Numerically trivial involutions of Enriques surfaces

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It is known that a nontrivial automorphism of a K3 surface acts nontrivially on its cohomology group ([1, Chap. VIII, Proposition (11.3)]). But this is not true for an Enriques surface S. An automorphism of S is said to be *numerically trivial* (resp. *cohomologically trivial*) if it acts on $H^2(S,\mathbb{Q})$ (resp. $H^2(S,\mathbb{Z})$) trivially. In this note, correcting [3], we classify the numerically trivial involutions of Kummer type.

Let S be a (minimal) *Enriques surface*, that is, a compact complex surface with $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$ and $2K_S \sim 0$, and σ a numerically trivial involution of S. Then σ lifts to an involution of the covering K3 surface $\tilde{S}$. More precisely, there are two lifts. One acts on $H^0(\tilde{S},\Omega^2)$ trivially and the other by -1. We denote them by σ_K and σ_R, respectively. Their product $\sigma_K\sigma_R$ is the covering involution ε of $\tilde{S} \to S$. We denote the anti-invariant part of the action of σ_R on $H^2(\tilde{S},\mathbb{Z})$ by N_R. Then N_R is isomorphic to either $U(2) \perp U(2)$ or $U \perp U(2)$ as a lattice ([3, Proposition (2.5)]). In the sequel we assume that $N_R \simeq U(2) \perp U(2)$ and call such σ *Kummer type*. The lattice $U \perp U$ is isomorphic to $M_2(\mathbb{Z})$, the group of 2×2 matrices of integral entries endowed with the bilinear form $(A.\,A) = 2\det A$. Hence, there exits a pair of elliptic curves E' and E'' such that $N_R(1/2)$ is isomorphic to $H^1(E',\mathbb{Z}) \otimes H^1(E'',\mathbb{Z})$ as a polarized Hodge structure. By the Torelli theorem for Kummer (or K3) surfaces ([1, Chap. VIII]), there exists an isomorphism ψ between $\tilde{S}$ and the Kummer surface of the product $E' \times E''$ such that the diagram

$$\begin{array}{ccccc} & \tilde{S} & \stackrel{\psi}{\to} & Km(E' \times E'') & \\ \sigma_R & \downarrow & & \downarrow & \mu \\ & \tilde{S} & \stackrel{\psi}{\to} & Km(E' \times E'') & \end{array} \quad (1)$$

is commutative, where μ is the involution induced by $(id_{E'}, -id_{E''})$.

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Example 1 ([3, Proposition (4.8)]) Let β be the involution of $Km(E' \times E'')$ induced by the translation of $E' \times E''$ by a 2-torsion point a with $a \notin E' \times 0 \cup 0 \times E''$. Then $\mu\beta$ has no fixed points and μ, or β, induces a cohomologically trivial involution of the Enriques surface $Km(E' \times E'')/\mu\beta$.

Let $\{p'_1, \ldots, p'_4\}$ and $\{p''_1, \ldots, p''_4\}$ be the branch of the double coverings $E' \to \mathbb{P}^1 \simeq E'/(-id)$ and $E'' \to \mathbb{P}^1 \simeq E''/(-id)$, respectively. Then the quotient $\tilde{S}/\sigma_R$ is isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 points (p'_i, p''_j), $1 \le i, j \le 4$. The above involution β is induced by an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 2 Assume that

(*) the ordered 4-tuples $(p'_1, \ldots, p'_4)$ and $(p''_1, \ldots, p''_4) \in (\mathbb{P}^1)^4$ are not projectively equivalent

and let β be the involution of $Km(E' \times E'')$ induced by the *standard Cremona involution* of $\mathbb{P}^1 \times \mathbb{P}^1$ with center the four points (p'_i, p''_i), $1 \le i \le 4$ (§1). Then $\mu\beta$ has no fixed points and μ induces a numerically trivial involution of the Enriques surface $Km(E' \times E'')/\mu\beta$ (Proposition 7).

This was overlooked in [3] and first found by Kondo. More precisely, the special case of Example 2 with $E \simeq E'' \simeq \mathbb{C}/(\mathbb{Z} + \mathbb{Z}e^{2\pi\sqrt{-1}/3})$ was studied in [2, (3.5)] as an Enriques surface whose automorphism group is finite. The following is the main result of this note:

Theorem 3 *Every numerically trivial involution of Kummer type of an Enriques surface is obtained in the way of Example 1 or 2.*

We have also the following since the involution of $Km(E' \times E'')/\mu\beta$ in Example 2 is not cohomologically trivial (Proposition 8).

Corollary 4 *Every cohomologically trivial involution of Kummer type is obtained in the way of Example 1.*

Notation U denotes the rank 2 lattice given by the symmetric matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The lattice obtained from a lattice L by replacing the bilinear form (.) with r(.), r being a suitable rational number, is denoted by $L(r)$.

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§1 Cremona involution of a quadric surface

The Enriques surface in Example 2 is closely related with a del Pezzo surface B of degree 4 and its small involution. [1] For our purpose it is most convenient to describe B as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$. We identify $\mathbb{P}^1 \times \mathbb{P}^1$ with a smooth quadric surface Q in $\mathbb{P}^3 = \mathbb{P}_{(x_1:x_2:x_3:x_4)}$.

Let $p_1 = (p'_1, p''_1), \ldots, p_4 = (p'_4, p''_4)$ be four points of $\mathbb{P}^1 \times \mathbb{P}^1$ which satisfy

$(**)$ $p'_1, \ldots, p'_4$ are distinct and $p''_1, \ldots, p''_4$ are distinct.

In terms of a smooth quadric, this is equivalent to

$(**')$ any line $\overline{p_i p_j}$, $1 \leq i < j \leq 4$, is not contained in Q.

We also assume the condition $(*)$ in the introduction, or equivalently,

$(*')$ $p_1, \ldots, p_4 \in Q \subset \mathbb{P}^3$ is not contained in a plane.

We take a system of homogeneous coordinates of $\mathbb{P}^3$ such that $p_1, \ldots, p_4$ are the coordinate points $(1:0:0:0), \ldots, (0:0:0:1)$. Then the equation of Q is of the form $\sum_{1 \leq i < j \leq 4} a_{ij} x_i x_j = 0$. By the assumption $(**')$, all coefficients a_{ij}'s are nonzero. Hence, replacing $x_1, \ldots, x_4$ by their suitable constant multiplications, we may and do assume that $Q \subset \mathbb{P}^3$ is defined by

$$a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2 + (x_1 + x_2 + x_3) x_4 = 0 \tag{2}$$

for some nonzero constants a_1, a_2 and $a_3 \in \mathbb{C}$.

Now we define a birational involution τ' of Q by

$$(x_1 : x_2 : x_3 : x_4) \mapsto \left(\frac{a_1}{x_1} : \frac{a_2}{x_2} : \frac{a_3}{x_3} : \frac{a_1 a_2 a_3}{x_4}\right)$$

and call it the *standard Cremona involution* of Q (or $\mathbb{P}^1 \times \mathbb{P}^1$) with center $p_1, \ldots, p_4$. The following is easily verified:

Lemma 5 (1) *The indeterminacy locus of $\tau' : Q \cdots \to Q$ is $\{p_1, \ldots, p_4\}$.*

(2) *For each $1 \leq i \leq 4$, the conic $C'_i : Q \cap \{x_i = 0\}$ is contracted to the point p_i by τ'.*

(3) *The fixed points of τ' are $(\varepsilon_1 \sqrt{a_1} : \varepsilon_2 \sqrt{a_2} : \varepsilon_2 \sqrt{a_3} : \sqrt{a_1 a_2 a_3})$, where all ε_i's are ± 1 and satisfy $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$.*

[1] An automorhism of a surface is *small* if all fixed points are isolated.

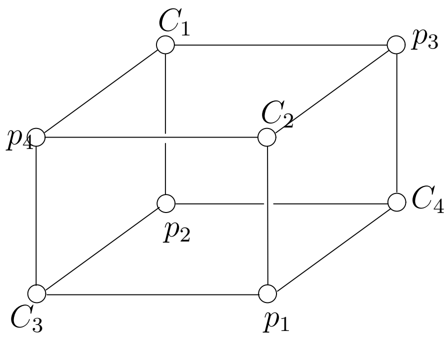
Let B be the blow-up of Q at $p_1, \dots, p_4$. Then B is a del Pezzo surface of degree 4 by $(*')$ and $(**')$. B contains 16 smooth rational curves of degree 1 with respect to the anti-canonical divisor $-K_B$:

0) the exceptional divisors over $p_1, \dots, p_4$,

1) the strict transforms of lines in Q passing through one of $p_1, \dots, p_4$, and

2) the strict transforms C_i of the four conics C_i', $1 \le i \le 4$, in the lemma.

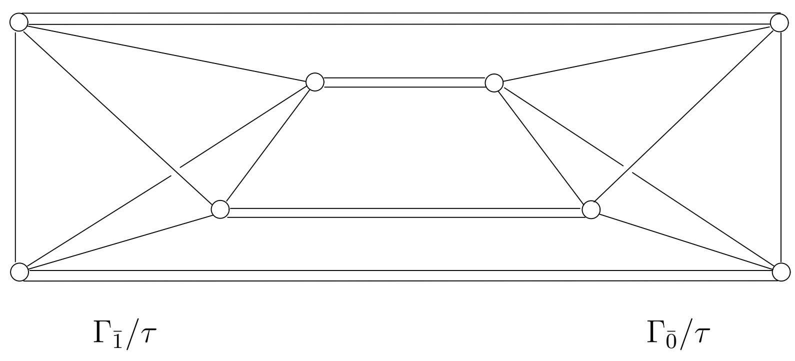
Consider the configuration of the eight curves 0) and 2). The dual graph $\Gamma_{\bar{0}}$ of this configuration is a cube:



The birational involution τ' induces an automorphism of B, which we denote by τ. τ sends each vertex of the cube $\Gamma_{\bar{0}}$ to its antipodal. The same holds for the configuration of the eight curves of 1), whose dual graph is denoted by $\Gamma_{\bar{1}}$. The following is easily verified:

$(***)$ for every curve m in $\Gamma_{\bar{0}}$ (resp. $\Gamma_{\bar{1}}$), there exists an antipodal pair of vertices n and n' in $\Gamma_{\bar{1}}$ (resp. $\Gamma_{\bar{0}}$) such that $(m.n) = (m.n') = 1$ and that m is disjoint from other curves in $\Gamma_{\bar{1}}$ (resp. $\Gamma_{\bar{0}}$).

Therefore, the graph $(\Gamma_{\bar{1}} \cup \Gamma_{\bar{0}})/\tau$ is as follows:



For the later use we compute the cohomological action of the standard Cremona involution. The second cohomology group $H^2(B, \mathbb{Z})$, or equivalently the Picard group of B, is the free abelian group with basis $\{h_1, h_2, e_1, \dots, e_4\}$, where h_1 and h_2 are the pull-backs of two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and $e_1, \dots, e_4$ are the classes of exceptional curves over $p_1, \dots, p_4$.

Lemma 6 *The action of the standard Cremona involution τ on $H^2(B, \mathbb{Z})$ is equal to the composite of the two reflections with respect to orthogonal (-2)-classes $h_1 - h_2$ and $h_1 + h_2 - e_1 - \cdots - e_4$.*

It is also convenient to treat B as the blow-up of the projective plane. Let q_4 and q_5 be the two intersection points of the line $l : x_1 + x_2 + x_3 = 0$ and the conic $C : a_1x_2x_3 + a_2x_1x_3 + a_3x_1x_2 = 0$ in the projective plane $\mathbb{P}^2 = \mathbb{P}_{(x_1:x_2:x_3)}$. By the equation (2), the surface B is the blow-up of $\mathbb{P}^2$ at the three coordinate points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and the two points q_4 and q_5. In this description the standard Cremona involution τ is induced by the quadratic Cremona transformation

$$(x_1 : x_2 : x_3) \mapsto (\frac{a_1}{x_1} : \frac{a_2}{x_2} : \frac{a_3}{x_3}) \tag{3}$$

which interchanges l and C. The cohomology group $H^2(B, \mathbb{Z})$ has $\{h, e'_1, \dots, e'_5\}$ as a standard basis. Here h is the pull-back of a line and $e'_1, \dots, e'_5$ are the classes of exceptional curves. The cohomological action of the transformation (3) on the blow-up of $\mathbb{P}^2$ at the three coordinate points is the reflection r with respect to $h - e'_1 - e'_2 - e'_3$. Since the transformation (3) interchanges q_4 and q_5, the cohomological action of τ is the composite of r and the reflection with respect to $e'_4 - e'_5$. This gives a proof of the lemma.

Let $\mathbb{P}^1_{(1)}$ and $\mathbb{P}^1_{(2)}$ be the projective lines whose inhomogenous coordinates are $y_1 = x_1/x_3$ and $y_2 = x_2/x_3$, respectively. Then the line l and the conic C are transformed to the curves

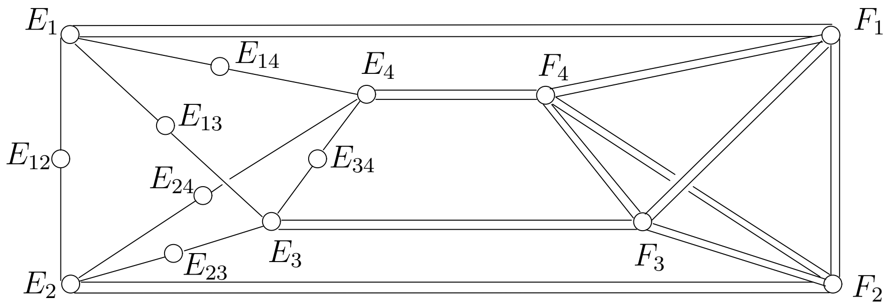
$$y_1 + y_2 + 1 = 0 \quad \text{and} \quad a_2y_1 + a_1y_2 + a_3y_1y_2 = 0 \tag{4}$$

of bidegree $(1, 1)$ on $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$, respectively. The del Pezzo surface B is blow-up of $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$ with center $(0, 0)$, (∞, ∞) and the intersection points of (4), and the involution τ is induced by the automorphism $(y_1, y_2) \mapsto (\frac{a_1}{a_3y_1}, \frac{a_2}{a_3y_2})$ of $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$.

§2 New numerically trivial involutions

We take the double cover of the del Pezzo surface B in the previous section with branch the union of all eight curves in $\Gamma_{\bar{1}}$. It has 12 nodes corresponding to the 12 edges of $\Gamma_{\bar{1}}$. Its minimal resolution is the Kummer surface $Km(E' \times E'')$ of product type. Here E' and E'' are the double covers of $\mathbb{P}^1$ with branch $p'_1, \ldots, p'_4$ and $p''_1, \ldots, p''_4$, respectively. The pull-back of each curve in $\Gamma_{\bar{0}}$ is a smooth rational curve on $Km(E' \times E'')$ by $(***)$. Hence $Km(E' \times E'')$ has 28 smooth rational curves: 12 come from nodes of the branch locus and the rest from the 16 curves on B.

The involution τ lifts to two involutions of $Km(E' \times E'')$. One is symplectic and hence has exactly 8 fixed points ([4]). Since τ has exactly 4 fixed points by Lemma 5, the other lift, denoted by ε, has no fixed points. Hence we obtain an Enriques surface $S = Km(E' \times E'')/\varepsilon$. The 28 smooth rational curves give rise to 14 smooth rational curves on S and the dual graph of their configuration is as follows:



(5)

Let σ be the involution of S induced by the covering involution of $Km(E' \times E'') \to B$. Then σ fixes these 14 smooth rational curves.

Proposition 7 *σ is numerically trivial.*

Proof. Let M_1 be the sublattice of $M = H^2(S, \mathbb{Z})/(\text{torsion})$ generated by the cohomology classes of 10 rational curves E_1, F_2, F_3, F_4 and E_{ij}, $1 \leq i < j \leq 4$. Then M_1 is the orthogonal (direct) sum of the five lattices $D = \langle E_1, E_{12}, E_{13}, E_{14} \rangle$, $F = \langle F_2, F_3, F_4 \rangle$, $\langle E_{23} \rangle$, $\langle E_{24} \rangle$ and $\langle E_{34} \rangle$. D is a negative definite root lattice of type D_4. The intersection form of F is $\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}$ and nondegenerate. Hence M_1 is of rank 10. Therefore, σ is numerically trivial. $\square$

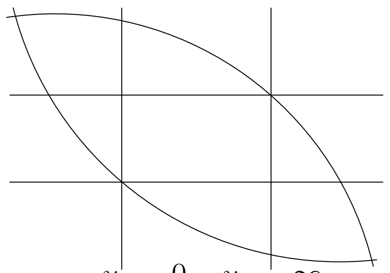
Proposition 8 *σ is not cohomologically trivial.*

Proof. We look at the subdiagram of (5) consisting of $E_1, \ldots, E_4$ and E_{12}, E_{13}, E_{14}. This diagram is of type $\tilde{E}_6$ and the complete linear system of

$$D = 3E_1 + E_2 + E_3 + E_4 + 2E_{12} + 2E_{13} + 2E_{14}$$

defines an elliptic fibration $\pi : S \longrightarrow \mathbb{P}^1$. Since S is an Enriques surface, π has two multiple fibers. Let G_1 and G_2 be their reduced parts. Since $(D.E_{23}) = 2$, G_i, $i = 1, 2$, meets E_{23} at exactly one point, say p_i. By our construction, the fixed point set of $\sigma|_{E_{23}}$ coincides with $E_{23} \cap D$. Hence we have $\sigma(p_1) = p_2$ and $\sigma(G_1) = G_2$. σ is cohomologically nontrivial since G_1 and G_2 differ by the nonzero 2-torsion K_S. $\square$

Remark 9 In terms of $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$ at the end of the previous section, the branch locus of $Km(E' \times E'')/B$ is as follows:



§3 Computation of the periods

In the sequel we fix a pair of elliptic curves E' and E''. Let σ be a numerically trivial involution of an Enriques surface S such that $\tilde{S}$, the universal cover, is the Kummer surface $Km := Km(E' \times E'')$ and that $\sigma_R = \mu$ as in (1). Let σ_K and ε be as in the introduction. We denote the anti-invariant parts of their action on $H^2(Km, \mathbb{Z})$ by N_K and N, respectively. In this section we compute the *period* of S, that is, the polarized Hodge structure of N for two examples in the introduction.

Since σ is numerically trivial, N contains both N_K and N_R. N_K is isomorphic to $E_8(2)$ ([3, Lemma (2.1)]) and the discriminant group of N is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 10}$. Since $N_R \simeq U(2) \perp U(2)$ by assumption, the orthogonal sum $N_K \perp N_R$ is of index two in N. Therefore, there exists a pair of nonzero 2-torsion elements $\alpha_K \in A_{N_K} = (\frac{1}{2}N_K)/N_K$ and $\alpha_R \in A_{N_R} = (\frac{1}{2}N_R)/N_R$ such that $N = N_K + N_R + \mathbb{Z}(x_K, x_R)$, where

$x_K \in \frac{1}{2}N_K$ and $x_R \in \frac{1}{2}N_R$ are representatives of α_K and α_R, respectively. This pair (α_K, α_R) is uniquely determined from the involution σ. We call it the *patching pair* of σ. Since N_K and N_R are orthogonal in N, we have $q_{N_K}(\alpha_K) + q_{N_R}(\alpha_R) = 0$ in $\mathbb{Z}/2\mathbb{Z}$.

Definition 10 A numerically trivial involution (of Kummer type) is *of even type* or *of odd type* according as the common quadratic value[2] $q_{N_K}(\alpha_K) = q_{N_R}(\alpha_R) \in \mathbb{Z}/2\mathbb{Z}$ of patching elements is 0 or 1.

N_K is orthogonal to $H^0(Km, \Omega^2) \subset N_R \otimes \mathbb{C}$ and $N_R(1/2)$ is isomorphic to $H^1(E', \mathbb{Z}) \otimes H^1(E'', \mathbb{Z})$ as a polarized Hodge structure. Hence the period of S is determined by the patching pair.

We recall a basic fact on the cohomology of the Kummer surface $Km(T)$ of a (2-dimensional) complex torus T. $Km(T)$ contains sixteen $(-2)\mathbb{P}^1$'s $\{E_a\}_{a \in T_2}$ parametrized by the 2-torsion subgroup $T_2 \simeq (\mathbb{Z}/2\mathbb{Z})^4$ of T. These generate a sublattice of rank 16 in the cohomology group $H^2(Km(T), \mathbb{Z})$. Since $Km(T)$ is the quotient of the blow-up of T at T_2, $H^2(Km(T), \mathbb{Z})$ contains the image of $H^2(T, \mathbb{Z}) = \bigwedge^2 H^1(T, \mathbb{Z})$ as a sublattice of rank 6. We denote these sublattices by Γ and Λ, respectively. These are orthogonal and generate a sublattice of finite index in $H^2(Km(T), \mathbb{Z})$. The lattice Λ is isomorphic to $U(2) \perp U(2) \perp U(2)$. The discriminant group A_Λ is $(\frac{1}{2}\Lambda)/\Lambda \simeq H^2(T, \mathbb{Z}/2\mathbb{Z})$ and the discriminant form q_Λ is essentially the cup product, that is, $q_\Lambda(\bar{y}) = (y \cup y)/2 \bmod 2$ for $y \in H^2(T, \mathbb{Z})$.

Let $P = \{0, a, b, c\} \subset T_2$ be a subgroup of order 4, or equivalently, a 2-dimensional subspace of T_2. We put $E_P = E_0 + E_a + E_b + E_c \in \Gamma$. We denote the Plücker coordinate of $P^\perp \subset T_2^\vee$ by $\pi_P \in \bigwedge^2 T_2^\vee \simeq H^2(T, \mathbb{Z}/2\mathbb{Z})$ and regard it as an element of $\Lambda/2\Lambda$. The following is easily verified ([1, Chap. VIII, §5]):

Lemma 11 *$(E_P$ mod $2) + \pi_P = 0$ holds in $H^2(Km(T), \mathbb{Z}/2\mathbb{Z})$.*

Now we return to the Kummer surface $Km = Km(E' \times E'')$ of product type. Two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ give two elliptic fibrations $Km \longrightarrow \mathbb{P}^1$. We denote the classes of these fibers by $\tilde{h}_1$ and $\tilde{h}_2 \in H^2(Km, \mathbb{Z})$. These $\tilde{h}_1$ and $\tilde{h}_2$ generate a rank 2 sublattice of Λ which is isomorphic to $U(2)$. Λ is the orthogonal (direct) sum of $\langle \tilde{h}_1, \tilde{h}_2 \rangle$ and N_R.

A subgroup P of order 4 of $(E' \times E'')_2$ is naturally associated with (S, σ) in the two examples:

[2] In [3, §2], it is erroneously stated that this common value is nonzero.

Observation 12 (1) Let $a = (a', a'') \in (E' \times E'')_2$ be a 2-torsion point as in Example 1 and we set $P := \{0, a, (a', 0), (0, a'')\}$. Then P is of order 4 and the Plücker coordinate π_P belongs to $N_R/2N_R$.

(2) Let $P \subset T_2$ be a subgroup of order 4 such that $P \cap ((E')_2 \times 0) = P \cap (0 \times (E'')_2) = 0$ and π_P the Plücker coordinate. Then $\pi_P - \tilde{h}_1 - \tilde{h}_2$ belongs to $N_R/2N_R$. Let β_P be the involution of Km induced by the standard Cremona involution $\beta_{0,P}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ with center the image of P. All (S, σ)'s of Example 2 are obtained from μ and β_P's.

Now we are ready to compute the patching pairs.

Lemma 13 *Let $\Pi \in \Lambda$ be a representative of $\pi_P \in \Lambda/2\Lambda$.*

(1) A numerically trivial involution σ of Example 1 is of even type and the patching pair is $(\Sigma/2, \Pi/2)$ with $\Sigma := E_0 - E_a + E_{(a',0)} - E_{(0,a'')}$.

(2) A numerically trivial involution σ of Example 2 is of odd type and the patching pair is $((\tilde{h}_1 + \tilde{h}_2 - E_P)/2, (\Pi - \tilde{h}_1 - \tilde{h}_2)/2)$.

Proof. (1) Since σ_K is induced by the translation of $E' \times E''$ by a, Σ belongs to N_K. By Lemma 11, $\Sigma + \Pi$ is divisible by 2. Hence the second half of (1) follows. Since π_P is the Plücker coordinate, $\frac{1}{2}(\pi_P \cup \pi_P) = 0 \in \mathbb{Z}/2\mathbb{Z}$ and σ is of even type.

(2) If σ is an involution of Example 2, then $\tilde{h}_1 + \tilde{h}_2 - E_P$ belongs to N_K by virtue of Lemma 6. The second half of (2) follows from this and Lemma 11. σ is of odd type since $\frac{1}{2}(\pi_P - \tilde{h}_1 - \tilde{h}_2) \cup (\pi_P - \tilde{h}_1 - \tilde{h}_2) = \frac{1}{2}(\pi_P \cup \pi_P) + \frac{1}{2}(\tilde{h}_1 + \tilde{h}_2) \cup (\tilde{h}_1 + \tilde{h}_2) = 1 \in \mathbb{Z}/2\mathbb{Z}$. $\square$

§4 Proof of Theorem 3

Let σ be a numerically trivial involution of an Enriques surface S and assume that it is of Kummer type. We shall show that S is isomorphic to an Enriques surface of Example 1 or 2 by the global Torelli theorem for Enriques surfaces ([1, Chap. VIII, Theorem (21.2)]). Since the group of numerically trivial automorphisms of S is cyclic ([3, (1.1)]), Theorem 3 follows from this.

Let $(\alpha_K, \alpha_R) \in A_{N_K} \times A_{N_R}$ be the patching pair of σ. Recall that $N_R(1/2)$ is isomorphic to $U \perp U$ as a lattice and isomorphic to $H^1(E', \mathbb{Z}) \otimes H^1(E'', \mathbb{Z})$ as a polarized Hodge structure. Hence $\alpha_R \in (\frac{1}{2}N_R)/N_R$ corresponds to $0 \neq a' \otimes a'' \in (E')_2 \otimes (E'')_2$ or to an isomorphism $\varphi : (E')_2 \xrightarrow{\sim}$

$(E'')_2$ according as σ is of even type or of odd type. In the former case the Enriques surface S is isomorphic to that described in Example 1 with $a = (a', a'')$ by Lemma 13 and the global Torelli theorem.

Assume that σ is of odd type.

Claim: There exists no isomorphism from E' to E'' whose restriction to the 2-torsion subgroups is φ.

Proof. Assume the contrary and let $\Phi \subset E' \times E''$ be the graph of such an isomorphism. Then $\Phi - E' \times 0 - 0 \times E''$ is a divisor of self-intersection -2 and its class belongs to $H^1(E', \mathbb{Z}) \otimes H^1(E'', \mathbb{Z}) \subset H^2(E' \times E'', \mathbb{Z})$. Hence $N_R \subset H^2(Km, \mathbb{Z})$ contains an algebraic cycle x_R of self-intersection number -4 such that $x_R/2$ represents α_R. Since $N_K \simeq E_8(2)$, α_N is represented by a (-4)-element $x_K \in N_K$. Then $x := (x_K + x_R)/2$ belongs to N by the definition of patching pairs and is algebraic since x_K is orthogonal to $H^0(\Omega^2) \subset N_R \otimes \mathbb{C}$. Since $(x^2) = -2$, x or $-x$ is effective by the Riemann-Roch theorem. This is a contradiction since $\varepsilon(x) = -x$. □

Let $P \subset T_2$ be the graph of φ. By Lemma 13 and the global Torelli theorem, the Enriques surface S is isomorphic to that obtained from the image of P as in (2) of Observation 12.

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