A UNIFIED WITTEN-RESHETIKHIN-TURAEV INVARIANT FOR INTEGRAL HOMOLOGY SPHERES

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Abstract. We construct an invariant J_M of integral homology spheres M with values in a completion $\widehat{\mathbb{Z}[q]}$ of the polynomial ring $\mathbb{Z}[q]$ such that the evaluation at each root of unity ζ gives the the $SU(2)$ Witten-Reshetikhin-Turaev invariant $\tau_\zeta(M)$ of M at ζ. Thus J_M unifies all the $SU(2)$ Witten-Reshetikhin-Turaev invariants of M. As a consequence, $\tau_\zeta(M)$ is an algebraic integer. Moreover, it follows that $\tau_\zeta(M)$ as a function on ζ behaves like an "analytic function" defined on the set of roots of unity. That is, the $\tau_\zeta(M)$ for all roots of unity are determined by a "Taylor expansion" at any root of unity, and also by the values at infinitely many roots of unity of prime power orders. In particular, $\tau_\zeta(M)$ for all roots of unity are determined by the Ohtsuki series, which can be regarded as the Taylor expansion at $q = 1$.

1. Introduction

In this paper we construct an invariant of integral homology spheres which unifies the $SU(2)$ Witten-Reshetikhin-Turaev invariants at all roots of unity, which we announced in a previous paper [17].

1.1. **The WRT invariant for integral homology spheres.** Witten [77] introduced the notion of Chern-Simons path integral which gives a quantum field theory interpretation of the Jones polynomial [30, 31] and predicts the existence of 3-manifold invariants. Using the quantum group $U_q(sl_2)$ at roots of unity, Reshetikhin and Turaev [73] gave a rigorous construction of 3-manifold invariants, which are believed to coincide with the Chern-Simons path integrals. These invariants are called the Witten-Reshetikhin-Turaev (WRT) invariants.

In the present paper, we focus on the WRT invariant for integral homology spheres, i.e., closed 3-manifolds M with trivial first homology groups. If we fix such M, the WRT invariant $\tau_\zeta(M) \in \mathbb{C}$ is defined for each root of unity ζ. (Unlike the case of general closed 3-manifolds, one does not have to specify a fourth root of ζ.) For an integer $r \geq 1$, $\tau_{\zeta_r}(M)$ for $\zeta_r = \exp \frac{2\pi\sqrt{-1}}{r}$ is also denoted by $\tau_r(M)$. In the literature, usually $\tau_1(M)$ is not defined, but for our purpose it is convenient to defined it as 1. For integral homology spheres, the version $\tau'_r(M)$ introduced by Kirby and Melvin [38] defined for odd $r \geq 3$ is equal to $\tau_r(M)$.

Let $\mathcal{Z} \subset \mathbb{C}$ denote the set of all roots of unity. Define the *WRT function* of M

$$\tau(M)\colon \mathcal{Z} \to \mathbb{C},$$

by $\tau(M)(\zeta) = \tau_\zeta(M)$. The behavior of the function $\tau(M)$ is of interest here.

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For $\tau_\zeta(M)$ with ζ roots of unity of *odd prime (power) orders*, there have been more extensive studies than for the other cases. Murakami [62] proved that if $\zeta \in \mathcal{Z}$ is of odd prime order, then $\tau_\zeta(M) \in \mathbb{Z}[\zeta]$, hence it is an algebraic integer. Ohtsuki [67] extracted from the $\tau_\zeta(M)$ for ζ of odd prime orders a power series invariant, known as the *Ohtsuki series* of M

$$\tau^O(M) = 1 + \sum_{n=1}^{\infty} \lambda_n(M)(q-1)^n \in \mathbb{Q}[[q-1]].$$

Lawrence [44] conjectured, and Rozansky [75] later proved, that $\tau^O(M) \in \mathbb{Z}[[q-1]]$ and that, for each $\zeta \in \mathcal{Z}$ of odd prime power order p^e, $\tau^O(M)|_{q=\zeta}$ converges p-adically to $\tau_\zeta(M)$. In this sense, the Ohtsuki series *unifies* the WRT invariants at roots of unity of odd prime power orders. (For generalizations of the above-mentioned results to rational homology spheres and to invariants associated to other Lie groups, see [49, 50, 59, 60, 63, 68, 76].)

The proofs of the above-mentioned results depends heavily on the fact that if $\zeta \in \mathcal{Z}$ is of prime power order, then $\zeta - 1$ is not a unit in the ring $\mathbb{Z}[\zeta]$. Otherwise, $\zeta - 1$ is a unit in $\mathbb{Z}[\zeta]$, and expansions in powers of $\zeta - 1$ do not work.

1.2. **The ring $\widehat{\mathbb{Z}[q]}$ of analytic functions on the set of roots of unity.** The invariant J_M of an integral homology sphere M which we construct in the paper takes values in a completion $\widehat{\mathbb{Z}[q]}$ of the polynomial ring $\mathbb{Z}[q]$, which was introduced in [17] and studied in [18]. One of the simplest definitions of $\widehat{\mathbb{Z}[q]}$ is

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q)_n),$$

where we set

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n).$$

The ring $\widehat{\mathbb{Z}[q]}$ may be regarded as the ring of "analytic functions defined on the set $\mathcal{Z}$ of roots of unity". This statement is justified by the following facts. (The following overlaps those in [17, 18].)

First of all, *each element of $\widehat{\mathbb{Z}[q]}$ can be evaluated at each root of unity.* That is, for each $\zeta \in \mathcal{Z}$, the evaluation map $\mathrm{ev}_\zeta\colon \mathbb{Z}[q] \to \mathbb{Z}[\zeta]$, $f(q) \mapsto f(\zeta)$, induces a (surjective) ring homomorphism

$$\mathrm{ev}_\zeta\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta],$$

since $\mathrm{ev}_\zeta((q)_n) = 0$ if $n \geq \mathrm{ord}(\zeta)$. It is often useful to write $f(\zeta) = \mathrm{ev}_\zeta(f(q))$ for $f(q) \in \widehat{\mathbb{Z}[q]}$.

Second, *each element of $\widehat{\mathbb{Z}[q]}$ can be regarded as a (set-theoretic) function on the set of roots of unity.* This means that each $f \in \widehat{\mathbb{Z}[q]}$ is determined uniquely by the values $\mathrm{ev}_\zeta(f) \in \mathbb{Z}[\zeta]$ for all $\zeta \in \mathcal{Z}$, or, equivalently, the function

$$\mathrm{ev}_{\mathcal{Z}}\colon \widehat{\mathbb{Z}[q]} \to \prod_{\zeta\in\mathcal{Z}} \mathbb{Z}[\zeta], \quad f(q) \mapsto (f(\zeta))_{\zeta\in\mathcal{Z}}. \tag{1.1}$$

is injective [18, Theorem 6.3]. Here, in a natural way, $\prod_{\zeta\in\mathcal{Z}} \mathbb{Z}[\zeta]$ can be regarded as a subring of the ring of $\mathbb{C}$-valued functions on $\mathcal{Z}$.

Third, *each element of $\widehat{\mathbb{Z}[q]}$ has a power series expansion in $q-\zeta$ for each root of unity, and any such power series determines $\widehat{\mathbb{Z}[q]}$*. In fact, for each $\zeta \in \mathcal{Z}$, the inclusion $\mathbb{Z}[q] \subset \mathbb{Z}[\zeta][q]$ induces a ring homomorphism

$$\iota_\zeta \colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q-\zeta]], \tag{1.2}$$

since, for each $i \geq 0$, $(q)_n$ is divisible by $(q-\zeta)^i$ in $\mathbb{Z}[\zeta][q]$ if $n \geq i\,\mathrm{ord}(\zeta)$. Since ι_ζ is injective [18, Theorem 5.2], each $f \in \widehat{\mathbb{Z}[q]}$ is determined by the power series $\iota_\zeta(f)$. $\iota_\zeta(f)$ may be regarded as the Taylor expansion of f, see Section 12.3.1.

Fourth, *each element of $\widehat{\mathbb{Z}[q]}$ is completely determined by its values on a subset $\mathcal{Z}' \subset \mathcal{Z}$ if $\mathcal{Z}'$ has a limit point. Otherwise, not completely.* To explain what this means, we introduce a topology on the set $\mathcal{Z}$, which is different from the usual one induced by the topology of $\mathbb{C}$. Two elements $\zeta, \xi \in \mathcal{Z}$ are said to be *adjacent* if $\zeta\xi^{-1}$ is of prime power order, or, equivalently, if $\zeta - \xi$ is not a unit in $\mathbb{Z}[\zeta, \xi]$. A subset $\mathcal{Z}' \subset \mathcal{Z}$ is defined to be *open* if, for each $\zeta \in \mathcal{Z}'$, all but finitely many elements adjacent to ζ is contained in $\mathcal{Z}'$. In this topology, an element $\xi \in \mathcal{Z}$ is a limit point of a subset $\mathcal{Z}' \subset \mathcal{Z}$ (i.e., $(U \setminus \{\xi\}) \cap \mathcal{Z}' \neq \emptyset$ for all neighborhood U of ξ) if and only if there are infinitely many $\zeta \in \mathcal{Z}'$ adjacent to ξ. We have the following.

Proposition 1.1 ([18, Theorem 6.3])**.** *If $\mathcal{Z}' \subset \mathcal{Z}$ has a limit point, then the ring homomorphism*

$$\mathrm{ev}_{\mathcal{Z}'} \colon \widehat{\mathbb{Z}[q]} \to \prod_{\zeta \in \mathcal{Z}'} \mathbb{Z}[\zeta], \quad f(q) \mapsto (f(\zeta))_{\zeta \in \mathcal{Z}}, \tag{1.3}$$

is injective.

If $\mathcal{Z}' \subset \mathcal{Z}$ has no limit point, then $\mathrm{ev}_{\mathcal{Z}'}$ is not injective, i.e., there is a non-zero "analytic function" $f \in \widehat{\mathbb{Z}[q]}$ vanishing on $\mathcal{Z}'$, see Proposition 12.2.

The above-explained properties of $\widehat{\mathbb{Z}[q]}$ are closely related to the integrality of $\widehat{\mathbb{Z}[q]}$. In fact, the completion $\widehat{\mathbb{Q}[q]} = \varprojlim_n \mathbb{Q}[q]/((q)_n)$ does not behave like $\widehat{\mathbb{Z}[q]}$, see [18, Section 7.5].

1.3. **A unified WRT invariant J_M with values in $\widehat{\mathbb{Z}[q]}$.** The following is the main result of the present paper, and follows from Theorems 10.2 and 11.1.

Theorem 1.2. *There is an invariant $J_M \in \widehat{\mathbb{Z}[q]}$ of an integral homology sphere M such that for any root of unity ζ we have*

$$\mathrm{ev}_\zeta(J_M) = \tau_\zeta(M). \tag{1.4}$$

The properties of the ring $\widehat{\mathbb{Z}[q]}$ explained in the last subsection implies that the WRT function $\tau(M)$ may be regarded as an analytic function defined on $\mathcal{Z}$. Let us describe some corollaries to Theorem 1.2 and properties of the ring $\widehat{\mathbb{Z}[q]}$. (Part of the discussion below overlaps those in [17, 18].)

An immediate consequence of Theorem 1.2 is the following generalization of Murakami's integrality result.

Corollary 1.3 (Conjectured by Lawrence [44])**.** *For any integral homology sphere M and for $\zeta \in \mathcal{Z}$, we have $\tau_\zeta(M) \in \mathbb{Z}[\zeta]$.*

Theorem 1.2 immediately implies *Galois equivariance* of $\tau_\zeta(M)$. Namely, we have

$$\tau_{\alpha(\zeta)}(M) = \alpha(\tau_\zeta(M)) \tag{1.5}$$

for $\alpha \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ and $\zeta \in \mathcal{Z}$. Here $\mathbb{Q}^{ab}$ denotes the maximal abelian extension of $\mathbb{Q}$, which is generated over $\mathbb{Q}$ by all roots of unity. ($\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ can be identified with the automorphism group of the group $\mathcal{Z} \cong \mathbb{Q}/\mathbb{Z}$.) It is well known that the Galois equivariance of $\tau_\zeta(M)$ is implied by Reshetikhin and Turaev's definition. Theorem 1.2 reexplains it in an apparent way.

Proposition 1.1 implies the following.

Theorem 1.4. *The invariant J_M is determined by the WRT function $\tau(M)$. (Thus J_M and $\tau(M)$ have the same strength in distinguishing two integral homology spheres.) Moreover, both J_M and $\tau(M)$ are determined by the values of $\tau_\zeta(M)$ for $\zeta \in \mathcal{Z}'$, where $\mathcal{Z}' \subset \mathcal{Z}$ is any infinite subset with a limit point in the sense explained in Section 1.2.*

The invariant J_M unifies not only the WRT invariants but also the Ohtsuki series $\tau^O(M)$. Namely, we have (Theorem 12.6)

$$\iota_1(J_M) = \tau^O(M). \tag{1.6}$$

Injectivity of ι_ζ in (1.2) for each $\zeta \in \mathcal{Z}$ implies that the J_M and hence $\tau(M)$ are determined by the power series expansion $\iota_\zeta(J_M) \in \mathbb{Z}[\zeta][[q-\zeta]]$. In particular, J_M is determined by $\tau^O(M)$, in view of 1.6. Thus J_M and $\tau^O(M)$ have the same strength in distinguishing integral homology spheres. As a consequence, the Le-Murakami-Ohtsuki invariant [52] determines J_M and $\tau(M)$, since it determines $\tau^O(M)$ (see [70]).

For further properties of J_M, see Sections 12 and 13.

1.4. **Organization of the paper.** In Section 2, we first recall the definition of the quantized enveloping algebra $U_h = U_h(sl_2)$ of the Lie algebra sl_2, which is a h-adically complete Hopf $\mathbb{Q}[[h]]$-algebra, and then introduce $\mathbb{Z}[q,q^{-1}]$-subalgebras $\mathcal{U}_q$, where $\mathbb{Z}[q,q^{-1}]$ is regarded as a subring of $\mathbb{Q}[[h]]$ by setting $q = \exp h$. The Hopf algebra structure of U_h induces a Hopf algebra structure on $\mathcal{U}_q$. We also introduce a $\mathbb{Z}[q,q^{-1}]$-subalgebra $\mathcal{U}_q^{\mathrm{ev}}$, which is the even part of $\mathcal{U}_q$ with respect to a natural $(\mathbb{Z}/2\mathbb{Z})$-grading of $\mathcal{U}_q$. We define completions $\tilde{\mathcal{U}}_q$ and $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ of the algebras $\mathcal{U}_q$ and $\mathcal{U}_q^{\mathrm{ev}}$, and also completed tensor products of copies of $\tilde{\mathcal{U}}_q$ and $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$. $\tilde{\mathcal{U}}_q$ is equipped with a complete Hopf algebra structure induced by the Hopf algebra structure of $\mathcal{U}_q$.

In Section 3, we first recall the ribbon Hopf algebra structure for U_h, and then the braided Hopf algebra structure $\underline{U_h}$, canonically defined for U_h. The main observation in this section (Theorem 3.1) is that $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ is equipped with a braided Hopf algebra structure inherited from that for $\underline{U_h}$.

In Section 4, we first recall from [20] the notion of bottom tangles. An n-component bottom tangle T is a tangle in a cube consisting of n arc components whose endpoints lie in a line in the bottom square of the cube in such a way that between the two endpoints of each arc there are no endpoints of other arcs. Then we adapt the universal invariants for bottom tangles associated to the ribbon Hopf algebra to the case of U_h. The universal invariant J_T of T takes values in the invariant part $\mathrm{Inv}(U_h^{\hat{\otimes} n})$ of the n-fold completed tensor product $U_h^{\hat{\otimes} n}$ of U_h, where the invariant part is considered with respect to the standard tensor product left U_h-module structure defined using the left adjoint action of U_h. The main observation in this section (Theorem 4.1) is that if T is an n-component, algebraically-split, 0-framed bottom tangle, then J_T is contained in the invariant part $\mathcal{K}_n$ of the n-fold completed tensor product $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ of $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$. (Recall that a link L is *algebraically split* if the linking number of any two distinct components are zero.) The proof of Theorem 4.1 uses the braided Hopf algebra structure of $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$.

For a 0-framed bottom knot (i.e., 1-component bottom tangle) T, the universal invariant J_T of T takes values in the center $Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$ of $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$. In view of a result proved in [19], this implies that we can express J_T as an infinite sum $\sum_{p\geq 0} a_p(T)\sigma_p$, where $a_p(T) \in \mathbb{Z}[q, q^{-1}]$, $p \geq 0$, and where $\sigma_p \in Z(\mathcal{U}_q^{\mathrm{ev}})$, $p \geq 0$, are a certain basis of $Z(\mathcal{U}_q^{\mathrm{ev}}) \subset Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$. Each $a_p(T)$ gives a $\mathbb{Z}[q, q^{-1}]$-valued invariant of T, which may be regarded also as an invariant of the closure $\mathsf{cl}(T)$ of T.

In Section 5, we recall the definition of the colored Jones polynomial $J_L(W_1, \dots, W_n)$ of a framed link $L = L_1 \cup \cdots \cup L_n$, where each component L_i of L is colored by a finite-dimensional, irreducible representation W_i of $U_h(sl_2)$. Then we recall from [20] a formulation of the colored Jones polynomial using universal invariant of bottom tangle:

$$J_L(W_1, \dots, W_n) = (\mathrm{tr}_q^{W_1} \otimes \cdots \otimes \mathrm{tr}_q^{W_n})(J_T),$$

where T is a bottom tangle whose closure is L, and where $\mathrm{tr}_q^{W_i} \colon U_h \to \mathbb{Q}[[h]]$ is the quantum trace in W_i.

Recall that for each $d \geq 0$ there is exactly one $(n+1)$-dimensional, irreducible representation of U_h up to isomorphism, denoted by V_d. We need extensions of the colored Jones polynomials for framed links whose components are colored by linear combinations (over $\mathbb{Q}(v)$, $\mathbb{Z}[v, v^{-1}]$, etc., where $v = q^{1/2} = \exp \frac{h}{2}$) of the V_d, which are defined naturally by multilinearity.

In Section 6, we relate the universal invariant $J_T \in Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$ and the $\mathbb{Z}[q, q^{-1}]$-valued invariants $a_p(T)$, $p \geq 0$, of a bottom knot T defined in Section 4 to the colored Jones polynomials of the closure $\mathsf{cl}(T)$ of T. Theorem 6.4 identifies $a_p(T)$ with $J_{\mathsf{cl}(T)}(P''_p)$, where P''_p is a $\mathbb{Q}(v)$-linear combination of $\mathsf{V}_0, \mathsf{V}_1, \dots, \mathsf{V}_p$.

In Section 7, we give some remarks on the universal invariant $J_T \in Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$ for a bottom knot. First of all, $Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$ is identified with a completion Λ of $\mathbb{Z}[q, q^{-1}, t + t^{-1}]$. For a knot $K = \mathsf{cl}(T)$ with T a bottom knot, we set $J_K(t, q) = J_T \in \Lambda$ by abuse of notation, which we call the *two-variable colored Jones invariant* of K. The normalized colored Jones polynomial $J_K(\mathsf{V}_n)/J_{\mathrm{unknot}}(\mathsf{V}_n) \in \mathbb{Z}[q, q^{-1}]$ is equal to the specialization $J_K(q^{n+1}, q)$ of $J_K(t, q)$. The specialization $J_K(1, q) \in \widehat{\mathbb{Z}[q]}$ can be regarded as a universal form for the Kashaev invariants of K. We relate the invariant $J_K(t, q)$ to Rozansky's integral version of the Melvin-Morton expansion of the colored Jones polynomials of K, and give several conjectures which generalizes Rozansky's rationality theorem.

In Section 8, we consider invariants of algebraically-split links. We define a $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{P}$ which is spanned by certain normalizations $\tilde{P}'_n$ of P''_n, and define a completion $\hat{\mathcal{P}}$ of $\mathcal{P}$. We show that for any n-component, algebraically-split, 0-framed link L and for any elements $x_1, \dots, x_n \in \hat{\mathcal{P}}$, there is a well-defined element $J_L(x_1, \dots, x_m) \in \widehat{\mathbb{Z}[q]}$, see Corollary 8.3. In the proof, results proved in the previous sections, such as Theorem 4.1, are used.

In Section 9, we define an element ω in the ring $\hat{\mathcal{P}}$. Theorem 9.4 states that for any $(m+1)$-component, algebraically-split, 0-framed link $L_1 \cup \cdots \cup L_m \cup K$ such that K is an unknot, and for $x_1, \dots, x_m \in \hat{\mathcal{P}}$, we have

$$J_{L \cup K}(x_1, \dots, x_m, \omega^{\mp 1}) = J_{L_{(K, \pm 1)}}(x_1, \dots, x_m).$$

Here $L_{(K,\pm1)}$ is the framed link in S^3 obtained from $L_1 \cup \cdots \cup L_m$ by ±1-framed surgery along K.

In Section 10, we prove the existence of an invariant $J_M \in \widehat{\mathbb{Z}[q]}$ of integral homology sphere M (Theorem 10.2). We outline the proof below. Recall that M can be expressed as the result of surgery along an algebraically-split framed link $L = L_1 \cup \cdots \cup L_m$ in S^3 with framings $f_1, \ldots, f_m \in \{\pm1\}$. Then J_M is defined by

$$J_M := J_{L^0}(\omega^{-f_1}, \omega^{-f_2}, \ldots, \omega^{-f_m}),$$

where L^0 is the framed link obtained from L by changing all the framings to 0. By Corollary 8.3, we have $J_M \in \widehat{\mathbb{Z}[q]}$. To prove that J_M does not depend on the choice of L, we use the twisting property of ω (Theorem 9.4) and a refined version of Kirby's calculus for algebraically-split, ±1-framed links (see Theorem 10.1), which was conjectured by Hoste [28] and proved in [21]. The proof of Theorem 10.2 does not involve any existence proof of the WRT invariants $\tau_\zeta(M)$ at roots of unity ζ, hence can be regarded as a new, unified proof for the existence for $\tau_\zeta(M)$, after establishing the specialization property (1.4).

In Section 11, we prove this specialization property (Theorem 11.1). We also give an alternative proof of the existence of J_M which uses the existence of $\tau_\zeta(M)$ but does not use Theorem 10.1.

In Section 12, we make several observations and give some applications. In Section 12.1, we observe the behavior of J_M under taking connected sums and orientation-reversal. In Section 12.2, we observe the failure of an approach to the conjecture that the WRT invariants $\tau_\zeta(M)$ at any infinitely many roots of unity determine J_M. In Section 12.3, we study the power series invariants $\iota_\zeta(J_M)$, including the Ohtsuki series $\tau^O(M) = \iota_1(J_M)$. In Section 12.4, we give some divisibility results for $J_M - 1$, etc., implied by well-known results for $\tau_\zeta(M)$ for ζ of small orders $1, 2, 3, 4, 6$, and give some applications of these results to the coefficients of the Ohtsuki series and the power series $\iota_{-1}(J_M)$. We also state a conjecture about the values of the eighth WRT invariant $\tau_8(M)$.

In Section 13, we first observe that for any complex number α, there is a formal specialization of J_M at $q = \alpha$. Motivated by this observation, for each prime p, we define a p-adic analytic version $\tau^p(M)$ of the WRT function, which is a p-adic analytic function from the unit circle in the field $\mathbb{C}_p$ of complex p-adic numbers into the valuation ring of $\mathbb{C}_p$. The mod p reduction of $\tau^p(M)$, denoted by $\tau^{\mathrm{mod}\, p}(M)$, is defined on the group of units, $\bar{\mathbb{F}}_p^\times$, in the algebraic closure $\bar{\mathbb{F}}_p$ of the field $\mathbb{F}_p$ of p elements, and takes values in $\bar{\mathbb{F}}_p$.

In Section 14, we compute some examples of J_M for integral homology spheres obtained as the result of surgery along the Borromean rings A with framings $1/a, 1/b, 1/c$ with $a, b, c \in \mathbb{Z}$, and some related knot and link invariants. First we compute the colored Jones polynomials of the Borromean rings. Then we compute the powers of the ribbon element in U_h and the powers of the twist element ω. This enables us to compute the invariants of the result of surgery from the Borromean rings by performing surgery along some (possibly all) of the three components by framings in $\{1/m \mid m \in \mathbb{Z}\}$.

In Section 15, we first generalize the universal invariant J_K of a knot in S^3 to knots in integral homology spheres (Theorem 15.3). Using this invariant, we prove that if two integral homology spheres M and M' are related by surgery along a knot with framing $1/m$ with $m \in \mathbb{Z}$, then J_M and $J_{M'}$ are congruent modulo

$q^{2m} - 1$ (Theorem 15.6). This result suggests that it would be natural to consider a generalization of Ohtsuki's theory of finite type invariants of integral homology spheres, involving $(1/m)$-surgeries ($m \in \mathbb{Z}$) along knots.

In Section 16, we give some remarks. In Section 16.1, we discuss the relationships between the unified WRT invariant J_M and another approach to unify the WRT invariants by realizing them as limiting values of holomorphic functions on the disk $|q| < 1$. In Sections 16.2 and 16.3, we mention generalizations of J_M to simple Lie algebras and rational homology spheres. In Section 16.4, we announce a generalization of J_M to certain cobordisms of surfaces, which includes homology cylinders.

2. The algebra $U_h(sl_2)$ and its subalgebras

In this section, we recall the definition and some properties of the quantized enveloping algebra $U_h(sl_2)$. Then we define subalgebras $\mathcal{U}_q$ and $\mathcal{U}_q^{\mathrm{ev}}$ of U_h, as well as their completions $\tilde{\mathcal{U}}_q$ and $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$, which we studied in [19].

2.1. q-integers.

Let h be an indeterminate, and set

$$v = \exp \frac{h}{2} \in \mathbb{Q}[[h]], \quad q = v^2 = \exp h \in \mathbb{Q}[[h]].$$

We have $\mathbb{Z}[q, q^{-1}] \subset \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}[[h]]$.

We use two systems of q-integer notations. One is the "q-version":

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$
$$[i]_q = \{i\}_q/\{1\}_q, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n}/\{n\}_q!,$$

for $i \in \mathbb{Z}$, $n \geq 0$. These are elements in $\mathbb{Z}[q, q^{-1}]$. (In later sections, we also use $(q)_n = (-1)^n \{n\}_q!$.) The other is the "balanced v-version":

$$\{i\} = v^i - v^{-i}, \quad \{i\}_n = \{i\}\{i-1\} \cdots \{i-n+1\}, \quad \{n\}! = \{n\}_n,$$
$$[i] = \{i\}/\{1\}, \quad [n]! = [n][n-1] \cdots [1], \quad \begin{bmatrix} i \\ n \end{bmatrix} = \{i\}_n/\{n\}!,$$

for $i \in \mathbb{Z}$, $n \geq 0$. These are elements of $\mathbb{Z}[q, q^{-1}] \sqcup v\mathbb{Z}[q, q^{-1}] \subset \mathbb{Z}[v, v^{-1}]$. These two families of notations are the same up to multiplication by powers of v. The former system is useful in clarifying that formulas are defined over $\mathbb{Z}[q, q^{-1}]$. The latter is sometimes useful in clarifying that formulas have symmetry under conjugation $v \leftrightarrow v^{-1}$.

2.2. The quantized enveloping algebra U_h.

We define $U_h = U_h(sl_2)$ as the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by the elements H, E, and F, satisfying the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$

where we set

$$K = v^H = \exp \frac{hH}{2}.$$

The algebra U_h has a complete Hopf algebra structure with the comultiplication $\Delta\colon U_h \to U_h \hat{\otimes} U_h$, the counit $\epsilon\colon U_h \to \mathbb{Q}[[h]]$ and the antipode $S\colon U_h \to U_h$ defined by

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \epsilon(H) = 0, \quad S(H) = -H,$$
$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \epsilon(E) = 0, \quad S(E) = -K^{-1}E,$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \epsilon(F) = 0, \quad S(F) = -FK.$$

(Here $\hat{\otimes}$ denotes the h-adically completed tensor product.)

For $p \in \mathbb{Z}$, let $\Gamma_p(U_h)$ denote the complete $\mathbb{Q}[[h]]$-submodule of U_h topologically spanned by the elements $F^i H^j E^k$ with $i, j, k \geq 0$, $k - i = p$. This gives a topological $\mathbb{Z}$-graded algebra structure for U_h

$$U_h = \hat{\oplus}_{p \in \mathbb{Z}} \Gamma_p(U_h).$$

(Here $\hat{\oplus}$ denotes h-adically completed direct sum.) The elements of $\Gamma_p(U_h)$ are said to be *homogeneous* of *degree* p. For a homogeneous element x of U_h, the degree of x is denoted by $|x|$.

2.3. **The subalgebras $\mathcal{U}_q$ and $\mathcal{U}_q^{\mathrm{ev}}$ of U_h.** Set

$$e = (v - v^{-1})E,$$
$$F^{(n)} = F^n/[n]!,$$
$$\tilde{F}^{(n)} = F^n K^n/[n]_q! = v^{-\frac{1}{2}n(n-1)} F^{(n)} K^n$$

for $n \geq 0$. Let $\mathcal{U}_q$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of U_h generated by K, K^{-1}, e, and $\tilde{F}^{(n)}$ for $n \geq 1$. The definition of $\mathcal{U}_q$ here is equivalent to that in [19, Section 11].

Let $\mathcal{U}_q^{\mathrm{ev}}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathcal{U}_q$ generated by K^2, K^{-2}, e, and $\tilde{F}^{(n)}$ for $n \geq 1$. ($\mathcal{U}_q^{\mathrm{ev}}$ is the same as $\mathcal{G}_0\mathcal{U}_q$ in [19].) $\mathcal{U}_q$ is equipped with a $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{Z}[q, q^{-1}]$-algebra structure

$$\mathcal{U}_q = \mathcal{U}_q^{\mathrm{ev}} \oplus K\mathcal{U}_q^{\mathrm{ev}}. \tag{2.1}$$

Later we need the following formulas in $\mathcal{U}_q$.

$$Ke = qeK, \quad K\tilde{F}^{(n)} = q^{-n}\tilde{F}^{(n)}K, \tag{2.2}$$

$$\tilde{F}^{(m)}\tilde{F}^{(n)} = q^{-mn}\begin{bmatrix} m+n \\ m \end{bmatrix}_q \tilde{F}^{(m+n)}, \tag{2.3}$$

$$e^m \tilde{F}^{(n)} = \sum_{p=0}^{\min(m,n)} q^{-n(m-p)} \begin{bmatrix} m \\ p \end{bmatrix}_q \tilde{F}^{(n-p)} \{H - m - n + 2p\}_{q,p} e^{m-p}. \tag{2.4}$$

Here, for $i \in \mathbb{Z}$ and $p \geq 0$, we set

$$\{H + i\}_{q,p} = \{H + i\}_q \{H + i - 1\}_q \cdots \{H + i - p + 1\}_q,$$

where

$$\{H + j\}_q = q^{H+j} - 1 = q^j K^2 - 1$$

for $j \in \mathbb{Z}$. Note that $\{H + j\}_q, \{H + i\}_{q,p} \in \mathbb{Z}[q, q^{-1}][K^2, K^{-2}]$.

The following is a "q-version" of [19, Proposition 3.1], which can be easily proved using (2.2)–(2.4).

Lemma 2.1. *$\mathcal{U}_q$ (resp. $\mathcal{U}_q^{\mathrm{ev}}$) is freely spanned over $\mathbb{Z}[q,q^{-1}]$ by the elements $\tilde{F}^{(i)}K^je^k$ (resp. $\tilde{F}^{(i)}K^{2j}e^k$) with $i,k\geq 0$ and $j\in\mathbb{Z}$.*

Recall from [19] that $\mathcal{U}_q$ inherits from U_h a Hopf $\mathbb{Z}[q,q^{-1}]$-algebra structure, which can be easily verified using the following formulas, which we also need later.

$$\Delta(K^i)=K^i\otimes K^i,\quad S^{\pm1}(K^i)=K^{-i}, \tag{2.5}$$

$$\Delta(e^n)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q e^{n-j}K^j\otimes e^j,\quad \Delta(\tilde{F}^{(n)})=\sum_{j=0}^{n}\tilde{F}^{(n-j)}K^j\otimes\tilde{F}^{(j)}, \tag{2.6}$$

$$S^{\pm1}(e^n)=(-1)^nq^{\frac{1}{2}n(n\mp1)}K^{-n}e^n,\quad S^{\pm1}(\tilde{F}^{(n)})=(-1)^nq^{-\frac{1}{2}n(n\mp1)}K^{-n}\tilde{F}^{(n)}, \tag{2.7}$$

$$\epsilon(K^i)=1,\quad \epsilon(e^n)=\epsilon(\tilde{F}^{(n)})=\delta_{n,0}. \tag{2.8}$$

For $n\geq 0$, the n-output comultiplication $\Delta^{[n]}\colon U_h\to U_h^{\hat{\otimes}n}$ is defined inductively by $\Delta^{[0]}=\epsilon$, and $\Delta^{[n+1]}=(\Delta^{[n]}\otimes\mathrm{id})\Delta$ for $n\geq 0$. For $x\in U_h$ and $n\geq 1$, we write

$$\Delta^{[n]}(x)=\sum x_{(1)}\otimes\cdots\otimes x_{(n)}.$$

2.4. **Adjoint action.** Let $\triangleright\colon U_h\hat{\otimes}U_h\to U_h$ denote the (left) adjoint action defined by

$$\triangleright(x\otimes y)=x\triangleright y=\sum x_{(1)}yS(x_{(2)})$$

for $x,y\in U_h$. We regard U_h as a left U_h-module via the adjoint action. Since we have $\mathcal{U}_q\triangleright\mathcal{U}_q\subset\mathcal{U}_q$, we may regard $\mathcal{U}_q$ as a left $\mathcal{U}_q$-module.

For each homogeneous element $x\in U_h$, we have

$$K^i\triangleright x=q^{i|x|}x\quad\text{for } i\in\mathbb{Z}, \tag{2.9}$$

$$e^n\triangleright x=\sum_{j=0}^{n}(-1)^jq^{\frac{1}{2}j(j-1)+j|x|}\begin{bmatrix}n\\j\end{bmatrix}_q e^{n-j}xe^j\quad\text{for } n\geq 0, \tag{2.10}$$

$$\tilde{F}^{(n)}\triangleright x=\sum_{j=0}^{n}(-1)^jq^{-\frac{1}{2}j(j-1)+j|x|}\tilde{F}^{(n-j)}x\tilde{F}^{(j)}\quad\text{for } n\geq 0. \tag{2.11}$$

Proposition 2.2. *$\mathcal{U}_q^{\mathrm{ev}}$ is a left $\mathcal{U}_q$-submodule of $\mathcal{U}_q$.*

Proof. It follows from (2.9)–(2.11) that if x is a homogeneous element of $\mathcal{U}_q^{\mathrm{ev}}$, then we have $y\triangleright x\in\mathcal{U}_q^{\mathrm{ev}}$ for $y=K^i,e^n,\tilde{F}^{(n)}$ with $i\in\mathbb{Z}$, $n\geq 0$. Since these elements generate $\mathcal{U}_q$, we have the assertion. □

2.5. **Filtrations and completions.** Here we introduce filtrations for the algebras $\mathcal{U}_q$ and $\mathcal{U}_q^{\mathrm{ev}}$, and also for the tensor powers $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$ of $\mathcal{U}_q^{\mathrm{ev}}$. These filtrations produce the associated completions. The definitions below are equivalent to those in [19].

First, we consider the filtration and the completion for $\mathcal{U}_q$. For $p\geq 0$, set

$$\mathcal{F}_p(\mathcal{U}_q)=\mathcal{U}_qe^p\mathcal{U}_q,$$

the two-sided ideal in $\mathcal{U}_q$ generated by e^p. Let $\tilde{\mathcal{U}}_q$ denote the "completion in U_h" of $\mathcal{U}_q$ with respect to the decreasing filtration $\{\mathcal{F}_p(\mathcal{U}_q)\}_{p\geq 0}$, i.e., $\tilde{\mathcal{U}}_q$ is the image of the homomorphism

$$\varprojlim_{p\geq 0}\mathcal{U}_q/\mathcal{F}_p(\mathcal{U}_q)\to U_h \tag{2.12}$$

induced by $\mathcal{U}_q \subset U_h$. (Conjecturally, (2.12) is injective, see [19, Conjecture 7.2].) Clearly, $\tilde{\mathcal{U}}_q$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_h.

Second, we consider $\mathcal{U}_q^{\mathrm{ev}}$. For $p \geq 0$, set

$$\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) = \mathcal{F}_p(\mathcal{U}_q) \cap \mathcal{U}_q^{\mathrm{ev}} = \mathcal{U}_q^{\mathrm{ev}} e^p \mathcal{U}_q^{\mathrm{ev}}.$$

We have a $(\mathbb{Z}/2\mathbb{Z})$-grading $\mathcal{F}_p(\mathcal{U}_q) = \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) \oplus K\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})$. Let $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ denote the "completion in U_h" of $\mathcal{U}_q^{\mathrm{ev}}$ with respect to $\{\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})\}_{p\geq 0}$, i.e., the image of

$$\varprojlim_{p\geq 0} \mathcal{U}_q^{\mathrm{ev}}/\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) \to U_h.$$

Note that $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\tilde{\mathcal{U}}_q$, and $\tilde{\mathcal{U}}_q$ has a $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{Z}[q,q^{-1}]$-algebra structure

$$\tilde{\mathcal{U}}_q = \tilde{\mathcal{U}}_q^{\mathrm{ev}} \oplus K\tilde{\mathcal{U}}_q^{\mathrm{ev}}$$

induced by (2.1).

Note that the elements of $\tilde{\mathcal{U}}_q$ (resp. $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$) are the elements of U_h that can be expressed as infinite sums $\sum_{i=0}^{\infty}\sum_{j=1}^{N_i} x_{i,j}e^i y_{i,j}$, where $N_0, N_1, \ldots \geq 0$, and $x_{i,j}, y_{i,j} \in \mathcal{U}_q$ (resp. $x_{i,j}, y_{i,j} \in \mathcal{U}_q^{\mathrm{ev}}$) for $i \geq 0$, $1 \leq j \leq N_i$.

Now, we define the filtration for $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$, $n \geq 1$, by

$$\mathcal{F}_p((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}) = \sum_{i=1}^{n} (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (i-1)} \otimes \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) \otimes (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (n-i)} \subset (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}.$$

Thus, an element of $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$ is in the pth filtration if and only if it is expressed as a sum of terms each having at least one tensor factor in the pth filtration. Define the "completed tensor product" $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n} = \tilde{\mathcal{U}}_q^{\mathrm{ev}} \tilde{\otimes} \cdots \tilde{\otimes} (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ to be the "completion in $U_h^{\hat{\otimes} n}$" of $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$ with respect to this filtration, i.e., the image of the homomorphism

$$\varprojlim_{p\geq 0} (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}/\mathcal{F}_p((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}) \to U_h^{\hat{\otimes} n}.$$

For $n = 0$, it is natural to set

$$\mathcal{F}_p((\mathcal{U}_q^{\mathrm{ev}})^{\otimes 0}) = \mathcal{F}_p(\mathbb{Z}[q,q^{-1}]) = \begin{cases} \mathbb{Z}[q,q^{-1}] & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} 0} = \mathbb{Z}[q,q^{-1}].$$

In what follows, we will also need the filtrations and completions of other iterated tensor products of $\mathcal{U}_q$ and $\mathcal{U}_q^{\mathrm{ev}}$, whose definitions should be obvious from the above definitions. For example, $\mathcal{U}_q \otimes \mathcal{U}_q^{\mathrm{ev}}$ has a filtration defined by

$$\mathcal{F}_p(\mathcal{U}_q \otimes \mathcal{U}_q^{\mathrm{ev}}) = \mathcal{F}_p(\mathcal{U}_q) \otimes \mathcal{U}_q^{\mathrm{ev}} + \mathcal{U}_q \otimes \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}).$$

Moreover, $\tilde{\mathcal{U}}_q \tilde{\otimes} \tilde{\mathcal{U}}_q^{\mathrm{ev}}$ is defined to be the image of $\varprojlim_{p\geq 0} \mathcal{F}_p(\mathcal{U}_q \otimes \mathcal{U}_q^{\mathrm{ev}}) \to U_h^{\hat{\otimes} 2}$.

2.6. **The Hopf algebra structure for $\tilde{\mathcal{U}}_q$.** In this subsection, we show that the $\mathbb{Z}[q,q^{-1}]$-algebra $\tilde{\mathcal{U}}_q$ inherits from U_h a complete Hopf algebra structure over $\mathbb{Z}[q,q^{-1}]$. (This fact is observed in [19] as a corollary to the case of a $\mathbb{Z}[v,v^{-1}]$-form $\tilde{\mathcal{U}} \cong \tilde{\mathcal{U}}_q \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[v,v^{-1}]$. However, it is convenient to provide a direct proof here.)

The Hopf $\mathbb{Z}[q,q^{-1}]$-algebra structure of $\mathcal{U}_q$ induces a complete Hopf algebra structure (with invertible antipode) over $\mathbb{Z}[q,q^{-1}]$ of $\tilde{\mathcal{U}}_q$, since (2.5)–(2.8) imply

$$\Delta(\mathcal{F}_p(\mathcal{U}_q)) \subset \mathcal{F}_{\lfloor \frac{p+1}{2} \rfloor}(\mathcal{U}_q^{\otimes 2}),$$
$$\epsilon(\mathcal{U}_q) \subset \mathbb{Z}[q,q^{-1}], \quad \epsilon(\mathcal{F}_1(\mathcal{U}_q)) = 0,$$
$$S^{\pm 1}(\mathcal{F}_p(\mathcal{U}_q)) \subset \mathcal{F}_p(\mathcal{U}_q),$$

for $p \geq 0$. Here $\lfloor \frac{p+1}{2} \rfloor$ denotes the largest integer smaller than or equal to $\frac{p+1}{2}$. The structure morphisms

$$\Delta\colon \tilde{\mathcal{U}}_q \to \tilde{\mathcal{U}}_q^{\tilde{\otimes} 2}, \quad \epsilon\colon \tilde{\mathcal{U}}_q \to \mathbb{Z}[q,q^{-1}], \quad S\colon \tilde{\mathcal{U}}_q \to \tilde{\mathcal{U}}_q$$

of $\tilde{\mathcal{U}}_q$ are equal to the restrictions of those of U_h to $\tilde{\mathcal{U}}_q$.

A consequence of the above fact is that if $f\colon U_h^{\hat{\otimes} i} \to U_h^{\hat{\otimes} j}$, $i,j \geq 0$, is a $\mathbb{Q}[[h]]$-module homomorphism obtained from finitely many copies of id_{U_h}, P_{U_h,U_h}, μ, η, Δ, ϵ and $S^{\pm 1}$ by taking completed tensor products and compositions, then we have $f(\tilde{\mathcal{U}}_q^{\tilde{\otimes} i}) \subset \tilde{\mathcal{U}}_q^{\tilde{\otimes} j}$. Here $P_{U_h,U_h}\colon U_h^{\hat{\otimes} 2} \to U_h^{\hat{\otimes} 2}$ is defined by $P_{U_h,U_h}(\sum x \otimes y) = \sum y \otimes x$.

It follows that the adjoint action $\triangleright\colon \mathcal{U}_q \otimes \mathcal{U}_q \to \mathcal{U}_q$ induces a left action

$$\triangleright\colon \tilde{\mathcal{U}}_q \tilde{\otimes} \tilde{\mathcal{U}}_q \to \tilde{\mathcal{U}}_q,$$

which is equal to the restriction of $\triangleright\colon U_h \hat{\otimes} U_h \to U_h$. By Proposition 2.2, $\triangleright$ restricts to

$$\triangleright\colon \tilde{\mathcal{U}}_q \tilde{\otimes} \tilde{\mathcal{U}}_q^{\mathrm{ev}} \to \tilde{\mathcal{U}}_q^{\mathrm{ev}}.$$

Thus, $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ is a left $\tilde{\mathcal{U}}_q$-submodule of $\tilde{\mathcal{U}}_q$.

3. Braided Hopf algebra structure for $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$

In this section, we recall a ribbon Hopf algebra structure for U_h and show that the associated braided Hopf algebra structure for U_h induces that for $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$.

3.1. **Ribbon structure for U_h.** The Hopf algebra U_h has a ribbon Hopf algebra structure as follows. The universal R-matrix and its inverse are given by

$$R = D\Big(\sum_{n \geq 0} v^{n(n-1)/2} \frac{(v-v^{-1})^n}{[n]!} F^n \otimes E^n\Big), \tag{3.1}$$

$$R^{-1} = \Big(\sum_{n \geq 0} (-1)^n v^{-n(n-1)/2} \frac{(v-v^{-1})^n}{[n]!} F^n \otimes E^n\Big) D^{-1}, \tag{3.2}$$

where

$$D = v^{\frac{1}{2} H \otimes H} = \exp(\frac{h}{4} H \otimes H) \in U_h^{\hat{\otimes} 2}.$$

In what follows, we use the following notations.

$$R = \sum \alpha \otimes \beta, \quad R^{-1} = \sum \bar{\alpha} \otimes \bar{\beta} \big(= \sum S(\alpha) \otimes \beta\big).$$

The ribbon element and its inverse are given by

$$\mathbf{r} = \sum S(\alpha)K^{-1}\beta = \sum \beta K S(\alpha), \quad \mathbf{r}^{-1} = \sum \alpha K \beta = \sum \beta K^{-1}\alpha.$$

The associated grouplike element $\kappa \in U_h$ defined by

$$\kappa = \bigl(\sum S(\beta)\alpha\bigr)\mathbf{r}^{-1}$$

satisfies $\kappa = K^{-1}$.

We also use the following notations.

$$D = \sum D_{[1]} \otimes D_{[2]}, \quad D^{-1} = \sum \bar{D}_{[1]} \otimes \bar{D}_{[2]}.$$

The following properties of D are freely used in what follows.

$$\sum D_{[2]} \otimes D_{[1]} = D, \quad (\Delta \otimes 1)(D) = D_{13}D_{23}, \tag{3.3}$$

$$(\epsilon \otimes 1)(D) = 1, \quad (S \otimes 1)(D) = D^{-1}, \tag{3.4}$$

$$D^{\pm 1}(1 \otimes x) = (K^{\pm |x|} \otimes x)D \quad \text{for homogeneous } x \in U_h, \tag{3.5}$$

where $D_{13} = \sum D_{[1]} \otimes 1 \otimes D_{[2]}$ and $D_{23} = \sum 1 \otimes D_{[1]} \otimes D_{[2]} = 1 \otimes D$.

We can easily obtain the following formulas.

$$R = D\Bigl(\sum_{n \ge 0} q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n\Bigr), \tag{3.6}$$

$$R^{-1} = D^{-1}\Bigl(\sum_{n \ge 0} (-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n\Bigr), \tag{3.7}$$

$$\mathbf{r} = \sum_{n \ge 0} (-1)^n \tilde{F}^{(n)} v^{-\frac{1}{2}H(H+2)} e^n, \tag{3.8}$$

$$\mathbf{r}^{-1} = \sum_{n \ge 0} q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-2n} v^{\frac{1}{2}H(H+2)} e^n. \tag{3.9}$$

We have $R^{\pm 1} \in D^{\pm 1}(\tilde{\mathcal{U}}_q \tilde{\otimes} \tilde{\mathcal{U}}_q)$, and $\mathbf{r}^{\pm 1} \in v^{\mp \frac{1}{2}H(H+2)} \tilde{\mathcal{U}}_q^{\mathrm{ev}}$.

3.2. **Braided Hopf algebra structure for U_h.** Let Mod_{U_h} denote the category of h-adically complete left U_h-modules and continuous left U_h-module homomorphisms. The category Mod_{U_h} is equipped with a standard braided category structure, where the braiding $\psi_{V,W} \colon V \otimes W \to W \otimes V$ of two objects V and W in Mod_{U_h} is defined by

$$\psi_{V,W}(v \otimes w) = \sum \beta w \otimes \alpha v \quad \text{for } v \in V,\, w \in W.$$

The inverse $\psi^{-1}_{V,W} \colon W \otimes V \to V \otimes W$ of $\psi_{V,W}$ is given by

$$\psi^{-1}_{V,W}(w \otimes v) = \sum S(\alpha)v \otimes \beta w \quad \text{for } v \in V,\, w \in W.$$

We regard U_h as an object of Mod_{U_h}, equipped with the adjoint action. For simplicity, we write $\psi = \psi_{U_h,U_h}$. Thus the braiding and its inverse for U_h satisfy

$$\psi(x \otimes y) = \sum (\beta \triangleright y) \otimes (\alpha \triangleright x),$$
$$\psi^{-1}(x \otimes y) = \sum (S(\alpha) \triangleright y) \otimes (\beta \triangleright x)$$

for $x, y \in U_h$.

Let $\underline{U_h} = (U_h, \mu, \eta, \underline{\Delta}, \epsilon, \underline{S})$ denote the transmutation [56, 57] of U_h, which is a standard braided Hopf algebra structure in Mod_{U_h} associated to U_h. Here

$$\mu\colon U_h^{\hat{\otimes}2} \to U_h, \quad \eta\colon \mathbb{Q}[[h]] \to U_h, \quad \epsilon\colon U_h \to \mathbb{Q}[[h]],$$

are the structure morphisms of U_h, and

$$\underline{\Delta}\colon U_h \to U_h^{\hat{\otimes}2}, \quad \underline{S}\colon U_h \to U_h$$

are the twisted versions of comultiplication and antipode defined by

$$\underline{\Delta}(x) = \sum x_{(1)} S(\beta) \otimes (\alpha \triangleright x_{(2)}), \tag{3.10}$$

$$\underline{S}(x) = \sum \beta\, S(\alpha \triangleright x) \tag{3.11}$$

for $x \in U_h$. The inverse $\underline{S}^{-1}\colon U_h \to U_h$ of $\underline{S}$ is given by

$$\underline{S}^{-1}(x) = \sum S^{-1}(\alpha \triangleright x)\beta. \tag{3.12}$$

3.3. **Braided Hopf algebra structure for $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$.** Unlike $\tilde{\mathcal{U}}_q$, the even part $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ of $\tilde{\mathcal{U}}_q$ does not inherit a Hopf algebra structure from U_h. However, we have the following.

Theorem 3.1. *The braided Hopf algebra structure of $\underline{U_h}$ induces a braided Hopf algebra structure with invertible antipode for $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$. In other words, for*

$$f \in \{\psi, \psi^{-1}, \mu, \eta, \underline{\Delta}, \epsilon, \underline{S}, \underline{S}^{-1}\}$$

with $f\colon U_h^{\hat{\otimes} i} \to U_h^{\hat{\otimes} j}$ $(i, j \geq 0)$ we have

$$f((\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} i}) \subset (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} j},$$

and the induced map $f\colon (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} i} \to (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} j}$ is continuous.

Corollary 3.2. *Suppose that $f\colon U_h^{\hat{\otimes} i} \to U_h^{\hat{\otimes} j}$, $i, j \geq 0$, is a $\mathbb{Q}[[h]]$-module homomorphism obtained from finitely many copies of*

$$1_{U_h}\colon U_h \to U_h, \quad \psi^{\pm 1}\colon U_h^{\hat{\otimes}2} \to U_h^{\hat{\otimes}2}, \quad \mu\colon U_h^{\hat{\otimes}2} \to U_h, \quad \eta\colon \mathbb{Q}[[h]] \to U_h,$$
$$\underline{\Delta}\colon U_h \to U_h^{\hat{\otimes}2}, \quad \epsilon\colon U_h \to \mathbb{Q}[[h]], \quad \underline{S}^{\pm 1}\colon U_h \to U_h$$

by taking iterated tensor products and compositions. Then we have

$$f((\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} i}) \subset (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} j}.$$

Theorem 3.1 follows immediately from the following.

Proposition 3.3. *If $f \in \{\psi, \psi^{-1}, \mu, \eta, \epsilon, \underline{S}, \underline{S}^{-1}\}$ with $f\colon U_h^{\hat{\otimes} i} \to U_h^{\hat{\otimes} j}$ $(i, j \geq 0)$, then we have*

$$f(\mathcal{F}_p((\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} i})) \subset \mathcal{F}_p((\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} j}) \tag{3.13}$$

for $p \geq 0$. Moreover, we have

$$\underline{\Delta}(\mathcal{F}_p(\tilde{\mathcal{U}}_q^{\mathrm{ev}})) \subset \mathcal{F}_{\lfloor \frac{p+1}{2} \rfloor}((\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes}2}) \tag{3.14}$$

for $p \geq 0$.

Proof. The assertion is obvious for $f = \mu, \eta, \epsilon$.

The case $f = \psi^{\pm 1}$ follows from the following formulas

$$\psi(x \otimes y) = \sum_{n=0}^{\infty} q^{-\frac{1}{2}n(n+1)+(|x|-n)|y|}(e^n \triangleright y) \otimes (\tilde{F}^{(n)} \triangleright x), \tag{3.15}$$

$$\psi^{-1}(x \otimes y) = \sum_{n=0}^{\infty} (-1)^n q^{-(|x|+n)|y|}(\tilde{F}^{(n)} \triangleright y) \otimes (e^n \triangleright x) \tag{3.16}$$

for homogeneous elements $x, y \in U_h$.

Consider the case $f = \underline{S}^{\pm 1}$. Using (3.11) and (3.12), we obtain

$$\underline{S}(x) = \sum_{n=0}^{\infty} q^{-\frac{1}{2}n(n+1)} e^n K^{|x|-n} S(\tilde{F}^{(n)} \triangleright x), \tag{3.17}$$

$$\underline{S}^{-1}(x) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)-n|x|} S^{-1}(\tilde{F}^{(n)} \triangleright x) K^{|x|-n} e^n. \tag{3.18}$$

Using these formulas, we see easily that $\underline{S}(\mathcal{F}_p(\tilde{\mathcal{U}}_q^{\mathrm{ev}})) \subset \mathcal{F}_p(\tilde{\mathcal{U}}_q)$. Hence, it remains to show that if $x, y \in \mathcal{U}_q^{\mathrm{ev}}$ are homogeneous, then each term in (3.17) and (3.18) is contained in $\mathcal{U}_q^{\mathrm{ev}}$. This follows, since for any homogeneous $x \in \mathcal{U}_q^{\mathrm{ev}}$ we have $K^{|x|} S^{\pm 1}(x) \in \mathcal{U}_q^{\mathrm{ev}}$ by (2.5) and (2.7).

Finally, we prove (3.14). By computation,

$$\underline{\Delta}(x) = \sum_{n=0}^{\infty} (-1)^n q^{-n} x_{(1)} K^{-|x_{(2)}|} e^n \otimes (\tilde{F}^{(n)} \triangleright x_{(2)}), \tag{3.19}$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. Using this formula, we easily see that $\underline{\Delta}(\mathcal{F}_p(\tilde{\mathcal{U}}_q^{\mathrm{ev}})) \subset \mathcal{F}_{\lfloor \frac{p+1}{2} \rfloor}(\tilde{\mathcal{U}}_q \otimes \tilde{\mathcal{U}}_q^{\mathrm{ev}})$. It suffices to show that if $x \in \mathcal{U}_q^{\mathrm{ev}}$ is homogeneous, then each term in (3.19) is contained in $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes 2}$. This follows, since we have

$$\Delta(\mathcal{U}_q^{\mathrm{ev}}) \subset \mathcal{U}_q^{\mathrm{ev}} \otimes (\mathcal{U}_q^{\mathrm{ev}})_0 + K\mathcal{U}_q^{\mathrm{ev}} \otimes (\mathcal{U}_q^{\mathrm{ev}})_1, \tag{3.20}$$

where $(\mathcal{U}_q^{\mathrm{ev}})_0$ (resp. $(\mathcal{U}_q^{\mathrm{ev}})_1$) denotes the $\mathbb{Z}[q, q^{-1}]$-submodule of $\mathcal{U}_q^{\mathrm{ev}}$ spanned by the homogeneous elements of even (resp. odd) degrees. The inclusion (3.20) can be easily verified using (2.5) and (2.6). □

Remark 3.4. $\tilde{\mathcal{U}}_q$ also has a braided Hopf algebra structure inherited from that of $\underline{U_h}$, i.e., Theorem 3.1 holds if we replace $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ with $\tilde{\mathcal{U}}_q$. Moreover, the $(\mathbb{Z}/2\mathbb{Z})$-grading for $\tilde{\mathcal{U}}_q$ is compatible with the braided Hopf algebra structure:

$$\psi^{\pm 1}(K^i \tilde{\mathcal{U}}_q^{\mathrm{ev}} \tilde{\otimes} K^j \tilde{\mathcal{U}}_q^{\mathrm{ev}}) \subset K^j \tilde{\mathcal{U}}_q^{\mathrm{ev}} \tilde{\otimes} K^i \tilde{\mathcal{U}}_q^{\mathrm{ev}},$$
$$\underline{\Delta}(K^i \tilde{\mathcal{U}}_q^{\mathrm{ev}}) \subset K^i \tilde{\mathcal{U}}_q^{\mathrm{ev}} \tilde{\otimes} K^i \tilde{\mathcal{U}}_q^{\mathrm{ev}}, \quad \underline{S}^{\pm 1}(K^i \tilde{\mathcal{U}}_q^{\mathrm{ev}}) \subset K^i \tilde{\mathcal{U}}_q^{\mathrm{ev}},$$

for $i, j \in \{0, 1\}$.

4. Universal sl_2 invariant of bottom tangles

In this section, we recall the definition of the universal invariant of bottom tangles, and prove necessary results.

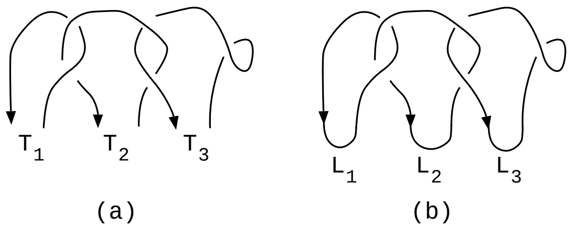

FIGURE 4.1. (a) A 3-component bottom tangle $T = T_1 \cup T_2 \cup T_3$.
(b) Its closure $\mathsf{cl}(L) = L_1 \cup L_2 \cup L_3$.

4.1. **Bottom tangles.** Here we recall from [20] the notion of bottom tangles.

An n-component *bottom tangle* $T = T_1 \cup \cdots \cup T_n$ is a framed tangle in a cube, which is drawn as a diagram in a rectangle as usual, consisting of n arcs $T_1, \ldots, T_n$ such that for each $i = 1, \ldots, n$ the component T_i runs from the $2i$th endpoint on the bottom to the $(2i-1)$st endpoint on the bottom, where the endpoints are counted from the left. For example, see Figure 4.1 (a). (Here and in what follows, we use the blackboard framing convention.)

For each $n \geq 0$, let BT_n denote the set of the isotopy classes of n-component bottom tangles. Set $\mathsf{BT} = \bigcup_{n\geq 0} \mathsf{BT}_n$.

The *closure* $\mathsf{cl}(T)$ of T is the n-component, oriented, ordered framed link in S^3, obtained from T by pasting a "$\cup$-shaped tangle" to each component of L, as depicted in Figure 4.1 (b). For any oriented, ordered framed link L, there is a bottom tangle whose closure is isotopic to L.

The *linking matrix* of a bottom tangle $T = T_1 \cup \cdots \cup T_n$ is defined as that of the closure T. Thus the linking number of T_i and T_j, $i \neq j$, is defined as the linking number of the corresponding components in $\mathsf{cl}(T)$, and the framing of T_i is defined as the framing of the closure of T_i.

A link or a bottom tangle is called *algebraically-split* if the linking matrix is diagonal.

For $n \geq 0$, let BT^0_n denote the subset of BT_n consisting of algebraically-split, 0-framed bottom tangles. Set $\mathsf{BT}^0 = \bigcup_{n\geq 0} \mathsf{BT}^0_n \subset \mathsf{BT}$.

4.2. **Universal sl_2 invariant of bottom tangles.** For each ribbon Hopf algebra H, there is a "universal invariant" of links and tangles from which one can recover the operator invariants, such as the colored Jones polynomials. Such universal invariants has been studied in [42, 43, 72, 53, 66, 33, 35, 34]. Here we need only the case of bottom tangles, which is described in [20].

For $T = T_1 \cup \cdots \cup T_n \in \mathsf{BT}_n$, we define the *universal sl_2 invariant* $J_T \in U_h^{\hat{\otimes} n}$ of T as follows. We choose a diagram for T, which is obtained from copies of *fundamental tangle*, see Figure 4.2, by pasting horizontally and vertically. For each copy of fundamental tangle in the diagram of T, we put elements of U_h with the rule described in Figure 4.3. We set

$$J_T = \sum J_{(T_1)} \otimes \cdots \otimes J_{(T_n)} \in U_h^{\hat{\otimes} n},$$

where for each $i = 1, \ldots, n$, the ith tensorand $J_{(T_i)}$ is defined to be the product of the elements put on the component T_i. Here the elements read off along each

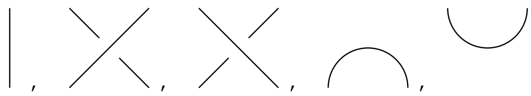

FIGURE 4.2. Fundamental tangles: vertical line, positive and negative crossings, local minimum and local maximum. Here the orientations are arbitrary.

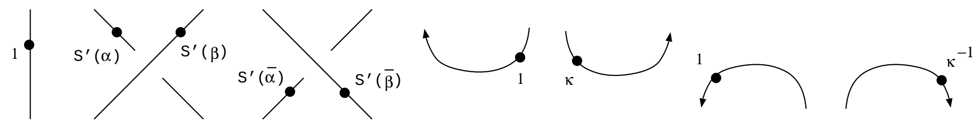

FIGURE 4.3. How to put elements of U_h on the strings. For each string in the positive and the negative crossings, "S'" should be replaced with id if the string is oriented downward, and by S otherwise.

component are written from right to left. Then J_T does not depend on the choice of diagram, and defines an isotopy invariant of bottom tangles.

4.3. **Universal sl_2-invariant of algebraically-split, 0-framed bottom tangles.** For any left U_h-module W, let $\mathrm{Inv}(W)$ denote the *invariant part* of W, defined by

$$\mathrm{Inv}(W) = \{w \in W \mid x \cdot w = \epsilon(x)w \quad \forall x \in U_h\}.$$

Recall that we regard U_h as a left U_h-module via the adjoint action. For $n \geq 0$, the completed tensor product $U_h^{\hat{\otimes} n}$ is equipped with a left U_h-module structure $\triangleright_n$ in the standard way: For $x = \sum x_1 \otimes \cdots \otimes x_n \in U_h^{\hat{\otimes} n}$ and $y \in U_h$ we have

$$y \triangleright_n x = \sum (y_{(1)} \triangleright x_1) \otimes \cdots \otimes (y_{(n)} \triangleright x_n).$$

In particular, $U_h^{\hat{\otimes} 0} = \mathbb{Q}[[h]]$ is given the trivial left U_h-module structure. For any subset $X \subset \mathrm{Inv}(U_h^{\hat{\otimes} n})$, we set

$$\mathrm{Inv}(X) = \mathrm{Inv}(U_h^{\hat{\otimes} n}) \cap X.$$

For $n \geq 0$, we set

$$\mathcal{K}_n = \mathrm{Inv}((\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}) \subset (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}.$$

One can easily see that $\mathcal{K}_n$ is the $\tilde{\mathcal{U}}_q$-invariant part of the $\tilde{\mathcal{U}}_q$-module $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$, i.e., we have

$$\mathcal{K}_n = \{x \in (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n} \mid y \triangleright_n x = \epsilon(y)x \quad \text{for all } y \in \tilde{\mathcal{U}}_q\}.$$

The main result of this subsection is the following.

Theorem 4.1. *If $T \in \mathsf{BT}_n^0$, $n \geq 0$, then we have $J_T \in \mathcal{K}_n$.*

To prove Theorem 4.1, we use the following two results from [20].

Proposition 4.2 (U_h case of [20, Proposition 8.2])**.** *For any n-component bottom tangle $T \in \mathsf{BT}_n$, we have $J_T \in \mathrm{Inv}(U_h^{\hat{\otimes} n})$.*

Proposition 4.3 (U_h case of [20, Corollary 9.15])**.** *Let $X_n \subset U_h^{\hat{\otimes} n}$, $n \geq 0$, be subsets satisfying the following conditions.*

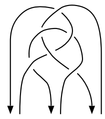

FIGURE 4.4. The Borromean tangle $B \in \mathsf{BT}^0_3$.

(1) $1 \in X_0$, $1 \in X_1$, *and* $J_B \in X_3$. *Here* $B \in \mathsf{BT}^0_3$ *is the* Borromean tangle *depicted in Figure 4.4.*
(2) *If* $x \in X_l$ *and* $y \in X_m$ *with* $l, m \geq 0$, *then* $x \otimes y \in X_{l+m}$.
(3) *For* $p, q \geq 0$ *and* $f \in \{\psi^{\pm 1}, \mu, \underline{\Delta}, \underline{S}\}$ *with* $f\colon U_h^{\hat{\otimes} i} \to U_h^{\hat{\otimes} j}$, *we have*

$$(1^{\otimes p} \otimes f \otimes 1^{\otimes q})(X_{p+i+q}) \subset X_{p+j+q}.$$

Then, for any $T \in \mathsf{BT}^0_n$, *we have* $J_T \in X_n$.

Proof of Theorem 4.1. By Proposition 4.2, it suffices to show that $J_T \in (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$. We have only to verify the conditions in Proposition 4.3, where we set $X_n = (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$.

The condition (2) is obvious.

The condition (3) follows from Corollary 3.2.

To prove (1), it suffices to prove $J_B \in (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} 3}$. Using Figure 4.4, we obtain

$$J_B = \sum \bar{\alpha}_3 \beta_1 \alpha_3 S^2(\bar{\beta}_1) \otimes \bar{\alpha}_1 \beta_2 \alpha_1 S^2(\bar{\beta}_2) \otimes \bar{\alpha}_2 S^{-2}(\beta_3) \alpha_2 \bar{\beta}_3,$$

where $R = \sum \alpha_i \otimes \beta_i$ and $R^{-1} = \sum \bar{\alpha}_i \otimes \bar{\beta}_i$ for $i = 1, 2, 3$. Using (3.6) and (3.7), we obtain

$$\begin{aligned} J_B = &\sum_{m_1, m_2, m_3, n_1, n_2, n_3 \geq 0} (-1)^{n_1+n_2+n_3} q^{-\frac{1}{2}m_1(m_1+1) - \frac{1}{2}m_2(m_2+1) - \frac{1}{2}m_3(m_3+1)} \\ &\tilde{F}^{(n_3)} \bar{D}'_3 e^{m_1} D''_1 D'_3 \tilde{F}^{(m_3)} S^2(\bar{D}''_1 e^{n_1}) \otimes \tilde{F}^{(n_1)} \bar{D}'_1 e^{m_2} D''_2 D'_1 \tilde{F}^{(m_1)} S^2(\bar{D}''_2 e^{n_2}) \\ &\otimes \tilde{F}^{(n_2)} \bar{D}'_2 S^{-2}(e^{m_3} D''_3) D'_2 \tilde{F}^{(m_2)} \bar{D}''_3 e^{n_3}, \end{aligned}$$

where $D = \sum D'_i \otimes D''_i$ and $D^{-1} = \sum \bar{D}'_i \otimes \bar{D}''_i$ for $i = 1, 2, 3$. We slide the tensor factors of the copies of $D^{\pm 1}$ using (3.5) so that these copies cancel at the cost of inserting powers of K. Thus we obtain

(4.1)
$$\begin{aligned} J_B = &\sum_{m_1, m_2, m_3, n_1, n_2, n_3 \geq 0} q^{m_3+n_3} (-1)^{n_1+n_2+n_3} q^{\sum_{i=1}^3 (-\frac{1}{2}m_i(m_i+1) - n_i + m_i m_{i+1} - 2m_i n_{i-1})} \\ &\tilde{F}^{(n_3)} e^{m_1} \tilde{F}^{(m_3)} e^{n_1} K^{-2m_2} \otimes \tilde{F}^{(n_1)} e^{m_2} \tilde{F}^{(m_1)} e^{n_2} K^{-2m_3} \otimes \tilde{F}^{(n_2)} e^{m_3} \tilde{F}^{(m_2)} e^{n_3} K^{-2m_1}, \end{aligned}$$

where the index i should be considered modulo 3. Each term in (4.1) is in $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes 3}$. For any $p \geq 0$, all but finitely many terms in (4.1) involve e^r with $r \geq p$, and therefore are contained in $\mathcal{F}_p((\mathcal{U}_q^{\mathrm{ev}})^{\otimes 3})$. Therefore, we have $J_B \in (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} 3}$. $\square$

4.4. **Universal sl_2 invariant of bottom knots.** By a *bottom knot*, we mean a 1-component bottom tangle. In what follows, we assume that bottom knots are given 0-framing. Thus the set of bottom knots (up to isotopy) is BT^0_1. By Theorem 4.1, for any bottom knot $T \in \mathsf{BT}^0_1$, we have

$$J_T \in \mathrm{Inv}(\tilde{\mathcal{U}}_q^{\mathrm{ev}}) = Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}}). \tag{4.2}$$

Let us recall from [19] the structure of $Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$. (In [19], $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ is denoted by $\mathcal{G}_0(\tilde{\mathcal{U}}_q)$.) Set

$$C = (v - v^{-1})^2 FE + vK + v^{-1}K^{-1} \in Z(U_h),$$

which is a well-known central element. We have

$$C = (v - v^{-1})\tilde{F}^{(1)}K^{-1}e + vK + v^{-1}K^{-1} \in vK\mathcal{U}_q^{\mathrm{ev}} \cap Z(U_h).$$

Hence $C^2 \in \mathcal{U}_q^{\mathrm{ev}} \cap Z(U_h) = Z(\mathcal{U}_q^{\mathrm{ev}})$. As a $\mathbb{Z}[q, q^{-1}]$-algebra, $Z(\mathcal{U}_q^{\mathrm{ev}})$ is freely generated by C^2, i.e., we have $Z(\mathcal{U}_q^{\mathrm{ev}}) \cong \mathbb{Z}[q, q^{-1}][C^2]$. For $p \geq 0$, set

$$\sigma_p = \prod_{i=1}^{p}(C^2 - (q^i + 2 + q^{-i})) \in Z(\mathcal{U}_q^{\mathrm{ev}}),$$

which is a monic polynomial of degree p in C^2. Therefore,

$$Z(\mathcal{U}_q^{\mathrm{ev}}) = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{\sigma_p \mid p \geq 0\}.$$

As for the center of the completion $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$, we have the following.

Theorem 4.4 ([19, Theorem 11.2]). *The isomorphism* $Z(\mathcal{U}_q^{\mathrm{ev}}) \cong \mathbb{Z}[q, q^{-1}][C^2]$ *induces an isomorphism*

$$Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}}) \cong \varprojlim_{p\geq 0} \mathbb{Z}[q, q^{-1}][C^2]/(\sigma_p).$$

Thus, each element in $Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$ *is uniquely expressed as an infinite sum* $\sum_{p\geq 0} a_p\sigma_p$, *where* $a_p \in \mathbb{Z}[q, q^{-1}]$ *for* $p \geq 0$.

This implies the following.

Theorem 4.5. *If* T *is a bottom knot, then* J_T *is uniquely expressed as*

$$J_T = \sum_{p\geq 0} a_p(T)\sigma_p, \tag{4.3}$$

where $a_p(T) \in \mathbb{Z}[q, q^{-1}]$ *for* $p \geq 0$.

Note that the $a_p(T)$ are invariants of a bottom knot T. In Section 6, we give a formula which express $a_p(T)$ using the colored Jones polynomials of the closure of T.

Remark 4.6. There is an obvious one-to-one correspondence between bottom knots and *string knots* (i.e., a string link consisting of just one arc component running from the above to the bottom). A bottom knot and the corresponding string knot have the same value of the universal invariant. Therefore, we have the result announced in [17, Theorem 2.1], [19, Theorem 1.2], which is the string knot version of Theorem 4.5.

5. Colored Jones polynomials

In this section, we recall the definition of the colored Jones polynomials of framed links.

5.1. **Finite-dimensional representations of U_h.** By a *finite-dimensional representation* of U_h, we mean a left U_h-module which is free of finite rank as a $\mathbb{Q}[[h]]$-module.

It is well known that, for each $n \geq 0$, there is exactly one irreducible finite-dimensional representation V_n of rank $n+1$ up to isomorphism, which corresponds to the $(n+1)$-dimensional irreducible representation of sl_2.

The structure of V_n is as follows. Let $\mathbf{v}_0^n \in \mathsf{V}_n$ denote a highest weight vector of V_n, which is characterized by $E\mathbf{v}_0^n = 0$, $H\mathbf{v}_0^n = n\mathbf{v}_0^n$ and $U_h\mathbf{v}_0^n = \mathsf{V}_n$. It is useful to define the other basis elements by

$$\mathbf{v}_i^n = \tilde{F}^{(i)}\mathbf{v}_0^n \quad \text{for } i = 1, \dots, n.$$

Then the action of $\mathcal{U}_q$ on V_n is given by

$$K^{\pm 1}\mathbf{v}_i^n = v^{n-2i}\mathbf{v}_i^n, \tag{5.1}$$

$$e^m\mathbf{v}_i^n = \{n-i+m\}_{q,m}\mathbf{v}_{i-m}^n, \tag{5.2}$$

$$\tilde{F}^{(m)}\mathbf{v}_i^n = q^{-mi}\begin{bmatrix} i+m \\ m \end{bmatrix}_q \mathbf{v}_{i+m}^n, \tag{5.3}$$

for $i = 0, \dots, n$ and $m \geq 0$, where we understand $\mathbf{v}_i^n = 0$ unless $0 \leq i \leq n$.

5.2. **Colored Jones polynomials.** Let $L = L_1 \cup \cdots \cup L_m$ be an m-component, framed, oriented, ordered link, and let $W_1, \dots, W_m \geq 0$ be finite-dimensional representations of U_h. We consider each W_i as a *color* attached to the component L_i. Then the colored Jones polynomial $J_L(W_1, \dots, W_m)$ of the colored link $(L; W_1, \dots, W_m)$ is defined as follows.

We choose a diagram of L which is obtained by pasting copies of the fundamental tangles, as in the definition of the universal sl_2 invariant in Section 4.2. To each copy of a fundamental tangle in L, we associate a left U_h-module homomorphism. To a vertical line contained in L_i, we associate to the left U_h-module W_i if the line is oriented downward, and the dual W_i^* of W_i if the line is oriented upward. To a positive crossing, we associate a braiding operator

$$\psi_{W,W'}\colon\ W \otimes W' \to W' \otimes W, \quad x \otimes y \mapsto \sum \beta y \otimes \alpha x,$$

where W (resp. W') are the left U_h-modules associated to the string which connects the upper left corner and the lower right corner (resp. the upper right corner and the lower left corner). To a negative crossing, we associate

$$\psi_{W',W}^{-1}\colon\ W \otimes W' \to W' \otimes W, \quad x \otimes y \mapsto \sum \bar{\alpha} y \otimes \bar{\beta} x,$$

where W and W' are as above. To the last four tangles in Figure 4.3, we associate the following operators

$$\begin{aligned}
\mathrm{ev}_W\colon\ W^* \otimes W \to \mathbb{Q}[[h]], &\quad f \otimes x \mapsto f(x),\\
\mathrm{ev}'_W\colon\ W \otimes W^* \to \mathbb{Q}[[h]], &\quad x \otimes f \mapsto f(\kappa x),\\
\mathrm{coev}_W\colon\ \mathbb{Q}[[h]] \to W \otimes W^*, &\quad 1 \mapsto \sum_i x_i \otimes x^i,\\
\mathrm{coev}'_W\colon\ \mathbb{Q}[[h]] \to W^* \otimes W, &\quad 1 \mapsto \sum_i x^i \otimes \kappa^{-1} x_i,
\end{aligned}$$

respectively, where $W = W_i$ if L_i is the component of L containing the fundamental tangle, and where the x_i are a basis of W and the $x^i \in W^*$ are the dual basis.

By tensoring and composing the operators, we obtain a left U_h-module homomorphism from $\mathbb{Q}[[h]]$ to $\mathbb{Q}[[h]]$, and we define $J_L(W_1,\dots,W_m)$ to be the trace of this homomorphism.

Note that "J_L" alone denotes the universal sl_2 invariant of L. The above definition of $J_L(W_1,\dots,W_n)$ is an abuse of notation, but it should not cause confusion.

It is well known that $J_L \in q^{p/4}\mathbb{Z}[q,q^{-1}] \subset \mathbb{Z}[q^{\pm 1/4}]$, where $p \in \mathbb{Z}/4\mathbb{Z}$ depends only on the linking matrix of L and on $n_1,\dots,n_m$. The invariant is normalized so that $J_\emptyset() = 1$, and $J_U(\mathsf{V}_n) = [n+1]$ for $n \ge 0$, where U denotes the 0-framed unknot.

5.3. **Quantum trace and the colored Jones polynomial.** If V is a finite-dimensional representation of U_h, then the *quantum trace* $\mathrm{tr}_q^V(x)$ in V of an element $x \in U_h$ is defined by

$$\mathrm{tr}_q^V(x) = \mathrm{tr}^V(\rho_V(\kappa x))\Big(= \mathrm{tr}^V(\rho_V(K^{-1}x))\Big) \in \mathbb{Q}[[h]],$$

where $\rho_V\colon U_h \to \mathrm{End}(V)$ denotes the left action of U_h on V, and $\mathrm{tr}^V\colon \mathrm{End}(V) \to \mathbb{Q}[[h]]$ denotes the trace in V. We have a continuous left H-module homomorphism

$$\mathrm{tr}_q^V\colon U_h \to \mathbb{Q}[[h]].$$

Let $L = L_1 \cup \cdots \cup L_m$ be an m-component, ordered, framed oriented link in S^3. Choose $T \in \mathsf{BT}_m$ such that $\mathsf{cl}(T)$ is isotopic to L. As explained in [20, Section 1.2] we have

$$J_L(W_1,\dots,W_m) = (\mathrm{tr}_q^{W_1} \otimes \mathrm{tr}_q^{W_2} \otimes \cdots \otimes \mathrm{tr}_q^{W_m})(J_T) \tag{5.4}$$

for finite dimensional representations $W_1,\dots,W_m$ of U_h.

5.4. **Representation rings.** For a commutative ring A with unit, let $\mathcal{R}_A$ denote the A-algebra

$$\mathcal{R}_A = \mathrm{Span}_A\{\mathsf{V}_n \mid n \ge 0\}.$$

with the multiplication induced by tensor product. Since each finite-dimensional representation of U_h is a direct sum of copies of V_n, $n \ge 0$, we may regard $\mathcal{R}_A$ as the *representation ring* of U_h over A.

By the well-known isomorphism of left U_h-modules

$$\mathsf{V}_m \otimes \mathsf{V}_n \cong \mathsf{V}_{|m-n|} \oplus \mathsf{V}_{|m-n|+2} \oplus \cdots \oplus \mathsf{V}_{m+n},$$

we have the identity in $\mathcal{R}_A$

$$\mathsf{V}_m\mathsf{V}_n = \mathsf{V}_{|m-n|} + \mathsf{V}_{|m-n|+2} + \cdots + \mathsf{V}_{m+n}.$$

As an A-algebra, $\mathcal{R}_A$ is freely generated by V_1, i.e., $\mathcal{R}_A \cong A[\mathsf{V}_1]$. Thus we identify $\mathcal{R}_A$ with $A[\mathsf{V}_1]$.

For $y = \sum_n a_n\mathsf{V}_n$ ($a_n \in \mathbb{Q}[[h]]$) and $x \in U_h$, we set

$$\mathrm{tr}_q^y(x) = \sum_n a_n \mathrm{tr}_q^{\mathsf{V}_n}(x).$$

Thus we have bilinear maps

$$\mathrm{tr}_q^-(-)\colon \mathcal{R}_{\mathbb{Z}[v,v^{-1}]} \times U_h \to \mathbb{Q}[[h]],$$
$$\mathrm{tr}_q^-(-)\colon \mathcal{R}_{\mathbb{Q}(v)} \times U_h \to \mathbb{Q}((h)),$$

where $\mathbb{Q}((h))$ denote the quotient field of $\mathbb{Q}[[h]]$.

Similarly, for a link L we have multilinear maps

$$J_L\colon \mathcal{R}_{\mathbb{Z}[v,v^{-1}]}\times\cdots\times\mathcal{R}_{\mathbb{Z}[v,v^{-1}]}\to\mathbb{Z}[v^{1/2},v^{-1/2}],$$
$$J_L\colon \mathcal{R}_{\mathbb{Q}(v)}\times\cdots\times\mathcal{R}_{\mathbb{Q}(v)}\to\mathbb{Q}(v^{1/2}).$$

If the framing of every component of L is even, then J_L above take values in $\mathbb{Z}[v,v^{-1}]$, $\mathbb{Q}(v)$, respectively.

6. Knots

In this section, we study the relationships between the universal invariant for a bottom knot T and the colored Jones polynomials for the closure of T.

6.1. The elements P_n and P''_n.

For each $n\geq 0$ set

$$P_n=\prod_{i=0}^{n-1}(\mathsf{V}_1-v^{2i+1}-v^{-2i-1})\in\mathcal{R}_{\mathbb{Z}[v,v^{-1}]},$$
$$P''_n=P_n/\{2n+1\}_{2n}\in\mathcal{R}_{\mathbb{Q}(v)}.$$

Since P_n is a monic polynomial in V_1 of degree n for each $n\geq 0$, it follows that the P_n form a basis of $\mathcal{R}_{\mathbb{Z}[v,v^{-1}]}$. We have the following base change formulas.

Lemma 6.1. *For $n\geq 0$, we have*

$$P_n=\sum_{i=0}^{n}(-1)^{n-i}\frac{[2i+2]}{[n+i+2]}\begin{bmatrix}2n+1\\ n+1+i\end{bmatrix}\mathsf{V}_i, \tag{6.1}$$

$$\mathsf{V}_n=\sum_{i=0}^{n}\begin{bmatrix}n+i+1\\ 2i+1\end{bmatrix}P_i. \tag{6.2}$$

Remark 6.2. A formula essentially the same as (6.2) formulated using the Kauffman bracket skein module of a solid torus has appeared in [16, Proposition 5.1.], [58, Equation 47]. (The former contained an incorrect sign.)

Proof. The proofs of (6.1) and (6.2) are by inductions on n using

$$P_n=P_{n-1}(\mathsf{V}_1-(v^{2n-1}+v^{-2n+1}))\quad(n\geq 1),$$
$$\mathsf{V}_n=\mathsf{V}_1\mathsf{V}_{n-1}-\mathsf{V}_{n-2}\quad(n\geq 2),$$

respectively. For an alternative proof of (6.2), see Remark 6.7. □

6.2. The reduced Jones polynomials of knots.

We prove the following result in Section 6.3.

Proposition 6.3. *For $m,n\geq 0$ we have*

$$\mathrm{tr}_q^{P''_m}(\sigma_n)=\delta_{m,n}. \tag{6.3}$$

Using Proposition 6.3 we obtain the following.

Theorem 6.4. *For a bottom knot $T\in\mathsf{BT}_1$ with closure $K=\mathsf{cl}(T)$, we have*

$$J_T=\sum_{n\geq 0}J_K(P''_n)\sigma_n. \tag{6.4}$$

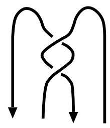

Figure 6.1. The bottom tangle $c_+ \in \mathsf{BT}_2$.

Proof. We express J_T as in Theorem 4.5. Then, applying $\mathrm{tr}_q^{P''_m}$ to (4.3), we have by Proposition 6.3

$$\mathrm{tr}_q^{P''_m}(J_T) = \sum_{p\geq 0} a_p(T)\mathrm{tr}_q^{P''_m}(\sigma_p) = a_p(T).$$

Therefore, we have (6.4). □

Theorems 4.5 and 6.4 implies that for a knot K, we have $J_K(P''_n) \in \mathbb{Z}[q,q^{-1}]$ for $n \geq 0$. We call $J_K(P''_n)$ the *nth reduced Jones polynomial* of K. It is clear that $J_K(P''_0) = 1$. For $n = 1$, we have

$$J_K(P''_1) = (J_K(\mathsf{V}_1) - [2])/\{2\}\{3\}.$$

The quantity $-q^{-2}J_K(P''_1)$ is known as the *reduced Jones polynomial* of K, and appears in the original paper of Jones [31, Proposition 12.5]. Some examples of $J_K(P''_n)$ are given in Section 14.3.

It is clear that the universal invariant J_T determines the colored Jones polynomials $J_K(\mathsf{V}_n)$, hence the reduced Jones polynomials $J_K(P''_n)$. Conversely, Theorem 6.4 implies that the universal invariant J_T is determined by the $J_K(P''_n)$, hence by the $J_K(\mathsf{V}_n)$ as is presumably well known.

By (6.2), we have

$$J_K(\mathsf{V}_n) = \sum_{i=0}^{n} \frac{\{n+1+i\}_{2i+1}}{\{1\}} J_K(P''_i), \tag{6.5}$$

where the sum may be replaced with $\sum_{i=0}^{\infty}$ since $\{n+1+i\}_{2i+1} = 0$ for $i > n$. (Conversely, one can use (6.1) to obtain a formula for the reduced Jones polynomials in terms of the colored Jones polynomials.) (6.5) implies the following.

Proposition 6.5. *If K is a knot, then for each $n \geq 0$ the nth colored Jones polynomial $J_K(\mathsf{V}_n)$ is determined modulo $(\{2n+1\}_{2n})$ by $J_K(\mathsf{V}_0), J_K(\mathsf{V}_1), \ldots, J_K(\mathsf{V}_{n-1})$.*

6.3. **Proof of Proposition 6.3.**

6.3.1. *The homomorphism $\xi\colon \mathcal{R}_{\mathbb{Q}[[h]]} \to Z(U_h)$.* Let c_+ denote the 2-component bottom tangle depicted in Figure 6.1. We have

$$J_{c_+} = (S\otimes 1)(R_{21}R) = \sum S(\alpha)S(\beta')\otimes \alpha'\beta \in \mathrm{Inv}(U_h^{\hat{\otimes} 2}),$$

where $R = \sum \alpha\otimes\beta = \sum \alpha'\otimes\beta'$.

Define a continuous $\mathbb{Q}[[h]]$-algebra homomorphism $\xi\colon \mathcal{R}_{\mathbb{Q}[[h]]} \to Z(U_h)$ by

$$\xi(y) := (1\otimes \mathrm{tr}_q^y)(J_{c_+}) = \sum S(\alpha)S(\beta')\mathrm{tr}_q^y(\alpha'\beta)$$

for $y \in \mathcal{R}_{\mathbb{Q}[[h]]}$. Indeed, we can verify

$$\xi(VV') = \xi(V)\xi(V') \tag{6.6}$$

graphically as depicted in Figure 6.2. We can also verify $\xi(\mathsf{V}_1) = C$ by computation.

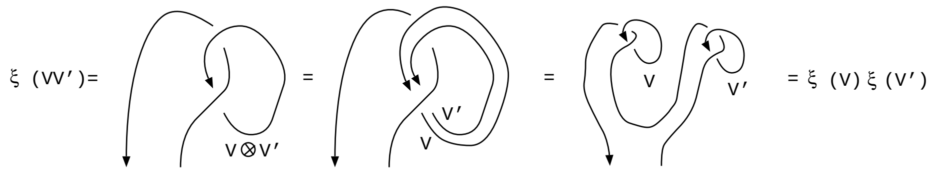

FIGURE 6.2. A graphical proof of (6.6).

Since the C^i, $i \geq 0$, are linearly independent in $Z(U_h)$, it follows that ξ is injective.

6.3.2. *The Hopf link pairing* $\langle , \rangle \colon \mathcal{R}_{\mathbb{Z}[v,v^{-1}]} \times \mathcal{R}_{\mathbb{Z}[v,v^{-1}]} \to \mathbb{Z}[v, v^{-1}]$. Define a symmetric bilinear form

$$\langle , \rangle \colon \mathcal{R}_{\mathbb{Q}[[h]]} \times \mathcal{R}_{\mathbb{Q}[[h]]} \to \mathbb{Q}[[h]]$$

by

$$\langle x, y \rangle := (\mathrm{tr}_q^x \otimes \mathrm{tr}_q^y)(J_{c_+}) = \mathrm{tr}_q^x(\xi(y)) = J_H(x, y), \tag{6.7}$$

for $x, y \in U_h$, where $H = H_1 \cup H_2 = \mathsf{cl}(c_+)$ denotes the 0-framed Hopf link with linking number -1.

It is well known (see [73, 38]) that for $m, n \geq 0$, we have

$$\langle \mathsf{V}_m, \mathsf{V}_n \rangle = [(m+1)(n+1)].$$

Therefore, $\langle , \rangle$ restricts to a bilinear form

$$\langle , \rangle \colon \mathcal{R}_{\mathbb{Z}[v,v^{-1}]} \times \mathcal{R}_{\mathbb{Z}[v,v^{-1}]} \to \mathbb{Z}[v, v^{-1}].$$

We also need the induced bilinear form

$$\langle , \rangle \colon \mathcal{R}_{\mathbb{Q}(v)} \times \mathcal{R}_{\mathbb{Q}(v)} \to \mathbb{Q}(v).$$

6.3.3. *The elements* S_n. Let $\mathcal{S}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathcal{R}_{\mathbb{Z}[q,q^{-1}]}$ generated by V_1^2. We have $\mathcal{S} = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{\mathsf{V}_{2i} \mid i \geq 0\}$.

For $n \geq 0$, set

$$S_n = \prod_{i=1}^{n} (\mathsf{V}_1^2 - (v^i + v^{-i})^2) = \prod_{i=1}^{n} (\mathsf{V}_2 - (q^i + 1 + q^{-i})) \in \mathcal{S}.$$

(The elements corresponding to S_n in the Kauffman bracket skein module of solid torus have been introduced in [16].) Clearly, we have

$$\xi(S_n) = \sigma_n \tag{6.8}$$

for $n \geq 0$. Observe that S_n is a monic polynomial in V_1^2 of degree n. Therefore, $\mathcal{S}$ is spanned over $\mathbb{Z}[q, q^{-1}]$ by the elements S_n, $n \geq 0$.

Proposition 6.6. *If* $m, n \geq 0$*, then we have*

$$\langle P_m, S_n \rangle = \delta_{m,n}\{2m+1\}_{2m}. \tag{6.9}$$

(As a consequence, the map $\langle , \rangle \colon \mathcal{R}_{\mathbb{Z}[v,v^{-1}]} \times \mathcal{S} \to \mathbb{Z}[v, v^{-1}]$ *is nondegenerate.)*

Proof. For $k \geq 0$, set $\mathsf{V}'_k = \mathsf{V}_k/[k+1]$. The map

$$\langle -, \mathsf{V}'_k \rangle \colon \mathcal{R}_{\mathbb{Z}[v,v^{-1}]} \to \mathbb{Z}[v, v^{-1}], \quad x \mapsto \langle x, \mathsf{V}'_k \rangle$$

is a $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism.

We will prove

$$\langle P_m, \mathsf{V}_{2n}\rangle = [2n+1]\{n+m\}_{2m}. \tag{6.10}$$

For $p \geq 0$, we have

$$\langle \mathsf{V}_1 - v^{2p+1} - v^{-2p-1}, \mathsf{V}'_{2n}\rangle = v^{2n+1} + v^{-2n-1} - v^{2p+1} - v^{-2p-1} = \{n-p\}\{n+p+1\}.$$

Since $\langle -, \mathsf{V}'_{2n}\rangle$ is a $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism, we have

$$\langle P_m, \mathsf{V}'_{2n}\rangle = \prod_{p=0}^{m-1} \langle \mathsf{V}_1 - v^{2p+1} - v^{-2p-1}, \mathsf{V}'_{2n}\rangle = \prod_{p=0}^{m-1} \{n-p\}\{n+p+1\} = \{n+m\}_{2m}.$$

Therefore, we have (6.10). Similarly, we can prove

$$\langle \mathsf{V}_m, S_n\rangle = \{m+n+1\}_{2n+1}/\{1\}. \tag{6.11}$$

By (6.10), $\langle P_m, \mathsf{V}_{2n}\rangle = 0$ when $0 \leq n < m$. Therefore, we have $\langle P_m, S_n\rangle = 0$ if $0 \leq n < m$. By (6.11), $\langle \mathsf{V}_m, S_n\rangle = 0$ when $0 \leq m < n$. Hence, we have $\langle P_m, S_n\rangle = 0$ when $0 \leq m < n$. By (6.10),

$$\langle P_m, S_m\rangle = \langle P_m, \mathsf{V}_{2m}\rangle = [2m+1]\{2m\}_{2m} = \{2m+1\}_{2m},$$

since $S_m - \mathsf{V}_{2m}$ is a linear combination of V_{2i} for $i = 0, 1, \ldots, m-1$. This completes the proof. □

Remark 6.7. One can prove (6.2) as follows. Assume $\mathsf{V}_n = \sum_{j=0}^{n} a_{n,j} P_j$, where $a_{n,j} \in \mathbb{Z}[v, v^{-1}]$ for $j = 0, \ldots, n$. Applying $\langle -, S_i\rangle$ to the both sides, we obtain $\langle \mathsf{V}_n, S_i\rangle = \sum_{j=0}^{n} a_{n,j}\langle P_j, S_i\rangle$. By (6.11) and (6.9), we have $\{n+i+1\}_{2i+1}/\{1\} = a_{n,i}\{2i+1\}_{2i}$, hence $a_{n,i} = \begin{bmatrix} n+i+1 \\ 2i+1 \end{bmatrix}$.

6.3.4. *Proof of Proposition 6.3.* For each $n \geq 0$ we have $\xi(S_n) = \sigma_n$. Therefore, by (6.8), (6.7) and (6.9), we have

$$\mathrm{tr}_q^{P_m}(\sigma_n) = \mathrm{tr}_q^{P_m}(\xi(S_n)) = \langle P_m, S_n\rangle = \delta_{m,n}\{2m+1\}_{2m},$$

hence the assertion.

7. Remarks on knot invariants

In this section we discuss some consequences of the results in Section 6. The reader may skip this section in the first reading, since it is not necessary for the proof of the existence of the invariants J_M for integral homology spheres.

Throughout this section, let T be a bottom knot and let $K = \mathsf{cl}(T)$ be the closure of T. By abuse of notation, we set $J_K = J_T \in Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$.

7.1. **The two-variable colored Jones invariant.** Part of the following overlaps [17, Section 3.3], where some results are stated without proofs.

It follows from the results in the previous sections that we have

$$J_K = J_T = \sum_{n\geq 0} J_K(P''_n)\sigma_n \in Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}}) \cong \varprojlim_k \mathbb{Z}[q, q^{-1}][C^2]/(\sigma_k).$$

Thus J_K may be regarded as an invariant with two variables q and C. It is useful to introduce variables t and α satisfying

$$\alpha^2 = C^2 - 4 = t + t^{-1} - 2.$$

Then $Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$ may be identified with

$$\Lambda := \varprojlim_k \mathbb{Z}[q,q^{-1}][t+t^{-1}]/(\sigma_k) \cong \varprojlim_k \mathbb{Z}[q,q^{-1}][\alpha^2]/(\sigma_k),$$

where

$$\sigma_k = \prod_{i=1}^k (t+t^{-1}-q^i-q^{-i}) = \prod_{i=1}^k (\alpha^2 - q^i - q^{-i}+2)$$

Remark 7.1. Λ can be naturally regarded as a subring of

$$\tilde{\Lambda} := \varprojlim_k \mathbb{Z}[q,q^{-1},t,t^{-1}]/(\sigma_k) \cong \varprojlim_k \mathbb{Z}[q,q^{-1},t,t^{-1}]/\Big(\prod_{-k\le i\le k,\ i\ne 0} (t-q^i)\Big).$$

In fact, Λ consists of the elements of $\tilde{\Lambda}$ which are invariant under the involutive continuous ring automorphism of $\tilde{\Lambda}$ defined by $f(q,t)\mapsto f(q,t^{-1})$.

By abuse of notation, we write

$$J_K(t,q) = J_K \in \Lambda.$$

For $i\in\mathbb{Z}$, let

$$\theta_i\colon\ \mathbb{Z}[q,q^{-1}][t+t^{-1}] \to \mathbb{Z}[q,q^{-1}]$$

be the unique $\mathbb{Z}[q,q^{-1}]$-algebra homomorphism such that $\theta_i(t+t^{-1}) = q^i+q^{-i}$. We have

$$\theta_i(\sigma_k) = \prod_{j=1}^k (q^i+q^{-i}-q^j-q^{-j}). \tag{7.1}$$

If $i\ne 0$, then θ_i induces a homomorphism

$$\hat{\theta}_i\colon\ \Lambda \to \mathbb{Z}[q,q^{-1}],$$

since it follows from (7.1) that $\theta_i(\sigma_j)=0$ for all $j\ge |i|$. Since we have $\theta_i=\theta_{-i}$, it follows that $\hat{\theta}_i = \hat{\theta}_{-i}$. By (6.5), it follows that for $i>0$

$$\theta_i(J_K) = J_K(q^i,q) = J_K(\mathsf{V}'_{i-1}), \tag{7.2}$$

where $\mathsf{V}'_{i-1} = \mathsf{V}_{i-1}/[i]$. Thus $J_K(q^i,q)\in\mathbb{Z}[q,q^{-1}]$ is the $(i-1)$st normalized colored Jones polynomial. Hence, it may be natural to call $J_K(t,q)\in\Lambda$ the *two-variable colored Jones invariant* of K. For the comparison with the well-known, two-variable power series expansion of the colored Jones polynomials, known as the Melvin-Morton expansion, see Section 7.2 below.

Now, consider the case $i=0$. θ_0 induces a continuous ring homomorphism

$$\hat{\theta}_0\colon\ \Lambda \to \widehat{\mathbb{Z}[q]},\quad f(t,q)\mapsto f(1,q),$$

since it follows from (7.1) that $\theta_0(\sigma_k) = (-1)^k(\{k\}!^2)\in\mathbb{Z}[q,q^{-1}]$ is divisible by ${(q)_k}^2$. For $\zeta\in\mathcal{Z}$ with $r=\mathrm{ord}(\zeta)$, by (7.2), we have

$$\mathrm{ev}_\zeta(J_K(1,q)) = J_K(1,\zeta) = J_K(\zeta^r,\zeta) = \mathrm{ev}_\zeta(J_K(q^r,q)) = \mathrm{ev}_\zeta(J_K(\mathsf{V}'_{r-1})).$$

The right-hand side is known as the *Kashaev invariant* of K at ζ [32, 64]. Hence, $J_K(1,q)\in\widehat{\mathbb{Z}[q]}$ may be regarded as a *unified Kashaev invariant*. Essentially the same observation has been made in [17, Section 3.3] (without proof). Using a different method, Huynh and Le [29] have given a proof of the existence of the unified Kashaev invariant in the above sense.

7.2. **Melvin-Morton expansions and Rozansky's rationality theorem.** Rozansky [74] proved that the colored Jones polynomials of a knot K can be repackaged into one invariant $J_K^{\mathrm{MMR}} \in \mathbb{Z}[[q-1,\alpha^2]]$, which is an integer-coefficient version of the Melvin-Morton expansion [61]. (In fact, Rozansky proved much stronger results there, see below.)

Let

$$\gamma\colon\ \Lambda \to \mathbb{Z}[[q-1,\alpha^2]]$$

be the ring homomorphism induced by $\mathrm{id}_{\mathbb{Z}[q,q^{-1},\alpha^2]}$. This is well-defined since $\sigma_{2k} \in (\alpha^{2k},(q-1)^{2k})$ in $\mathbb{Z}[q,q^{-1},\alpha^2]$ for $k \geq 0$. As mentioned in [17, Section 3.3], $\gamma(J_K) \in \mathbb{Z}[[q-1,\alpha^2]]$ is equal to the Rozansky's integral version of the Melvin-Morton expansion.

The first half of the following proposition was mentioned in [17] without proof, and implies that J_K and J_K^{MMR} have the same strength.

Proposition 7.2. *The map γ is injective and non-surjective.*

Proof. Let us prove injectivity. Let $f = \sum_{n=0}^{\infty} f_n(q)\sigma_n \in \Lambda$, where $f_n(q) \in \mathbb{Z}[q,q^{-1}]$ for $n \geq 0$, and suppose that $\gamma(f) = 0$. We can express each σ_n as

$$\sigma_n = \sum_{k=0}^{n} g_{n,k}(q)(2-q-q^{-1})^{2n-2k}\alpha^{2k},$$

where $g_{n,k}(q) \in \mathbb{Z}[q,q^{-1}]$. Hence, we obtain

$$\gamma(f) = \sum_{k=0}^{\infty}\Big(\sum_{n=k}^{\infty} f_n(q)g_{n,k}(q)(2-q-q^{-1})^{2n-2k}\Big)\alpha^{2k}.$$

Since $\gamma(f) = 0$, we have

$$\sum_{n=k}^{\infty} f_n(q)g_{n,k}(q)(2-q-q^{-1})^{2n-2k} = 0 \tag{7.3}$$

for all $k \geq 0$.

One can show that $g_{n,k}(q)$ is not divisible by $q-1$. (Indeed, one can use induction on n to show that $g_{n,k}(1) \in \mathbb{Z}$ is positive.) Using this fact and (7.3), one can prove using induction that for any $n, d \geq 0$, $f_n(q)$ is divisible by $(q-1)^d$. Hence $f_n(q) = 0$ for all n.

Non-surjectivity of γ can be verified in several ways. For example, let $u\colon \mathbb{Z}[[q-1,\alpha^2]] \to \mathbb{Z}[[q-1]]$ be the continuous $\mathbb{Z}[[q-1]]$-algebra homomorphism such that $u(\alpha^2) = 0$. Then we have

$$u\gamma(\Lambda) = \mathbb{Z}[q,q^{-1}] \subsetneq \mathbb{Z}[[q-1]] = u(\mathbb{Z}[[q-1,\alpha^2]]).$$

It follows that γ is not surjective. □

Rozansky [74] also proved that the coefficient of $(q-1)^n$ of the integral Melvin-Morton expansion of a knot K is a rational function in α^2 with denominator a power of the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t+t^{-1}]$ of K. (This result generalizes a conjecture of Melvin and Morton [61], proved in [2].) More precisely, Rozansky proved that J_K^{MMR} has the following power series expansion

$$J_K^{\mathrm{MMR}} = \sum_{n\geq 0} \frac{P_{K,n}(t)}{\Delta_K(t)^{2n+1}}(q-1)^n \in \mathbb{Z}[t+t^{-1}, \frac{1}{\Delta_K(t)}][[q-1]],$$

where $P_{K,n}(t) \in \mathbb{Z}[t+t^{-1}]$ for $n \geq 0$. The $P_{K,n}(t)$ are uniquely determined by K and n. Thus $\gamma(J_K)$ is contained in a much smaller ring than $\mathbb{Z}[[q-1,\alpha^2]]$. Melvin and Morton's conjecture (essentially) states that

$$J_K^{\mathrm{MMR}}\big|_{q=1} = \frac{1}{\Delta_K(t)}.$$

Thus we have $P_{K,0}(t) = 1$.

The following conjecture (supported by some computer calculations) generalizes Rozansky's rationality theorem.

Conjecture 7.3. *Let K be a knot and let $d\colon \mathbb{N} \to \{0,1,2,\ldots\}$ be a function which vanishes for all but finitely many elements of $\mathbb{N}$. Set*

$$\Delta_K^d(t) = \prod_{r\in\mathbb{N}, d(r)>0} \Delta_K(t^r)^{2d(r)-1} \in \mathbb{Z}[t+t^{-1}].$$

Then we have

$$\Delta_K^d(t) J_K \in \mathbb{Z}[t+t^{-1}, q, q^{-1}] + \Phi_d(q)\Lambda.$$

Conjecture 7.3 implies that J_K^{MMR} is contained in the image of the natural (injective) homomorphism

$$\varprojlim_k \mathbb{Z}[q, q^{-1}, t+t^{-1}, \frac{1}{\Delta_K(t^i)} \quad (i \geq 1)]/((q)_k) \to \mathbb{Z}[[q-1,\alpha^2]].$$

Conjecture 7.3 also implies the following conjecture, in which the case $\zeta = 1$ is Rozansky's rationality theorem.

Conjecture 7.4. *Let $\zeta \in \mathcal{Z}$ with $r = \mathrm{ord}(\zeta)$. Let*

$$\gamma_\zeta\colon \Lambda \to \mathbb{Z}[\zeta][[\alpha^2, q-\zeta]]$$

be the homomorphism induced by $\mathbb{Z}[q,q^{-1},\alpha^2] \subset \mathbb{Z}[\zeta][q,q^{-1},\alpha^2]$. Define $P_{K,\zeta,k}(t) \in \mathbb{Z}[\zeta][[\alpha^2]]$ for $k \geq 0$ by

$$\gamma_\zeta(J_K) = \sum_{k\geq 0} \frac{P_{K,\zeta,k}(t)}{\Delta_K(t^r)^{2k+1}}(q-\zeta)^k.$$

Then we have $P_{K,\zeta,k}(t) \in \mathbb{Z}[\zeta][t+t^{-1}]$.

Using the injectivity of $\gamma = \gamma_1$, one can prove that the homomorphism γ_ζ is injective.

Conjecture 7.4 implies that

$$J_K(t,\zeta) = \frac{P_{K,\zeta,0}(t)}{\Delta_K(t^r)} \in \frac{1}{\Delta_K(t^r)}\mathbb{Z}[\zeta][t+t^{-1}].$$

Some computer calculations support the following conjecture.

Conjecture 7.5. *We have*

$$J_K(t,-1) = \frac{\Delta_K(-t)}{\Delta_K(t^2)}.$$

Thus we have $P_{K,-1,0}(t) = \Delta_K(-t)$ in the notation of Conjecture 7.4.

Presumably, $P_{K,\zeta_r,0}(t)$ with $\zeta_r = \exp\frac{2\pi\sqrt{-1}}{r}$ is the rth Akutsu-Deguchi-Ohtsuki invariant [1], after suitable change of variables. (Recall that the second Akutsu-Deguchi-Ohtsuki invariant of a knot is the Alexander polynomial [65].) If this is true, then it follows that the Akutsu-Deguchi-Ohtsuki invariants of a knot K determine J_K, hence also the colored Jones polynomials $J_K(\mathsf{V}_m)$, $m \geq 0$.

8. Algebraically-split links

In this section, we prove an integrality result for the colored Jones polynomials of algebraically-split, 0-framed links. We also give some conjectures for boundary links in the last subsection.

8.1. The algebra $\mathcal{P}$ and its completion $\hat{\mathcal{P}}$.

For $n \geq 0$, set

$$P'_n = \frac{P_n}{\{n\}!} \in \mathcal{R}_{\mathbb{Q}(v)},$$
$$\tilde{P}'_n = v^{-\frac{1}{2}n(n-1)} P'_n \in \mathcal{R}_{\mathbb{Q}(v)}.$$

We have $P'_0 = 1$. Define a $\mathbb{Z}[q, q^{-1}]$-submodule $\mathcal{P}$ of $\mathcal{R}_{\mathbb{Q}(v)}$ by

$$\mathcal{P} = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{\tilde{P}'_n \mid n \geq 0\}.$$

Lemma 8.1. *$\mathcal{P}$ is a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathcal{R}_{\mathbb{Q}(v)}$.*

Proof. By induction, we obtain

$$P_m P_n = \sum_{i=0}^{\min(m,n)} \frac{\{m\}_i \{n\}_i \{m+n\}_i}{\{i\}!} P_{m+n-i}.$$

Hence,

$$P'_m P'_n = \sum_{i=0}^{\min(m,n)} \frac{\{m+n\}!}{\{i\}!\{m-i\}!\{n-i\}!} P'_{m+n-i}. \tag{8.1}$$

The coefficient in the right-hand side is contained in $v^d\mathbb{Z}[q, q^{-1}]$, where $d \equiv \binom{m+n+1}{2} - \binom{i+1}{2} - \binom{m-i+1}{2} - \binom{n-i+1}{2} \equiv \binom{i+1}{2} \pmod 2$. Since $\binom{m}{2} + \binom{n}{2} - \binom{m+n-i}{2} \equiv \binom{i+1}{2} \pmod 2$, we have $v^{\binom{m}{2}} P'_m v^{\binom{n}{2}} P'_n \in \mathcal{P}$. Therefore, we have $\mathcal{P}\mathcal{P} \subset \mathcal{P}$. Since we have $1 = P'_0 \in \mathcal{P}$, we have the assertion. ∎

For $k \geq 0$, set

$$\mathcal{P}_k = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{\tilde{P}'_n \mid n \geq k\} \subset \mathcal{P}.$$

We have

$$\mathcal{P} = \mathcal{P}_0 \supset \mathcal{P}_1 \supset \mathcal{P}_2 \supset \cdots.$$

It follows from (8.1) that each $\mathcal{P}_k$ is an ideal in $\mathcal{P}$. Set

$$\hat{\mathcal{P}} = \varprojlim_{k \geq 0} \mathcal{P}/\mathcal{P}_k,$$

which inherits a $\mathbb{Z}[q, q^{-1}]$-algebra structure from $\mathcal{P}$. Each element $x \in \hat{\mathcal{P}}$ can be uniquely expressed as an infinite sum $x = \sum_{k=0}^{\infty} x_k \tilde{P}'_k$, where $x_k \in \mathbb{Z}[q, q^{-1}]$ for $k \geq 0$. Since $\bigcap_{k\geq 0} \mathcal{P}_k = \{0\}$, we may regard $\mathcal{P}$ as a subalgebra of $\hat{\mathcal{P}}$.

8.2. **Integrality for algebraically-split, 0-framed links.** The following theorem is proved in Section 8.3.

Theorem 8.2. *Let L be an m-component, algebraically-split, 0-framed link. For $i = 1, \dots, m$, let $x_i \in \mathcal{P}_{k_i}$, $k_i \geq 0$. Then we have*

$$J_L(x_1, \dots, x_m) \in \frac{\{2k+1\}_{q,k+1}}{\{1\}_q}\mathbb{Z}[q, q^{-1}],$$

where $k = \max(k_1, \dots, k_m)$.

Theorem 8.2 immediately implies the following.

Corollary 8.3. *Let L be an m-component, algebraically-split, 0-framed link. Then the $\mathbb{Q}(v)$-multilinear map $J_L\colon \mathcal{R}_{\mathbb{Q}(v)} \times \cdots \times \mathcal{R}_{\mathbb{Q}(v)} \to \mathbb{Q}(v)$ restricts to a $\mathbb{Z}[q, q^{-1}]$-multilinear map*

$$J_L\colon \mathcal{P} \times \cdots \times \mathcal{P} \to \mathbb{Z}[q, q^{-1}],$$

which induces a $\mathbb{Z}[q, q^{-1}]$-multilinear map

$$J_L\colon \hat{\mathcal{P}} \times \cdots \times \hat{\mathcal{P}} \to \widehat{\mathbb{Z}[q]}.$$

Remark 8.4. It follows from (6.2) that for an m-component, algebraically-split, 0-framed link L, the colored Jones polynomial $J_L(\mathsf{V}_{n_1}, \dots, \mathsf{V}_{n_m})$, $n_1, \dots, n_m \geq 0$, can be expressed as a linear combination of the $J_L(P'_{k_1}, \dots, P'_{k_m})$, $k_1, \dots, k_m \geq 0$ as follows:

$$J_L(\mathsf{V}_{n_1}, \dots, \mathsf{V}_{n_m}) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} \prod_{i=1}^{m} \Big(\begin{bmatrix} n_i + k_i + 1 \\ 2k_i + 1 \end{bmatrix} \{k_i\}! \Big) J_L(P'_{k_1}, \dots, P'_{k_m}). \tag{8.2}$$

Here the sums may be replaced with infinite sums $\sum_{k_i=0}^{\infty}$.

8.3. **Proof of Theorem 8.2.** The following lemma is proved in the next subsection.

Lemma 8.5. *If $x \in \mathcal{U}_q^{\mathrm{ev}}$ and $y \in \mathcal{P}$, then we have $\mathrm{tr}_q^y(x) \in \mathbb{Z}[q, q^{-1}]$.*

Using Lemma 8.5, we obtain the following generalization of (4.2).

Proposition 8.6. *If T is an m-component, algebraically-split, 0-framed bottom tangle, and if $x_2, \dots, x_m \in \mathcal{P}$, then we have*

$$(1 \otimes \mathrm{tr}_q^{x_2} \otimes \cdots \otimes \mathrm{tr}_q^{x_m})(J_T) \in Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}}).$$

Proof. By Lemma 8.5,

$$(1 \otimes \mathrm{tr}_q^{x_2} \otimes \cdots \otimes \mathrm{tr}_q^{x_m})((\mathcal{U}_q^{\mathrm{ev}})^{\otimes m}) \subset \mathcal{U}_q^{\mathrm{ev}}. \tag{8.3}$$

Let $N \geq 0$ be such that

$$x_2, \dots, x_m \in \mathrm{Span}_{\mathbb{Q}(v)}\{P'_0, P'_1, \dots, P'_N\} = \mathrm{Span}_{\mathbb{Q}(v)}\{\mathsf{V}_0, \mathsf{V}_1, \dots, \mathsf{V}_N\}. \tag{8.4}$$

If $p > N$ then we have $\mathrm{tr}_q^{x_i}(\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})) = 0$ for $i = 2, \dots, m$, since e^p acts as zero on $\mathsf{V}_0, \mathsf{V}_1, \dots, \mathsf{V}_N$. Therefore, we have

$$(1 \otimes \mathrm{tr}_q^{x_2} \otimes \cdots \otimes \mathrm{tr}_q^{x_m})(\mathcal{F}_p((\mathcal{U}_q^{\mathrm{ev}})^{\otimes m})) \subset \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}). \tag{8.5}$$

Now, (8.3), (8.5) and Theorem 4.1 imply the assertion. □

Proof of Theorem 8.2. We may assume $k_1 = k = \max(k_1, \dots, k_m)$ without loss of generality. (Otherwise, change the order of components of L.) Choose a bottom tangle $T \in \mathsf{BT}_m$ such that $\mathsf{cl}(T) = L$. By Proposition 8.6,

$$J_L(x_1, \dots, x_m) = (\mathrm{tr}_q^{x_1} \otimes \cdots \otimes \mathrm{tr}_q^{x_m})(J_T) = \mathrm{tr}_q^{x_1}(y),$$

where we set $y = (1 \otimes \mathrm{tr}_q^{x_2} \otimes \cdots \otimes \mathrm{tr}_q^{x_m})(J_T)$, which is contained $Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}})$ by Proposition 8.6. By the arguments in Section 4.4,

$$y = \sum_{p \ge 0} a_p \sigma_p,$$

where $a_p \in \mathbb{Z}[q, q^{-1}]$ for $p \ge 0$. Moreover, we have

$$x_1 = \sum_{p=k}^{N} b_p \tilde{P}'_p,$$

where $b_p \in \mathbb{Z}[q, q^{-1}]$ and $N \ge k$. Therefore, we have

$$\mathrm{tr}_q^{x_1}(y) = \sum_{p=k}^{N} a_p b_p \mathrm{tr}_q^{\tilde{P}'_p}(\sigma_p).$$

By Proposition 6.3, we have $\mathrm{tr}_q^{\tilde{P}'_p}(\sigma_p) \in \frac{\{2p+1\}_{q,p+1}}{\{1\}_q}\mathbb{Z}[q, q^{-1}]$, hence the assertion. □

8.4. **Proof of Lemma 8.5.** To prove Proposition 8.5, we need some lemmas.

Lemma 8.7. *If $y, y' \in \mathcal{R}_{\mathbb{Q}(v)}$ and $x \in U_h$, then we have*

$$\mathrm{tr}_q^{yy'}(x) = \sum \mathrm{tr}_q^{y}(x_{(1)}) \mathrm{tr}_q^{y'}(x_{(2)}),$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

Proof. The assertion follows from the well-known identity for any two finite-dimensional representations W, W'

$$\mathrm{tr}_q^{W \otimes W'}(x) = (\mathrm{tr}_q^{W} \otimes \mathrm{tr}_q^{W'})\Delta(x).$$

□

Lemma 8.8. *(1) If $n, l, l' \ge 0$, $l \ne l'$ and $j \in \mathbb{Z}$, then we have $\mathrm{tr}_q^{P_n}(\tilde{F}^{(l)} K^{2j} e^{l'}) = 0$.*

(2) If $0 \le n < l$ and $j \in \mathbb{Z}$, then we have $\mathrm{tr}_q^{P_n}(\tilde{F}^{(l)} K^{2j} e^{l}) = 0$.

(3) If $0 \le l \le n$ and $j \in \mathbb{Z}$, then we have

$$\mathrm{tr}_q^{P_n}(\tilde{F}^{(l)} K^{2j} e^{l}) = v^n q^{-nj+2lj+l^2-ln} \{n\}_q! \{n-l\}_q! \begin{bmatrix} j+n-1 \\ n-l \end{bmatrix}_q \begin{bmatrix} j-1 \\ n-l \end{bmatrix}_q. \tag{8.6}$$

Proof. The assertion (1) follows from the fact that $l \ne l'$ implies that $\tilde{F}^{(l)} K^{2j} e^{l'}$ acts nilpotently on each V_n, $n \ge 0$.

The assertion (2) follows from the fact that P_n is a linear combination of $\mathsf{V}_0, \dots, \mathsf{V}_n$, and that if $0 \le n < l$, then e^l acts as 0 on $\mathsf{V}_0, \dots, \mathsf{V}_n$.

We prove (3) by induction on n. The case $n = l = 0$ follows from

$$\mathrm{tr}_q^{P_0}(K^{2j}) = \epsilon(K^{2j}) = 1.$$

Let $n \geq 1$. Set $p_n = \mathsf{V}_1 - v^{2n-1} - v^{-2n+1}$, so that $P_n = P_{n-1}p_n$. Using Lemma 8.7, we have

$$\operatorname{tr}_q^{P_n}(\tilde{F}^{(l)}K^{2j}e^l) = \operatorname{tr}_q^{P_{n-1}p_n}(\tilde{F}^{(l)}K^{2j}e^l) = \sum \operatorname{tr}_q^{P_{n-1}}(x_{(1)})\operatorname{tr}_q^{p_n}(x_{(2)}), \tag{8.7}$$

where

$$\begin{aligned}\sum x_{(1)} \otimes x_{(2)} &= \Delta(\tilde{F}^{(l)}K^{2j}e^l)\\ &= \sum_{r=0}^{l}\sum_{s=0}^{l} \begin{bmatrix} l \\ s \end{bmatrix}_q (\tilde{F}^{(l-r)}K^rK^{2j}e^{l-s}K^s \otimes \tilde{F}^{(r)}K^{2j}e^s).\end{aligned}$$

Since p_n is a linear combination of V_0 and V_1, we have $\operatorname{tr}_q^{p_n}(F^{(r)}K^{2j}e^s) = 0$ unless $(r,s) = (0,0)$ or $(1,1)$. Therefore, the right-hand side of (8.7) equals

$$\operatorname{tr}_q^{P_{n-1}}(\tilde{F}^{(l)}K^{2j}e^l)\operatorname{tr}_q^{p_n}(K^{2j}) + q^{-l+1}[l]_q\operatorname{tr}_q^{P_{n-1}}(\tilde{F}^{(l-1)}K^{2j+2}e^{l-1})\operatorname{tr}_q^{p_n}(\tilde{F}^{(1)}K^{2j}e). \tag{8.8}$$

By straightforward computation,

$$\begin{aligned}\operatorname{tr}_q^{p_n}(K^{2j}) &= v^{-2j+1}\{j-n\}_q\{j+n-1\}_q,\\ \operatorname{tr}_q^{p_n}(\tilde{F}^{(1)}K^{2j}e) &= v^{2j+1}\{1\}_q.\end{aligned}$$

Using these identities and the inductive hypothesis, we can verify that (8.8) equals the right-hand side of (8.6). This completes the proof. □

Proof of Lemma 8.5. It suffices to prove that if $n \geq 0$ and $x \in \mathcal{U}_q^{\mathrm{ev}}$, then we have $\operatorname{tr}_q^{P'_n}(x) \in v^{\frac{1}{2}n(n-1)}\mathbb{Z}[q,q^{-1}]$. We may assume without loss of generality that $x = \tilde{F}^{(l)}K^{2j}e^k$ with $j \in \mathbb{Z}$, $k,l \geq 0$. By Lemma 8.8, $\operatorname{tr}_q^{P_n}(x) \in v^n\{n\}_q!\mathbb{Z}[q,q^{-1}] = v^{\frac{1}{2}n(n-1)}\{n\}!\mathbb{Z}[q,q^{-1}]$. Therefore, we have the assertion. □

8.5. **Remarks on boundary links and boundary bottom tangles.** In this subsection, we give some remarks on the case of boundary links. The reader may skip this subsection since it is not necessary in the rest of the paper.

Let $\bar{U}_q^{\mathrm{ev}}$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathcal{U}_q^{\mathrm{ev}}$ generated by the elements K^2, K^{-2}, e and $f := (q-1)FK$. Define a decreasing filtration $\mathcal{F}_k(\bar{U}_q^{\mathrm{ev}})$, $k \geq 0$, by

$$\mathcal{F}_k(\bar{U}_q^{\mathrm{ev}}) = \bar{U}_q^{\mathrm{ev}} \cap \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) = \bar{U}_q^{\mathrm{ev}} \cap \mathcal{F}_p(\mathcal{U}_q).$$

Let $(\bar{U}_q^{\mathrm{ev}})\breve{}$ denote the completion in U_h of $\bar{U}_q^{\mathrm{ev}}$ with respect to the filtration $\mathcal{F}_k(\bar{U}_q^{\mathrm{ev}})$, i.e., we set

$$(\bar{U}_q^{\mathrm{ev}})\breve{} = \operatorname{Image}\Big(\varprojlim_k \bar{U}_q^{\mathrm{ev}}/\mathcal{F}_k(\bar{U}_q^{\mathrm{ev}}) \to U_h\Big).$$

Clearly, $(\bar{U}_q^{\mathrm{ev}})\breve{}$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$. In the similar way to the case of $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes}n}$, we can define the completed tensor products $(\bar{U}_q^{\mathrm{ev}})\breve{}^{\,\tilde{\otimes}n} = (\bar{U}_q^{\mathrm{ev}})\breve{}\,\tilde{\otimes}\cdots\tilde{\otimes}(\bar{U}_q^{\mathrm{ev}})\breve{} \subset U_h^{\hat{\otimes}n}$. Let $\operatorname{Inv}((\bar{U}_q^{\mathrm{ev}})\breve{}^{\,\tilde{\otimes}n})$ denote the invariant part of $(\bar{U}_q^{\mathrm{ev}})\breve{}^{\,\tilde{\otimes}n}$. We have

$$\operatorname{Inv}((\bar{U}_q^{\mathrm{ev}})\breve{}^{\,\tilde{\otimes}1}) = \operatorname{Inv}((\bar{U}_q^{\mathrm{ev}})\breve{}) = Z((\bar{U}_q^{\mathrm{ev}})\breve{}) = Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}}).$$

Here one can easily verify the last identity using [19, Section 9].

A bottom tangle $T = T_1 \cup \cdots \cup T_n$ with 0-framings in a cube $[0,1]^3$ is called a *boundary bottom tangle* if there are n disjoint, connected, oriented surface $F_1, \ldots, F_n$ in $[0,1]^3$ such that for each $i = 1, \ldots, n$ the boundary ∂F_i of F_i is the union of T_i and the line segment in $\partial[0,1]^3$ bounded by the two endpoints of T_i. (See also [20, Section 9.3].)

The following conjecture generalizes the case $n = 1$ of Theorem 4.1. (Note that a bottom knot is a 1-component, boundary bottom tangle.)

Conjecture 8.9. *Let $T \in \mathsf{BT}_n$ be an n-component, boundary bottom tangle. Then we have $J_T \in \mathrm{Inv}((\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n})$.*

Here we give an outline of a possible proof of Conjecture 8.9. We have not worked out all the details yet, but the following idea seems useful. The main tool is [20, Theorem 9.9], which reduces the problem to showing that $Y^{\otimes g}(J_{T'}) \in (\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} g}$ for any bottom tangle $T' \in \mathsf{BT}_{2g}$ (with any linking matrix), $g \geq 0$. Here $Y\colon U_h \hat{\otimes} U_h \to U_h$ is the commutator morphism for the transmutation of U_h (see [20, Section 9.3] for the definition).

Conjecture 8.9 implies the following variant of Theorem 8.2.

Conjecture 8.10. *Let L be an m-component, 0-framed, boundary link. For $i = 1, \ldots, m$, let $x_i \in \mathcal{P}_{k_i}$, $k_i \geq 0$. Then we have*

$$J_L(x_1, \ldots, x_m) \in \frac{\{2k_1+1\}_{q,k_1+1}}{\{1\}_q} I_{k_2} I_{k_3} \cdots I_{k_m}, \tag{8.9}$$

Here, for $k \geq 0$, I_k is the ideal in $\mathbb{Z}[q, q^{-1}]$ generated by the elements $\{k-l\}_q!\{l\}_q!$, $l = 0, \ldots, k$. (One can permute $k_1, \ldots, k_m$ in (8.9).)

Proof that Conjecture 8.9 implies Conjecture 8.10. We can express L as the closure of an m-component, boundary bottom tangle T. We have

$$y := (\mathrm{id} \otimes \mathrm{tr}_q^{x_2} \otimes \cdots \otimes \mathrm{tr}_q^{x_m})(J_T) \in I_{k_2} I_{k_3} \cdots I_{k_m} (\bar{U}_q^{\mathrm{ev}})\tilde{},$$

since J_T is an infinite sum of elements of $(\bar{U}_q^{\mathrm{ev}})^{\otimes m}$ by Conjecture 8.9 and we have $\mathrm{tr}_q^{x_i}(\bar{U}_q^{\mathrm{ev}}) \subset I_{k_i}$ by Lemma 8.8. By (5.4), we have

$$J_L(x_1, \ldots, x_m) = (\mathrm{tr}_q^{x_1} \otimes \cdots \otimes \mathrm{tr}_q^{x_m})(J_T) = \mathrm{tr}_q^{x_1}(y).$$

The rest of the proof is similar to that for Theorem 8.2. □

Remark 8.11. One can prove partial results of Conjecture 8.10 by focusing on the divisibility of link invariants by powers of $q-1$ or $q+1$. It is not difficult to prove using the theory of Goussarov-Vassiliev finite type link invariants that if L is an m-component, boundary link, then $J_L(P_{k_1}, \ldots, P_{k_m})$ is divisible by $(q-1)^{2(k_1+\cdots+k_m)}$ (hence $J_L(P'_{k_1}, \ldots, P'_{k_m})$ is divisible by $(q-1)^{k_1+\cdots+k_m}$). One can also prove that $J_L(P_{k_1}, \ldots, P_{k_m})$ is divisible by $(q+1)^{k_1+\cdots+k_m}$. The latter assertion follows from the fact that the Jones polynomial (with colors V_1) of n-component, boundary link is divisible by $(q+1)^n$, which we plan to prove in a paper in preparation [22] using skein theory.

9. Twists

In this section, we introduce an element $\omega \in \hat{\mathcal{P}}$ satisfying a "twisting property".

9.1. The twist element $\omega \in \hat{\mathcal{P}}$.

Define two elements $\omega_+, \omega_- \in \hat{\mathcal{P}}$ by

$$\omega_\pm = \sum_{n=0}^{\infty} (\pm 1)^n v^{\pm\frac{1}{2}n(n+3)} P'_n.$$

Set $\mathcal{S}_{\mathbb{Q}(v)} = \mathcal{S} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(v) \subset \mathcal{R}_{\mathbb{Q}(v)}$. By Proposition 6.6, the bilinear map $\langle , \rangle \colon \mathcal{P} \times \mathcal{S}_{\mathbb{Q}(v)} \to \mathbb{Q}(v)$ induces a bilinear map

$$\langle , \rangle \colon \hat{\mathcal{P}} \times \mathcal{S}_{\mathbb{Q}(v)} \to \mathbb{Q}(v).$$

Proposition 9.1. *For each $x \in \mathcal{S}_{\mathbb{Q}(v)}$, we have*

$$\langle \omega_\pm, x \rangle = J_{U_\pm}(x), \tag{9.1}$$

where $U_\pm$ is a ± 1-framed unknot.

Proof. We may assume $x = \mathsf{V}'_{2k} = \mathsf{V}_{2k}/[2k+1]$, $k \geq 0$, without loss of generality.
By (6.10), we have

$$\langle \omega_\pm, \mathsf{V}'_{2k} \rangle = \sum_{n \geq 0} (\pm 1)^n v^{\pm \frac{1}{2} n(n+3)} \langle P'_n, \mathsf{V}'_{2k} \rangle = \sum_{n=0}^{k} (\pm 1)^n v^{\pm \frac{1}{2} n(n+3)} \frac{\{k+n\}_{2n}}{\{n\}!}. \tag{9.2}$$

Since V_{2k} is an irreducible U_h-module, we have $\mathbf{r}^{\mp 1} \mathbf{v}_i^{2k} = J_{U_\pm}(\mathsf{V}'_{2k}) \mathbf{v}_i^{2k}$ for all $i = 0, \ldots, 2k$. (Recall that $\mathbf{r} \in U_h$ denotes the ribbon element, and $\mathbf{v}_i^{2k}$ for $i = 0, \ldots, 2k$ denote the basis elements of the irreducible left U_h-module V_{2k}.) Consider the case $i = k$. By straightforward calculations using (3.8) and (3.9), we obtain

$$\mathbf{r}^{\mp 1} \mathbf{v}_k^{2k} = \sum_{n=0}^{k} (\pm 1)^n v^{\pm \frac{1}{2} n(n+3)} \frac{\{k+n\}_{2n}}{\{n\}!} \mathbf{v}_k^{2k}.$$

Hence, $J_{U_\pm}(\mathsf{V}'_{2k})$ is equal to the right-hand side of (9.2). Hence, we have (9.1) for $x = \mathsf{V}'_{2k}$. □

Proposition 9.2. *ω_+ and ω_- are inverse to each other in the algebra $\hat{\mathcal{P}}$.*

Proof. A direct proof using (8.1) is possible. Here we give another proof.
By Proposition 9.1 and the fact that $\langle -, \mathsf{V}'_{2p} \rangle \colon \mathcal{S}_{\mathbb{Q}(v)} \to \mathbb{Q}(v)$ is a $\mathbb{Q}(v)$-algebra homomorphism, we have

$$\langle \omega_+ \omega_-, \mathsf{V}'_{2p} \rangle = \langle \omega_+, \mathsf{V}'_{2p} \rangle \langle \omega_-, \mathsf{V}'_{2p} \rangle = q^{p(p+1)} q^{-p(p+1)} = 1 = \langle 1, \mathsf{V}'_{2p} \rangle$$

for each $p \geq 0$. Hence, we have $\langle \omega_+ \omega_-, x \rangle = \langle 1, x \rangle$ for all $x \in \mathcal{S}_{\mathbb{Q}(v)}$. Since the map

$$\hat{\mathcal{P}} \to \mathrm{Fun}(\mathcal{S}_{\mathbb{Q}(v)}, \mathbb{Q}(v)), \quad x \mapsto (y \mapsto \langle x, y \rangle),$$

is injective (by Proposition 6.6), it follows that $\omega_+ \omega_- = 1$. (Here $\mathrm{Fun}(\mathcal{S}_{\mathbb{Q}(v)}, \mathbb{Q}(v))$ denotes the set of functions on $\mathcal{S}_{\mathbb{Q}(v)}$ with values in $\mathbb{Q}(v)$.) □

Set $\omega = \omega_+$. We have $\omega_- = \omega^{-1}$.

Remark 9.3. The twist element ω was announced first in [16] in a formulation using skein theory, and then in [17] in the present form (without proofs). Later, Masbaum [58] gave a proof using skein theory.

9.2. **Twisting theorem.**

Theorem 9.4. *Let $L_1 \cup \cdots \cup L_m \cup K$ be an $(m+1)$-component, algebraically-split, 0-framed link such that K is an unknot. Set $L = L_1 \cup \cdots \cup L_m$, and let $L_{(K,\pm 1)}$ denote the framed link in S^3 obtained from L by ± 1-framed surgery along K. Then, for any $x_1, \ldots, x_m \in \hat{\mathcal{P}}$, we have*

$$J_{L \cup K}(x_1, \ldots, x_m, \omega^{\mp 1}) = J_{L_{(K,\pm 1)}}(x_1, \ldots, x_m). \tag{9.3}$$

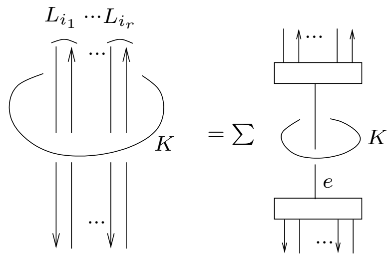

FIGURE 9.1. A fusing identity.

Proof. By Theorem 8.2, it suffices to prove (9.3) for $x_1, \dots, x_m \in \mathcal{P}$. In fact, we will show (9.3) for $x_1, \dots, x_m \in \mathcal{R}_{\mathbb{Q}(v)}$. (Well-definedness of the left-hand side of (9.3) in this case follows from the argument below.) Apply the standard "fusing" argument to the strands running through the unknot K, see Figure 9.1. In the left-hand side, the strands running through K are paired in such a way that each pair contains two segments from the same component of L in the opposite orientations. By fusing these strands, we obtain a linear combination of trivalent colored graphs, where each term has exactly one edge e running through K, colored by an even number. Hence, there is $y \in \mathcal{S}_{\mathbb{Q}(v)}$ such that

$$J_{L\cup K}(x_1, \dots, x_m, \omega^{\mp 1}) = J_H(y, \omega^{\mp 1}) = \langle \omega^{\mp 1}, y\rangle,$$

where H denotes the Hopf link with 0-framings. By Proposition 9.1, the right-hand side is equal to $J_{U_\mp}(y)$, where $U_\mp$ denotes the ∓ 1-framed unknot. Since we have

$$J_{U_\mp}(y) = J_{L_{(K,\pm 1)}}(x_1, \dots, x_m),$$

we have (9.3). □

10. THE INVARIANT J_M

In this section, we construct an invariant $J_M \in \widehat{\mathbb{Z}[q]}$ of an integral homology sphere M.

10.1. **Admissible framed links and the Hoste moves.** A framed link L in S^3 is said to be *admissible* if L is algebraically split and ± 1-framed. Surgery on S^3 along an admissible framed link yields an integral homology sphere. Conversely, each integral homology sphere M can be obtained as the result of surgery on S^3 along an admissible framed link.

In the definition of the Reshetikhin-Turaev invariant [73] and its variants, one uses Kirby's theorem [37] or its variant by Fenn and Rourke [6]. Fenn and Rourke's theorem states that two framed links in S^3 yields orientation-preserving homeomorphic results of surgery if and only if they are related by a sequence of isotopies and *Fenn-Rourke moves*, where a Fenn-Rourke move is either surgery on a unknotted, ± 1-framed component, or the inverse operation. See Figure 10.1.

We use the following version of Fenn-Rourke's theorem, which was conjectured by Hoste [28].

Theorem 10.1 ([21]). *Two admissible framed links L and L' in S^3 yield orientation-preserving homeomorphic results of surgery if and only if L and L' are related by*

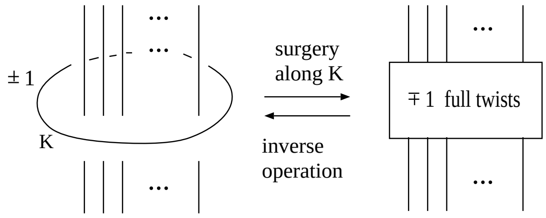

FIGURE 10.1. A Fenn-Rourke move.

a sequence of isotopies and Hoste moves. *Here a Hoste move is defined to be a Fenn-Rourke move between two admissible framed links.*

10.2. **Definition of J_M.** Let M be an integral homology sphere. Choose an admissible framed link $L = L_1 \cup \cdots \cup L_m$ in S^3 such that $S^3_L \cong M$. For $i = 1, \dots, m$, let $f_i = \pm 1$ denote the framing of L_i. Let $L^0 = L^0_1 \cup \cdots \cup L^0_m$ denote the link L with 0 framings. Set

$$J_M = J_{L^0}(\omega^{-f_1}, \omega^{-f_2}, \dots, \omega^{-f_m}) \in \widehat{\mathbb{Z}[q]}. \tag{10.1}$$

Theorem 10.2. *For an integral homology sphere M, the element $J_M \in \widehat{\mathbb{Z}[q]}$ defined in* (10.1) *does not depend on the choice of L. Hence, the correspondence $M \mapsto J_M$ defines an invariant of integral homology spheres with values in $\widehat{\mathbb{Z}[q]}$.*

Proof. Let $I(L)$ denote the right-hand side of (10.1). Clearly, $I(L)$ is an isotopy invariant of admissible framed links. By Theorem 10.1, it suffices to prove that $I(L)$ is invariant under Hoste moves. Observe that $I(L)$ is invariant under permutation of components. Therefore, it suffices to show that if the last component L_m of L is an unknot, then

$$J_{L^0}(\omega^{-f_1}, \dots, \omega^{-f_{m-1}}, \omega^{-f_m}) = J_{(L^0 \setminus L^0_m)_{L_m}}(\omega^{-f_1}, \dots, \omega^{-f_{m-1}}),$$

where $(L^0 \setminus L^0_m)_{L_m}$ is obtained from $L^0 \setminus L^0_m$ by performing surgery along L_m with framing f_m. This identity follows from Theorem 9.4. Hence the assertion. □

In Section 11.4, we give an alternative proof of Theorem 10.2.

10.3. **Some remarks.** If M and L are as above, then we have the following formula for J_M:

$$J_M = \sum_{k_1,\dots,k_m=0}^{\infty} \Big(\prod_{i=1}^{m} (-f_i)^{k_i} v^{\frac{-f_i}{2} k_i(k_i+3)} \Big) J_{L^0}(P'_{k_1}, \dots, P'_{k_m}). \tag{10.2}$$

In particular, if L is a knot (i.e., $m = 1$), then we have

$$J_M = J_{L^0}(\omega^{\mp 1}) = \sum_{k=0}^{\infty} (\mp 1)^k v^{\mp \frac{1}{2} k(k+3)} \frac{\{2k+1\}_{k+1}}{\{1\}} J_{L^0}(P''_k). \tag{10.3}$$

In (10.2), the term for $k_1 = \cdots = k_m = 0$ is

$$J_{L^0}(P'_0, \dots, P'_0) = J_{L^0}(1, \dots, 1) = 1.$$

It follows from Theorem 8.2 that the other terms are divisible by $(q^2-1)(q^3-1)/(q-1)$ in $\mathbb{Z}[q, q^{-1}]$. Hence, we have the following result, which we improve in Section 12.4.2.

Lemma 10.3. *Let M be an integral homology sphere. Then $J_M - 1$ is divisible by $(q^2-1)(q^3-1)/(q-1)$ in $\widehat{\mathbb{Z}[q]}$.*

Remark 10.4. It follows from Theorem 8.2 that

$$J_M \mod I_k \in \widehat{\mathbb{Z}[q]}/I_k \cong \mathbb{Z}[q]/I_k,$$

where $I_k = \frac{\{2k+1\}_{q,k+1}}{\{1\}_q}\mathbb{Z}[q,q^{-1}]$, is a finite linear combination of $J_{L^0}(P'_{k_1},\dots,P'_{k_m})$ for $k_1,\dots,k_m \in \{0,1,\dots,k-1\}$, hence is a finite linear combination of the colored Jones polynomial $J_{L^0}(\mathsf{V}_{k_1},\dots,\mathsf{V}_{k_m})$ for $k_1,\dots,k_m \in \{0,1,\dots,k-1\}$. In particular, for $k=2$, $J_M \mod I_2$ is a linear combination of the Jones polynomials (with colors V_1) of all the sublinks of L^0.

11. Specializations at roots of unity

In this section, we prove the following.

Theorem 11.1. *Let M be an integral homology sphere and let ζ be a primitive rth root of unity, $r \in \mathbb{N}$. Then we have*

$$\mathrm{ev}_\zeta(J_M) = \tau_\zeta(M), \tag{11.1}$$

where $\tau_\zeta(M)$ denotes the sl_2 WRT invariant of M at ζ.

11.1. **Definition of $\tau_\zeta(M)$.** We briefly recall the definition of the sl_2 WRT invariant $\tau_\zeta(M)$ of a closed 3-manifold M at $\zeta \in \mathcal{Z}$. For the details, see [73, 38].

If $\zeta = 1$, then we set $\tau_\zeta(M) = 1$ for any closed 3-manifold M.

Let $r \geq 2$ and let $\zeta^{1/4}$ be a primitive $4r$th root of unity. Thus ζ is a primitive rth root of unity. (The following construction depends not only on ζ but also $\zeta^{1/2}$ or $\zeta^{1/4}$, but we suppress them from the notation. For integral homology spheres, $\tau_\zeta(M)$ depends only on ζ and M, not on $\zeta^{1/4}$.)

Let

$$A_r = \mathbb{Z}[q^{1/4},q^{-1/4}]_{(\Phi_{4r}(q^{1/4}))},$$

denote the localization of the ring $\mathbb{Z}[q^{1/4},q^{-1/4}]$ by the principal ideal generated by the $4r$th cyclotomic polynomial $\Phi_{4r}(q^{1/4})$ in $q^{1/4}$ (which is equal to $\Phi_{2r}(q^{1/2})$). We have

$$A_r = \Big\{ \frac{f(q^{1/4})}{g(q^{1/4})} \in \mathbb{Q}(q^{1/4}) \;\Big|\; f(q^{1/4}), g(q^{1/4}) \in \mathbb{Z}[q^{1/4}],\ g(\zeta^{1/4}) \neq 0 \Big\}.$$

We extend $\mathrm{ev}_\zeta\colon \mathbb{Z}[q,q^{-1}] \to \mathbb{Z}[\zeta]$ to the $\mathbb{Q}$-algebra homomorphism

$$\mathrm{ev}_\zeta\colon A_r \to \mathbb{Q}(\zeta^{1/4})$$

defined by $\mathrm{ev}_\zeta(f(q^{1/4})) = f(\zeta^{1/4})$.

Set

$$\Omega_r = \sum_{i=0}^{r-2} [i+1]\mathsf{V}_i \in \mathcal{R}_{\mathbb{Z}[v,v^{-1}]}.$$

Let L be an m-component link in S^3. Set

$$I_r(L) = J_L(\Omega_r,\dots,\Omega_r) \in \mathbb{Z}[q^{1/4},q^{-1/4}],$$
$$I_\zeta(L) = \mathrm{ev}_\zeta(I_r(L)) \in \mathbb{Z}[\zeta^{1/4},\zeta^{-1/4}].$$

For an unknot $U_\pm$ of framing ± 1, we have $I_\zeta(U_\pm) \neq 0$. Set

$$\tau_\zeta(L) = \frac{I_\zeta(L)}{I_\zeta(U_+)^{\sigma_+(L)} I_\zeta(U_-)^{\sigma_-(L)}} \in \mathbb{Q}(\zeta^{1/4}), \tag{11.2}$$

where $U_\pm$ denotes an unknot with framing ± 1, and $\sigma_+(L)$ (resp. $\sigma_-(L)$) denotes the number of positive (resp. negative) eigenvalues of the linking matrix of L. It is known that $\tau_\zeta(L)$ is invariant under Kirby moves, i.e., stabilization and handle slides. Hence, by setting $\tau_\zeta(M) = \tau_\zeta(L)$ for a 3-manifold M with $S^3_L \cong M$, we obtain a 3-manifold invariant $\tau_\zeta(M)$.

11.2. **WRT invariants of integral homology spheres.** Define $t\colon \mathcal{R}_{\mathbb{Q}(q^{1/4})} \to \mathcal{R}_{\mathbb{Q}(q^{1/4})}$ to be the unique $\mathbb{Q}(q^{1/4})$-module isomorphism, called the *twist operator*, characterized by

$$J_U(t^{\pm 1}(x)) = J_{U_\pm}(x) \quad \text{for } x \in \mathcal{R}_{\mathbb{Q}(q^{1/4})},$$

where U is the 0-framed unknot. We have $t(\mathsf{V}_n) = q^{\frac{n(n+2)}{4}} \mathsf{V}_n$.

Set

$$\Omega_{r,\pm 1} = t^{\mp 1}(\Omega_r)/I_r(U_\mp).$$

An important property of $\Omega_{r,\pm 1}$ is the following:

$$\mathrm{ev}_\zeta(\langle \Omega_{r,\pm 1}, x\rangle) = \mathrm{ev}_\zeta(J_{U_\pm}(x)) \tag{11.3}$$

for $x \in \mathcal{R}_{\mathbb{Q}(q^{1/4})}$.

For admissible framed links, we can simplify (11.2) as follows. Let $L = L_1 \cup \cdots \cup L_m$ be an admissible framed link with framings $f_1, \ldots, f_m \in \{\pm 1\}$. Let $L^0 = L^0_1 \cup \cdots \cup L^0_m$ denote L with 0 framings. Then we have

$$\tau_\zeta(M) = \tau_\zeta(L) = \mathrm{ev}_\zeta(J_{L^0}(\Omega_{r,-f_1}, \ldots, \Omega_{r,-f_m})).$$

11.3. **Proof of Theorem 11.1.** First consider the case $r = 1$, $\zeta = 1$. We have $\tau_1(M) = 1$ by the definition. We also have $\mathrm{ev}_1(J_M) = 1$ by Lemma 10.3. Hence, we have (11.1) in this case.

Let $r \geq 2$. Let L be as in the last subsection. By the definition of J_M, we have

$$\mathrm{ev}_\zeta(J_M) = \mathrm{ev}_\zeta(J_{L^0}(\omega^{-f_1}, \omega^{-f_2}, \ldots, \omega^{-f_m})) \tag{11.4}$$

Set $d = \lfloor \frac{r-2}{2} \rfloor$. It follows from Theorem 8.2 that if $x_1, \ldots, x_m \in \mathcal{P}$ and we have $x_i \in \mathcal{P}_{d+1}$ for some $i \in \{1, \ldots, m\}$, then we have $\mathrm{ev}_\zeta(J_L(x_1, \ldots, x_m)) = 0$. Hence, by (11.4), we have

$$\mathrm{ev}_\zeta(J_M) = \mathrm{ev}_\zeta(J_{L^0}(\omega_{r,-f_1}, \omega_{r,-f_2}, \ldots, \omega_{r,-f_m})),$$

where $\omega_{r,\pm 1}$ is the truncation of $\omega^{\pm 1}$ given by

$$\omega_{r,\pm 1} = \sum_{n=0}^{d} (\pm 1)^n v^{\pm \frac{1}{2} n(n+3)} P'_n \in \mathcal{P}.$$

Note that $\omega_{r,\pm 1} \in \mathrm{Span}_{A_r}\{P_0, P_1, \ldots, P_d\} \subset \mathcal{R}_{A_r}$. We also have $\Omega_{r,\pm 1} \in \mathcal{R}_{A_r}$.

Lemma 11.2. *Let $L = L_1 \cup \cdots \cup L_m \cup K$ be an $(m+1)$-component, algebraically-split, 0-framed link. Then, for $x_1, \ldots, x_m \in \mathcal{R}_{A_r}$, we have*

$$\mathrm{ev}_\zeta(J_{L \cup K}(x_1, \ldots, x_m, \omega_{r,\pm 1})) = \mathrm{ev}_\zeta(J_{L \cup K}(x_1, \ldots, x_m, \Omega_{r,\pm 1})). \tag{11.5}$$

Proof. If K is an unknot, then by Theorem 9.4 and the definition of $\omega_{r,\pm1}$ we have

$$\mathrm{ev}_\zeta(J_{L\cup K}(x_1,\dots,x_m,\omega_{r,\pm1})) = \mathrm{ev}_\zeta(J_{L_{(K,\mp1)}}(x_1,\dots,x_m)), \tag{11.6}$$

where $L_{(K,\mp1)}$ is the result from L by surgery along K with framing ∓1. By (11.3) and the standard fusing argument, we also have

$$\mathrm{ev}_\zeta(J_{L\cup K}(x_1,\dots,x_m,\Omega_{r,\pm1})) = \mathrm{ev}_\zeta(J_{L_{(K,\mp1)}}(x_1,\dots,x_m)).$$

Hence, we have (11.5).

Now, we consider the general case. A crossing change of two strands in K can be done by a Hoste move. Applying this move finitely many times, we get a framed link $L\cup K'\cup E_1\cup\dots\cup E_t$, $t\geq 0$, such that all the linking numbers between two components of $L\cup K'\cup E_1\cup\dots\cup E_t$ are zero, K' is a 0-framed unknot, $E_1\cup\dots\cup E_t$ is an admissible framed unlink with framings $g_1,\dots,g_t\in\{\pm1\}$, and $(L\cup K')_{E_1\cup\dots\cup E_t}\cong L\cup K$. Using (11.6) iteratively, we obtain

$$\begin{aligned}&\mathrm{ev}_\zeta(J_{L\cup K}(x_1,\dots,x_m,y))\\ &\quad=\mathrm{ev}_\zeta(J_{L\cup K'\cup E_1^0\cup\dots\cup E_m^0}(x_1,\dots,x_m,y,\omega_{r,-g_1},\dots,\omega_{r,-g_t})),\end{aligned}$$

where E_i^0 is E_i with 0-framing, and y is either $\omega_{r,\pm1}$ or $\Omega_{r,\pm1}$. Since K' is an unknot, the assertion reduces to the special case proved above. □

11.4. **An alternative proof of Theorem 10.2.** Here we give an alternative proof of Theorem 10.2 which does not use Theorem 10.1, but uses the existence of the invariant $\tau_\zeta(M)$.

Let M and L be as in Section 10.2. Let $(J_M)_L$ denote the right-hand side of (10.1). Let L' be another link such that $(S^3)_{L'}=M$. We have to show that $(J_M)_L=(J_M)_{L'}$. By Theorem 11.1, we have $\mathrm{ev}_\zeta((J_M)_L)=\tau_\zeta(M)=\mathrm{ev}_\zeta((J_M)_{L'})$ for each $\zeta\in\mathcal{Z}$. (Note that we do not need the invariance of J_M in the proof of Theorem 11.1.) Hence, Proposition 1.1 implies $(J_M)_L=(J_M)_{L'}$. This completes the proof.

Similar idea has been used in [3, 51]. See Section 16.3.

12. Some properties of J_M

In this section, we give some properties of J_M and some applications.

Throughout this section, M denotes an integral homology sphere.

12.1. **Connected sum and orientation reversal.**

Proposition 12.1. *(1) The invariant J_M is multiplicative under connected sum, i.e., for any two integral homology spheres M and M' we have $J_{M\sharp M'}=J_MJ_{M'}$, where $M\sharp M'$ denotes the connected sum of M and M'. We also have $J_{S^3}=1$.*

(2) If $-M$ denotes the mirror image of an integral homology sphere, then we have $J_{-M}=\overline{J_M}$. Here $\overline{J_M}\in\widehat{\mathbb{Z}[q]}$ is the conjugate of J_M, i.e., the image of J_M under the involutive ring automorphism $\widehat{\mathbb{Z}[q]}\to\widehat{\mathbb{Z}[q]}$, $f(q)\mapsto f(q^{-1})$.

Proof. The assertions follows from the corresponding, well-known properties of $\tau_\zeta(M)$

$$\tau_\zeta(M\sharp M')=\tau_\zeta(M)\tau_\zeta(M),\quad \tau_\zeta(S^3)=1,\quad \tau_\zeta(-M)=\tau_{\zeta^{-1}}(M)$$

for any $\zeta\in\mathcal{Z}$, and from Proposition 1.1. (Alternatively, one can use the similar properties for the Ohtsuki series $\tau^O(M)$ and the injectivity of $\iota_1\colon\widehat{\mathbb{Z}[q]}\to\mathbb{Z}[[q-1]]$.

One can also prove the assertions directly using properties of the colored Jones polynomials.) □

12.2. **On determination of J_M by infinitely many WRT invariants.** In this subsection, we prove the following.

Proposition 12.2. *If $\mathcal{Z}' \subset \mathcal{Z}$ has no limit point, then the ring homomorphism* $\mathrm{ev}_{\mathcal{Z}'}$ *in* (1.3) *is not injective.*

Here, recall from Section 1.2 that a "limit point" of a subset $\mathcal{Z}' \subset \mathcal{Z}$ is an element $\xi \in \mathcal{Z}$ such that there are infinitely many $\zeta \in \mathcal{Z}'$ adjacent to ξ. (Recall that ξ and ζ are adjacent if $\mathrm{ord}(\zeta\xi^{-1})$ is a prime power.) Any finite subset $\mathcal{Z}' \subset \mathcal{Z}$ has no limit point. There are many infinite subset $\mathcal{Z}' \subset \mathcal{Z}$ without limit points, for example $\{6^i \mid i \geq 0\}$.

Proposition 12.2 shows that Conjecture 6.1 in our previous paper [18] is false. It follows that we can not prove, using only properties of $\widehat{\mathbb{Z}[q]}$, that the $\tau_\zeta(M)$ for any infinitely many $\zeta \in \mathcal{Z}$ would determine J_M and hence $\tau(M)$. It should be remarked that it is still open whether the WRT invariants at any infinitely many roots of unity determine J_M or not.

Proposition 12.2 follows easily from Proposition 12.3 below. Two elements $m, n \in \mathbb{N}$ are said to be *adjacent* to each other (written $m \Leftrightarrow n$) if $m/n = p^e$ with p prime and $e \in \mathbb{Z}$. For a subset $S \subset \mathbb{N}$, a limit point of S is defined to be an element $m \in \mathbb{N}$ which is adjacent to infinitely many elements of S. Note that if $\zeta, \xi \in \mathcal{Z}$ are adjacent, then $\mathrm{ord}(\zeta)$ and $\mathrm{ord}(\xi)$ are adjacent. (One can define a topology on $\mathbb{N}$ where $S \subset \mathbb{N}$ is open if for each $m \in S$, S contains all but finitely many elements adjacent to m. The map $\mathrm{ord}\colon \mathcal{Z} \to \mathbb{N}$, $\zeta \mapsto \mathrm{ord}(\zeta)$, turns out to be a continuous map.)

Proposition 12.3. *If $S \subset \mathbb{N}$ is a subset with no limit point, then the ring homomorphism*

$$\mathrm{ev}_S\colon \widehat{\mathbb{Z}[q]} \to \prod_{m\in S} \mathbb{Z}[q]/(\Phi_m(q)), \quad f(q) \mapsto (f(q) \mod (\Phi_m(q)))_{m\in S}, \tag{12.1}$$

is not injective.

Proof. Let $S = \{m_1, m_2, \ldots\}$ with $1 \leq m_1 < m_2 < \cdots$. Choose $a \in \mathbb{N} \setminus S$. We construct $f(q) \in \widehat{\mathbb{Z}[q]}$ such that $f(q) \in (\Phi_{m_i}(q))$ for all $i \geq 1$ and $f(q) \notin (\Phi_a(q))$. For this purpose, it suffices to construct a sequence $f_i(q) \in \mathbb{Z}[q]$, $i \geq 1$, such that

(1) $f_i(q) \to 1$ in $\widehat{\mathbb{Z}[q]}$ as $i \to \infty$ (i.e., for each j there is i such that $f_k(q) - 1$ is divisible by $(q)_j$ for all $k \geq i$),
(2) $f_i(q) \in (\Phi_{m_i}(q))$ for $i \geq 1$,
(3) $f_i(q) \notin (\Phi_a(q))$ for $i \geq 1$.

Suppose there is such a sequence $f_i(q)$. Set $f(q) = \prod_{i\geq 1} f_i(q)$, which is well defined by (1). By (2) we have $f(q) \in (\Phi_{m_i}(q))$ for all $i \geq 1$. (1) and (3) implies that $f(q) \notin (\Phi_a(q))$, hence $f(q) \neq 0$.

Now we show that there is a sequence $f_i(q)$ as above. For $i \geq 0$, set

$$b_i = \max(\{0\} \cup \{b \in \mathbb{N} \mid b < m_i;\ b \not\Leftrightarrow m_j \text{ for all } j \geq i\}).$$

The assumption in the statement implies:

(a) $0 \leq b_1 \leq b_2 \leq \cdots$ and $\lim_i b_i = \infty$,
(b) if $1 \leq n \leq b_i$ and $i \leq j$, then m_j and n are not adjacent.

It follows from (b) that, for $i \geq 1$, there is $u_i(q) \in \mathbb{Z}[q]$ such that $(q)_{b_i} u_q(q) \equiv 1 \mod (\Phi_{m_i}(q))$. Set

$$f_i(q) = \begin{cases} \Phi_{m_i}(q) & \text{if } b_i < a, \\ 1-(q)_{b_i} u_i(q) & \text{if } b_i \geq a, \end{cases}$$

(2) and (3) can be easily verified. (1) follows from (a). □

Proof of Proposition 12.2. Set $S = \{\mathrm{ord}(\zeta) \mid \zeta \in \mathcal{Z}'\} \subset \mathbb{N}$. By assumption, S has no limit point. Hence, by Proposition 12.3, ev_S is not injective. It follows that $\mathrm{ev}_{\mathcal{Z}'}$ is not injective, since it factors through ev_S. □

Remark 12.4. By slightly modifying the definition of $f_i(q)$ in the proof of Proposition 12.3, we see that if $S \subset \mathbb{N}$ has no limit point and if $d\colon S \to \{0,1,\dots\}$ is any function, then the ring homomorphism

$$g_{S,d}\colon \widehat{\mathbb{Z}[q]} \to \prod_{m\in S} \mathbb{Z}[q]/(\Phi_m(q)^{d(m)}), \quad f(q) \mapsto (f(q) \mod (\Phi_m(q)^{d(m)}))_{m\in S},$$

is not injective. (It suffices to replace $f_i(q)$ with the $d(m_i)$th power of the original definition of $f_i(q)$.) Using this, one can show that if $\mathcal{Z}' \subset \mathcal{Z}$ has no limit point and if $d\colon \mathcal{Z}' \to \{0,1,\dots\}$ is any function, then the ring homomorphism

$$g_{\mathcal{Z}',d}\colon \widehat{\mathbb{Z}[q]} \to \prod_{\zeta\in\mathcal{Z}'} \mathbb{Z}[q]/((q-\zeta)^{d(m)}), \quad f(q) \mapsto (f(q) \mod ((q-\zeta)^{d(\zeta)}))_{\xi\in\mathcal{Z}'},$$

is not injective.

Remark 12.5. In the situation of Proposition 12.3, suppose moreover that the elements of S are pairwise non-adjacent. Then the homomorphism ev_S is surjective. Since $S = \{6^n \mid n \geq 0\}$ is such an example of S and it is infinite, it follows that $\widehat{\mathbb{Z}[q]}$ is *not* Noetherian.

12.3. **Power series invariants.** In this subsection, we study the power series expansions $\iota_\zeta(J_M) \in \mathbb{Z}[\zeta][[q-\zeta]]$, in particular the Ohtsuki series $\tau^O(M) = \iota_1(J_M)$ (see Theorem 12.6 below).

12.3.1. *Derivatives and power series expansions.* Like polynomials and analytic functions, the elements of $\widehat{\mathbb{Z}[q]}$ can be differentiated arbitrarily many times. In fact, the derivation $\frac{d}{dq}\colon \mathbb{Z}[q] \to \mathbb{Z}[q]$ is continuous with respect to the topology defined by the ideals $(q)_n\mathbb{Z}[q]$, $n \geq 0$, since we have

$$\frac{d}{dq}(q)_{2n} \in (q)_n\mathbb{Z}[q]. \tag{12.2}$$

Therefore, $\frac{d}{dq}$ induces a continuous $\mathbb{Z}$-linear derivation

$$\frac{d}{dq}\colon \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}. \tag{12.3}$$

This implies that there are higher derivations

$$\frac{1}{k!}\frac{d^k}{dq^k} = \frac{1}{k!}(\frac{d}{dq})^k\colon \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}.$$

for all $k \geq 0$. (Recall that $(\frac{d}{dq})^k(\mathbb{Z}[q]) \subset k!\mathbb{Z}[q]$. This implies that $(\frac{d}{dq})^k(\widehat{\mathbb{Z}[q]}) \subset k!\widehat{\mathbb{Z}[q]}$.) Suppose $f(q) \in \widehat{\mathbb{Z}[q]}$ and $\zeta \in \mathcal{Z}$. Define the "Taylor series at ζ" of $f(q)$ by

$$T_\zeta(f(q)) = \sum_{k\geq 0} \frac{1}{k!}\frac{d^k f}{dq^k}(\zeta)(q-\zeta)^k \in \mathbb{Z}[\zeta][[q-\zeta]].$$

Then one can easily check $T_\zeta(f(q)) = \iota_\zeta(f(q))$.

12.3.2. *The Ohtsuki series.* As mentioned in the introduction, according to Lawrence's conjecture proved by Rozansky, the Ohtsuki series $\tau^O(M)$ for an integral homology sphere can be defined as the unique element $\tau^O(M) \in \mathbb{Z}[[q-1]]$ such that for each root of unity ζ of odd prime power order r, $\tau^O(M)|_{q=\zeta}$ converges p-adically to $\tau_\zeta(M) \in \mathbb{Z}[\zeta]$, i.e., we have

$$\mathrm{ev}_\zeta(\tau^O(M)) = \tau_\zeta(M) \tag{12.4}$$

in $\mathbb{Z}_p[\zeta] \cong \varprojlim_n \mathbb{Z}[\zeta]/(p^n) \cong \varprojlim_n \mathbb{Z}[\zeta]/((\zeta-1)^n)$, where

$$\mathrm{ev}_\zeta\colon\ \mathbb{Z}[[q-1]] \to \mathbb{Z}_p[\zeta]$$

denote the ring homomorphism induced by the evaluation map

$$\mathrm{ev}_\zeta\colon\ \mathbb{Z}[q] \to \mathbb{Z}[\zeta], \quad f(q) \mapsto f(\zeta).$$

These properties characterize $\tau^O(M)$.

Theorem 12.6. *For every integral homology sphere M, we have $\iota_1(J_M) = \tau^O(M)$.*

Proof. For each root of unity ζ of prime power order $r = p^e$, we have the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{Z}[q]} & \xrightarrow{\iota_1} & \mathbb{Z}[[q-1]] \\ {\scriptstyle \mathrm{ev}_\zeta}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_\zeta} \\ \mathbb{Z}[\zeta] & \xrightarrow[i]{} & \mathbb{Z}_p[\zeta]. \end{array}$$

Here i is the inclusion. By the commutativity of the diagram and by Theorem 11.1, we have $\mathrm{ev}_\zeta\,\iota_1(J_M) = i\,\mathrm{ev}_\zeta(J_M) = i(\tau_\zeta(M))$. Therefore, by (12.4), we have $\mathrm{ev}_\zeta(\iota_1(J_M)) = \mathrm{ev}_\zeta(\tau^O(M))$. The assertion follows from Lemma 12.7. $\square$

Lemma 12.7. *The map*

$$\mathbb{Z}[[q-1]] \to \prod_{p:odd\ prime} \mathbb{Z}_p[\zeta_p], \quad x \mapsto (\mathrm{ev}_{\zeta_p}(x))_p,$$

is injective. Here $\zeta_p = \exp\frac{2\pi\sqrt{-1}}{p}$.

Proof. Write $x = \sum_{n\geq 0} x_n(q-1)^n$. We assume that $\mathrm{ev}_{\zeta_p}(x) = 0$ for all p, and we see that $x_n = 0$ by induction on n. Let $n \geq 0$ and suppose that $x_0 = \cdots = x_{n-1} = 0$. Then, for every odd prime p, we have $\mathrm{ev}_{\zeta_p}(x) \equiv x_n(\zeta-1)^n \mod (\zeta-1)^{n+1}$. Therefore, x_n is divisible by p. It follows that $x_n = 0$. $\square$

12.3.3. *Expansions at roots of unity.* For every root of unity ζ, $\iota_\zeta(J_M) \in \mathbb{Z}[\zeta][[q-\zeta]]$ may be considered as a generalization of the Ohtsuki series $\tau^O(M) = \iota_1(J_M)$. Define $\lambda_{\zeta,k}(M) \in \mathbb{Z}[\zeta]$, $k \geq 0$, by

$$\iota_\zeta(J_M) = \sum_{k\geq 0} \lambda_{\zeta,k}(M)(q-\zeta)^k.$$

Question 12.8. Recall that the Ohtsuki series is considered to be related to the perturbative expansion of the Chern-Simons path integral. Are there any "physical" interpretation of the other power series $\iota_\zeta(J_M)$?

Let us consider the case $\zeta = -1$. We have

$$\iota_{-1}(J_M) = \sum_{k\geq 0} \lambda_{-1,k}(M)(q+1)^k \in \mathbb{Z}[[q+1]] \tag{12.5}$$

where $\lambda_{-1,k}(M) \in \mathbb{Z}$. It follows from Lemma 10.3 that $\lambda_{-1,0}(M) = 1$. Thus the first interesting coefficient is $\lambda_{-1,1}(M) \in \mathbb{Z}$, which is additive under connected sum, and satisfies $\lambda_{-1,1}(-M) = -\lambda_{-1,1}(M)$ by Proposition 12.1. Some computations show that $\lambda_{-1,1}$ is nontrivial (i.e., non-vanishing for some M), and linearly independent over $\mathbb{Q}$ to the Casson invariant $\lambda(M) = \frac{1}{6}\lambda_1(M)$ of M. (However, there is a congruence relation between them, see Remark 12.22 below.) See Remarks 12.21 and 12.22 for some other properties of $\lambda_{-1,k}(M)$. An interesting problem is to identify the topological information carried by $\lambda_{-1,1}(M)$.

12.3.4. *Relations between two power series expansions at different roots of unity.* Let $\zeta, \xi \in \mathcal{Z}$, where $\operatorname{ord}(\zeta/\xi) = p^e$ is a prime power. We consider how the two power series expansions $\iota_\zeta(J_M) \in \mathbb{Z}[\zeta][[q-\zeta]]$ and $\iota_\xi(J_M) \in \mathbb{Z}[\xi][[q-\xi]]$ are related. Note that these two formal power series rings can be regarded as subrings of

$$R_{\zeta,\xi} := \varprojlim_{i,j} \mathbb{Z}[\zeta,\xi][q]/((q-\zeta)^i, (q-\xi)^j) \cong \mathbb{Z}_p[\zeta,\xi][[q-\zeta]] \cong \mathbb{Z}_p[\zeta,\xi][[q-\xi]].$$

In $R_{\zeta,\xi}$ we have

$$\begin{aligned}\iota_\zeta(J_M) &= \sum_{k\geq 0} \lambda_{\zeta,k}(q-\zeta)^k \\ &= \sum_{k\geq 0} \lambda_{\zeta,k}((q-\xi)+(\xi-\zeta))^k \\ &= \sum_{i\geq 0}\Big(\sum_{j\geq 0}\binom{i+j}{i}(\xi-\zeta)^j\lambda_{\zeta,i+j}\Big)(q-\xi)^i\end{aligned}$$

Hence, we have

$$\lambda_{\xi,k} = \sum_{j\geq 0}\binom{k+j}{k}(\xi-\zeta)^j\lambda_{\zeta,k+j} \tag{12.6}$$

for $k \geq 0$. This formula describes how the coefficients of the power series expansion at ζ determines those at ξ. Note that each $\lambda_{\xi,k}$ is determined modulo a given power of p by finitely many coefficients in the power series expansion at ζ.

Setting $k = 0$ in (12.6), we have

$$\tau_\xi(M) = \sum_{j\geq 0}(\xi-\zeta)^j\lambda_{\zeta,j} \tag{12.7}$$

Note that if $\zeta = 1$ and $\mathrm{ord}(\xi)$ is an odd prime power, then (12.7) implies that the Lawrence's p-adic convergence conjecture mentioned in the introduction. Thus (12.7) generalizes Lawrence's conjecture.

(12.7) implies the following.

Proposition 12.9. *Let $\zeta, \xi \in \mathcal{Z}$ with $\mathrm{ord}(\zeta/\xi)$ a prime power. Then we have*

$$\tau_\zeta(M) \equiv \tau_\xi(M) \pmod{(\xi - \zeta)}$$

in $\mathbb{Z}[\zeta, \xi]$.

Corollary 12.10. *Let $\zeta \in \mathcal{Z}$ with $\mathrm{ord}(\zeta) = dp^e$, where $d \in \{1, 2, 3, 4, 6\}$, $e \geq 0$, and p is a prime. Then we have $\tau_\zeta(M) \neq 0$.*

Proof. It is well known that we have $\tau_\xi(M) \neq \pm 1$ for $\xi \in \mathcal{Z}$ with $\mathrm{ord}(\xi) \in \{1, 2, 3, 4, 6\}$ (see Section 12.4.2). This fact and Proposition 12.9 implies the assertion, since $(\xi - \zeta) \neq (1)$ in $\mathbb{Z}[\zeta, \xi]$. □

The case $d = 1$ and p odd in Corollary 12.10 follows also from Lawrence's conjecture proved by Rozansky, and presumably has been well known.

Conjecture 12.11 (Non-vanishing Conjecture)**.** *For any integral homology sphere M, we have $\tau_\zeta(M) \neq 0$ for every root of unity ζ.*

See Conjectures 13.4 and 13.9 for stronger statements.

12.4. **Divisibility properties and applications to power series invariants.**

12.4.1. *Divisibility in the ring $\widehat{\mathbb{Z}[q]}$.* Here, we make some basic observations about divisibility for elements of $\widehat{\mathbb{Z}[q]}$ by products of powers of cyclotomic polynomials, which we will freely use in the rest of this section.

Lemma 12.12. *Let $f(q) \in \widehat{\mathbb{Z}[q]}$, $\zeta \in \mathcal{Z}$, $n = \mathrm{ord}\,\zeta$, and $k \geq 0$. Then the following conditions are equivalent.*

1. *$f(q)$ is divisible by $\Phi_n(q)^k$ in $\widehat{\mathbb{Z}[q]}$.*
2. *$f(q)$ is divisible by $(q - \zeta)^k$ in $\widehat{\mathbb{Z}[q]} \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]$.*
3. *$\frac{d^i f}{dq^i}(\zeta) = 0$ for $i = 0, 1, \ldots, k-1$.*

Proof. It is easy to check that (2) and (3) are equivalent, and that (1) implies (2). To prove (2) implies (1), use the fact that $f(q)$ is divisible by $(q - \zeta')^k$ for each $\zeta' \in \mathcal{Z}$ with $\mathrm{ord}(\zeta') = n$ (by Galois equivariance). The details are left to the reader. □

Given $f(q) \in \widehat{\mathbb{Z}[q]}$ with $f(q) \neq 0$, let $\mathrm{ord}_n(f(q)) \geq 0$ denote the largest integer l such that $f(q)$ is divisible by $\Phi_n(q)^l$. (Existence of such integer follows from injectivity of the natural homomorphism

$$\widehat{\mathbb{Z}[q]} \to \varprojlim_k \mathbb{Z}[q]/(\Phi_n(q)^k),$$

see [18, Corollary 4.1].) If $f(q) = 0$, then set $\mathrm{ord}_n(f(q)) = \infty$ for all $n \in \mathbb{N}$. It is not difficult to show that if $d\colon \mathbb{N} \to \{0, 1, \ldots\}$ is a function such that $d(n) \leq \mathrm{ord}_n(f(q))$ for all $n \in \mathbb{N}$, and such that $d(n) = 0$ for all but finitely many $n \in \mathbb{N}$, then $f(q)$ is divisible by $\prod_{n \in \mathbb{N}} \Phi_n(q)^{d(n)}$ in $\widehat{\mathbb{Z}[q]}$.

A consequence of this fact is the following.

Proposition 12.13. *The ring* $\widehat{\mathbb{Z}[q]}$ *is* not *a unique factorization domain.*

Proof. One can easily check that each cyclotomic polynomial $\Phi_n(q)$, $n \in \mathbb{N}$, is a prime element in $\widehat{\mathbb{Z}[q]}$. (In fact, the principal ideal $(\Phi_n(q))$ in $\widehat{\mathbb{Z}[q]}$ is a prime ideal.)

Let $f(q) \in \widehat{\mathbb{Z}[q]}$ be as defined in the proof of Proposition 12.3. Thus, there is a sequence $1 \leq m_1 < m_2 < \cdots$ such that $f(q) \in (\Phi_{m_i}(q)) \subset \widehat{\mathbb{Z}[q]}$ for all $i \geq 1$, where $\{m_1, m_2, \ldots\}$ has no limit points (see the proof of Proposition 12.3). The observation above implies that $f(q)$ is divisible by $\Phi_{m_1}(q) \cdots \Phi_{m_i}(q)$ for all $i \geq 1$. Since $\Phi_{m_i}(q)$ are prime elements, it follows that $\widehat{\mathbb{Z}[q]}$ is not a unique factorization domain. □

12.4.2. *Divisibility results.*

Proposition 12.14. *For any integral homology sphere M, $J_M - 1$ is divisible by $q^6 - 1$ in $\widehat{\mathbb{Z}[q]}$.*

Proof. In Lemma 10.3, we showed that $J_M - 1$ is divisible by

$$\frac{(q^2-1)(q^3-1)}{q-1} = \Phi_1(q)\Phi_2(q)\Phi_3(q) = \frac{q^6-1}{\Phi_6(q)}.$$

Kirby, Melvin and Zhang [39] proved that $\tau_6(M) = 1$ for any integral homology sphere M. Therefore, $J_M - 1$ is divisible by $\Phi_6(q)$, hence the assertion. □

Set

$$\tilde{J}_M = q^{-\lambda_1(M)} J_M \in \widehat{\mathbb{Z}[q]},$$

where $\lambda_1(M) = 6\lambda(M) \in 6\mathbb{Z}$ is the first coefficient of the Ohtsuki series $\tau^O(M) \in \mathbb{Z}[[q-1]]$. Clearly, $\iota_1(\tilde{J}_M - 1) \in (q-1)^2\mathbb{Z}[[q-1]]$. By Proposition 12.14, $\tilde{J}_M - 1$ is divisible by $q^6 - 1 = (q-1)(q+1)(q^2+q+1)(q^2-q+1)$. Kirby and Melvin [38] proved that $\tau_4(M) = \tau_{\sqrt{-1}}(M) = (-1)^{\chi(M)} \in \{\pm 1\}$, where $\chi(M) \in \mathbb{Z}/2\mathbb{Z}$ is the Rochlin invariant of M, which is the mod 2 reduction of $\lambda(M)$. Therefore, we have $\mathrm{ev}_{\sqrt{-1}}(\tilde{J}_M - 1) = 0$. Hence, $\tilde{J}_M - 1$ is divisible by $q^2 + 1$. These divisibility properties of $\tilde{J}_M - 1$ imply the following.

Proposition 12.15. *For any integral homology sphere M, $\tilde{J}_M - 1$ is divisible by*

$$\Phi_1(q)^2\Phi_2(q)\Phi_3(q)\Phi_4(q)\Phi_6(q) = (q-1)(q^6-1)(q^2+1).$$

12.4.3. *Applications to the Ohtsuki series.* Here we apply Propositions 12.14 and 12.15 to the Ohtsuki series $\tau^O(M)$, and obtain some congruence relations on the coefficients of $\tau^O(M)$, which generalize well-known results by Murakami [62] and Ohtsuki [67], and by Lin and Wang [54, 55] for the first and the second coefficients.

We write

$$\tau^O(M) = \iota_1(J_M) = 1 + \lambda_1(M)\hbar + \lambda_2(M)\hbar^2 + \cdots \in \mathbb{Z}[[\hbar]],$$

where we set $\hbar = q - 1$. We often write $\lambda_i = \lambda_i(M)$ in the following.

The following gives an infinite set of congruence relations for the coefficients $\lambda_i(M)$ of the Ohtsuki series of M.

Proposition 12.16. *For $k \geq 0$, we have*

$$\sum_{i=0}^{k} a_i \lambda_{k-i+1}(M) \equiv 0 \pmod{\mathbb{Z}}. \tag{12.8}$$

Here $a_0 = 1/6$, $a_1 = -1/4$, $a_2 = 17/72$, $a_3 = -25/144$, $\dots \in \mathbb{Z}[\frac{1}{6}]$ *are determined by*

$$\sum_{i\geq 0} a_i \hbar^i = \frac{1}{(q+1)(q^2+q+1)} = \frac{1}{6+9\hbar+5\hbar^2+\hbar^3}.$$

In particular, each $\lambda_{k+1}(M) \mod 6$, $k \geq 0$, *is determined by* $\lambda_i(M)$, $i = 1, \dots, k$.

Proof. Set

$$f(\hbar) = \hbar^{-1}(\tau^O(M) - 1) = \sum_{k\geq 0} \lambda_{k+1}\hbar^k \in \mathbb{Z}[[\hbar]],$$
$$a(\hbar) = (q+1)(q^2+q+1) = 6+9\hbar+5\hbar^2+\hbar^3.$$

Using Proposition 12.14, we see that there is $g(\hbar) = \sum_{k\geq 0} g_k\hbar^k \in \mathbb{Z}[[\hbar]]$ such that $f(\hbar) = a(\hbar)g(\hbar)$, hence $g(\hbar) = f(\hbar)a(\hbar)^{-1}$. (Here we used the divisibility of $J_M - 1$ by $(q-1)(q+1)(q^2+q+1)$, not by $q^6 - 1$.) We have

$$g_k = \sum_{i=0}^{k} a_i \lambda_{k-i+1}.$$

Since $g_k \in \mathbb{Z}$ for all $k \geq 0$, we have (12.8). Since $a_0 = 1/6$, the latter assertion follows. □

By (12.8) and $a_0 = \frac{1}{6}$, we have

$$\lambda_{k+1} \equiv -\sum_{i=1}^{k} 6a_i\lambda_{k-i+1} \pmod 6. \tag{12.9}$$

For small values of k, (12.9) implies the following.

(1) $\lambda_1 \equiv 0 \pmod 6$. This is known by Murakami [62] and Ohtsuki [67].
(2) $\lambda_2 \equiv \frac{3}{2}\lambda_1 \equiv \frac{1}{2}\lambda_1 \pmod 6$. This is known by Lin and Wang [54]. See also Corollary 12.20 below.
(3) $\lambda_3 \equiv \frac{3}{2}\lambda_2 - \frac{17}{12}\lambda_1 \pmod 6$.
(4) $\lambda_4 \equiv \frac{3}{2}\lambda_3 - \frac{17}{12}\lambda_2 + \frac{25}{24}\lambda_1 \pmod 6$.

Remark 12.17. One can use divisibility of $J_M - 1$ by $q^6 - 1$ instead of $(q-1)(q+1)(q^2+q+1) = (q^6-1)/\Phi_6(q)$. In this case, we obtain a result equivalent to Proposition 12.16 as a consequence of the fact that $\Phi_6(q) = q^2 - q + 1$ is invertible in $\mathbb{Z}[[q-1]]$.

The above-described idea can be applied also to Proposition 12.15 to obtain better results. We write

$$\iota_1(J_M - q^{\lambda_1(M)}) = \tau^O(M) - q^{\lambda_1(M)} = \sum_{k\geq 2} \lambda'_k(M)\hbar^k.$$

We have

$$\lambda'_k(M) = \lambda_k(M) - \binom{\lambda_1(M)}{k} \in \mathbb{Z} \tag{12.10}$$

for $k \geq 2$.

Proposition 12.18. *For $k \geq 0$, we have*

$$\sum_{i=0}^{k} b_i \lambda'_{k-i+2}(M) \equiv 0 \pmod{\mathbb{Z}}. \tag{12.11}$$

Here $b_0 = 1/12$, $b_1 = -5/24$, $b_2 = 41/144$, $b_3 = -77/288, \dots \in \mathbb{Z}[\frac{1}{6}]$ *are determined by*

$$\sum_{i\geq 0} b_i \hbar^i = \frac{1}{(q+1)(q^2+q+1)(q^2+1)} = \frac{1}{12+30\hbar+34\hbar^2+21\hbar^3+7\hbar^4+\hbar^5}.$$

In particular, for $k \geq 0$, $\lambda'_{k+2}(M) \mod 12$ *is determined by* $\lambda'_i(M)$, $i = 2, \dots, k+1$.

Proof. The proof is similar to that for Proposition 12.16, where we use Proposition 12.15 instead of Proposition 12.14. The details are left to the reader. □

Corollary 12.19. *For* $k \geq 0$, $\lambda_{k+2}(M) \mod 12$ *is determined by* $\lambda_i(M)$, $i = 1, \dots, k+1$.

Proof. This immediately follows from Proposition 12.18 and (12.10). □

The case $k = 0$ of (12.11) is equivalent to the following.

Corollary 12.20 (Lin and Wang [55])**.** *We have*

$$\lambda_2(M) \equiv 3\lambda(M) = \frac{1}{2}\lambda_1(M) \pmod{12}.$$

Proof. We have $\lambda_2(M) \equiv \binom{\lambda_1(M)}{2} = \binom{6\lambda(M)}{2} = 18\lambda(M)^2 - 3\lambda(M) \equiv 6\lambda(M) - 3\lambda(M) \equiv 3\lambda(M) \pmod{12}$. □

Remark 12.21. Propositions 12.14 and 12.15 can be applied to the power series $\iota_{-1}(J_M) \in \mathbb{Z}[[q+1]]$ and one can obtain results similar to Propositions 12.16 and 12.18. In particular, we have the following.

(1) $\lambda_{-1,1}(M) \in 6\mathbb{Z}$.
(2) $\lambda_{-1,2}(M) \equiv \frac{1}{2}\lambda_{-1,1}(M) \pmod{12}$.
(3) For $k \geq 2$, $\lambda_{-1,k}(M) \mod 12$ is determined by $\lambda_{-1,i}(M)$, $i = 1, 2, \dots, k-1$.

Here $\lambda_{-1,k}(M) \in \mathbb{Z}$ is defined as in (12.5).

Remark 12.22. As for the relation between $\lambda_1(M)$ and $\lambda_{-1,1}(M)$, we have

$$\lambda_1(M) \equiv \lambda_{-1,1}(M) \pmod{12}. \tag{12.12}$$

Hence, like the Casson invariant, the mod 2 reduction of the integer-valued invariant $\frac{1}{6}\lambda_{-1,1}$ equals the Rochlin invariant of M. (12.12) follows from $\lambda_1(M) \equiv \lambda_{-1,1}(M) \equiv 0 \pmod 3$ and $\lambda_1(M) \equiv \lambda_{-1,1}(M) \pmod 4$. (Here, the latter congruence follows from (12.6) for $\zeta = 1$, $\xi = -1$, $k = 1$.)

12.4.4. *The eighth WRT invariant* $\tau_8(M)$. For each $r \in \mathbb{N}$, it would be interesting to know the range of $\tau_r(M)$. Using the modified version

$$\tilde{\tau}_r(M) = \mathrm{ev}_{\zeta_r}(\tilde{J}_M) = \zeta_r^{-6\lambda(M)}\tau_r(M) \in \mathbb{Z}[\zeta_r],$$

one can expect to obtain sharper results than using $\tau_r(M)$. Theorem 12.15 implies

$$\tilde{\tau}_r(M) - 1 \in (\zeta_r - 1)(\zeta_r^6 - 1)(\zeta_r^2 + 1)\mathbb{Z}[\zeta_r]. \tag{12.13}$$

If r is of the form ip^e, where $i \in \{1, 2, 3, 4, 6\}$ and p^e is a prime power, (12.13) gives some restriction on the value of $\tilde{\tau}_r(M)$.

For example, we here consider $\tau_8(M)$. We have $\zeta_8 = \exp\frac{2\pi\sqrt{-1}}{8} = \frac{1+\sqrt{-1}}{\sqrt{2}}$, and

$$\tilde{\tau}_8(M) = \sqrt{-1}^{\lambda(M)}\tau_8(M) \in \mathbb{Z}[\zeta_8].$$

Proposition 12.23. *For any integral homology sphere M, we have*

$$\tilde{\tau}_8(M) - 1 \in 2(\zeta_8 - 1)\mathbb{Z}[\zeta_8] = \mathrm{Span}_{\mathbb{Z}}\{4, 2\sqrt{2}, 2 + 2\sqrt{-1}, 2 + \sqrt{2} + \sqrt{-2}\}.$$

Proof. By (12.13), it follows that $\tilde{\tau}_8(M)-1$ is divisible in $\mathbb{Z}[\zeta_8]$ by $(\zeta_8-1)(\zeta_8^4-1) = -2(\zeta_8 - 1)$. Hence, we have the assertion. □

Note that $\mathbb{Z}[\zeta_8]/2(\zeta_8 - 1)\mathbb{Z}[\zeta_8] \cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^3$. Computations suggest the following, a much stronger restriction on $\tilde{\tau}_8$.

Conjecture 12.24. *For any integral homology sphere M, we have*

$$\tilde{\tau}_8(M) - 1 \in \mathrm{Span}_{\mathbb{Z}}\{4, 2\sqrt{2}\}.$$

(Consequently, $\tilde{\tau}_8(M)$ is real. Hence, $\tau_8(M)$ is either real or purely imaginary.)

Remark 12.25. Conjecture 12.24 implies that the $\mathbb{Q}$-vector subspace G_8 of $\mathbb{Q}(\zeta_8)$ generated by $\tilde{\tau}_8(M)-1$ for all integral homology spheres M is strictly smaller than $\mathbb{Q}(\zeta_8)$. More generally, let $G_r \subset \mathbb{Q}(\zeta_r)$ be the $\mathbb{Q}$-vector subspace generated by the $\tilde{\tau}_r(M) - 1$. Then $G_r \subsetneq \mathbb{Q}(\zeta_r)$ if $r = 1, 2, 3, 4, 6$. Conjecturally, $G_r = \mathbb{Q}(\zeta_r)$ if $r \neq 1, 2, 3, 4, 6, 8$.

13. The p-adic and the mod p WRT functions

In this section, we first observe that the invariant $J_M \in \widehat{\mathbb{Z}[q]}$ can be formally evaluated at any element in a commutative, unital ring, and in particular at any complex number. Then we introduce the p-adic analytic versions and mod p versions of the WRT functions.

Most constructions in this section can be applied to any invariants of links and 3-manifolds which take values in $\widehat{\mathbb{Z}[q]}$.

13.1. **Formal evaluations in rings.** Let R be a commutative, unital ring, and let $\alpha \in R$. Let $\mathrm{ev}_\alpha\colon \mathbb{Z}[q] \to R$ be the ring homomorphism satisfying $\mathrm{ev}_\alpha(q) = \alpha$. Then ev_α induces a ring homomorphism

$$\hat{\mathrm{ev}}_\alpha\colon \widehat{\mathbb{Z}[q]} \to \hat{R}^\alpha := \varprojlim_n R/((\alpha)_n),$$

where

$$(\alpha)_n = (1-\alpha)(1-\alpha^2)\cdots(1-\alpha^n) \in R$$

for $n \geq 0$.

Using $\hat{\mathrm{ev}}_\alpha$, we can produce various "reduced version" $\hat{\mathrm{ev}}_\alpha(J_M)$ of the invariant $J_M \in \widehat{\mathbb{Z}[q]}$, which may be regarded as the "value at α" of the WRT function of M.

13.2. **Evaluations at complex numbers.** Let $\alpha \in \mathbb{C}$. Then $\mathrm{ev}_\alpha\colon \mathbb{Z}[q] \to \mathbb{Z}[\alpha]$ induces

$$\hat{\mathrm{ev}}_\alpha\colon \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[\alpha]}^\alpha = \varprojlim_n \mathbb{Z}[\alpha]/((\alpha)_n).$$

Thus, one can formally evaluate J_M at any complex number. We can easily verify the following.

(1) If $\alpha = 0$, then $\widehat{\mathbb{Z}[0]}^0 = 0$.

(2) If α is transcendental, then $\hat{\mathrm{ev}}_\alpha\colon \widehat{\mathbb{Z}[q]} \xrightarrow{\cong} \widehat{\mathbb{Z}[\alpha]}^{\alpha}$.
(3) If $\alpha \in \mathcal{Z}$, then $\widehat{\mathbb{Z}[\alpha]}^{\alpha} \cong \mathbb{Z}[\alpha]$.

The case $\alpha \in \mathbb{Q}$ is of special interest.

Proposition 13.1. *If $a/b \in \mathbb{Q} \setminus \{\pm 1\}$ with $a, b \in \mathbb{Z}, (a, b) = 1$, then we have*

$$\widehat{\mathbb{Z}[a/b]}^{a/b} \cong \varprojlim_{m\in\mathbb{N},(m,ab)=1} \mathbb{Z}/m\mathbb{Z} \cong \prod_{p\in P_{ab}} \mathbb{Z}_p, \tag{13.1}$$

where, for $n \in \mathbb{N}$, P_n denote the set of primes p coprime with n.

Proof. We have

$$\widehat{\mathbb{Z}[a/b]}^{a/b} = \varprojlim_n \mathbb{Z}[1/b]/((a/b)_n) \cong \varprojlim_n \mathbb{Z}[1/b]/((b, a)_n),$$

where we set

$$(b, a)_n = b^{\frac{1}{2}n(n+1)}(a/b)_n = (b - a)(b^2 - a^2) \cdots (b^n - a^n) \in \mathbb{Z}.$$

The families of ideals in $\mathbb{Z}[1/b]$, $X := \{((b, a)_n)\}_{n\geq 0}$ and $Y := \{(m)\}_{m\in\mathbb{N},(m,ab)=1}$, are cofinal with each other. (Indeed, $a/b \neq 1$ implies $X \subset Y$. Conversely, for $m \in Y$, we have $b^n - a^n \equiv 0 \mod m$, where $n \geq 1$ is the least common multiple of the orders of b and a in $(\mathbb{Z}/m\mathbb{Z})^\times$. Then $(b, a)_n$ is divisible by m.) Therefore,

$$\widehat{\mathbb{Z}[a/b]}^{a/b} \cong \varprojlim_{m\in\mathbb{N},(m,ab)=1} \mathbb{Z}[1/b]/(m) \cong \varprojlim_{m\in\mathbb{N},(m,ab)=1} \mathbb{Z}/m\mathbb{Z}.$$

Here the latter isomorphism follows from invertibility of b in $\mathbb{Z}/m\mathbb{Z}$ for each m. The latter isomorphism in (13.1) is standard. □

By Proposition 13.1, we may regard $\mathrm{ev}_{a/b}(J_M)$ as an element in $\prod_{p\in P_{ab}} \mathbb{Z}_p$. Hence, for each $p \in P_{a,b}$, we can extract a $\mathbb{Z}_p$-valued function

$$\tau^p(M)\colon U_p(\mathbb{Q}) \to \mathbb{Z}_p, \tag{13.2}$$

where $U_p(\mathbb{Q})$ is the set of $a/b \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$, $(a, b) = 1$, such that p is coprime with ab.

Note that $U_p(\mathbb{Q}) = \mathbb{Q} \cap U(\mathbb{Z}_p)$, where $U(\mathbb{Z}_p)$ is the group of units in the ring $\mathbb{Z}_p$. We extend the domain of $\tau^p(M)$ in the next subsection.

13.3. **The p-adic WRT functions.** In this subsection, we show that the WRT function of an integral homology sphere can be extended to a p-adic analytic function on the unit circle in the field $\mathbb{C}_p$ of complex p-adic numbers.

The field $\mathbb{C}_p$ is defined as follows (see [40, 13] for details). Let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $|\cdot|_p\colon \mathbb{Q}_p \to \mathbb{R}$ be the p-adic norm, defined by

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\mathrm{ord}_p(x) \in \mathbb{Z}$ is the p-adic ordinal. Let $\bar{\mathbb{Q}}_p$ denote the algebraic closure of $\mathbb{Q}_p$. It is well known that the norm $|\cdot|_p$ on $\mathbb{Q}_p$ uniquely extends to $\bar{\mathbb{Q}}_p$. Then $\mathbb{C}_p$ is defined to be the completion of $\bar{\mathbb{Q}}_p$ with respect to the norm $|\cdot|_p$. The norm on $\mathbb{C}_p$ induced by $|\cdot|_p$ on $\bar{\mathbb{Q}}_p$ is denoted again by $|\cdot|_p$. It is well known that $\mathbb{C}_p$ is algebraically closed.

Let $\mathcal{O}_p$ and $\mathcal{P}_p$ denote the valuation ring and the valuation ideal, respectively, of $\mathbb{C}_p$, i.e.,

$$\mathcal{O}_p = \{x \in \mathbb{C}_p\colon\ |x|_p \le 1\}, \quad \mathcal{P}_p = \{x \in \mathbb{C}_p\colon\ |x|_p < 1\}.$$

Let

$$U(\mathcal{O}_p) = \mathcal{O}_p \setminus \mathcal{P}_p = \{x \in \mathbb{C}_p\colon\ |x|_p = 1\}$$

denote the unit circle of $\mathcal{O}_p$, which consists of the multiplicative units in $\mathcal{O}_p$. In particular, all the roots of unity in $\mathbb{C}_p$ are contained in $U(\mathcal{O}_p)$. We may assume that the set $\mathcal{Z}(\subset \bar{\mathbb{Q}} \subset \mathbb{C})$ of complex roots of unity as a subset of $U(\mathcal{O}_p)$ (by choosing a field homomorphism $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p(\subset \mathbb{C}_p)$).

Lemma 13.2. *For each $x \in U(\mathcal{O}_p)$, the sequence $(x)_n$, $n \ge 0$, of elements in $\mathcal{O}_p$ is converging to* 0 *in $\mathcal{O}_p$.*

Proof. Since $x \in U(\mathcal{O}_p)$, there is $\zeta \in \mathcal{Z}$ such that $x - \zeta \in \mathcal{P}_p$. Set $r = \operatorname{ord}(\zeta)$. Then

$$\begin{aligned} 1 - x^r &= 1 - (\zeta + (x-\zeta))^r \\ &= 1 - \sum_{i=0}^{r} \binom{r}{i} \zeta^{r-i}(x-\zeta)^i \\ &= -\sum_{i=1}^{r} \binom{r}{i} \zeta^{r-i}(x-\zeta)^i \in \mathcal{P}_p. \end{aligned}$$

Since $(x)_{rk}$ is divisible by $(1-x^r)^k$ for each $k \ge 0$, it follows that $(x)_n$ converges to 0 as $n \to \infty$. □

Lemma 13.2 implies that for $x \in U(\mathcal{O}_p)$, the ring homomorphism $\mathbb{Z}[q] \to \mathcal{O}_p$, $f(q) \mapsto f(x)$, induces a continuous ring homomorphism

$$\operatorname{ev}_x\colon\ \widehat{\mathbb{Z}[q]} \to \mathcal{O}_p, \quad f(q) \mapsto f(x).$$

Gathering the ev_x for all $x \in U(\mathcal{O}_p)$, we obtain a ring homomorphism

$$\operatorname{ev}_{U(\mathcal{O}_p)}\colon\ \widehat{\mathbb{Z}[q]} \to \operatorname{Fun}(U(\mathcal{O}_p), \mathcal{O}_p)$$

such that $\operatorname{ev}_{U(\mathcal{O}_p)}(f)(x) = \operatorname{ev}_x(f(q))$. For $f \in \widehat{\mathbb{Z}[q]}$, the function

$$\hat{f} = \operatorname{ev}_{U(\mathcal{O}_p)}(f)\colon\ U(\mathcal{O}_p) \to \mathcal{O}_p \tag{13.3}$$

is p-adic analytic of radius of convergence ≥ 1 everywhere in $U(\mathcal{O}_p)$. I.e., $\hat{f}$ has a converging power series expansion at any point $x \in U(\mathcal{O}_p)$ which coincides $\hat{f}$ for all $y \in U(\mathcal{O}_p)$ with $|y - x|_p < 1$.

It is easy to see that the restriction of $\hat{f}$ to

$$U(\mathbb{Z}_p) = \{x \in \mathbb{Z}_p\colon\ |x|_p = 1\} = U(\mathcal{O}_p) \cap \mathbb{Q}_p$$

takes values in $\mathbb{Z}_p$.

For any subset $V \subset U(\mathcal{O}_p)$, $\operatorname{ev}_{U(\mathcal{O}_p)}$ induces a ring homomorphism

$$\operatorname{ev}_V\colon\ \widehat{\mathbb{Z}[q]} \to \operatorname{Fun}(V, \mathcal{O}_p).$$

Proposition 13.3. *Let $V \subset U(\mathcal{O}_p)$ be an infinite subset. Suppose that there is a point $x \in U(\mathcal{O}_p)$ and a sequence $x_i \in V \setminus x$, $i = 0, 1, \ldots$, such that* $\lim_{i\to\infty} x_i = x$ *with respect to the p-adic topology. Then* ev_V *is injective.*

Proof. Suppose $\mathrm{ev}_V(f) = 0$, $f \in \widehat{\mathbb{Z}[q]}$. Define $\hat{f}$ as in (13.3). Set $D(x,1) = \{y \in \mathcal{O}_p\colon |y-x|_p < 1\}$. The restriction $\hat{f}_x = \hat{f}|_{D(x,1)}\colon D(x,1) \to \mathcal{O}_p$ can be expressed as a power series at $q = x$, convergent on $D(x,1)$. The assumption implies that $\hat{f}_x = 0$, i.e., $\hat{f}(x) = 0$ for all $x \in D(x,1)$.

Since $x \in U(\mathcal{O}_p)$, we have $D(x,1) \cap \mathcal{Z} \neq \emptyset$. Moreover, if $\zeta \in D(x,1) \cap \mathcal{Z}$ and $\xi \in \mathcal{Z}$ with $\mathrm{ord}(\xi)$ a power of p, then $\zeta\xi \in D(x,1) \cap \mathcal{Z}$. Since for each $\zeta \in D(x,1) \cap \mathcal{Z}$ we have $f(\zeta) = \hat{f}(\zeta) = 0$, it follows from Proposition 1.1 that $f = 0$. □

Now we give applications of the above-mentioned facts to the WRT invariants. For each integral homology sphere M, define the *p-adic WRT function* of M by

$$\tau^p(M) = \mathrm{ev}_{U(\mathcal{O}_p)}(J_M)\colon U(\mathcal{O}_p) \to \mathcal{O}_p, \tag{13.4}$$

which is p-adic analytic of radius of convergence ≥ 1 everywhere in $U(\mathcal{O}_p)$. $\tau^p(M)$ restricts to

$$\tau^p(M)\colon U(\mathbb{Z}_p) \to \mathbb{Z}_p.$$

Note that these invariants generalize the one given in (13.2).

If V is as in Proposition 13.3, then J_M is determined by the restriction of $\tau^p(M)$ to V. In particular, for $V = U(\mathcal{O}_p)$, J_M (hence $\tau(M)$) is determined by $\tau^p(M)\colon U(\mathcal{O}_p) \to \mathcal{O}_p$ and also by its restriction to $U(\mathbb{Z}_p)$.

The following generalizes Conjecture 12.11.

Conjecture 13.4 (p-adic Non-vanishing Conjecture)**.** *For any integral homology sphere M, for any prime p, $\tau^p(M)\colon U(\mathcal{O}_p) \to \mathcal{O}_p$ is nowhere zero.*

If Conjecture 13.4 is true for at least one prime p, then Conjecture 12.11 is true.

As a partial result of Conjecture 13.4, we have the following result, closely related to Corollary 12.10.

Proposition 13.5. *If $\zeta \in \mathcal{Z} \subset U(\mathcal{O}_p)$ is a root of unity of order* 1, 2, 3, 4 *or* 6, *then $\tau^p(M)$ has no zero on $D(\zeta,1) = \{x \in U(\mathcal{O}_p)\colon |x-\zeta|_p < 1\}$.*

Proof. The assertion follows from $\tau_\zeta(M) = \pm 1$ and the fact that the power series expansion of $\tau^p(M)$ at $q = \zeta$ has coefficients in $\mathcal{O}_p$. □

Remark 13.6. By Corollary 12.10, $\tau^p(M)$ is non-vanishing at $q = \zeta\xi$, where $\xi \in \mathcal{Z}$ is of any prime power order. It follows from the analyticity that $\tau^p(M)$ is non-vanishing on some neighborhood of $\zeta\xi$.

13.4. **The mod p WRT functions.** Let p be a prime and let $\mathbb{F}_p$ be the field of p elements. Let $\bar{\mathbb{F}}_p$ denote the algebraic closure of $\mathbb{F}_p$, and set $\bar{\mathbb{F}}_p^\times = \mathbb{F}_p \setminus \{0\}$.

For $f(q) \in \widehat{\mathbb{Z}[q]}$ and $x \in \bar{\mathbb{F}}_p^\times$, the value $f(x) \in \bar{\mathbb{F}}_p$ of $f(q)$ at x is well defined, since we have $(x)_m = 0$ whenever $m \geq \mathrm{ord}(x)$. (Recall that any $x \in \bar{\mathbb{F}}_p^\times$ is a root of unity.) Note that $f(x)$ is contained in the (finite) subfield of $\bar{\mathbb{F}}_p$ generated by x over $\mathbb{F}_p$. For $f(q) \in \widehat{\mathbb{Z}[q]}$, define a function

$$\tilde{f}\colon \bar{\mathbb{F}}_p^\times \to \bar{\mathbb{F}}_p$$

by $\tilde{f}(x) = f(x)$. Then $\tilde{f}$ is Galois equivariant, i.e., for any $\alpha \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$, we have

$$f(\alpha(x)) = \alpha(f(x)).$$

The correspondence $f(q) \mapsto \tilde{f}$ defines a ring homomorphism

$$\mathrm{ev}_{\bar{\mathbb{F}}_p^\times}\colon \widehat{\mathbb{Z}[q]} \to \mathrm{Fun}_g(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p),$$

where $\mathrm{Fun}_g(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p)$ denotes the ring of all Galois-equivariant functions from $\bar{\mathbb{F}}_p^\times$ to $\bar{\mathbb{F}}_p$.

Clearly, $\mathrm{ev}_{\bar{\mathbb{F}}_p^\times}$ is not injective. In fact, it factors through the surjective homomorphism

$$\widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{F}_p[q]} := \varprojlim_n \mathbb{F}_p[q]/((q)_n) \big(\cong \widehat{\mathbb{Z}[q]} \otimes_{\mathbb{Z}} \mathbb{F}_p\big)$$

induced by $\mathbb{Z}[q] \to \mathbb{F}_p[q]$. Thus there is a ring homomorphism

$$s'_{\bar{\mathbb{F}}_p^\times} \colon \widehat{\mathbb{F}_p[q]} \to \mathrm{Fun}_g(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p)$$

induced by $\mathrm{ev}_{\bar{\mathbb{F}}_p^\times}$.

Proposition 13.7. *$s'_{\bar{\mathbb{F}}_p^\times}$ is surjective, but not injective. Moreover, we have an isomorphism*

$$\mathrm{Fun}_g(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p) \cong \prod_{r\in\mathbb{N}_p} \mathbb{F}_p[q]/(\Phi_r(q)). \tag{13.5}$$

Proof. Set $\mathbb{N}_p = \{r \in \mathbb{N} \mid r \text{ coprime with } p\}$. We have the following isomorphisms

$$\widehat{\mathbb{F}_p[q]} \cong \varprojlim_{r\in\mathbb{N}_p, k\geq 0} \mathbb{F}_p[q]/((q^r-1)^k) \cong \prod_{r\in\mathbb{N}_p} \varprojlim_{k\geq 0} \mathbb{F}_p[q]/(\Phi_r(q)^k).$$

The first one follows, since in $\mathbb{F}_p[q]$ the families of ideals, $\{((q)_n)\}_{n\geq 0}$ and $\{((q^r-1)^k)\}_{r\in\mathbb{N}_p,k\geq 0}$ are cofinal with each other. (This is a consequence of the identity $q^{rp^e}-1 = (q^r-1)^{p^e}$ in $\mathbb{F}_p[q]$ for $r \in \mathbb{N}_p$ and $e \geq 1$.) The second isomorphism can be verified using the Chinese Remainder Theorem in a way similar to the arguments in [18, Section 7.5].

For each $r \in \mathbb{N}_p$, there is a surjective, non-injective homomorphism

$$\varprojlim_{k\geq 0} \mathbb{F}_p[q]/(\Phi_r(q)^k) \to \mathbb{F}_p[q]/(\Phi_r(q)),$$

induced by $\mathbb{F}_p[q] \to \mathbb{F}_p[q]/(\Phi_r(q))$. Hence, there is a surjective, non-injective homomorphism

$$\prod_{r\in\mathbb{N}_p} \varprojlim_{k\geq 0} \mathbb{F}_p[q]/(\Phi_r(q)^k) \to \prod_{r\in\mathbb{N}_p} \mathbb{F}_p[q]/(\Phi_r(q)).$$

Thus it remains to show the second assertion. Note that $\mathrm{Fun}_g(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p)$ has a natural direct sum decomposition

$$\mathrm{Fun}_g(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p) \cong \prod_{r\in\mathbb{N}_p} \mathrm{Fun}_g(\mathcal{Z}_r(\bar{\mathbb{F}}_p), \bar{\mathbb{F}}_p),$$

where $\mathcal{Z}_r(\bar{\mathbb{F}}_p) \subset \bar{\mathbb{F}}_p$ is the set of primitive rth roots of unity, and $\mathrm{Fun}_g(\mathcal{Z}_r(\bar{\mathbb{F}}_p), \bar{\mathbb{F}}_p)$ is the ring of Galois-equivariant functions from $\mathcal{Z}_r(\bar{\mathbb{F}}_p)$ into $\bar{\mathbb{F}}_p$. Since there is a natural $\mathbb{F}_p$-algebra isomorphism

$$\mathbb{F}_p[q]/(\Phi_r(q)) \cong \mathrm{Fun}_g(\mathcal{Z}_r(\bar{\mathbb{F}}_p), \bar{\mathbb{F}}_p), \quad f(q) \mod (\Phi_r(q)) \mapsto (x \mapsto f(x)),$$

we have the isomorphism (13.5). □

Now, set

$$\tau^{\bmod p}(M) = \mathrm{ev}_{\bar{\mathbb{F}}_p^\times}(J_M) \in \mathrm{Fun}_g(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p). \tag{13.6}$$

We call it the *mod p WRT function* of M. For $x \in \bar{\mathbb{F}}_p^\times$,

$$\tau_x(M) := \tau^{\operatorname{mod} p}(M)(x)$$

may be regarded as the WRT invariant at x.

$\tau^{\operatorname{mod} p}(M)$ is determined by $J_M \mod p \in \widehat{\mathbb{F}_p[q]}$, but Proposition 13.7 implies that this fact gives no information on the range of $\tau^{\operatorname{mod} p}(M)$.

However, the $\tau^{\operatorname{mod} p}(M)$ for infinitely many primes p can determine J_M (and hence the WRT invariants) by the following.

Proposition 13.8. *If P is an infinite set of primes, then the ring homomorphism*

$$(\operatorname{ev}_{\bar{\mathbb{F}}_p^\times})_{p\in P}\colon\ \widehat{\mathbb{Z}[q]} \to \prod_{p\in P} \operatorname{Fun}(\bar{\mathbb{F}}_p^\times, \bar{\mathbb{F}}_p)$$

is injective.

Proof. Suppose $(\operatorname{ev}_{\bar{\mathbb{F}}_p^\times})_{p\in P}(f(q)) = 0$, $f(q) \in \widehat{\mathbb{Z}[q]}$. It suffices to prove that for any $\zeta \in \mathcal{Z}$, we have $\operatorname{ev}_\zeta(f(q)) = 0$. For each $p \in P$ coprime with $n := \operatorname{ord}(\zeta)$, choose a ring homomorphism $\pi_{\zeta,p}\colon\ \mathbb{Z}[\zeta] \to \bar{\mathbb{F}}_p$, which maps ζ to a primitive nth root of unity in $\bar{\mathbb{F}}_p$. By assumption, $\pi_{\zeta,p} \operatorname{ev}_\zeta(f(q)) = 0$. This implies that $\operatorname{ev}_\zeta(f(q)) \in \mathbb{Z}[\zeta]$ is divisible by p. Since there are infinitely many such p, it follows that $\operatorname{ev}_\zeta(f(q)) = 0$. Hence, the assertion follows from Proposition 1.1. □

It is well known that $\bar{\mathbb{F}}_p$ is isomorphic to the residue class field $\mathcal{O}_p/\mathcal{P}_p$. Thus, understanding the property of the function $\tau^{\operatorname{mod} p}(M)$ may be considered as the first step in understanding the properties of $\tau^p(M)$. Let

$$\pi\colon\ \mathcal{O}_p \to \mathcal{O}_p/\mathcal{P}_p \cong \bar{\mathbb{F}}_p,$$

denote the projection. Then we have

$$\tau_{\pi(y)}(M) = \pi(\tau_y(M)) \tag{13.7}$$

for $y \in U(\mathcal{O}_p)$.

The following conjecture is supported by some computer calculations.

Conjecture 13.9 (mod p Non-vanishing Conjecture)**.** *For any integral homology sphere M and for any prime p, $\tau^{\operatorname{mod} p}(M)\colon\ \bar{\mathbb{F}}_p^\times \to \bar{\mathbb{F}}_p$ is nowhere zero.*

Using (13.7), one can easily see that Conjecture 13.9 implies Conjecture 13.4.

Remark 13.10. One can apply the construction in this section to any invariants of 3-manifolds and links with values in $\widehat{\mathbb{Z}[q]}$ to obtain functions like (13.4) and (13.6). For the case of the normalized Jones polynomials $\theta_2(J_T) = J_K(\mathsf{V}'_1) \in \mathbb{Z}[q, q^{-1}] \subset \widehat{\mathbb{Z}[q]}$ for $i \geq 1$ and the unified Kashaev invariant $\theta_0(J_T) \in \widehat{\mathbb{Z}[q]}$, the analogues of Conjectures 13.4 and 13.9 do not hold. Probably, these conjecture do not hold in general for the normalized colored Jones polynomials $\theta_n(J_T) = J_K(\mathsf{V}'_{n-1})$ for $n \geq 3$.

14. Examples

In this section, we compute the invariants of some links and integral homology spheres.

14.1. **Borromean rings.** As in Section 4.3, let $B \in \mathsf{BT}_3^0$ denote the Borromean tangle. The closure $A = \mathsf{cl}(B)$ is the Borromean rings.

Proposition 14.1. *For $i, i' \geq 0$, we have*

$$(1 \otimes \mathrm{tr}_q^{P'_i} \otimes \mathrm{tr}_q^{P'_{i'}})(J_B) = \delta_{i,i'}(-1)^i \sigma_i.$$

Proof. Let $U_h^0 \cong \mathbb{Q}[H][[h]]$ denote the h-adic closure of the $\mathbb{Q}[[h]]$-subalgebra of U_h generated by H. Recall that the *Harish-Chandra map* of U_h is the continuous $\mathbb{Q}[[h]]$-module homomorphism $\varphi\colon U_h \to U_h^0$ determined by $\varphi(F^i H^j E^k) = \delta_{i,0}\delta_{k,0}H^j$ for $i, j, k \geq 0$.

By (4.1), for any $x, x' \in \mathcal{R}_{\mathbb{Q}[[h]]}$, we have

$$(\varphi \otimes \mathrm{tr}_q^x \otimes \mathrm{tr}_q^{x'})(J_B) = (\varphi \otimes \mathrm{tr}_q^x \otimes \mathrm{tr}_q^{x'})(\sum_{m_1,m_2,m_3,n_1,n_2,n_3 \geq 0} B_{m_1,m_2,m_3,n_1,n_2,n_3}),$$

where $B_{m_1,m_2,m_3,n_1,n_2,n_3}$ denotes the summand in (4.1). It follows from properties of φ and tr_q^x that $(\varphi \otimes \mathrm{tr}_q^x \otimes \mathrm{tr}_q^{x'})(B_{m_1,m_2,m_3,n_1,n_2,n_3}) = 0$ unless $n_1 = n_3 = 0$ and $m_1 = m_3 = m_2 + n_2$. Hence, setting $k = m_1$ and $l = n_2$, we have

$$\begin{aligned}
&(\varphi \otimes \mathrm{tr}_q^x \otimes \mathrm{tr}_q^{x'})(J_B)\\
&= \sum_{k\geq 0}\sum_{l=0}^{k}(\varphi \otimes \mathrm{tr}_q^x \otimes \mathrm{tr}_q^{x'})(B_{k,l,k,0,k-l,0})\\
&= \sum_{k\geq 0}\sum_{l=0}^{k}(-1)^l q^{-\frac{1}{2}(k-l)(k-l+1)-l-4kl+2k^2}\\
&\quad \varphi(e^k\tilde{F}^{(k)}K^{-2k+2l})\mathrm{tr}_q^x(e^{k-l}\tilde{F}^{(k)}e^l K^{-2k})\mathrm{tr}_q^{x'}(\tilde{F}^{(l)}e^k\tilde{F}^{(k-l)}K^{-2k}).
\end{aligned}$$

Using the identity $\mathrm{tr}_q^x(yy') = \mathrm{tr}_q^x(y'S^2(y))$ for $y, y \in U_h$, we obtain

$$\begin{aligned}
&\mathrm{tr}_q^x(e^{k-l}\tilde{F}^{(k)}e^l K^{-2k})\mathrm{tr}_q^{x'}(\tilde{F}^{(l)}e^k\tilde{F}^{(k-l)}K^{-2k})\\
&= q^{-2k+2l+3kl+l^2}\begin{bmatrix}k\\ l\end{bmatrix}_q \mathrm{tr}_q^x(\tilde{F}^{(k)}K^{-2k}e^k)\mathrm{tr}_q^{x'}(\tilde{F}^{(k)}K^{-2k}e^k).
\end{aligned}$$

Now, set $x = P_i$ and $x' = P_{i'}$. By Lemma 8.8, we have for $i \geq 0$

$$\mathrm{tr}_q^{P_i}(\tilde{F}^{(k)}K^{-2k}e^k) = \delta_{i,k}v^k q^{-k^2}\{k\}_q!.$$

Moreover, using (2.4), we obtain

$$\varphi(e^k\tilde{F}^{(k)}K^{-2k+2l}) = \varphi(e^k\tilde{F}^{(k)})K^{-2k+2l} = \{H\}_{q,k}K^{-2k+2l}.$$

Using the above formulas, we obtain

$$(\varphi \otimes \mathrm{tr}_q^{P_i} \otimes \mathrm{tr}_q^{P_{i'}})(J_B) = \delta_{i,i'}\Big(\sum_{l=0}^{i}(-1)^l q^{\frac{1}{2}l(l+3)-\frac{1}{2}i(i+3)}\begin{bmatrix}i\\ l\end{bmatrix}_q K^{2l-2i}\Big)\{H\}_{q,i}(\{i\}_q!)^2.$$

Using the identities

$$\begin{aligned}
&\sum_{l=0}^{i}(-1)^l q^{\frac{1}{2}l(l+3)-\frac{1}{2}i(i+3)}\begin{bmatrix}i\\ l\end{bmatrix}_q K^{2l-2i} = \{-H-2\}_{q,i},\\
&\varphi(\sigma_i) = \{H\}_i\{H+i+1\}_i = (-1)^i\{H\}_i\{-H-2\}_i,
\end{aligned}$$

we obtain

$$\begin{aligned}(\varphi\otimes \mathrm{tr}_q^{P_i}\otimes \mathrm{tr}_q^{P_{i'}})(J_B) &= \delta_{i,i'}\{-H-2\}_{q,i}\{H\}_{q,i}(\{i\}_q!)^2\\ &= \delta_{i,i'}\{-H-2\}_i\{H\}_i(\{i\}!)^2=\delta_{i,i'}(-1)^i\varphi(\sigma_i)(\{i\}!)^2.\end{aligned}$$

Hence, we have $(\varphi\otimes \mathrm{tr}_q^{P'_i}\otimes \mathrm{tr}_q^{P'_{i'}})(J_B)=\delta_{i,i'}(-1)^i\varphi(\sigma_i)$. Then the assertion follows from the well-known fact that φ is injective on the center $Z(U_h)$. □

Propositions 14.1 implies the following.

Corollary 14.2. *For $i,j,k\geq 0$, we have*

$$J_A(P'_i,P'_j,P'_k)=\begin{cases}(-1)^i\{2i+1\}_{i+1}/\{1\} & \text{if } i=j=k,\\ 0 & \text{otherwise,}\end{cases}\tag{14.1}$$

(14.2)

$$J_A(\mathsf{V}_i,\mathsf{V}_j,\mathsf{V}_k)=\sum_{p=0}^{\min(i,j,k)}(-1)^p\begin{bmatrix}i+1+p\\2p+1\end{bmatrix}\begin{bmatrix}j+1+p\\2p+1\end{bmatrix}\begin{bmatrix}k+1+p\\2p+1\end{bmatrix}(\{p\}!)^2\{2p+1\}_{2p}.$$

Proof. For (14.1), use Proposition 6.3. For (14.2), use (8.2). □

Corollary 14.2 was first stated using the Kauffman bracket in [16] without proof. One can easily modify the arguments in [58] to obtain a skein-theoretic proof of Corollary 14.2.

14.2. **Powers of the ribbon element and the twist element.** Here we give formulas for the powers of the ribbon element $\mathbf{r}$ and the twist element ω.

For $p,n\geq 0$, set

$$S(n,p)=\{(i_1,\dots,i_p)\mid i_1,\dots,i_p\geq 0,\quad i_1+\dots+i_p=n\}.$$

For $\mathbf{i}=(i_1,\dots,i_p)\in S(n,p)$, set $\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_q=\frac{[n]_q!}{[n_1]_q!\cdots[n_p]_q!}$, $f(\mathbf{i})=\sum_{j=1}^{p-1}(s_j^2+s_j)$ with $s_j=\sum_{k=1}^j i_k$, and $g(\mathbf{i})=\sum_{j=1}^p(p-j)i_j$.

Proposition 14.3. *For $p\geq 0$, we have*

$$\mathbf{r}^{-p}=v^{\frac{p}{2}H(H+2)}\sum_{n\geq 0}v^{\frac12 n(n+3)}\Big(\sum_{\mathbf{i}\in S(n,p)}\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_q q^{f(\mathbf{i})}K^{2g(\mathbf{i})}\Big)K^nF^{(n)}e^n,$$

$$\mathbf{r}^{p}=v^{-\frac{p}{2}H(H+2)}\sum_{n\geq 0}(-1)^n v^{-\frac12 n(n+3)}\Big(\sum_{\mathbf{i}\in S(n,p)}\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_{q^{-1}} q^{-f(\mathbf{i})}K^{-2g(\mathbf{i})}\Big)K^{-n}F^{(n)}e^n,$$

where $\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_{q^{-1}}\Big(=q^{-\sum_{1\leq j<k\leq p}i_ji_k}\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_q\Big)$ *is the conjugate of* $\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_q$.

Proof. We only prove the first formula. The other can be similarly proved.

Define $a_{p,n}\in U_h^0$ by

$$\mathbf{r}^{-p}=v^{\frac{p}{2}H(H+2)}\sum_{n\geq 0}v^{\frac12 n(n+3)}a_{p,n}K^nF^{(n)}e^n.\tag{14.3}$$

We have $a_{0,n}=\delta_{n,0}$ and $a_{1,n}=1$.

Since $\mathbf{r}$ is central, we have

$$\begin{aligned}\mathbf{r}^{-p-1} &= v^{\frac{p}{2}H(H+2)}\sum_{i\geq 0} v^{\frac{1}{2}i(i+3)}a_{p,i}K^iF^{(i)}\Big(v^{\frac{1}{2}H(H+2)}\sum_{j\geq 0} v^{\frac{1}{2}j(j+3)}K^jF^{(j)}e^j\Big)e^i\\ &= v^{\frac{p+1}{2}H(H+2)}\sum_{k\geq 0} v^{\frac{1}{2}k(k+3)}\Big(\sum_{i=0}^{k} a_{p,i}\begin{bmatrix}k\\ i\end{bmatrix}_q q^{i^2+i}K^{2i}\Big)K^kF^{(k)}e^k.\end{aligned}$$

Hence, we have

$$a_{p+1,k} = \sum_{i=0}^{k} a_{p,i}\begin{bmatrix}k\\ i\end{bmatrix}_q q^{i^2+i}K^{2i}.$$

By induction using this formula, we can verify for $p \geq 0$

$$a_{p,n} = \sum_{\mathbf{i}\in S(n,p)}\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_q q^{f(\mathbf{i})}K^{2g(\mathbf{i})}. \tag{14.4}$$

□

We can derive formulas for the powers of ω from Proposition 14.3 as follows.

Proposition 14.4. *For $p \in \mathbb{Z}$, we have*

$$\omega^p = \sum_{n\geq 0}\omega_{p,n}P'_n, \tag{14.5}$$

where

$$\omega_{p,n} = \begin{cases} v^{\frac{1}{2}n(n+3)}\sum_{\mathbf{i}\in S(n,p)}\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_q q^{f(\mathbf{i})} & \text{for } p \geq 0,\\ (-1)^n v^{-\frac{1}{2}n(n+3)}\sum_{\mathbf{i}\in S(n,-p)}\begin{bmatrix}n\\ \mathbf{i}\end{bmatrix}_{q^{-1}} q^{-f(\mathbf{i})} & \text{for } p \leq 0.\end{cases}$$

Proof. Since the proof is similar to that of Proposition 9.1, we give only a sketch proof. We consider only the case $p \geq 0$; the other case follows by symmetry.

We can express ω^p as in (14.5), where the $\omega_{p,n}$ are in $\mathbb{Z}[v, v^{-1}]$. For each $k \geq 0$, we have

$$\langle \omega^p, \mathsf{V}'_{2k}\rangle = \sum_{n\geq 0}\omega_{p,n}\langle P'_n, \mathsf{V}'_{2k}\rangle = \sum_{n=0}^{k}\omega_{p,n}\{k+n\}_{2n}/\{n\}!.$$

By computing the action of $\mathbf{r}^{-p}$ on the weight 0 vector $\mathbf{v}^{2k}_k \in \mathsf{V}_{2k}$ using Proposition 14.3, we obtain

$$\mathrm{tr}_q^{\mathsf{V}'_{2k}}(\mathbf{r}^{-p}) = \sum_{n=0}^{k} v^{\frac{1}{2}n(n+3)}\epsilon(a_{p,n})\{k+n\}_{2n}/\{n\}!,$$

where $a_{p,n}$ is given by (14.3) and (14.4). Since $\langle \omega^p, \mathsf{V}'_{2k}\rangle = \mathrm{tr}_q^{\mathsf{V}'_{2k}}(\mathbf{r}^{-p})$, we have

$$\sum_{n=0}^{k}\omega_{p,n}\{k+n\}_{2n}/\{n\}! = \sum_{n=0}^{k} v^{\frac{1}{2}n(n+3)}\epsilon(a_{p,n})\{k+n\}_{2n}/\{n\}!$$

for all $k \geq 0$. Since $\{k+n\}_{2n}/\{n\}! \neq 0$ whenever $0 \leq n \leq k$, it follows that $\omega_{p,n} = v^{\frac{1}{2}n(n+3)}\epsilon(a_{p,n})$ for all $n \geq 0$, hence the assertion. □

A skein-theoretic version of Proposition 14.4 has been given by Masbaum [58].

14.3. **Surgeries along the Borromean rings.** For $i, j, k \in \mathbb{Z}$, we define L_i, $K_{i,j}$ and $M_{i,j,k}$ as follows. Let A_1, A_2, A_3 denote the components of the Borromean rings A.

- $L_i = (A_1 \cup A_2)_{A_3;-1/i} \subset S^3$, the two component link obtained from $A_1 \cup A_2$ by surgery along A_3 with framing $-1/i$.
- $K_{i,j} = (A_1)_{A_2 \cup A_3;-1/i,-1/j} \subset S^3$, the knot obtained from A_1 by surgery along A_2 and A_3 with framings $-1/i$ and $-1/j$, respectively.
- $M_{i,j,k} = (S^3)_{A_1 \cup A_2 \cup A_3;-1/i,-1/j,-1/k}$, the integral homology sphere obtained from S^3 by surgery along A_1, A_2, A_3 with framings $-1/i$, $-1/j$, $-1/k$, respectively.

Proposition 14.5. *For $i, j, k \in \mathbb{Z}$ and $l, m \geq 0$, we have*

$$J_{L_i}(P'_l, P'_m) = \delta_{l,m}\omega_{i,l}(-1)^l\{2l+1\}_{l+1}/\{1\}, \tag{14.6}$$

$$J_{K_{i,j}}(P'_l) = \omega_{i,l}\omega_{j,l}(-1)^l\{2l+1\}_{l+1}/\{1\}, \tag{14.7}$$

$$J_{M_{i,j,k}} = \sum_{l\geq 0}\omega_{i,l}\omega_{j,l}\omega_{k,l}(-1)^l\{2l+1\}_{l+1}/\{1\}, \tag{14.8}$$

$$J_{L_i}(\mathsf{V}_l, \mathsf{V}_m) = \sum_{s=0}^{\min(l,m)}\begin{bmatrix} l+s+1 \\ 2s+1 \end{bmatrix}\begin{bmatrix} m+s+1 \\ 2s+1 \end{bmatrix}\omega_{i,s}(-1)^s\{s\}!\{2s+1\}!/\{1\}, \tag{14.9}$$

$$\begin{aligned} J_{K_{i,j}}(\mathsf{V}_l) &= \sum_{s=0}^{l}\begin{bmatrix} l+s+1 \\ 2s+1 \end{bmatrix}\omega_{i,s}\omega_{j,s}(-1)^s\{2s+1\}!/\{1\} \\ &= \sum_{s=0}^{l}(-1)^s\omega_{i,s}\omega_{j,s}\frac{\{l+s+1\}\{l+s\}\cdots\{l-s+1\}}{\{1\}}. \end{aligned} \tag{14.10}$$

Proof. The first three identities follows from $J_{L_i}(P'_l, P'_m) = J_A(P'_l, P'_m, \omega^i)$, $J_{K_{i,j}}(P'_l) = J_A(P'_l, \omega^i, \omega^j)$, and $J_{M_{i,j,k}} = J_A(\omega^i, \omega^j, \omega^k)$. Use (6.2) for the others. □

By (14.7), we have

$$J_{K_{i,j}}(P''_l) = (-1)^l\omega_{i,l}\omega_{j,l}.$$

The special case of Proposition 14.5 for $K_{i,j}$, $M_{i,j,k}$ with $i, j, k \in \{\pm 1\}$ appeared in [16, 17, 50] (without proofs). The case for $K_{i,\pm 1}$, $i \in \mathbb{Z}$, (and essentially $L_{\pm 1}$) has been proved by Masbaum [58] using skein theory.

15. Knots in integral homology spheres

Let K be a 0-framed knot in an integral homology sphere M. In this section, we will define an invariant $J_{(M,K)} \in Z(\hat{\mathcal{U}}_q^{\mathrm{ev}})$ of the pair (M, K). Here $Z(\hat{\mathcal{U}}_q^{\mathrm{ev}})$ denotes the $\widehat{\mathbb{Z}[q]}$-subalgebra of $Z(U_h)$ consisting of the infinite sums $\sum_{n\geq 0} a_n\sigma_n$ with $a_n \in \widehat{\mathbb{Z}[q]}$ for $n \geq 0$. (The notation $Z(\hat{\mathcal{U}}_q^{\mathrm{ev}})$ comes from the fact that $Z(\hat{\mathcal{U}}_q^{\mathrm{ev}})$ is equal to the center of the even part $\hat{\mathcal{U}}_q^{\mathrm{ev}}$ of the completion $\hat{\mathcal{U}}_q$ of $\mathcal{U}_q$ defined in [19]. We will not need this fact in what follows.) Clearly, we have $Z(\tilde{\mathcal{U}}_q^{\mathrm{ev}}) \subset Z(\hat{\mathcal{U}}_q^{\mathrm{ev}})$.

In what follows, the invariant $J_{(M,K)}$ is constructed using a generalization $J_{(M,K)}$ of the invariant $J_K(P''_m)$ of knots in S^3 studied in Section 6. $J_{(M,K)}(P''_m)$ and its generalization to algebraically-split links have been studied by Garoufalidis and Le [7]. (In [17, Section 4], we announced a "universal sl_2 invariant for links in integral homology spheres". The invariant $J_{(M,K)}$ may be considered as a special case of this. We do not consider the link case here.)

15.1. **Bottom knots with links.** Let $T = T_0 \cup L_1 \cup \cdots \cup L_l$ be a tangle in a cube consisting of a bottom knot T_0 and an algebraically-split, 0-framed link $L_1 \cup \cdots \cup L_l$, where $\mathrm{lk}(T_0, L_i) = 0$ for $i = 1, \dots, l$. For $x_1, \dots, x_l \in \mathcal{P}$, we define $J_T(x_1, \dots, x_l)$ as follows. Let $T' = T_0 \cup T_1 \cup \cdots \cup T_l$ denote an $(l+1)$-component bottom tangle such that T is obtained from T' by closing the components $T_1, \dots, T_l$. Set

$$J_T(x_1, \dots, x_l) = (1 \otimes \mathrm{tr}_q^{x_1} \otimes \cdots \otimes \mathrm{tr}_q^{x_l})(J_{T'}),$$

which is well-defined, i.e., does not depend on the choice of T'.

For $m \geq 0$, define $F_m(Z(\mathcal{U}_q^{\mathrm{ev}}))$ as the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $Z(\mathcal{U}_q^{\mathrm{ev}})$ consisting of the elements $\sum_{k \geq 0} a_k \sigma_k$, where $a_k \in \mathbb{Z}[q, q^{-1}]$ for $k \geq 0$ and if $0 \leq k \leq m$ then a_k is divisible by $\frac{\{2m+1\}_{q,m+1}}{\{2k+1\}_{q,k+1}}$. The $F_m(Z(\mathcal{U}_q^{\mathrm{ev}}))$ are a descending filtration of ideals in $Z(\mathcal{U}_q^{\mathrm{ev}})$. We have a natural isomorphism

$$\varprojlim_m Z(\mathcal{U}_q^{\mathrm{ev}})/F_m(Z(\mathcal{U}_q^{\mathrm{ev}})) \cong Z(\hat{\mathcal{U}}_q^{\mathrm{ev}}).$$

(Indeed, both are naturally isomorphic to $\varprojlim_{k,l} Z(\mathcal{U}_q^{\mathrm{ev}})/((q)_k, \sigma_l)$.)

Proposition 15.1. *Let T be as above. Let $m_1, \dots, m_l \geq 0$. Then we have*

$$J_T(\tilde{P}'_{m_1}, \dots, \tilde{P}'_{m_l}) \in F_m(Z(\mathcal{U}_q^{\mathrm{ev}})),$$

where $m = \max(m_1, \dots, m_l)$.

Proof. Let $g_k \in \mathbb{Z}[q, q^{-1}]$, $k \geq 0$, be such that $\sum_{k \geq 0} g_k \sigma_k = J_T(\tilde{P}'_{m_1}, \dots, \tilde{P}'_{m_l})$. By an argument similar to the proof of Theorem 6.4, we have $g_k = \mathrm{tr}_q^{P''_k}(J_T(\tilde{P}'_{m_1}, \dots, \tilde{P}'_{m_l}))$. Let L denote the link obtained from T by closing K. Then we have

$$g_k = \mathrm{tr}_q^{P''_k}(J_T(\tilde{P}'_{m_1}, \dots, \tilde{P}'_{m_l})) = J_L(P''_k, \tilde{P}'_{m_1}, \dots, \tilde{P}'_{m_l}).$$

It follows from Theorem 8.2 that if $0 \leq k \leq m$ then g_k is divisible by $\frac{\{2m+1\}_{q,m+1}}{\{2k+1\}_{q,k+1}}$. Hence, we have the assertion. □

Corollary 15.2. *Let T be as above. Then $J_T \colon \mathcal{P} \times \cdots \times \mathcal{P} \to Z(\mathcal{U}_q^{\mathrm{ev}})$, $(x_1, \dots, x_l) \mapsto J_T(x_1, \dots, x_l)$, induces a well-defined $\mathbb{Z}[q, q^{-1}]$-multilinear map*

$$J_T \colon \hat{\mathcal{P}} \times \cdots \times \hat{\mathcal{P}} \to Z(\hat{\mathcal{U}}_q^{\mathrm{ev}}).$$

Let $L = \mathsf{cl}(T_0) \cup L_1 \cup \cdots \cup L_l$ be the link obtained from T closing T_0. Then we set

$$J_L(*, x_1, \dots, x_l) = J_T(x_1, \dots, x_l) \in Z(\hat{\mathcal{U}}_q^{\mathrm{ev}})$$

for $x_1, \dots, x_l \in \hat{\mathcal{P}}$. Thus "$*$" means "keep the element in this place".

15.2. **The invariant $J_{(M,K)}$.** Let (M, K) be a pair of an integral homology sphere M and a 0-framed knot $K \subset M$. Choose an algebraically-split framed link $L = K' \cup L_1 \cup \cdots \cup L_l$ in S^3 such that

(1) the framing of L_i is $f_i \in \{\pm 1\}$,
(2) $(S^3, K')_{L_1 \cup \cdots \cup L_l} \cong (M, K)$.

L is called a *surgery presentation* for (M, K).

Theorem 15.3. *Set*

$$J_{(M,K)} = J_L(*, \omega^{-f_1}, \dots, \omega^{-f_l}) \in Z(\hat{\mathcal{U}}_q^{\mathrm{ev}}). \tag{15.1}$$

Then $J_{(M,K)}$ is a well-defined invariant of the pair (M, K).

Proof. (Sketch) The shortest way to prove the well-definedness of $J_{(M,K)}$ (i.e., independence from the choice of L) may be to consider the elements $\mathrm{ev}_\zeta(J_{(M,K)}(\mathsf{V}'_k))$ for $k \geq 0$ and $\zeta \in \mathcal{Z}$, where $\mathsf{V}'_k = \mathsf{V}_k/[k+1]$. One can prove that this quantity is equal to the normalized WRT invariant of (M, K) at ζ with K colored by the V'_k. It follows from Proposition 1.1 that $J_{(M,K)}(\mathsf{V}'_k)$ is a well-defined invariant of (M, K). Therefore, $J_{(M,K)}(P''_k)$ for $k \geq 0$ are also well-defined, hence so is $J_{(M,K)}$. □

Remark 15.4. Another way to prove Theorem 15.3 is to use a "pair version" of Theorem 10.1: *Two surgery presentations (in the above sense) of pairs of integral homology spheres and knots yield orientation-preserving homeomorphic results of surgery if and only if they are related by a sequence of stabilizations and Hoste moves. Here a Hoste move is defined to be a Fenn-Rourke move (see Figure 10.1) between two surgery presentations.* (One can generalize this result to to links and tangles (in a homology ball).)

In view of this result, one has only to show that the right-hand side of (15.1) is invariant under a Hoste move. To prove this invariance, one has to consider the "evaluations" at V_k for $k \geq 0$ as in the proof of Theorem 15.3, but one does not have to consider the evaluations at roots of unity.

Remark 15.5. Yet another, conceptually more interesting, way to prove Theorem 15.3 is to extend the domain of definition of the invariant to "bottom tangles in homology handlebodies". See Section 16.4 for some explanation.

15.3. **Surgery along a knot with framing** $1/m$**.** Kirby and Melvin proved periodicity results for the WRT invariant of integral homology spheres obtained from S^3 by surgery along a knot with framings $1/m$ (Corollaries 8.15 and 8.26 of [38]). We have the following generalization.

Theorem 15.6. *Let K be a knot in an integral homology sphere M. Let $M' = M_{K;1/m}$ be the result of surgery from M along K with framing $1/m$ with $m \in \mathbb{Z}$. Then we have*

$$J_{M'} \equiv J_M \pmod{(q^{2m} - 1)}.$$

Proof. One way to prove the assertion is first to prove $\tau_\zeta(M') = \tau_\zeta(M)$ for all $\zeta \in \mathcal{Z}$ with $\mathrm{ord}(\zeta)|2m$ and then to use Section 12.4.1. The former is a natural generalization of Kirby and Melvin's result, and is not difficult to prove. Here we give a direct proof.

We express M as the result of surgery S^3_L along an admissible framed link $L = L_1 \cup \cdots \cup L_l$ in S^3, with framings $f_1, \dots, f_l \in \{\pm 1\}$. We may assume that K corresponds to a knot $K' \subset S^3 \setminus L$ such that $K' \cup L$ is algebraically split. By puncturing S^3 at a point of K', we obtain a bottom knot T and a link L in a cube, so that closing T yields $K' \cup L$. We have

$$\begin{aligned}
J_{M'} - J_M &= J_{K \cup L^0}(\omega^{-m}, \omega^{-f_1}, \dots, \omega^{-f_l}) - J_{K \cup L^0}(1, \omega^{-f_1}, \dots, \omega^{-f_l}) \\
&= J_{K \cup L^0}(\omega^{-m} - 1, \omega^{-f_1}, \dots, \omega^{-f_l}) \\
&= \mathrm{tr}_q^{\omega^{-m}-1}(J_{T \cup L^0}(\omega^{-f_1}, \dots, \omega^{-f_l})),
\end{aligned}$$

where L^0 is L with 0 framings. In view of the last subsection, we have

$$J_{T\cup L^0}(\omega^{-f_1},\dots,\omega^{-f_l}) = \sum_{k\geq 0} a_k\sigma_k,$$

where $a_k \in \widehat{\mathbb{Z}[q]}$ for $k \geq 0$. It suffices to prove that $\mathrm{tr}_q^{\omega^{-m}-1}(\sigma_k) \in (q^{2n}-1)\widehat{\mathbb{Z}[q]}$ for each $k \geq 0$. We have

$$\mathrm{tr}_q^{\omega^{-m}-1}(\sigma_k) = \langle \omega^{-m}-1, S_k\rangle = \langle \omega^{-m}-1, \sum_{i=0}^{k} b_{k,i}\mathsf{V}_{2i}\rangle,$$

where $b_{k,i} \in \mathbb{Z}[q,q^{-1}]$. We have

$$\langle \omega^{-m}-1, \mathsf{V}_{2i}\rangle = (q^{-i(i+1)m}-1)[2i+1] \in (q^{2m}-1)\mathbb{Z}[q,q^{-1}].$$

Hence, the assertion follows. □

15.4. **Cyclotomic finite type invariants of integral homology spheres.** Kricker and Spence [41] (see also [70]) proved that for integral homology spheres the Ohtsuki series $\tau^O(M)$ modulo $(q-1)^{k+1}$ is a finite type invariant of degree $3k$ in the sense of Ohtsuki [69]. It would be natural to ask the topological meaning of J_M mod $\Phi_d(q)$ for $d \in \mathrm{Fun}^0(\mathbb{N},\mathbb{Z}_+)$, where $\mathrm{Fun}^0(\mathbb{N},\mathbb{Z}_+)$ denote the set of functions from $\mathbb{N} = \{1,2,\dots\}$ into $\mathbb{Z}_+ = \{0,1,\dots\}$ vanishing for all but finitely many elements of $\mathbb{N}$. J_M mod $\Phi_d(q)$ is not an Ohtsuki finite type invariant in general. Thus it would be interesting to have a variant of the notion of Ohtsuki finite type invariants in which J_M mod $\Phi_d(q)$ is of "finite type". In what follows, we give a conjectural definition of such a notion of finite type.

Let $\mathcal{M}$ denote the set of orientation-preserving homeomorphism classes of integral homology spheres. Let $\mathbb{Z}\mathcal{M}$ denote the free abelian group generated by $\mathcal{M}$, which has a standard ring structure with multiplication induced by connected sum. The *Ohtsuki filtration*

$$\mathbb{Z}\mathcal{M} = F_0 \supset F_1 \supset F_2 \supset \cdots$$

is defined as follows. For $k \geq 0$, let F_k denote the $\mathbb{Z}$-submodule of $\mathbb{Z}\mathcal{M}$ spanned by the alternating sums

$$[M; L_1,\dots,L_k] = \sum_{L'\subset\{L_1,\dots,L_k\}} (-1)^{|L'|} M_{L'}$$

for all pairs of $M \in \mathcal{M}$ and algebraically-split, ± 1-framed links $L = \{L_1,\dots,L_k\}$ in M. Here the sum runs over all the sublinks L' of L, $|L'|$ denotes the number of components of L', and $M_{L'}$ denotes the result from M of surgery along L'. A $\mathbb{Z}$-linear map $f\colon \mathbb{Z}\mathcal{M} \to A$ with A an abelian group is called an *Ohtsuki invariant of type* k if $f|_{F_{k+1}} = 0$.

Now we generalize the notion of finite type. For $d \in \mathrm{Fun}^0(\mathbb{N},\mathbb{Z}_+)$, let F_d denote the $\mathbb{Z}$-submodule of $\mathbb{Z}\mathcal{M}$ generated by the alternating sums $[M; L_1,\dots,L_k]$, where $M \in \mathcal{M}$ and $L = \{L_1,\dots,L_k\}$ is an algebraically-split framed link in M with framings in $\{1/m \mid m \in \mathbb{Z}, m \neq 0\}$ such that for each $n \in \mathbb{N}$ there is exactly $d(n)$ components of L of framing $\pm n$. One sees easily that the F_d, $d \in \mathrm{Fun}^0(\mathbb{N},\mathbb{Z}_+)$, form an inverse system of $\mathbb{Z}$-submodules of $\mathbb{Z}\mathcal{M}$. Note that for $k \geq 0$, F_k is equal to F_{d_k}, where $d_k(n) = k\delta_{n,1}$.

A $\mathbb{Z}$-linear map $f\colon \mathbb{Z}\mathcal{M} \to A$ with A an abelian group is said to be of *cyclotomic finite type* if $f|F_d = 0$ for some d.

Theorem 15.6 implies that for all $m \in \mathbb{N}$, $J_M \mod (q^m - 1)$, is of cyclotomic finite type. It follows that for all $\zeta \in \mathcal{Z}$, the WRT invariant $\tau_\zeta(M)$ is of cyclotomic finite type.

Conjecture 15.7. *For each $d \in \mathrm{Fun}^0(\mathbb{N}, \mathbb{Z}_+)$, there is $d' \in \mathrm{Fun}^0(\mathbb{N}, \mathbb{Z}_+)$ such that the ring homomorphism*

$$J\colon \mathbb{Z}\mathcal{M} \to \widehat{\mathbb{Z}[q]}, \quad M \mapsto J_M,$$

maps $F_{d'}$ into $\Phi_d(q)\widehat{\mathbb{Z}[q]}$. Thus $J_M \mod (\Phi_d(q))$ is of cyclotomic finite type. Consequently, J induces a continuous ring homomorphism

$$J\colon \widehat{\mathbb{Z}\mathcal{M}} \to \widehat{\mathbb{Z}[q]},$$

where

$$\widehat{\mathbb{Z}\mathcal{M}} = \varprojlim_{d \in \mathrm{Fun}^0(\mathbb{N}, \mathbb{Z}_+)} \mathbb{Z}\mathcal{M}/F_d.$$

If M is an integral homology sphere and if $d \in \mathrm{Fun}^0(\mathbb{N}, \mathbb{Z}_+)$, then there is an integral homology sphere M' not homeomorphic to M, such that $M - M' \in F_d$. Indeed, if L is an algebraically-split, Brunnian framed link in M with framings in $\{1/m \mid m \in \mathbb{Z}, m \neq 0\}$ such that for each $n \in \mathbb{N}$ there are at least $d(n)$ components of L of framings $\pm n$, then one can show that $M_L - M \in F_d$ by modifying a well-known arguments in the study of Ohtsuki finite type invariants. For a certain type of Brunnian links (e.g., iterated Bing doubles of Borromean rings), one can show that M_L and M are not homeomorphic, using the Le-Murakami-Ohtsuki invariant [52].

Recall that for integral homology spheres M the Le-Murakami-Ohtsuki invariant $Z^{\mathrm{LMO}}(M)$ is a "universal finite type invariant" [48], i.e., every Ohtsuki finite type invariant factors through $Z^{\mathrm{LMO}}(M)$. It would be interesting to have a "cyclotomic finite type" version $Z^{\mathrm{cyc}}(M)$ of the Le-Murakami-Ohtsuki invariant which recovers *via ring homomorphisms* the unified sl_2 invariant J_M (as well as the invariant for the other simple Lie algebras, see Section 16.2 below) and the Le-Murakami-Ohtsuki invariant $Z^{\mathrm{LMO}}(M)$. As we mentioned in the introduction, $Z^{\mathrm{LMO}}(M)$ determines J_M. However, this determination is not via a ring homomorphism, i.e., there is no natural homomorphism from the ring $\mathcal{A}(\emptyset)$, in which the Le-Murakami-Ohtsuki invariant takes values, into $\widehat{\mathbb{Z}[q]}$.

Remark 15.8. Similarly to the Cochran and Melvin's generalization [4] of the Ohtsuki finite type invariants, one can generalize the above definition of "cyclotomic finite type" to oriented, connected 3-manifolds by using surgeries along null-homologous, algebraically-split framed links with framings in $\{1/m \mid m \in \mathbb{Z}, m \neq 0\}$. It would be natural to expect that the generalization of J_M to rational homology spheres defined in [3, 51] (see Section 16.3 below) modulo $(\Phi_d(q))$, $d \in \mathrm{Fun}^0(\mathbb{N}, \mathbb{Z}_+)$, would be of "cyclotomic finite type" in the above-explained sense.

16. Concluding remarks

In this section, we give some remarks and discussions.

16.1. **The WRT invariants as limiting values of holomorphic functions.** As explained in the introduction, the invariant J_M for an integral homology sphere M unifies the WRT invariants $\tau_\zeta(M)$ for all the roots of unity in an algebraic way. It should be remarked that there is another *analytic* approach to "unify" the WRT invariants by realizing the radial limiting values of a holomorphic function defined on the unit disk $\{z \in \mathbb{C};\ |z| < 1\}$, studied in [44, 45, 46, 47, 24, 25, 26, 27].

Unfortunately, the invariant $J_M \in \widehat{\mathbb{Z}[q]}$ does not immediately determine such a holomorphic function, i.e., there is no natural homomorphism from the ring $\widehat{\mathbb{Z}[q]}$ to the ring of holomorphic function on $|q| < 1$. However, some expressions for J_M may define well-defined holomorphic functions for on $|q| < 1$ as we explain below for the Poincaré homology sphere $\Sigma(2,3,5)$.

We have

$$J_{\Sigma(2,3,5)} = \sum_{n\geq 0} q^n \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n+1})}{1-q} \in \widehat{\mathbb{Z}[q]}. \tag{16.1}$$

The modified WRT invariant $W(\zeta) = \zeta(\zeta-1)\tau_\zeta(\Sigma(2,3,5))$, $\zeta \in \mathcal{Z}$, defined by Lawrence and Zagier [47], is unified into

$$W(q) = q(q-1)J_{\Sigma(2,3,5)} \in \widehat{\mathbb{Z}[q]}.$$

For our purpose, it is useful to modify it further as

$$\begin{aligned} B(q) &= 1 - W(q) = 1 + q(1-q)J_{\Sigma(2,3,5)} \\ &= \sum_{n\geq 0} q^n (q^n;q)_n, \end{aligned} \tag{16.2}$$

where $(q^n;q)_n = (1-q^n)(1-q^{n+1})\cdots(1-q^{2n-1})$. A beautiful observation by Lawrence and Zagier [47] is that the series

$$A(q) = \sum_{n\geq 1} \chi_+(n) q^{\frac{n^2-1}{120}} = 1 + q + q^3 + q^7 - q^8 - \cdots \in \mathbb{Z}[[q]],$$

which converges on $|q| < 1$, converges radially at each root of unity ζ to $2(1 - W(\zeta)) = 2B(\zeta)$. They also showed that the radial asymptotic expansion of $A(q)$ at $q = 1$ gives 2 times the Ohtsuki series $\tau^O(M)$.

Note that the formula (16.2) for $B(q)$ can also define a power series in q. (Actually, (16.2) defines an element of $\varprojlim_n \mathbb{Z}[q]/(q^n(q)_n) \cong \widehat{\mathbb{Z}[q]} \times \mathbb{Z}[[q]]$.) Let $B(q)_{\mathbb{Z}[[q]]} \in \mathbb{Z}[[q]]$ denote this element. Hikami [26] proved

$$A(q) = B(q)_{\mathbb{Z}[[q]]} \in \mathbb{Z}[[q]].$$

Thus, there are two ways to obtain the value of the (modified) WRT invariants from $B(q)$. One is just to evaluate at roots of unity, and the other is to expand it in q, take the radial limit at roots of unity and then multiply by $\frac{1}{2}$. The meaning of the factor $\frac{1}{2}$ is not clear to the author. It would be natural to expect that the radial asymptotic expansion at each root of unity ζ is 2 times the power series $\iota_\zeta(B(q)) \in \mathbb{Z}[\zeta][[q-\zeta]]$.

It would be natural to expect that certain formulas (16.1) for J_M of integral homology spheres M may give a power series in q whose radial asymptotic behavior at a root of unity ζ is closely related to the "algebraic behavior near ζ" obtained by the map $\iota_\zeta\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q-\zeta]]$. For example, for an integral homology sphere $M_{i,j,k}$ with $i,j,k > 0$ defined in Section 14.3, the formula 14.8 of $J_{M_{i,j,k}}$ defines an element of $\mathbb{Z}[[q]]$ which converges on $|q|<1$. Note that the Poincaré homology sphere $\Sigma(2,3,5)$ is the special case $M_{1,1,1}$.

16.2. **Generalizations to simple Lie algebra.** In a joint work with Le [23], for each simple Lie algebra $\mathfrak{g}$, we will generalize the invariant $J_M \in \widehat{\mathbb{Z}[q]}$ of integral homology sphere M to an invariant $J^{\mathfrak{g}}_M \in \widehat{\mathbb{Z}[q]}$. This invariant can be characterized by the property that for each root of unity ζ such that the quantum $\mathfrak{g}$ invariant $\tau^{\mathfrak{g}}_M$ is well-defined, we have $\mathrm{ev}_\zeta(J^{\mathfrak{g}}_M)=\tau^{\mathfrak{g}}_\zeta$. This specialization property holds also for the projective quantum $\mathfrak{g}$ invariant. See [71, Conjecture 7.29] for a precise statement.

16.3. **Rational homology spheres.** The invariant J_M has been generalized by Beliakova, Blanchet and Le [3] to closed 3-manifold whose first homology group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and by Le [51] to rational homology spheres. These invariants take values in modifications of the ring $\widehat{\mathbb{Z}[q]}$. We briefly describe these invariants below. See the original papers for the details.

In [3], it is proved that there is an invariant of closed 3-manifolds M with $H_1M \cong \mathbb{Z}/2\mathbb{Z}$ with values in the completion

$$\widehat{\mathbb{Z}[v]}_2 := \varprojlim_n \mathbb{Z}[q^{1/2}]/\Big(\prod_{i=1}^{n}(1+(-q^{1/2})^i)\Big),$$

which specializes to the a normalized version of the $SO(3)$ quantum invariant for all $q^{1/2}$ a root of unity of order $\not\equiv 2(4)$. (In the notation of [18], we have $\widehat{\mathbb{Z}[v]}_2 = \mathbb{Z}[q^{1/2}]^{\{n\in\mathbb{N}\ \mid\ n\not\equiv 2(4)\}}$.) A generalization to closed spin manifolds with $H_1\cong\mathbb{Z}/2\mathbb{Z}$ is also given.

In [51], it is proved that for a rational homology sphere M with $\max\{\mathrm{ord}(g)\mid g\in H_1M\}=d$, there is an invariant I_M of M with values in

$$\hat{\Lambda}_d := \varprojlim_{g\in \mathrm{Fun}^0(\mathbb{N}_d,\mathbb{Z}_+)} \mathbb{Z}[1/d][q^{1/d}]/\Big(\prod_{n\in\mathbb{N}_d}\big(\Phi_n(q^{1/d})^{g(n)}\big)\Big).$$

where $\mathbb{N}_d=\{n\in\mathbb{N}\colon n \text{ coprime with } d\}$, and $\mathrm{Fun}^0(\mathbb{N}_d,\mathbb{Z}_+)$ denotes the set of functions from $\mathbb{N}_d$ to $\mathbb{Z}_+=\{0,1,\ldots\}$ vanishing for all but finitely many elements of $\mathbb{N}_d$. The invariant I_M specializes (essentially) to the $SO(3)$ quantum invariant for q any root of unity of order odd and coprime with d.

For M with $H_1(M;\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}$, the Beliakova-Blanchet-Le invariant is better than Le's invariant in the sense that the former is defined in a smaller subring, i.e., has stronger integrality. It would be interesting to generalize these two invariants to an invariant of all rational homology spheres which is defined in smaller rings than the $\hat{\Delta}_d$.

16.4. **Bottom tangles in homology handlebodies.** An interesting generalization of integral homology spheres (more precisely, integral homology 3-balls) are *homology cylinders* over a surface [12, 15] (see also [8, 14]). (In [15] they are called "homologically-trivial homology cobordisms".) For $g\ge 0$, let $\mathcal{H}_{g,1}$ denote the monoid of homology cylinders over a surface $\Sigma_{g,1}$ of genus 1 with one boundary component. For $g=0$, $\mathcal{H}_{0,1}$ is identified with the monoid of integral homology spheres. The Torelli group $\mathcal{I}_{g,1}$ for $\Sigma_{g,1}$ is regarded as a subgroup of $\mathcal{H}_{g,1}$ via the mapping cylinder construction.

In [20, Section 14.5], we defined the "category of bottom tangles in handlebodies" $\mathcal{B}$ and the "category of bottom tangles in homology handlebodies" $\bar{\mathcal{B}}$. Here $\mathcal{B}$ is a subcategory of $\bar{\mathcal{B}}$, and $\bar{\mathcal{B}}$ may be regarded as a subcategory of the category $\mathcal{C}$ of cobordisms of surfaces with connected boundaries as introduced by Crane and Yetter [5] and by Kerler [36]. These categories are braided categories and generated by a braided Hopf algebra and some additional morphisms. Homology cylinders are contained in $\bar{\mathcal{B}}$ as morphisms.

In a future paper, using a completion of the algebra $\mathcal{U}_q^{\mathrm{ev}}$, we will construct a braided functor $\bar{J}$ from $\bar{\mathcal{B}}$ to a certain braided category defined over $\widehat{\mathbb{Z}[q]}$. This functor maps each integral homology 3-ball M' to the multiplication map $\widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$, $x \mapsto xJ_M$, where M is the integral homology sphere obtained from M' by capping off the boundary with a ball. Thus, $\bar{J}$ may be regarded as a generalization of J_M into a large class of 3-manifolds with boundary, which contains all homology cylinders. This functor restricts to a representation of the Torelli group of each $\Sigma_{g,1}$. It is expected that the functor $\bar{J}$ would lead to some refinements of the integral structures in topological quantum field theories [9, 11, 10].

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References

[1] Y. Akutsu, T. Deguchi and T. Ohtsuki, Invariants of colored links. *J. Knot Theory Ramifications* **1** (1992), no. 2, 161–184.
[2] D. Bar-Natan and S. Garoufalidis, On the Melvin-Morton-Rozansky conjecture. *Invent. Math.* **125** (1996), no. 1, 103–133.
[3] A. Beliakova, C. Blanchet and T. Le, Laplace transform and universal sl(2) invariants. preprint, math.QA/0509394.
[4] T. D. Cochran and P. Melvin, Finite type invariants of 3-manifolds. *Invent. Math.* **140** (2000), 45–100.
[5] L. Crane and D. Yetter, On algebraic structures implicit in topological quantum field theories. *J. Knot Theory Ramifications* **8** (1999), no. 2, 125–163.
[6] R. Fenn and C. Rourke, On Kirby's calculus of links. *Topology* **18** (1979), no. 1, 1–15.
[7] S. Garoufalidis and T. T. Q. Le, Is the Jones polynomial of a knot really a polynomial?. preprint, math.GT/0601139.
[8] S. Garoufalidis and J. Levine, Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism. *Graphs and patterns in mathematics and theoretical physics*, 173–203, Proc. Sympos. Pure Math., 73, Amer. Math. Soc., Providence, RI, 2005.
[9] P. M. Gilmer, Integrality for TQFTs. *Duke Math. J.* **125** (2004), no. 2, 389–413.
[10] P. M. Gilmer and G. Masbaum, Integral lattices in TQFT. preprint, math.QA/0411029.
[11] P. M. Gilmer, G. Masbaum and P. van Wamelen, Integral bases for TQFT modules and unimodular representations of mapping class groups. *Comment. Math. Helv.* **79** (2004), no. 2, 260–284.
[12] M. Goussarov, Finite type invariants and n-equivalence of 3-manifolds. *C. R. Acad. Sci. Paris Sèr. I Math.* **329** (1999), no. 6, 517–522.
[13] F. Q. Gouvêa, *p-adic numbers. An introduction. Second edition.* Universitext. Springer-Verlag, Berlin, 1997.
[14] N. Habegger, Milnor, Johnson, and tree level perturbative invariants. preprint.
[15] K. Habiro, Claspers and finite type invariants of links. *Geom. Topol.* **4** (2000), 1–83.
[16] K. Habiro, On the colored Jones polynomials of some simple links. *Recent progress towards the volume conjecture (Kyoto, 2000)*, RIMS Kokyuroku, No. 1172, (2000), 34–43.
[17] K. Habiro, On the quantum sl_2 invariants of knots and integral homology spheres. *Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs, 4, (2002), 55-68.

[18] K. Habiro, Cyclotomic completions of polynomial rings. *Publ. Res. Inst. Math. Sci.* **40** (2004), 1127–1146.

[19] K. Habiro, An integral form of the quantized enveloping algebra of sl_2 and its completions. preprint math.GT/0605313.

[20] K. Habiro, Bottom tangles and universal invariants. to appear in *Alg. Geom. Topol.*

[21] K. Habiro, Refined Kirby calculus for integral homology spheres. preprint, math.GT/0509039.

[22] K. Habiro, Spanning surfaces and the Jones polynomial. in preparation.

[23] K. Habiro and T. T. Q. Le, in preparation.

[24] K. Hikami, Quantum invariant, modular form, and lattice points. *Int. Math. Res. Not.* **2005** (2005), no. 3, 121–154.

[25] K. Hikami, On the quantum invariant for the Brieskorn homology spheres. *Internat. J. Math.* **16** (2005), no. 6, 661–685.

[26] K. Hikami, Mock (false) theta functions as quantum invariants. *Regul. Chaotic Dyn.* **10** (2005), no. 4, 509–530.

[27] K. Hikami, Quantum invariants, modular forms, and lattice points II. preprint, math.QA/0604091.

[28] J. Hoste, A formula for Casson's invariant. *Trans. Amer. Math. Soc.* **297** (1986), no. 2, 547–562.

[29] V. Huynh and T. T. Q. Le, On the colored Jones polynomial and the Kashaev invariant. preprint, math.GT/0503296.

[30] V. F. R. Jones, A new knot polynomial and von Neumann algebras. *Notices Amer. Math. Soc.* **33** (1986), no. 2, 219–225.

[31] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* **126** (1987), 335–388.

[32] R. M. Kashaev, A link invariant from quantum dilogarithm. *Modern Phys. Lett. A* **10** (1995), 1409–1418.

[33] L. H. Kauffman, Gauss codes, quantum groups and ribbon Hopf algebras. *Rev. Math. Phys.* **5** (1993), no. 4, 735–773.

[34] L. Kauffman and D. E. Radford, Oriented quantum algebras, categories and invariants of knots and links. *J. Knot Theory Ramifications* **10** (2001), no. 7, 1047–1084.

[35] T. Kerler, Genealogy of non-perturbative quantum invariants of 3–manifolds: The surgical family. *Geometry and physics (Aarhus, 1995)*, 503–547, Lect. Notes Pure Appl. Math., 184, Dekker, 1997.

[36] T. Kerler, Bridged links and tangle presentations of cobordism categories. *Adv. Math.* **141** (1999), no. 2, 207–281.

[37] R. Kirby, A calculus for framed links in S^3. *Invent. Math.* **45** (1978), no. 1, 35–56.

[38] R. Kirby and P. Melvin, The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2, C)$. *Invent. Math.* **105** (1991), no. 3, 473–545.

[39] R. Kirby, P. Melvin and X. Zhang, Quantum invariants at the sixth root of unity. *Comm. Math. Phys.* **151** (1993), no. 3, 607–617.

[40] N. Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions. Second edition.* Graduate Texts in Mathematics, 58. Springer-Verlag, New York, 1984.

[41] A. Kricker and B. Spence, Ohtsuki's invariants are of finite type. *J. Knot Theory Ramifications* **6** (1997), no. 4, 583–597.

[42] R. J. Lawrence, A universal link invariant using quantum groups. *Differential geometric methods in theoretical physics (Chester, 1988)*, 55–63, World Sci. Publishing, 1989.

[43] R. J. Lawrence, A universal link invariant, *The interface of mathematics and particle physics (Oxford, 1988)*, Inst. Math. Appl. Conf. Ser. New Ser., 24, 151–156, Oxford University Press, 1990.

[44] R. J. Lawrence, Asymptotic expansions of Witten-Reshetikhin-Turaev invariants for some simple 3-manifolds. *J. Math. Phys.* **36** (1995), 6106–6129.

[45] R. J. Lawrence, Witten-Reshetikhin-Turaev invariants of 3-manifolds as holomorphic functions. *Geometry and physics (Aarhus, 1995)*, Lect. Notes Pure Appl. Math. 184, 363-377 Marcel Dekker, 1997.

[46] R. Lawrence and L. Rozansky, Witten-Reshetikhin-Turaev invariants of Seifert manifolds. *Commun. Math. Phys.* **205**, no.2, 287-314 (1999).

[47] R. Lawrence and D. Zagier, Modular forms and quantum invariants of 3-manifolds. *Asian J. Math.* **3** (1999), 93–107.

[48] T. T. Q. Le, An invariant of integral homology 3-spheres which is universal for all finite type invariants. *Solitons, geometry, and topology: on the crossroad*, 75–100, Amer. Math. Soc. Transl. Ser. 2, 179, Amer. Math. Soc., Providence, RI, 1997.

[49] T. T. Q. Le, On perturbative $PSU(n)$ invariants of rational homology 3-spheres. *Topology* **39** (2000), no. 4, 813–849.

[50] T. T. Q. Le, Quantum invariants of 3-manifolds: integrality, splitting, and perturbative expansion. *Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, 1999). Topology Appl.* **127** (2003), no. 1-2, 125–152.

[51] T. T. Q. Le, Strong integrality of quantum invariants of 3-manifolds. preprint, math.GT/0512433.

[52] T. T. Q. Le, J. Murakami and T. Ohtsuki, On a universal perturbative invariant of 3-manifolds. *Topology* **37** (1998), no. 3, 539–574.

[53] H. C. Lee, Tangles, links, and twisted quantum groups. *Physics, geometry and topology (Banff, 1989)*, 623–655, NATO Adv. Sci. Inst. Ser. B Phys., 238, Plenum, 1990.

[54] X. Lin and Z. Wang, On Ohtsuki's invariants of integral homology 3-spheres. *Acta Math. Sinica* **15** (1999), no. 3, 293–316.

[55] X.-S. Lin and Z. Wang, Fermat limit and congruence of Ohtsuki invariants. *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, 321–333, Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry, 1999.

[56] S. Majid, Algebras and Hopf algebras in braided categories. *Advances in Hopf algebras*, Marcel Dekker Lec. Notes Pure and Applied Math., 158, (1994) 55–105.

[57] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, 1995.

[58] G. Masbaum, Skein-theoretical derivation of some formulas of Habiro. *Alg. Geom. Top.* **3** (2003), 537–556.

[59] G. Masbaum and J. D. Roberts, A simple proof of integrality of quantum invariants at prime roots of unity. *Math. Proc. Cambridge Philos. Soc.* **121** (1997), no. 3, 443–454.

[60] G. Masbaum and H. Wenzl, Integral modular categories and integrality of quantum invariants at roots of unity of prime order. *J. Reine Angew. Math.* **505** (1998), 209–235.

[61] P. M. Melvin and H. R. Morton, The coloured Jones function. *Comm. Math. Phys.* **169** (1995), 501–520.

[62] H. Murakami, Quantum $SU(2)$-invariants dominate Casson's $SU(2)$-invariant. *Math. Proc. Cambridge Philos. Soc.* **115** (1994), no. 2, 253–281.

[63] H. Murakami, Quantum $SO(3)$-invariants dominate the $SU(2)$-invariant of Casson and Walker. *Math. Proc. Cambridge Philos. Soc.* **117** (1995), no. 2, 237–249.

[64] H. Murakami and J. Murakami, The coloured Jones polynomials and the simplicial volume of a knot. *Acta Math.* **186** (2001), no. 1, 85–104.

[65] J. Murakami, A state model for the multi-variable Alexander polynomial. *Pacific J. Math.* **157** (1993), 109–135.

[66] T. Ohtsuki, Colored ribbon Hopf algebras and universal invariants of framed links. *J. Knot Theory Ramifications* **2** (1993), no. 2, 211–232.

[67] T. Ohtsuki, A polynomial invariant of integral homology 3-spheres. *Math. Proc. Cambridge Philos. Soc.* **117** (1995), no. 1, 83–112.

[68] T. Ohtsuki, A polynomial invariant of rational homology 3-spheres. *Invent. Math.* **123** (1996), no. 2, 241–257.

[69] T. Ohtsuki, Finite type invariants of integral homology 3-spheres. *J. Knot Theory Ramifications* **5** (1996), no. 1, 101–115.

[70] T. Ohtsuki, The perturbative $SO(3)$ invariant of rational homology 3-spheres recovers from the universal perturbative invariant. *Topology* **39** (2000), 1103–1135.

[71] T. Ohtsuki (ed.), Problems on invariants of knots and 3-manifolds. *Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs, 4, (2002), 377–572.

[72] N. Yu. Reshetikhin, Quasitriangle Hopf algebras and invariants of tangles. (Russian) , *Algebra Anal.* **1** (1989), no. 2, 169–188; English translation *Leningr. Math. J.* **1** (1990), no. 2, 491–513.

[73] N. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103** (1991), no. 3, 547–597.

[74] L. Rozansky, The universal R-matrix, Burau representation, and the Melvin-Morton expansion of the colored Jones polynomial. *Adv. Math.* **134** (1998), 1–31.

[75] L. Rozansky, On p-adic properties of the Witten-Reshetikhin-Turaev invariant. preprint, math.QA/9806075.

[76] T. Takata and Y. Yokota, The $PSU(N)$ invariants of 3-manifolds are algebraic integers. *J. Knot Theory Ramifications* **8** (1999), no. 4, 521–532.
[77] E. Witten, Quantum field theory and the Jones polynomial. *Comm. Math. Phys.* **121** (1989), no. 3, 351–399.

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