Matroids on Convex Geometries (cg-matroids)

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Abstract

We consider matroidal structures on convex geometries, which we call cg-matroids. The concept of a cg-matroid is closely related to but different from that of a supermatroid introduced by Dunstan, Ingleton, and Welsh in 1972. Distributive supermatroids or poset matroids are supermatroids defined on distributive lattices or sets of order ideals of posets. The class of cg-matroids includes distributive supermatroids (or poset matroids). We also introduce the concept of a strict cg-matroid, which turns out to be exactly a cg-matroid that is also a supermatroid. We show characterizations of cg-matroids and strict cg-matroids by means of the exchange property for bases and the augmentation property for independent sets. We also examine submodularity structures of strict cg-matroids.

Keywords: Matroids, convex geometries, base exchange property, supermatroids

1. Introduction

Dunstan, Ingleton, and Welsh [6] introduced the concept of a *supermatroid* in 1972 as a generalization of the concept of an ordinary matroid and integral polymatroid ([29, 8]; also see [26, 27, 28, 22]). Supermatroids have been investigated in the literature such as [9, 10, 12, 14, 17, 18, 19]. Distributive supermatroids or poset matroids are supermatroids defined on distributive lattices or sets of order ideals of partially ordered sets (posets). Faigle [11] investigated their geometric strucure and examined a greedy algorithm on them. Tardos [25] showed a matroid-type intersection theorem for distributive supermatroids, and Peled and Srinivasan [23] also considered a generalization of the matroid independent matching problem for distributive supermatroids. Moreover, Barnabei, Nicoletti, and Pezzoli [3, 4] studied distributive supermatroids in more detail. Also see a related general framework in [13].

We generalize the concept of a distributive supermatroid (or a poset matroid) by considering a convex geometry, instead of a poset, as the underlying combinatorial structure on which we define a matroidal structure, which we call a *cg-matroid*. For a cg-matroid we define independent sets, bases, and other related concepts, and examine their combinatorial structural properties. We show characterizations of cg-matroids by means of the exchange property for bases and the augmentation property for independent sets. We also introduce the concept of a strict cg-matroid; strict cg-matroids will turn out to be exactly cg-matroids that are also supermatroids. In other words, strict cg-matroids are exactly supermatroids defined on the lattices of closed sets of convex geometries. We also examine submodularity structures of strict cg-matroids.

In Section 2 we give definitions and some preliminaries on convex geometries. We define a cg-matroid and associated concepts of bases, independent sets, etc. of a cg-matroid in Section 3. Moreover, in Section 4 we introduce the concept of a strict cg-matroid and give a characterization of strict cg-matroids. We also give some remarks on the dual exchange property for cg-matroids in Section 5.

2. Definitions and Preliminaries on Convex Geometries

In this section we give some definitions and preliminaries on convex geometries (see [7, 15] for more details).

Let E be a nonempty finite set and $\mathcal{F}$ be a family of subsets of E. The pair $(E, \mathcal{F})$ is called a *closure space* on E if it satisfies the following two conditions:

(F0) $\emptyset,\ E \in \mathcal{F}$.

(F1) $X, Y \in \mathcal{F} \Longrightarrow X \cap Y \in \mathcal{F}$.

The set E is called the *ground set* of the closure space $(E, \mathcal{F})$, and each member of $\mathcal{F}$ is called a *closed set*. Moreover, we call the closure space $(E, \mathcal{F})$ a *convex geometry* if it satisfies the following condition:

(F2) $\forall X \in \mathcal{F} \setminus \{E\},\ \exists e \in E \setminus X\colon\ X \cup \{e\} \in \mathcal{F}$.

Condition (F2) is equivalent to the following chain condition:

(F2)$'$ Every maximal chain $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = E$ in $\mathcal{F}$ has length $n = |E|$.

Next we define an operator $\tau : 2^E \to 2^E$ associated with the closure space $(E, \mathcal{F})$. For any $X \in 2^E$ define

$$\tau(X) = \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}. \tag{2.1}$$

That is, $\tau(X)$ is the unique minimal closed set containing X. The operator τ satisfies the following properties (cl0)$\sim$(cl3):

(cl0) $\tau(\emptyset) = \emptyset$.

(cl1) $X \subseteq \tau(X)$ for $X \in 2^E$ (Extensionality).

(cl2) $X \subseteq Y \Longrightarrow \tau(X) \subseteq \tau(Y)$ for any $X, Y \in 2^E$ (Monotonicity).

(cl3) $\tau(\tau(X)) = \tau(X)$ for any $X \in 2^E$ (Idempotence).

In general, any operator $\tau : 2^E \to 2^E$ satisfying the four conditions given above is called a *closure operator*. Conversely, given a closure operator τ, define $\mathcal{F} = \{X \in 2^E \mid \tau(X) = X\}$. Then $\mathcal{F}$ forms a closure space on E. Hence, for a finite set E and a closure operator τ on E we also call the pair (E, τ) a *closure space*.

In terms of closure operator, a closure space (E, τ) is a convex geometry if and only if it satisfies the following property, called the *anti-exchange property*:

(AE) $X \subseteq E,\ p \in E \setminus \tau(X),\ q \in \tau(X \cup \{p\}) \setminus \{p\} \implies p \notin \tau(X \cup \{q\})$.

Example 2.1.

(a) Given a finite set E of points in a Euclidean space $\mathbf{R}^k$, the convex hull operator in $\mathbf{R}^k$ gives a closure operator τ on 2^E. We then get a convex geometry on E, called a *convex shelling*.

(b) Let E be the vertex set of a tree T. The vertex sets of subtrees of T form the closed sets of a convex geometry, called a *tree shelling*.

(c) For a poset $\mathcal{P}$, (order) ideals of $\mathcal{P}$ gives the closed sets of a convex geometry, called a *poset shelling*. It is well-known that a convex geometry $(E, \mathcal{F})$ is a poset shelling if and only if $\mathcal{F}$ is closed with respect to set union.

□

Every convex geometry forms a graded lattice with respect to set-inclusion, where the lattice operations *join* $\vee$ and *meet* $\wedge$ are given by

$$X \vee Y = \tau(X \cup Y), \qquad X \wedge Y = X \cap Y \tag{2.2}$$

for any $X, Y \in \mathcal{F}$.

Now, we define dual operators $\mathrm{ex} : 2^E \to 2^E$ and $\mathrm{ex}^* : 2^E \to 2^E$, associated with a convex geometry $(E, \mathcal{F})$ or, more generally, a closure space (E, τ). The first one, ex, is the *extreme-point operator* of the closure space (E, τ) defined by

$$\mathrm{ex}(X) = \{e \mid e \in X,\ e \notin \tau(X \setminus \{e\})\} \tag{2.3}$$

for any $X \in 2^E$. An element in $\mathrm{ex}(X)$ is called an *extreme point* of X. The second operator, ex^*, is the *co-extreme-point operator* of (E, τ) defined by

$$\mathrm{ex}^*(X) = \{e \mid e \in E \setminus \tau(X),\ \tau(X) \cup \{e\} = \tau(X \cup \{e\})\} \tag{2.4}$$

for any $X \in 2^E$.

The extreme-point operator ex satisfies the following properties (ex0)∼(ex4).

(ex0) $\mathrm{ex}(\{e\}) = \{e\}$ for every $e \in E$ (Singleton Identity).

(ex1) $\mathrm{ex}(X) \subseteq X$ for every $X \in 2^E$ (Intensionality).

(ex2) $X \subseteq Y \subseteq E \implies \mathrm{ex}(Y) \cap X \subseteq \mathrm{ex}(X)$ (Chernoff property).

(ex3) $X \subseteq E,\ p, q \in E \setminus X,\ p \notin \mathrm{ex}(X \cup \{p\}),\ q \in \mathrm{ex}(X \cup \{q\})$
$\implies q \in \mathrm{ex}(X \cup \{p\} \cup \{q\})$.

(ex4) $\mathrm{ex}(Y) \subseteq X \subseteq Y \subseteq E \implies \mathrm{ex}(X) \subseteq \mathrm{ex}(Y)$ (Aizerman's Axiom).

It is known (see [2]) that conditions (ex0)∼(ex3) completely characterize the extreme-point operator ex for closure spaces, while conditions (ex0)∼(ex2) and (ex4) completely characterize the extreme-point operator ex for convex geometries. Note that extreme-point operators are also investigated as choice functions; see [1, 16, 21, 5] (also see [7, 24]).

The following facts are fundamental, but their proofs are easy so that we omit them.

Let $(E, \mathcal{F})$ be a closure space on E.

- For any closed set $X \in \mathcal{F}$

$$\mathrm{ex}(X) = \{e \mid e \in X,\ X \setminus \{e\} \in \mathcal{F}\}. \tag{2.5}$$

- For any closed set $X \in \mathcal{F}$

$$\mathrm{ex}^*(X) = \{e \mid e \in E \setminus X,\ X \cup \{e\} \in \mathcal{F}\}. \tag{2.6}$$

- Let (E, τ) be a closure space. For any $X \in 2^E$ and $e \in \mathrm{ex}(\tau(X))$,

$$\tau(X \setminus \{e\}) \subseteq \tau(X) \setminus \{e\}, \tag{2.7}$$
$$\mathrm{ex}(\tau(X)) \setminus \{e\} \subseteq \mathrm{ex}(\tau(X) \setminus \{e\}), \tag{2.8}$$
$$\mathrm{ex}(\tau(X)) \subseteq X, \tag{2.9}$$
$$\tau(X \cup \{e'\}) = \tau(X) \qquad (e' \in \tau(X)). \tag{2.10}$$

The following two lemmas are useful and will be used in the following argument.

Lemma 2.2. *Let $(E, \mathcal{F})$ be a convex geometry. For any $X, Y \in \mathcal{F}$, $\mathrm{ex}(\tau(X \cup Y)) \subseteq \mathrm{ex}(X) \cup \mathrm{ex}(Y)$.*

Proof. From (2.9),

$$\mathrm{ex}(\tau(X \cup Y)) \subseteq X \cup Y. \tag{2.11}$$

Also, from (ex2)

$$\mathrm{ex}(\tau(X \cup Y)) \cap X \subseteq \mathrm{ex}(X), \qquad \mathrm{ex}(\tau(X \cup Y)) \cap Y \subseteq \mathrm{ex}(Y). \tag{2.12}$$

Hence, from (2.11) and (2.12) we have $\mathrm{ex}(\tau(X \cup Y)) \subseteq \mathrm{ex}(X) \cup \mathrm{ex}(Y)$. $\square$

Lemma 2.3. *Let $(E, \mathcal{F})$ be a convex geometry. For any $X, Y \in \mathcal{F}$ with $X \nsubseteq Y$, we have $\mathrm{ex}(\tau(X \cup Y)) \cap \mathrm{ex}(X) \nsubseteq Y$.*

Proof. For any $X, Y \in \mathcal{F}$ with $X \nsubseteq Y$ there exists an element $e \in \mathrm{ex}(\tau(X \cup Y))$ such that $e \notin Y$. Such an element e must belong to $\mathrm{ex}(X)$, due to (2.9) and Lemma 2.2. $\square$

3. Matroids on Convex Geometries (cg-matroids)

In this section we define a matroid on a convex geometry, called a cg-matroid. The concept of a cg-matroid is closely related to but different from that of a supermatroid introduced by Dunstan, Ingleton, and Welsh [6]. Their relationship will be made clear in Section 4.

3.1. Definition

Let E be a nonempty finite set and $(E, \mathcal{F})$ be a convex geometry on E with a family $\mathcal{F}$ of closed sets. Let $\tau : 2^E \to \mathcal{F}$ be the closure operator associated with convex geometry $(E, \mathcal{F})$.

Definition 3.1 (Matroid on a convex geometry). For a convex geometry $(E, \mathcal{F})$ and a family $\mathcal{B} \subseteq \mathcal{F}$, suppose that $\mathcal{B}$ satisfies the following three conditions:

(B0) $\mathcal{B} \neq \emptyset$.

(B1) $B_1, B_2 \in \mathcal{B}\,,\ B_1 \subseteq B_2 \implies B_1 = B_2$.

(BM) (Middle Base Property)
For any $B_1, B_2 \in \mathcal{B}$ and $X, Y \in \mathcal{F}$ with $X \subseteq B_1,\ B_2 \subseteq Y$, and $X \subseteq Y$,
there exists $B \in \mathcal{B}$ such that $X \subseteq B \subseteq Y$.

Then we call $(E, \mathcal{F}; \mathcal{B})$ a *matroid on the convex geometry* $(E, \mathcal{F})$ (or a *cg-matroid* for short). Each $B \in \mathcal{B}$ is called a *base*, and $\mathcal{B}$ the *family of bases* of cg-matroid $(E, \mathcal{F}; \mathcal{B})$. □

Note that a cg-matroid $(E, \mathcal{F}; \mathcal{B})$ is an ordinary matroid when $\mathcal{F} = 2^E$ and that $(E, \mathcal{F}; \mathcal{B})$ is a poset matroid (or a distributive supermatroid) when $\mathcal{F}$ is the set of order ideals of a poset on E.

Example 3.2. For a convex geometry $(E, \mathcal{F})$, let k be an integer such that $0 \leq k \leq |E|$, and define

$$\mathcal{B}(k) = \{X \mid X \in \mathcal{F},\ |X| = k\}. \tag{3.1}$$

We can easily see that $(E, \mathcal{F}; \mathcal{B}(k))$ satisfies (B0), (B1), and (BM) and is a cg-matroid on $(E, \mathcal{F})$, which we call a *uniform cg-matroid of rank* k. A uniform cg-matroid of rank 0 is called *trivial* and that of rank $|E|$ *free*.

The family of subtrees, of fixed size, of a tree is an example of such a uniform cg-matroid. □

3.2. Bases and an exchange property

We examine properties of bases of a cg-matroid $(E, \mathcal{F}; \mathcal{B})$ on a convex geometry $(E, \mathcal{F})$.

Theorem 3.3. *For any cg-matroid $(E, \mathcal{F}; \mathcal{B})$ all the bases in $\mathcal{B}$ have the same cardinality, i.e.,*

$(\mathrm{B1})'$ $B_1, B_2 \in \mathcal{B} \implies |B_1| = |B_2|$.

Proof. Let $B_1, B_2 \in \mathcal{B}$. Suppose $|B_1| \geq |B_2|$. We show the present theorem by induction on $k = |\tau(B_1 \cup B_2)| - |B_2|$.

First, suppose $k = 0$. Then, since $|B_2| = |\tau(B_1 \cup B_2)|$ and $B_2 \subseteq \tau(B_1 \cup B_2)$, we have $B_2 = \tau(B_1 \cup B_2)$. Since B_2 is a closed set, i.e., $\tau(B_2) = B_2$, it follows that $B_1 \subseteq B_2$. Hence, $B_1 = B_2$ (due to (B1)) and $|B_1| = |B_2|$.

Next, for an integer $k \geq 0$ suppose that $|B_1| = |B_2|$ holds for any $B_1, B_2 \in \mathcal{B}$ such that $|\tau(B_1 \cup B_2)| - |B_2| = k$. Consider any distinct $B_1, B_2 \in \mathcal{B}$ such that $|\tau(B_1 \cup B_2)| - |B_2| = k + 1$. Since $B_1 \not\subseteq B_2$, we see from Lemma 2.3 that there exists an element $\hat{e} \in \mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus B_2$. Then, from (2.5),

$$B_1 \setminus \{\hat{e}\} \in \mathcal{F}, \qquad \tau(B_1 \cup B_2) \setminus \{\hat{e}\} \in \mathcal{F}. \tag{3.2}$$

Note that

$$B_1 \setminus \{\hat{e}\} \subseteq B_1, \qquad B_2 \subseteq \tau(B_1 \cup B_2) \setminus \{\hat{e}\}, \tag{3.3}$$

and also

$$B_1 \setminus \{\hat{e}\} \subseteq \tau(B_1 \cup B_2) \setminus \{\hat{e}\}. \tag{3.4}$$

It follows from (3.2)∼(3.4) and (BM) that there exists $\hat{B} \in \mathcal{B}$ such that

$$B_1 \setminus \{\hat{e}\} \subseteq \hat{B} \subseteq \tau(B_1 \cup B_2) \setminus \{\hat{e}\}, \tag{3.5}$$

where note that $\hat{e} \notin \hat{B}$.

Now, from (3.5) and the monotonicitity property (cl2) of τ we have

$$\tau(\hat{B} \cup B_2) \subseteq \tau(B_1 \cup B_2). \tag{3.6}$$

Since $\hat{e} \in \mathrm{ex}(\tau(B_1 \cup B_2))$ and from (3.5) $\hat{e} \notin \tau(\hat{B} \cup B_2)$, we have from (3.6)

$$|\tau(\hat{B} \cup B_2)| < |\tau(B_1 \cup B_2)|. \tag{3.7}$$

It follows from the induction assumption that $|\hat{B}| = |B_2|$.

Furthermore, since $\hat{e} \notin \hat{B}$ and $\hat{e} \in B_1$, from (3.5) and (B1) we have $B_1 \setminus \{\hat{e}\} \subsetneq \hat{B}$. Consequently, $|B_1| \leq |\hat{B}|$. Since $|B_1| \geq |B_2| = |\hat{B}|$, we thus have $|B_1| = |B_2|$. □

Theorem 3.4 (Exchange Property). *A cg-matroid* $(E, \mathcal{F}; \mathcal{B})$ *satisfies*

(BE) (Exchange Property)
For any $B_1, B_2 \in \mathcal{B}$ *and any* $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \setminus B_2$,
there exists $e_2 \in \tau(B_1 \cup B_2) \setminus B_1$ *such that* $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

Proof. Consider any $B_1, B_2 \in \mathcal{B}$ and any $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \setminus B_2$. Here note that

$$e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \setminus B_2 \Longrightarrow e_1 \in \mathrm{ex}(B_1), \tag{3.8}$$

due to Lemma 2.2. Then, by the same argument as in (3.2)∼(3.5), there exists $B \in \mathcal{B}$ such that $B_1 \setminus \{e_1\} \subseteq B \subseteq \tau(B_1 \cup B_2) \setminus \{e_1\}$. Since from Theorem 3.3 we have $|B_1| = |B|$, it follows that there exists $e_2 \in \tau(B_1 \cup B_2) \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} = B \in \mathcal{B}$. □

To get the converse of Theorem 3.4 we first show the following.

Lemma 3.5. *Let $(E, \mathcal{F})$ be a convex geometry. If $\mathcal{B} \subseteq \mathcal{F}$ satisfies* (B0) *and* (BE)*, then it also satisfies* (B1)$'$*, i.e., all elements of $\mathcal{B}$ as subsets of E have the same cardinality.*

Proof. The proof given here is similar to that of Theorem 3.3. Consider any $B_1, B_2 \in \mathcal{B}$ such that $|B_1| \geq |B_2|$. We show the present lemma by induction on the number $k = |\tau(B_1 \cup B_2)| - |B_2|$.

First, when $k = 0$, we have $B_1 = B_2$ as in the proof of Theorem 3.3, and hence $|B_1| = |B_2|$.

Next, for some $k \geq 0$, suppose that $|B_1| = |B_2|$ holds for any $B_1, B_2 \in \mathcal{B}$ such that $|\tau(B_1 \cup B_2)| - |B_2| \leq k$. Consider any $B_1, B_2 \in \mathcal{B}$ such that $|B_1| \geq |B_2|$ and $|\tau(B_1 \cup B_2)| - |B_2| = k + 1$. From Lemma 2.3, there exists an element $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus B_2$. Then, from (BE) there exists an element $e_2 \in \tau(B_1 \cup B_2) \setminus B_1$ such that

$$B' \equiv (B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}. \tag{3.9}$$

Since $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus B_2$ and $e_2 \in \tau(B_1 \cup B_2) \setminus B_1$, we have

$$\begin{aligned} \tau(B' \cup B_2) &= \tau(B_1 \cup B_2 \cup \{e_2\} \setminus \{e_1\}) \\ &\subseteq \tau(\tau(B_1 \cup B_2) \cup \{e_2\} \setminus \{e_1\}) \\ &= \tau(\tau(B_1 \cup B_2) \setminus \{e_1\}) \\ &= \tau(B_1 \cup B_2) \setminus \{e_1\}. \end{aligned} \tag{3.10}$$

Hence we have $|\tau(B' \cup B_2)| < |\tau(B_1 \cup B_2)|$, which implies $|B_2| = |B'|(= |B_1|)$ due to the induction assumption.

Consequently, (B1)$'$ holds. □

Now, we have

Theorem 3.6. *Let $(E, \mathcal{F})$ be a convex geometry. If $\mathcal{B} \subseteq \mathcal{F}$ satisfies* (B0) *and* (BE)*, then it also satisfies* (B0), (B1) *and* (BM)*, and hence $(E, \mathcal{F}; \mathcal{B})$ is a cg-matroid.*

Proof. Lemma 3.5 implies (B1), so that we show (BM) by induction on the number $k = |\tau(B_1 \cup B_2) \setminus Y|$.

Consider any $B_1, B_2 \in \mathcal{B}$ and $X, Y \in \mathcal{F}$ such that $X \subseteq B_1$, $B_2 \subseteq Y$, and $X \subseteq Y$. Suppose $|\tau(B_1 \cup B_2) \setminus Y| = 0$, i.e., $\tau(B_1 \cup B_2) \subseteq Y$. Then we have $X \subseteq B_1 \subseteq \tau(B_1 \cup B_2) \subseteq Y$ and take $B = B_1$.

Next, for an integer $k \geq 0$, suppose that for any $B_1, B_2 \in \mathcal{B}$ and $X, Y \in \mathcal{F}$ such that

$$X \subseteq B_1, \quad B_2 \subseteq Y, \quad X \subseteq Y, \quad |\tau(B_1 \cup B_2) \setminus Y| \leq k, \tag{3.11}$$

there exists $B \in \mathcal{B}$ such that $X \subseteq B \subseteq Y$. Consider any $B_1, B_2 \in \mathcal{B}$ and $X, Y \in \mathcal{F}$ such that $X \subseteq B_1,\ B_2 \subseteq Y$, and $X \subseteq Y$, and suppose $|\tau(B_1 \cup B_2) \setminus Y| = k + 1$. There are two cases, (Case I) and (Case II), to be considered.

(Case I) If $\mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \subseteq Y$, then from Lemma 2.2 and $B_2 \subseteq Y$ we have $\mathrm{ex}(\tau(B_1 \cup B_2)) \subseteq Y$, so that $\tau(B_1 \cup B_2) \subseteq Y$. We thus have $X \subseteq B_1 \subseteq \tau(B_1 \cup B_2) \subseteq Y$, and take $B = B_1$.

(Case II) Suppose that $\mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus Y \neq \emptyset$. Choose any $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus Y$. Note that $e_1 \notin B_2$ and $e_1 \notin X$ since $e_1 \notin Y$. It follows from (BE) that there exists

$$e_2 \in \tau(B_1 \cup B_2) \setminus B_1 \tag{3.12}$$

such that

$$B' \equiv (B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}. \tag{3.13}$$

Also note that $B' \cup B_2 \subseteq \tau(B_1 \cup B_2)$ and $e_1 \in \tau(B_1 \cup B_2) \setminus (B' \cup B_2)$, where recall that $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2))$ and $e_1 \notin B' \cup B_2$. Hence we have

$$\tau(B' \cup B_2) \subseteq \tau(B_1 \cup B_2) \setminus \{e_1\}. \tag{3.14}$$

Since $e_1 \notin Y$, we have from (3.14)

$$\tau(B' \cup B_2) \setminus Y \subsetneq \tau(B_1 \cup B_2) \setminus Y. \tag{3.15}$$

Since $e_1 \notin X$ and hence $X \subseteq B'$, it follows from the induction assumption that there exists $B \in \mathcal{B}$ such that $X \subseteq B \subseteq Y$.

This completes the proof. □

Combining the preceding two theorems, we have one of our main results.

Theorem 3.7. *For any convex geometry $(E, \mathcal{F})$ and $\mathcal{B} \subseteq \mathcal{F}$, $(E, \mathcal{F}; \mathcal{B})$ is a cg-matroid if and only if $\mathcal{B}$ satisfies* (B0) *and* (BE)*.*

Moreover, we have the following.

Theorem 3.8 (Multiple-Exchange Property). *For any cg-matroid $(E, \mathcal{F}; \mathcal{B})$, we have*

(BmE) (Multiple-Exchange Property)
For any $B_1, B_2 \in \mathcal{B}$ and any $S \subseteq B_1 \setminus B_2$ such that $\tau(B_1 \cup B_2) \setminus S \in \mathcal{F}$, there exists $T \subseteq \tau(B_1 \cup B_2) \setminus B_1$ such that $|T| = |S|$ and $(B_1 \setminus S) \cup T \in \mathcal{B}$.

Proof. We prove this theorem by induction on the number $k = |S|$.

When $k = 1$, (BmE) is just (BE), and hence (BmE) holds.

Next, suppose that (BmE) holds when $k = n (\geq 1)$. Consider the case when $k = n+1$. For any $B_1, B_2 \in \mathcal{B}$ and any $S \subseteq B_1 \setminus B_2$ such that $|S| = n+1$ and $\tau(B_1 \cup B_2) \setminus S \in \mathcal{F}$, considering a maximal chain of $\mathcal{F}$ that includes $\tau(B_1 \cup B_2)$ and $\tau(B_1 \cup B_2) \setminus S$, we see that there exists $e \in S \cap \tau(B_1 \cap B_2)$ such that $(\tau(B_1 \cup B_2) \setminus S) \cup \{e\} \in \mathcal{F}$. Hence, putting $S' = S \setminus \{e\}$, we have $\tau(B_1 \cup B_2) \setminus S' \in \mathcal{F}$, $S' \subseteq B_1 \setminus B_2$, and $|S'| = |S| - 1$. From the induction assumption, there exists $T' \subseteq B_2 \setminus B_1$ such that $|T'| = |S'|$ and $B_1' \equiv (B_1 \setminus S') \cup T' \in \mathcal{B}$. Note that $e \in B_1' \setminus B_2$ and $e \in \mathrm{ex}(\tau(B_1 \cup B_2) \setminus S')$.

Now, we show

$$\tau(B_1' \cup B_2) \subseteq \tau(B_1 \cup B_2) \setminus S'. \tag{3.16}$$

Because $B_2 \cap S' = \emptyset$, we have $B_2 \subseteq \tau(B_1 \cup B_2) \setminus S' \in \mathcal{F}$. Also, using $S' \cap T' = \emptyset$ and $T' \subseteq \tau(B_1 \cup B_2)$, we have $B_1' = (B_1 \setminus S') \cup T' = (B_1 \cup T') \setminus S' \subseteq \tau(B_1 \cup B_2) \setminus S'$. So we have $B_1' \cup B_2 \subseteq \tau(B_1 \cup B_2) \setminus S' \in \mathcal{F}$, from which the desired relation follows.

Then from (ex2) we have

$$\mathrm{ex}(\tau(B_1 \cup B_2) \setminus S') \cap \tau(B_1' \cup B_2) \subseteq \mathrm{ex}(\tau(B_1' \cup B_2)). \tag{3.17}$$

Here, e belongs to the set in the left-hand side, so that $e \in \mathrm{ex}(\tau(B_1' \cup B_2))$. Since B_1', B_2, and e satisfy the condition of (BE), there exists $e' \in \tau(B_1' \cup B_2) \setminus B_1'$ such that $(B_1' \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$.

Then, since $e' \notin B_1'$, we have $e' \notin T'$. And note that $S' \cap T' = \emptyset$, $e \in B_1 \setminus S'$. Hence we have $(B_1' \setminus \{e\}) \cup \{e'\} = (((B_1 \setminus S') \cup T') \setminus \{e\}) \cup \{e'\} = (B_1 \setminus S) \cup (T' \cup \{e'\}) \in \mathcal{B}$, where note that $S = S' \cup \{e\}$. Putting $T = T' \cup \{e'\}$, we get $T \subseteq \tau(B_1 \cup B_2) \setminus B_1$, $|T| = |T'| + 1 = |S'| + 1 = |S|$, and $(B_1 \setminus S) \cup T \in \mathcal{B}$.

The present theorem thus holds. □

It follows from the above theorem that (BE) and (BmE) are equivalent under (B0).

3.3. Independent sets

Let us define a family of independent sets for a cg-matroid, similarly as for ordinary matroids.

Definition 3.9 (Independent set). Let $(E, \mathcal{F})$ be a convex geometry and $(E, \mathcal{F}; \mathcal{B})$ be a cg-matroid with a family $\mathcal{B}$ of bases. For a closed set $I \in \mathcal{F}$, if there exists a base $B \in \mathcal{B}$ such that $I \subseteq B$, then we call I an *independent set* of the cg-matroid $(E, \mathcal{F}; \mathcal{B})$. □

Denote by $\mathcal{I}$ the family of independent sets of a cg-matroid $(E, \mathcal{F}; \mathcal{B})$.

Theorem 3.10. *The family $\mathcal{I}$ of independent sets of a cg-matroid $(E, \mathcal{F}; \mathcal{B})$ with a family $\mathcal{B}$ of bases satisfies the following three conditions:*

(I0) $\emptyset \in \mathcal{I}$.

(I1) $I_1 \in \mathcal{F},\ I_2 \in \mathcal{I},\ I_1 \subseteq I_2 \implies I_1 \in \mathcal{I}$.

(IA) (Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ *with* $|I_1| < |I_2|$ *and* I_2 *being maximal in* $\mathcal{I}$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ *such that* $I_1 \cup \{e\} \in \mathcal{I}$.

Proof. We can easily see from (B0) and the definition of independent sets that (I0) and (I1) hold. Let us show (IA). For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ and I_2 being maximal in $\mathcal{I}$ there exists a base B_1 such that $I_1 \subsetneq B_1$, and I_2 itself is a base because of its maximality. Hence, by the middle base property (BM) there exists a base B such that $I_1 \subsetneq B \subseteq \tau(I_1 \cup I_2)$. Since there exists a chain of subsets in $\mathcal{F}$ containing I_1, B, and $\tau(I_1 \cup I_2))$, there exists $e \in B \setminus I_1 (\subseteq \tau(I_1 \cup I_2) \setminus I_1)$ such that $I_1 \cup \{e\} \subseteq B$. Hence (IA) holds. □

Remark 3.11. It should be emphasized that in Condition (IA) the maximality of I_2 is required. The maximality is not necessary for characterizing independent sets of ordinary matroids, but (IA) without the maximality of I_2 does not always hold for cg-matroids. In Section 4 we consider cg-matroids whose families of independent sets satisfy (IA) without the maximality of I_2. □

Conversely,

Theorem 3.12 ($\mathcal{I} \to \mathcal{B}$). *Let* $(E, \mathcal{F})$ *be a convex geometry. Suppose that* $\mathcal{I} \subseteq \mathcal{F}$ *satisfies* (I0), (I1) *and* (IA). *Define*

$$\mathcal{B} = \{I \in \mathcal{I} \mid I \text{ is maximal in } \mathcal{I}\}. \tag{3.18}$$

Then, $\mathcal{B}$ *is a family of bases of a cg-matroid on* $(E, \mathcal{F})$.

To show this theorem we employ the following lemma.

Lemma 3.13. *The family* $\mathcal{B}$ *given by* (3.18) *is equicardinal, i.e., it satisfies*

$(\mathrm{B1})'$ $B_1, B_2 \in \mathcal{B} \implies |B_1| = |B_2|$.

Proof. If we have $|B_1| < |B_2|$ for some $B_1, B_2 \in \mathcal{B}$, then from (IA) there exists $e \in \tau(B_1 \cup B_2) \setminus B_1$ such that $B_1 \cup \{e\} \in \mathcal{I}$, which contradicts the maximality of B_1 in $\mathcal{I}$. □

Proof of Theorem 3.12. Property (B0) follows from (I0), and (B1) from (B1)′. We show (BE). Consider any $B_1, B_2 \in \mathcal{B}$ and $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus B_2$. We see from (I1) that $B_1 \setminus \{e_1\} \in \mathcal{I}$. Since from (B1)′ $|B_1 \setminus \{e_1\}| < |B_2|$, it follows from (IA) that there exists $e_2 \in \tau((B_1 \setminus \{e_1\}) \cup B_2) \setminus (B_1 \setminus \{e_1\})$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{I}$. Here since $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus B_2$, we have

$$\begin{aligned} \tau((B_1 \setminus \{e_1\}) \cup B_2) \setminus (B_1 \setminus \{e_1\}) &= \tau((B_1 \cup B_2) \setminus \{e_1\}) \setminus (B_1 \setminus \{e_1\}) \\ &= (\tau(B_1 \cup B_2) \setminus \{e_1\}) \setminus (B_1 \setminus \{e_1\}) \\ &= \tau(B_1 \cup B_2) \setminus B_1 \end{aligned} \tag{3.19}$$

And we have $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$ because of its maximum cardinality. We thus have (BE). $\square$

From Theorems 3.10 and 3.12, if $\mathcal{I}$ satisfies (I0), (I1), and (IA), we also denote by $(E, \mathcal{F}; \mathcal{I})$ a cg-matroid with a family $\mathcal{I}$ of independent sets.

4. Strict cg-matroids

It seems to be difficult to define the rank function of a general cg-matroid in a meaningful way, so that we shall introduce a subclass of cg-matroids, called strict cg-matroids, for which we define rank functions.

4.1. The strict augmentation property

Let us consider the following augmentation property that is stronger than (IA) given in Theorem 3.10. Note that we do not require that I_2 is maximal in $\mathcal{I}$.

(IsA) (Strict Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Definition 4.1 (Strict cg-matroid). Let $(E, \mathcal{F})$ be a convex geometry. If $\mathcal{I} \subseteq \mathcal{F}$ satisfies (I0), (I1) and (IsA), then we call $(E, \mathcal{F}; \mathcal{I})$ a *strict cg-matroid* with a family $\mathcal{I}$ of independent sets. $\square$

By definition, any strict cg-matroid is a cg-matroid. It should also be noted that in the case of matroids, i.e., when $\mathcal{F} = 2^E$, the set of axioms (I0), (I1), and (IA) and that of (I0), (I1), and (IsA) are equivalent. But in the case of cg-matroids they are not equivalent; the following example shows a cg-matroid that is not a strict cg-matroid.

Example 4.2. Let $E = \{1, 2, 3, 4, 5\}$ and $(E, \mathcal{F})$ be the convex shelling of the five points in the plane given in Figure 1. Define $\mathcal{B} = \{\{1, 2, 3\}, \{2, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}$. Then $(E, \mathcal{F}; \mathcal{B})$ satisfies the conditions of the cg-matroid with a family $\mathcal{B}$ of bases. But this is not a strict cg-matroid. For, $I_1 = \{1\}$ and $I_2 = \{4, 5\}$ are, respectively, subsets of $B_1 = \{1, 2, 3\}$ and $B_2 = \{2, 4, 5\}$, so that they are independent sets, i.e., $I_1, I_2 \in \mathcal{I}$. Since $|I_1| < |I_2|$ and $\tau(I_1 \cup I_2) \setminus I_1 = \{4, 5\}$, it follows from (IsA) that $\{1, 4\}$ or $\{1, 5\}$ should be an independent set. But neither $\{1, 4\}$ nor $\{1, 5\}$ is included in any member of $\mathcal{B}$. Hence the present cg-matroid does not satisfy (IsA). $\square$

Remark 4.3. A uniform cg-matroid is a strict cg-matroid. $\square$

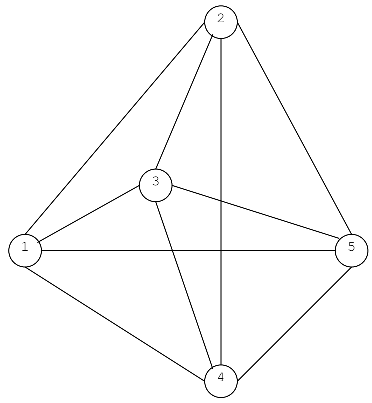

Figure 1: An example of five points in the plane.

First, we show the following characterization.

Theorem 4.4 (Local Augmentation Property). *Let $(E, \mathcal{F})$ be a convex geometry. Suppose that $\mathcal{I} \subseteq \mathcal{F}$ satisfies* (I0) *and* (I1). *Then the strict augmentation property* (IsA) *is equivalent to the following property.*

(ILA) (Local Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| + 1 = |I_2|$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Proof. The implication, (IsA) ⇒ (ILA), is trivial. We show the converse, (ILA) ⇒ (IsA). Consider $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$. Then there exists $I \in \mathcal{F}$ such that $I \subseteq I_2$ and $|I| = |I_1| + 1$. From (I1), we have $I \in \mathcal{I}$. Hence, from (ILA), there exists $e \in \tau(I_1 \cup I) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$. Since $I \subseteq I_2$, we have $\tau(I_1 \cup I) \setminus I_1 \subseteq \tau(I_1 \cup I_2) \setminus I_1$, and hence $e \in \tau(I_1 \cup I_2) \setminus I_1$. We thus have (IsA). □

Next, we give another characterization of the strict cg-matroids, which reveals the exact relationship between the concept of a strict cg-matroid and that of a supermatroid introduced by Dunstan, Ingleton, and Welsh [6].

Lemma 4.5. *Let $(E, \mathcal{F}; \mathcal{I})$ be a strict cg-matroid with a family $\mathcal{I}$ of independent sets. Then $\mathcal{I}$ satisfies the following property.*

(IS) *For each $X \in \mathcal{F}$, all the maximal elements of $\mathcal{I}^{(X)} \equiv \{X \cap I \mid I \in \mathcal{I}\}$ have the same cardinality (as subsets of E).*

Proof. Take any $X \in \mathcal{F}$. Suppose that $X \cap I_1$ and $X \cap I_2$ $(I_1, I_2 \in \mathcal{I})$ are maximal in $\mathcal{I}^{(X)}$ and that $|X \cap I_1| < |X \cap I_2|$. Since $X \cap I_i \in \mathcal{F}$ and $X \cap I_i \subseteq I_i$ $(i = 1, 2)$, we have $X \cap I_1, X \cap I_2 \in \mathcal{I}$. Hence, from (IsA) there exists $e \in \tau((X \cap I_1) \cup (X \cap I_2)) \setminus (X \cap I_1)$

such that $I_0 \equiv (X \cap I_1) \cup \{e\} \in \mathcal{I}$, which contradicts the maximality of $X \cap I_1$ in $\mathcal{I}^{(X)}$, since $e \in X \setminus I_1$. (Here note that $\tau((X \cap I_1) \cup (X \cap I_2)) \subseteq \tau(X) = X$ and $X \cap I_1 \subsetneq (X \cap I_1) \cup \{e\} = X \cap ((X \cap I_1) \cup \{e\}) = X \cap I_0 \in \mathcal{I}$.) □

Conversely, we have the following.

Lemma 4.6. *Let $(E, \mathcal{F})$ be a convex geometry. Suppose that $\mathcal{I} \subseteq \mathcal{F}$ satisfies* (I0), (I1), *and* (IS)*. Then, $\mathcal{I}$ also satisfies* (IsA)*, and hence $(E, \mathcal{F}; \mathcal{I})$ is a strict cg-matroid.*

Proof. Suppose that $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$. Consider $X = \tau(I_1 \cup I_2)$ in (IS). Then, $I_i = \tau(I_1 \cup I_2) \cap I_i \in \mathcal{I}^{(\tau(I_1 \cup I_2))}$ $(i = 1, 2)$. From the assumption that $|I_1| < |I_2|$, we see that I_1 is not maximal in $\mathcal{I}^{(\tau(I_1 \cup I_2))}$. Hence, there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}^{(\tau(I_1 \cup I_2))} \subseteq \mathcal{I}$, where the last inclusion follows from (I1). □

Axioms (I0), (I1), and (IS) are exactly those for what is called a *supermatroid* [6] when restricted on the lattices of closed sets of convex geometries. Hence the above two lemmas establish the following.

Theorem 4.7. *The concept of a strict cg-matroid is equivalent to that of a supermatroid on the lattice of closed sets of a convex geometry.* □

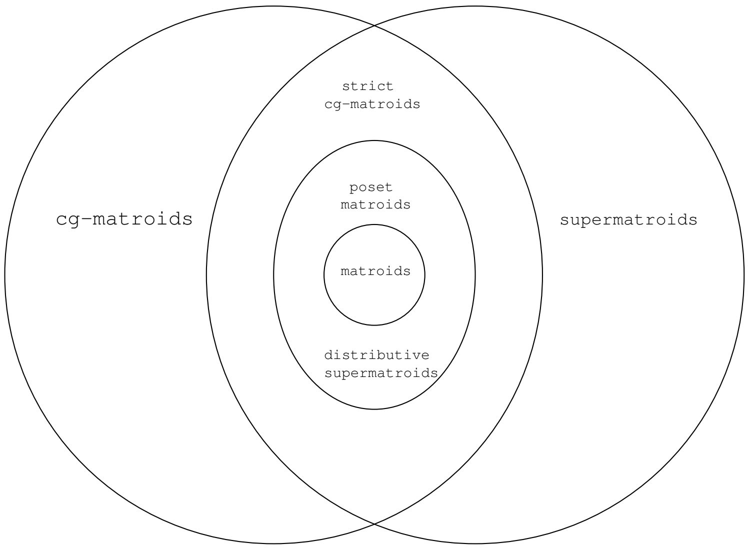

Figure 2: Generalizations of matroids.

Recall that for a convex geometry $(E, \mathcal{F})$, if $\mathcal{F}$ is closed with respect to the set union, then it is distributive and is represented as the set of ideals of a poset. Also note that the class of distributive cg-matroids (or poset matroids) is strictly included in the class of strict cg-matroids.

See Figure 2 for the relationship among the relevant concepts.

4.2. Rank functions

Now we define rank functions of strict cg-matroids. Since strict cg-matroids are supermatroids, some of the following results on rank functions are subsumed by those in [6].

We denote the set of nonnegative integers by $\mathbb{Z}_+$.

Definition 4.8 (Rank function of a strict cg-matroid). Let $(E, \mathcal{F}; \mathcal{I})$ be a strict cg-matroid with a family $\mathcal{I}$ of independent sets. Define a function $\rho : 2^E \to \mathbb{Z}_+$ as

$$\rho(X) = \max\{|I| \mid I \in \mathcal{I}, I \subseteq X\} \quad (X \in 2^E). \tag{4.1}$$

We call the function ρ the *rank function* of the strict cg-matroid $(E, \mathcal{F}; \mathcal{I})$. We call $\rho(X)$ the *rank* of X. □

We examine some properties of the rank function $\rho : \mathcal{F} \to \mathbb{Z}_+$ such as submodularity, which is a fundamental and crucial property of rank functions of ordinary matroids (see for more details [8, 13, 20]).

We first show a useful property of strict cg-matroids.

Theorem 4.9. *A strict cg-matroid $(E, \mathcal{F}; \mathcal{I})$ with a family $\mathcal{I}$ of independent sets satisfies the following property.*

(IE) (Extension Property)
For any $X \in \mathcal{F}$ and $I \in \mathcal{I}$ with $I \subseteq X$,
there exists $I^+ \in \mathcal{I}$ such that $I \subseteq I^+ \subseteq X$ and $\rho(I^+) = \rho(X)$.

Proof. Suppose that $|I| < \rho(X)$ and $\rho(X) = |I_X|$ for an $I_X \in \mathcal{I}$ with $I_X \subseteq X$. Since $I, I_X \subseteq X$ and $X \in \mathcal{F}$, we have $\tau(I \cup I_X) \subseteq X$. Hence, applying (IsA) $|I_X \setminus I|$ times, we get a desired independent set I^+. □

Then we consider the following "local" properties.

(RL0) $\rho(\emptyset) = 0$.

(RL1) $X \in \mathcal{F},\ e \in \mathrm{ex}^*(X) \implies \rho(X) \le \rho(X \cup \{e\}) \le \rho(X) + 1$.

(RLS) (Local Submodularity)
For any $X \in \mathcal{F}$ and $e_1, e_2 \in \mathrm{ex}^*(X)$ such that $X \cup \{e_1, e_2\} \in \mathcal{F}$,
if $\rho(X) = \rho(X \cup \{e_1\}) = \rho(X \cup \{e_2\})$, then $\rho(X) = \rho(X \cup \{e_1, e_2\})$.

Theorem 4.10. *The rank function $\rho : \mathcal{F} \to \mathbb{Z}_+$ of a strict cg-matroid $(E, \mathcal{F}; \mathcal{I})$ satisfies properties* (RL0), (RL1), *and* (RLS).

Proof. (RL0) follows from (I0).

Next we show (RL1). Suppose that $\rho(X) = |I|$ for an $I \in \mathcal{I}$. Since $I \subseteq X \cup \{e\}$, we have $\rho(X) \le \rho(X \cup \{e\})$. Also suppose that $\rho(X \cup \{e\}) = |I'|$ for an $I' \in \mathcal{I}$. If $\rho(X \cup \{e\}) > \rho(X) + 1 (= |I| + 1)$, then we have $e \in I'$ (otherwise $I' \subseteq X$ and $|I'| > |I|$, which contradicts the definition of $\rho(X)$). Now, $e \in \mathrm{ex}^*(X)$ implies $e \in \mathrm{ex}(X \cup \{e\})$. It follows from (ex2) that $\mathrm{ex}(X \cup \{e\}) \cap I' \subseteq \mathrm{ex}(I')$, and hence $e \in \mathrm{ex}(I')$. This implies $I'' \equiv I' \setminus \{e\} \in \mathcal{I}$ and $I'' \subseteq X$, which contradicts the assumption that $\rho(X) < \rho(X \cup \{e\}) - 1$. We thus have property (RL1).

Finally, we show (RLS). Suppose that $\rho(X) = \rho(X \cup \{e_1\}) = \rho(X \cup \{e_2\})$. Then, from (RL1), we have $\rho(X) \le \rho(X \cup \{e_1, e_2\}) \le \rho(X) + 1$. Suppose to the contrary that $\rho(X \cup \{e_1, e_2\}) = \rho(X) + 1$. Then there exist $I, I' \in \mathcal{I}$ such that (1) $I \subseteq X$ and $\rho(X) = |I|$ and (2) $I' \subseteq X \cup \{e_1, e_2\}$ and $\rho(X \cup \{e_1, e_2\}) = |I'| (= |I| + 1)$. Since $|I'| > |I|$, from (IsA) there exists $\hat{e} \in \tau(I' \cup I) \setminus I$ such that $I'' \equiv I \cup \{\hat{e}\} \in \mathcal{I}$. Here, since $\tau(I' \cup I) \subseteq X \cup \{e_1, e_2\}$, we must have $\hat{e} \in X$ or $\hat{e} = e_1$ or $\hat{e} = e_2$, which leads us to $I'' \subseteq X$ or $I'' \subseteq X \cup \{e_1\}$ or $I'' \subseteq X \cup \{e_2\}$. This contradicts the assumption on $\rho(X)$ or $\rho(X \cup \{e_1\})$ or $\rho(X \cup \{e_2\})$. We thus have shown (RLS). □

For any function $\rho : \mathcal{F} \to \mathbb{Z}_+$ that satisfies (RL0), (RL1), and (RLS), let us define

$$\mathcal{I}(\rho) = \{X \in \mathcal{F} \mid \rho(X) = |X|\}. \tag{4.2}$$

We may expect that $\mathcal{I}(\rho)$ would give a strict cg-matroid. But, unfortunately, this is not true as seen from the following example.

Example 4.11. Let $E = \{1, 2, 3, 4\}$. Consider a tree with a vertex set E and an edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ that forms a path of length three. See Figure 3. Let $(E, \mathcal{F})$ be the tree shelling of the tree, i.e., $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. Define a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ as follows: $\rho(\emptyset) = 0$, $\rho(\{1\}) = \rho(\{2\}) = \rho(\{3\}) = \rho(\{4\}) = \rho(\{2, 3\}) = 1$, $\rho(\{1, 2\}) = \rho(\{3, 4\}) = \rho(\{1, 2, 3\}) = \rho(\{2, 3, 4\}) = 2$, $\rho(\{1, 2, 3, 4\}) = 3$. Then the function $\rho : \mathcal{F} \to \mathbb{Z}_+$ satisfies (RL0), (RL1), and (RLS), and we have $\mathcal{I}(\rho) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\}$. But the obtained $\mathcal{I}(\rho)$ is not a strict cg-matroid. □

Next, we consider some "global" properties.

(RG0) $0 \le \rho(X) \le |X|$ for any $X \in \mathcal{F}$.

(RG1) $X, Y \in \mathcal{F}$, $X \subseteq Y \Longrightarrow \rho(X) \le \rho(Y)$.

(RGS) (Global Submodularity)
For any $X, Y \in \mathcal{F}$ such that $X \cup Y \in \mathcal{F}$,
$\rho(X) + \rho(Y) \ge \rho(X \cup Y) + \rho(X \cap Y)$.

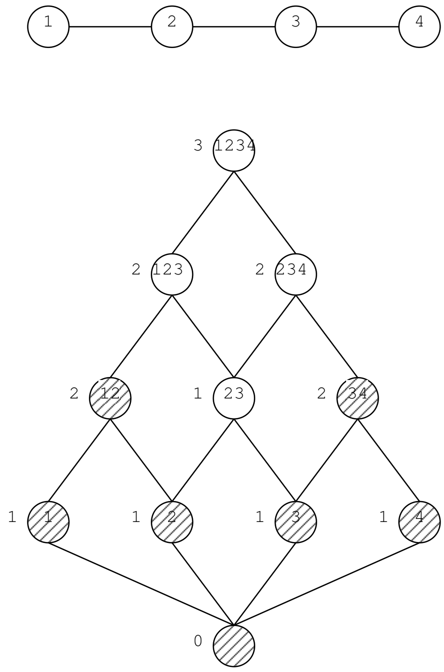

Figure 3: A path of length three and its tree shelling.

Theorem 4.12. *The rank function* $\rho : \mathcal{F} \to \mathbb{Z}_+$ *of a strict cg-matroid* $(E, \mathcal{F}; \mathcal{I})$ *satisfies properties* (RG0)*,* (RG1)*, and* (RGS)*.*

Proof. We can easily see that the definition of rank function ρ implies (RG0) and (RG1). We show (RGS). Consider any $X, Y \in \mathcal{F}$ such that $X \cup Y \in \mathcal{F}$. Then $X \cap Y \in \mathcal{F}$, and there exists $I \in \mathcal{I}$ such that $\rho(X \cap Y) = |I|$ and $I \subseteq X \cap Y$. The extension property (IE) implies the following (1) and (2).

(1) There exists $J_1 \subseteq X \setminus I$ such that $I \cup J_1 \in \mathcal{I}$, $\rho(X) = |I \cup J_1|$ and $I \cup J_1 \subseteq X$.

(2) There exists $J_2 \subseteq E \setminus X$ such that $I \cup J_1 \cup J_2 \in \mathcal{I}$, $\rho(X \cup Y) = |I \cup J_1 \cup J_2|$, and $I \cup J_1 \cup J_2 \subseteq X \cup Y$.

Then, from (I1) and the definition of $\rho(X)$, we have $J_2 \subseteq Y \setminus X$. Therefore, we get $\rho(X \cup Y) - \rho(X) + \rho(X \cap Y) = |I| + |J_1| + |J_2| - (|I| + |J_1|) + |I| = |I| + |J_2|$.

Next, consider $\rho(Y)$. Since $\tau(I \cup J_2) \subseteq Y$ and $\tau(I \cup J_2) \subseteq I \cup J_1 \cup J_2 \in \mathcal{I}$, from (I1) we get $\tau(I \cup J_2) \in \mathcal{I}$. We thus have $\rho(Y) \geq |\tau(I \cup J_2)| \geq |I \cup J_2| = |I| + |J_2|$.

Hence, we have $\rho(X \cup Y) - \rho(X) + \rho(X \cap Y) = |I| + |J_2| \leq \rho(Y)$, i.e., $\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y)$. □

Again the above-mentioned three properties do not completely characterize rank functions of strict cg-matroids. In fact, consider Example 4.11 again. The function $\rho : \mathcal{F} \to \mathbb{Z}_+$ defined there also satisfies (RG0), (RG1), and (RGS).

Example 4.13. Let $(E, \mathcal{F}; \mathcal{B})$ be a unirorm cg-matroid of rank 3 on the tree shelling of a path of length three, i.e., $E = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$, and $\mathcal{B} = \{\{1, 2, 3\}, \{2, 3, 4\}\}$ (see Figure 4). Then, from Remark 4.3, $(E, \mathcal{F}; \mathcal{B})$ is a strict cg matroid with a family $\mathcal{B}$ of bases.

For $X = \{1\}$ and $Y = \{4\}$, we have $X \wedge Y = \emptyset$ and $X \vee Y = \{1, 2, 3, 4\}$. Since $\rho(X) = 1$, $\rho(Y) = 1$, $\rho(X \vee Y) = 3$, and $\rho(X \wedge Y) = 0$, we have $\rho(X) + \rho(Y) < \rho(X \vee Y) + \rho(X \wedge Y)$. □

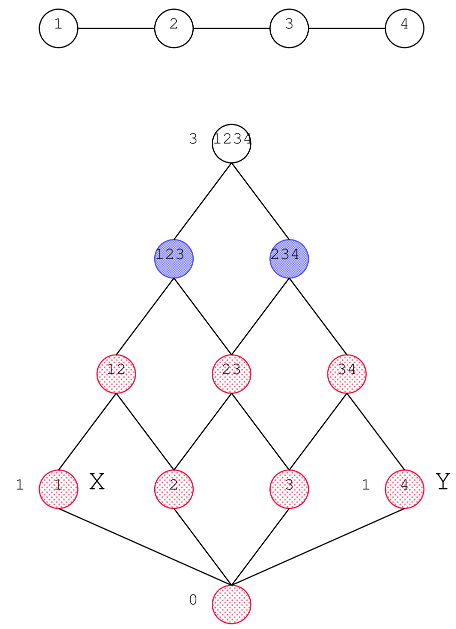

Figure 4: A strict cg-matroid that does not satisfy the submodularity on the lattice.

Remark 4.14. It follows from Example 4.13 that the rank function ρ of a strict cg-matroid $(E, \mathcal{F}; \mathcal{I})$ does not always satisfy the submodularity on the lattice $\mathcal{F}$:

- $\rho(X) + \rho(Y) \geq \rho(X \vee Y) + \rho(X \wedge Y) \quad$ for any $X, Y \in \mathcal{F}$,

where $X \vee Y = \tau(X \cup Y)$ and $X \wedge Y = X \cap Y$. Hence strict cg-matroids are not submodular supermatroids which are defined in [12]. □

5. Concluding Remarks

We have introduced the concept of a cg-matroid, a matroidal structure defined on a convex geometry, and have shown characterizations of cg-matroids by means of an exchange property for bases and an augmentation property of independent sets. We have also defined a strict cg-matroid, which turns out to be a cg-matroid that is at the same time a supermatroid on the lattice of closed sets of the undelying convex geometry, and examined the submodularity property of the rank function of a strict cg-matroid.

The problem of linear and nonlinear optimization over cg-matroids is left for future work. Also we should examine how polyhedral characterizations of (a special class of) cg-matroids would be possible.

Finally, we give some remarks on dual exchange properties for cg-matroids. The family of bases of an ordinary matroid $(E, \mathcal{B})$ satisfies the following dual exchange property.

(BE*) (Dual Exchange Property for ordinary matroids)
For any $B_1, B_2 \in \mathcal{B}$ and $e_2 \in B_2 \setminus B_1$,
there exists $e_1 \in B_1 \setminus B_2$ such that $(B_1 \cup \{e_2\}) \setminus \{e_1\} \in \mathcal{B}$.

We can show the following for cg-matroids (we omit its proof).

(Dual Exchange Property) *Any cg-matroid $(E, \mathcal{F}; \mathcal{B})$ satisfies*

(BE*1) *For any $B_1, B_2 \in \mathcal{B}$ and any $e_2 \in \mathrm{ex}^*(B_1) \cap B_2$,
there exists $e_1 \in \mathrm{ex}(B_1) \setminus B_2$ such that $B_1 \cup \{e_2\} \setminus \{e_1\} \in \mathcal{B}$.*

(BE*2) *For any $B_1, B_2 \in \mathcal{B}$ and any $e_2 \in \mathrm{ex}^*(B_1) \cap \tau(B_1 \cup B_2)$,
there exists $e_1 \in (\mathrm{ex}(B_1) \cup \{e_2\}) \setminus B_2$ such that $(B_1 \cup \{e_2\}) \setminus \{e_1\} \in \mathcal{B}$.*

(BE*3) *For any $B_1, B_2 \in \mathcal{B}$ and any $e_2 \in \mathrm{ex}^+(B_1) \cap \tau(B_1 \cup B_2)$,
there exists $e_1 \in \mathrm{ex}(B_1)$ such that $(B_1 \cup \{e_2\}) \setminus \{e_1\} \in \mathcal{B}$,
where the operator $\mathrm{ex}^+ : \mathcal{B} \to 2^E$ is defined by*
$$\mathrm{ex}^+(B) = \{e \mid e \in E \setminus B, e \in B' \subseteq B \cup \{e\} \text{ for some } B' \in \mathcal{B}\}$$
for any base $B \in \mathcal{B}$.

(BE*3)′ *For any $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$,
we have $\mathrm{ex}^+(B_1) \cap \tau(B_1 \cup B_2) \neq \emptyset$.*

Unfortunately the dual exchange properties given above do not characterize cg-matroids as seen from the following examples.

Example 5.1. Let $(E, \mathcal{F})$ be the convex shelling of nine points in the plane given in Figure 5. Define $\mathcal{B} = \{\{1, 2, 3\}, \{7, 8, 9\}\}$. Then $\mathcal{B}$ satisfies conditions (BE*1) and (BE*2), but it is not a cg-matroid. □

Example 5.2. Let $(E, \mathcal{F})$ be the convex shelling of eight points in the plane given in Figure 6. Define $\mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{5, 7, 8\}, \{6, 7, 8\}\}$. Then $\mathcal{B}$ satisfies conditions (BE*3) and (BE*3)′, but it is not a cg-matroid. □

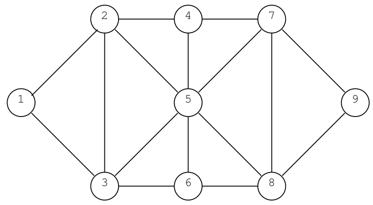

Figure 5: An example of nine points in the plane.

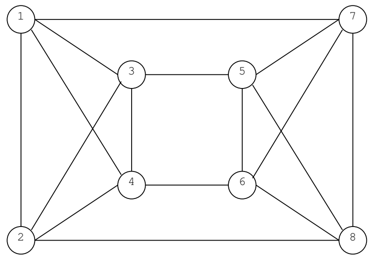

Figure 6: An example of eight points in the plane.

Remark 5.3. A shortcoming of (BE*1) is that if $\mathrm{ex}^*(B_1) \cap B_2 = \emptyset$, then condition (BE*1) is void, while that of (BE*2) is that there is a possibility of $e_1 = e_2$, which makes condition (BE*2) trivial. □

It is still open to characterize cg-matroids by means of a dual exchange property.

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