) \neq 0$ (due to ii)), one has $\frac{1}{c} = 0$ which is impossible. □

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.