ON C_n-MOVES FOR LINKS

JEAN-BAPTISTE MEILHAN AND AKIRA YASUHARA

Abstract. A C_n-move is a local move on links defined by Habiro and Goussarov, which can be regarded as a 'higher order crossing change'. We use Milnor invariants with repeating indices to provide several classification results for links up to C_n-moves, under certain restrictions. Namely, we give a classification up to C_4-moves of 2-component links, 3-component Brunnian links and n-component C_3-trivial links, and we classify n-component link-homotopically trivial Brunnian links up to C_{n+1}-moves.

1. Introduction

A C_n-move is a local move on links as illustrated in Figure 1.1, which can be regarded as a kind of 'higher order crossing change' (in particular, a C_1-move is a crossing change). These local moves were introduced by Habiro [5] and independently by Goussarov [3].

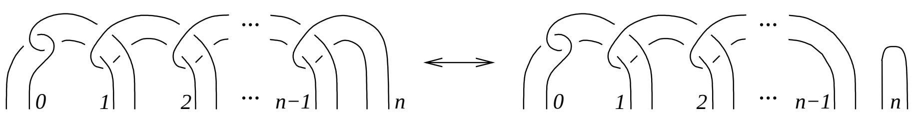

Figure 1.1. A C_n-move involves $n + 1$ strands of a link, labeled here by integers between 0 and n.

The C_n-move generates an equivalence relation on links, called *C_n-equivalence*. This notion can also be defined using the theory of claspers (see §2). The C_n-equivalence relation becomes finer as n increases, i.e., the C_m-equivalence implies the C_k-equivalence for $m > k$. It is well known that the C_n-equivalence allows to approximate the topological information carried by Goussarov-Vassiliev invariants. Namely, two links cannot be distinguished by any Goussarov-Vassiliev invariant of order less than n if they are C_n-equivalent [3, 6].

Links are classified up to C_2-equivalence [19]. C_3-classifications of links with 2 or 3-components, or of algebraically split links are done by Taniyama and the second author [21]. These results involve Milnor $\overline{\mu}$ invariants (of length ≤ 3) with distinct indices. (For the definition of Milnor invariants, see §3.) In this paper, we use Milnor $\overline{\mu}$ invariants with (possibly) repeating indices to provide several classification results for higher values of k, under certain restrictions.

First, we consider the case $k = 4$. We obtain the following for *C_3-trivial links*, i.e., links which are C_3-equivalent to the unlink.

Date: July 5, 2006.

2000 *Mathematics Subject Classification.* 57M25, 57M27.

Key words and phrases. C_n-moves, Milnor invariants, string links, Brunnian links, claspers.

The first author is supported by a Postdoctoral Fellowship and a Grant-in-Aid for Scientific Research of the Japan Society for the Promotion of Science. The second author is partially supported by Grant-in-Aid for Scientific Research (C) (#18540071) of the Japan Society for the Promotion of Science.

Theorem 1.1. *Let L and L' be two n-component C_3-trivial links. Then L and L' are C_4-equivalent if and only if they satisfy the following properties*

(1) $\overline{\mu}_L(I) = \overline{\mu}_{L'}(I)$ *for all multi-index I with $|I| = 4$,*
(2) *no Vassiliev knot invariant of order ≤ 3 can distinguish the i^{th} component of L and the i^{th} component of L', for all $1 \leq i \leq n$.*

Here, a multi-index I is a sequence of non-necessarily distinct integers in $\{1, ..., n\}$, and $|I|$ denotes the number of entries in I. Note that this result, together with [19] and [21], imply the following.

Corollary 1.2. *An n-component link L is C_4-trivial if and only if $\overline{\mu}_L(I) = 0$ for all multi-index I with $|I| \leq 4$ and any Vassiliev knot invariant of order ≤ 3 vanishes for each component.*

For 2-component links, we obtain a refinement of a result of H.A. Miyazawa [17, Thm. 1.5].

Proposition 1.3. *Let L and L' be two 2-component links. Then L and L' are C_4-equivalent if and only if they are not distinguished by any Vassiliev invariant of order ≤ 3.*

On the other hand, we consider Brunnian links. Recall that a link L in the 3-sphere S^3 is *Brunnian* if every proper sublink of L is trivial. In particular, all trivial links are Brunnian. It is known that an n-component link is Brunnian if and only if it can be turned into the unlink by a sequence of C_{n-1}-moves of a specific type, called C^a_{n-1}-*moves*, involving *all* the components [7]. Also, two n-component Brunnian links are C_n-equivalent if and only if their Milnor invariants $\overline{\mu}(\sigma(1), ..., \sigma(n-2), n-1, n)$ coincide for all σ in the symmetric group S_{n-2} [8]. Here, we consider the next stage, namely C_{n+1}-moves for n-component Brunnian links.

Recall that two links are *link-homotopic* if they are related by a sequence of isotopies and self-crossing changes, i.e., crossing changes involving two strands of the same component. For n-component Brunnian links, the link-homotopy coincides with the C_n-equivalence [18, 8]. Given $k \in \{1, ..., n\}$ and a bijection τ from $\{1, ..., n-1\}$ to $\{1, ..., n\} \setminus \{k\}$, set

$$\mu_\tau(L) := \overline{\mu}_L(\tau(1), ..., \tau(n-1), k, k).$$

We obtain the following.

Theorem 1.4. *Two n-component link-homotopically trivial Brunnian links L and L' are C_{n+1}-equivalent if and only if $\mu_\tau(L) = \mu_\tau(L')$ for all $k \in \{1, ..., n\}$, $\tau \in \mathcal{B}(k)$, where $\mathcal{B}(k)$ denotes the set of all bijections τ from $\{1, ..., n-1\}$ to $\{1, ..., n\} \setminus \{k\}$ such that $\tau(1) < \tau(n-1)$.*

In the case of 3-component Brunnian links, we have the following improvement of Theorem 1.4.

Theorem 1.5. *Two 3-component Brunnian links L and L' are C_4-equivalent if and only if $\overline{\mu}_L(123) = \overline{\mu}_{L'}(123)$, $\overline{\mu}_L(1233) = \overline{\mu}_{L'}(1233)$, $\overline{\mu}_L(1322) = \overline{\mu}_{L'}(1322)$ and $\overline{\mu}_L(2311) = \overline{\mu}_{L'}(2311)$.*[1]

The rest of the paper is organized as follows. In Section 2 we recall elementary notions of the theory of claspers. In Section 3 we recall the definition of Milnor invariants for (string) links and give some lemmas. In Section 4 we consider Brunnian string links. The main result of this section is Proposition 4.3, which gives a

[1]Note that $\overline{\mu}_L(ijkk)$ denotes here the *residue class* of the integer $\mu_L(ijkk)$ (defined in §3) modulo $\overline{\mu}_L(ijk)$.

set of generators for the abelian group of C_{n+1}-equivalence classes of n-component Brunnian string links. In Section 5 we use results of Section 4 to prove Theorems 1.4 and 1.5. In Section 6 we prove Theorem 1.1 and Proposition 1.3.

Acknowledgments. The authors wish to thank Kazuo Habiro for helpful comments and conversations.

2. Claspers and local moves on links

2.1. **A brief review of clasper theory.** Let us briefly recall from [6] the basic notions of clasper theory for (string) links. In this paper, we essentially only need the notion of C_k-tree. For a general definition of claspers, we refer the reader to [6].

Definition 1. Let L be a link in S^3. An embedded disk F in S^3 is called a *tree clasper* for L if it satisfies the following (1), (2) and (3):
(1) F is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.
(2) The disks have either 1 or 3 incident edges, called *leaves* or *nodes* respectively.
(3) L intersects F transversely and the intersections are contained in the union of the interior of the leaves.
The *degree* of G is the number of the leaves *minus* 1.

A degree k tree clasper is called a C_k-tree. A C_k-tree is *simple* if each leaf intersects L at one point.

We will make use of the drawing convention for claspers of [6, Fig. 7], except for the following: a $\oplus$ (resp. $\ominus$) on an edge represents a positive (resp. negative) half-twist. (This replaces the convention of a circled S (resp. S^{-1}) used in [6]).

Given a C_k-tree G for a link L in S^3, there is a procedure to construct, in a regular neighborhood of G, a framed link $\gamma(G)$. There is thus a notion of *surgery along* G, which is defined as surgery along $\gamma(G)$. There exists a canonical diffeomorphism between S^3 and the manifold $S^3_{\gamma(G)}$: surgery along the C_k-tree G can thus be regarded as a local move on L in S^3. We say that the resulting link L_G in S^3 is obtained by surgery on L along G. In particular, surgery along a simple C_k-tree as illustrated in Figure 2.1 is equivalent to band-summing a copy of the $(k+1)$-component Milnor's link L_{k+1} (see [15, Fig. 7]), and is equivalent to a C_k-move as defined in the introduction (Figure 1.1). A C_k-tree G having the shape of the tree

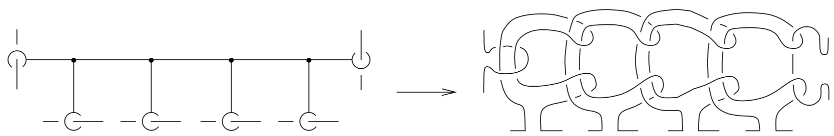

Figure 2.1. Surgery along a simple C_5-tree.

clasper in Figure 2.1 is called *linear*, and the left-most and right-most leaves of G in Figure 2.1 are called the *ends* of G.

The C_k-equivalence (as defined in the introduction) coincides with the equivalence relation on links generated by surgery along C_k-trees and isotopies. We use the notation $L \sim_{C_k} L'$ for two C_k-equivalent links L and L'.

2.2. **Some lemmas.** In this subsection we give some basic results of calculus of claspers, whose proof can be found in [6] or [13]. For convenience, we give the statements for string links. Recall that a string link is a pure tangle, without closed components (see [4] for a precise definition). Denote by $SL(n)$ the set of

n-component string links up to isotopy with respect to the boundary. $SL(n)$ has a monoid structure with composition given by the *stacking product*, denoted by $\cdot$, and with the trivial n-component string link $\mathbf{1}_n$ as unit element.

Lemma 2.1. *Let T be a union of C_k-trees for a string link L, and let T' be obtained from T by passing an edge across L or across another edge of T, or by sliding a leaf over a leaf of another component of T (see Figure 2.2). Then $L_T \sim_{C_{k+1}} L_{T'}$.*

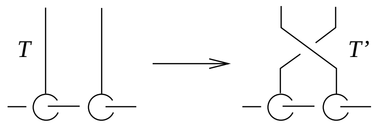

FIGURE 2.2. Sliding a leaf over another leaf.

Lemma 2.2. *Let T be a C_k-tree for $\mathbf{1}_n$ and let $\overline{T}$ be a C_k-trees obtained from T by adding a half-twist on an edge. Then $(\mathbf{1}_n)_T \cdot (\mathbf{1}_n)_{\overline{T}} \sim_{C_{k+1}} \mathbf{1}_n$.*

Lemma 2.3. *Consider some C_k-trees T and T' (resp. T_I, T_H and T_X) for $\mathbf{1}_n$ which differ only in a small ball as depicted in Figure 2.3, then $(\mathbf{1}_n)_T \cdot (\mathbf{1}_n)_{T'} \sim_{C_{k+1}} \mathbf{1}_n$ (resp. $(\mathbf{1}_n)_{T_I} \sim_{C_{k+1}} (\mathbf{1}_n)_{T_H} \cdot (\mathbf{1}_n)_{T_X}$).*

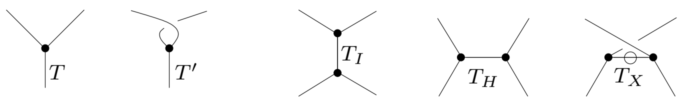

FIGURE 2.3. The AS and IHX relations for C_k-trees.

Lemma 2.4. *Let G be a C_k-tree for $\mathbf{1}_n$. Let f_1 and f_2 be two disks obtained by splitting a leaf f of G along an arc α as shown in figure 2.4 (i.e., $f = f_1 \cup f_2$ and $f_1 \cap f_2 = \alpha$). Then, $(\mathbf{1}_n)_G \sim_{C_{k+1}} (\mathbf{1}_n)_{G_1} \cdot (\mathbf{1}_n)_{G_2}$, where G_i denotes the C_k-tree for $\mathbf{1}_n$ obtained from G by replacing f by f_i ($i = 1, 2$).*

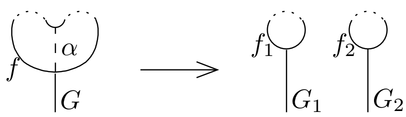

FIGURE 2.4. Splitting a leaf.

2.3. **C_k^a-trees and C_k^a-equivalence.**

Definition 2. Let L be an m-component link in a 3-manifold M. For $k \geq m-1$, a (simple) C_k-tree T for L in M is a *(simple) C_k^a-tree* if it satisfies the following:

(1) For each disk-leaf f of T, $f \cap L$ is contained in a single component of L,
(2) T intersects *all* the components of L.

The *C_k^a-equivalence* is an equivalence relation on links generated by surgeries along C_k^a-trees and isotopies. The next result shows the relevance of this notion in the study of Brunnian (string) links.

Theorem 2.5 ([7, 18])**.** *Let L be an n-component link in S^3. Then L is Brunnian if and only if it is C_{n-1}^a-equivalent to the n-component trivial link.*

Further, it is known that, for n-component Brunnian links the C_n-equivalence coincides with the C_n^a-equivalence (and with the link-homotopy) [18], see also [8].

3. On Milnor invariants

3.1. **A short definition.** J. Milnor defined in [15] a family of invariants of oriented, ordered links in S^3, known as Milnor's $\overline{\mu}$-invariants.

Given an n-component link L in S^3, denote by π the fundamental group of $S^3\setminus L$, and by π_q the q^{th} subgroup of the lower central series of π. We have a presentation of π/π_q with n generators, given by a meridian m_i of the i^{th} component of L. So for $1 \leq i \leq n$, the longitude l_i of the i^{th} component of L is expressed modulo π_q as a word in the m_i's (abusing notations, we still denote this word by l_i).

The *Magnus expansion* $E(l_i)$ of l_i is the formal power series in non-commuting variables $X_1, ..., X_n$ obtained by substituting $1+X_j$ for m_j and $1-X_j+X_j^2-X_j^3+...$ for m_j^{-1}, $1 \leq j \leq n$. We use the notation $E_k(l_i)$ to denote the degree k part of $E(l_i)$ (where the degree of a monomial in the X_j is simply defined by the sum of the powers).

Let $I = i_1 i_2 ... i_{k-1} j$ be a multi-index (i.e., a sequence of possibly repeating indices) among $\{1, ..., n\}$. Denote by $\mu_L(I)$ the coefficient of $X_{i_1}...X_{i_{k-1}}$ in the Magnus expansion $E(l_j)$. *Milnor invariant* $\overline{\mu}_L(I)$ is the residue class of $\mu_L(I)$ modulo the greatest common divisor of all Milnor invariants $\mu_L(J)$ such that J is obtained from I by removing at least one index and permuting the remaining indices cyclicly. $|I| = k$ is called the *length* of Milnor invariant $\overline{\mu}_L(I)$.

The indeterminacy comes from the choice of the meridians m_i. Equivalently, it comes from the indeterminacy of representing the link as the closure of a string link [4]. Indeed, $\mu(I)$ is a well-defined invariant for string links. Furthermore, $\mu(I)$ is known to be a Goussarov-Vassiliev invariant of degree $|I|-1$ for string links [1, 12].

3.2. **Some lemmas.** Let us first recall a result due to Habiro.

Lemma 3.1 ([6]). *Milnor invariants of length k for (string) links are invariants of C_k-equivalence.*

Next we state a simple lemma that will be used in the following.

Lemma 3.2. *Let L be an n-component string link which is obtained from $\mathbf{1}_n$ by surgery along a union F of C_k-trees which is disjoint from the j^{th} component of $\mathbf{1}_n$. Then $\mu_L(I) = 0$, for all multi-index I containing j and satisfying $|I| \leq k+1$.*

Proof. Consider a diagram of $\mathbf{1}_n$ together with F. The diagram contains several crossings between an edge of F and the j^{th} component of $\mathbf{1}_n$. Denote by F_o (resp. F_u) the union of C_k-trees obtained from F by performing crossing changes so that the j^{th} component of $\mathbf{1}_n$ overpasses (resp. underpasses) all edges. By Lemma 2.1, we have $L \sim_{C_{k+1}} U_{F_o} \sim_{C_{k+1}} U_{F_u}$. The result then follows from Lemma 3.1 and the following observation.

Consider the diagram D of a string link K. If the i^{th} component of K overpasses all the other components in D, it follows from the definition of Milnor invariants that $\mu_K(I) = 0$ for any sequence I with the last index i. Similarly, if the i^{th} component of K underpasses all the other components in D, then $\mu_K(I) = 0$ for any sequence I with an index i which is not equal to the last one. □

We have the following simple additivity property.

Lemma 3.3. *Let L and L' be two n-component string links such that all Milnor invariants of L (resp. L') of length $\leq m$ (resp. $\leq m'$) vanish. Then $\mu_{L\cdot L'}(I) = \mu_L(I) + \mu_{L'}(I)$ for all I of length $\leq m+m'$.*

Proof. Milnor invariant of $L \cdot L'$ is computed by taking the Magnus expansion of the k^{th} longitude L_k of $L \cdot L'$. Denote respectively by l_i and m_i (resp. l'_i and m'_i) the i^{th} meridian and longitude of L (resp. L') ; $1 \leq i \leq n$. We have $L_k = l_k \cdot \tilde{l}'_k$, where $\tilde{l}'_k$ is obtained from l'_k by replacing m'_i with $M_i = l_i^{-1} m_i l_i$ for each i. So $E(L_k) = E(l_k) \cdot E(\tilde{l}'_k)$, where $E(\tilde{l}'_k)$ is obtained from $E(l'_k)$ by substituting $\tilde{X}_i$ for X_i in $E(l'_k)$, where $\tilde{X}_i := E(M_i) - 1$.

The Magnus expansion of l_i is the form $E(l_i) = 1 + (\text{terms of order} \geq m)$, so

$$\begin{aligned} E(M_i) &= E(l_i^{-1})E(m_i)E(l_i) \\ &= E(l_i^{-1})E(l_i) + E(l_i^{-1})X_i E(l_i) \\ &= 1 + X_i + \ (\text{terms of order} \geq m+1). \end{aligned}$$

So $E(\tilde{l}'_k)$ is obtained from $E(l'_k) = \sum_{j \geq m'} E_j(l'_k)$ by replacing each X_i by $X_i +$ (terms of order $\geq m+1$) for all i. It follows that

$$E(\tilde{l}'_k) = 1 + \sum_{m+m'-1 \geq j \geq m'} E_j(l'_k) + (\text{ terms of order} \geq (m+m'))\,.$$

It follows that $E(L_k) = E(l_k)E(\tilde{l}'_k)$ has the form

$$1 + \sum_{m+m'-1 \geq j \geq m} E_j(l_k) + \sum_{m+m'-1 \geq j \geq m'} E_j(l'_k) + (\text{ terms of order} \geq (m+m'))\,,$$

which implies that all Milnor invariants of length $\leq m+m'$ of $L \cdot L'$ are additive. □

4. C_{n+1}-moves for n-component Brunnian string links

An n-component string link L is Brunnian if every proper substring link of L is the trivial string link. In particular, any trivial string link is Brunnian. n-component Brunnian string links form a submonoid of $SL(n)$, denoted by $BSL(n)$.

Recall that, given $L \in SL(n)$, the *closure* $\text{cl}(L)$ of L is an n-component link in S^3 [4]. By [7], an n-component link is Brunnian if and only if it is the closure of a certain Brunnian string link.

4.1. n-component Brunnian string links up to C_n-equivalence.

Let $BSL(n)/C_n$ denote the abelian group of C_n-equivalence classes of n-component Brunnian string links. Habiro and the first author gave in [8] a basis for $BSL(n)/C_n$ as follows.

Let σ be an element in the symmetric group S_{n-2}. Denote by L_σ the n-component string link obtained from $\mathbf{1}_n$ by surgery along the C^a_{n-1}-tree T_σ shown in Figure 4.1. Likewise, denote by $(L_\sigma)^{-1}$ the n-component string link obtained from the C^a_{n-1}-tree $\overline{T}_\sigma$, which is obtained from T_σ by adding a positive half-twist in the edge e (see Figure 4.1).

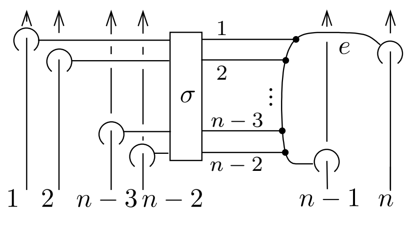

FIGURE 4.1. The simple C^a_n-tree T_σ. Here, the numbering of the edges just indicates how $\sigma \in S_{n-1}$ acts on the edges of T_σ (a similar notation is used in Fig. 4.2).

Let $\mu_\sigma(L)$ denote the Milnor invariant $\mu_L(\sigma(1), ..., \sigma(n-2), n-1, n)$ for any element $\sigma \in S_{n-2}$.

Proposition 4.1 ([8]). *Let L be an n-component Brunnian string link. Then*

$$L \sim_{C_n} \prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(L)}.$$

Remark 4.2. Recall from [8, 18] that the C_n-equivalence, the link-homotopy and the C_n^a-equivalence coincide on $BSL(n)$.

4.2. **n-component Brunnian string links up to C_{n+1}-equivalence.** In this section, we study the quotient $BSL(n)/C_{n+1}$. Note that $BSL(n)/C_{n+1}$ is a finitely generated abelian group (this is shown by using the same arguments as in the proof of [6, Lem. 5.5]).

Given $k \in \{1, ..., n\}$, consider a bijection τ from $\{1, ..., n-1\}$ to $\{1, ..., n\} \setminus \{k\}$. Denote by V_τ the n-component string link obtained from $\mathbf{1}_n$ by surgery along the C_n^a-tree G_τ shown in Figure 4.2. Denote by $\overline{G}_\tau$ the C_n^a-tree for $\mathbf{1}_n$ obtained from G_τ by adding a positive half-twist in the edge e (see Figure 4.1). Let $(V_\tau)^{-1}$ be the n-component string link obtained from $\mathbf{1}_n$ by surgery along $\overline{G}_\tau$.

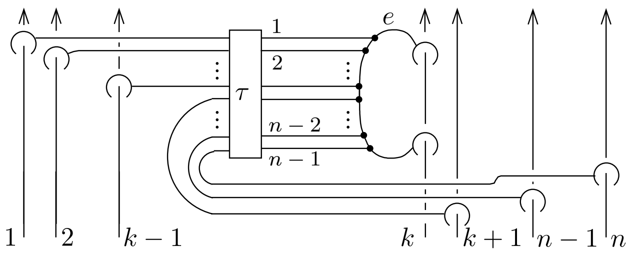

FIGURE 4.2. The simple C_n^a-tree G_τ.

Set

$$\mu_\tau(L) := \mu_L(\tau(1), ..., \tau(n-1), k, k)$$

Denote by $\mathcal{B}(k)$ the set of all bijections τ from $\{1, ..., n-1\}$ to $\{1, ..., n\} \setminus \{k\}$ such that $\tau(1) < \tau(n-1)$, and denote by ρ a bijection from $\{1, ..., n-1\}$ to itself defined by $\rho(i) = n - i$. We have the following.

Proposition 4.3. *Let L be an n-component Brunnian string link. Then*

$$L \sim_{C_{n+1}} \left(\prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(L)} \right) \cdot L_1 \cdot ... \cdot L_n, \tag{4.1}$$

where, for each k ($1 \le k \le n$), L_k is the n-component Brunnian string link

$$\prod_{\tau \in \mathcal{B}(k)} (V_\tau)^{n_\tau(L)} \cdot (V_{\tau\rho})^{n'_\tau(L)},$$

such that, for any $\tau \in \mathcal{B}(k)$ ($k = 1, ..., n$), $n_\tau(L)$ and $n'_\tau(L)$ are two integers satisfying

$$n_\tau(L) + (-1)^{n-1} n'_\tau(L) = \mu_\tau(L_1 \cdot ... \cdot L_n). \tag{4.2}$$

Proof. By Proposition 4.1 and Remark 4.2, L is obtained from the n-component string link

$$L_0 := \prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(L)}$$

by surgery along a disjoint union F of simple C_n^a-trees. By Lemma 2.1, we have

$$L \sim_{C_{n+1}} L_0 \cdot (\mathbf{1}_n)_{G_1} \cdot ... \cdot (\mathbf{1}_n)_{G_p},$$

where, G_j $(1 \leq j \leq p)$ are simple C_n^a-trees for $\mathbf{1}_n$. Denote by k_j the (unique) element of $\{1, ..., n\}$ such that G_j intersects twice the k_j^{th} component of $\mathbf{1}_n$ $(1 \leq j \leq p)$. We can use the AS and IHX relations for tree claspers to replace, up to C_{n+1}-equivalence, each of these C_n^a-trees with a union of linear C_n^a-trees whose ends intersect the k_j^{th} component. More precisely, by lemmas 2.3, 2.2 and 2.1 we have for each $1 \leq j \leq p$

$$(\mathbf{1}_n)_{G_j} \sim_{C_{n+1}} \prod_{i=1}^{m_j} (V_{\nu_{ij}})^{\varepsilon_{ij}},$$

where $\varepsilon_{ij} \in \mathbf{Z}$ and where ν_{ij} is a bijection from $\{1, ..., n-1\}$ to $\{1, ..., n\} \setminus \{k_j\}$. Since there exists, for each such ν_{ij}, a unique element τ of $\mathcal{B}(k_j)$ such that ν_{ij} is equal to either τ or $\tau\rho$, it follows that L is C_{n+1}-equivalent to an n-component string link of the form given in (4.1). It remains to prove (4.2).

First, let us compute $\mu_\tau(V_\eta)$ for all $\tau \in \mathcal{B}(k)$ and $\eta \in \mathcal{B}(l)$; $k, l = 1, ..., n$. By [16, Theorem 7], we have

$$\mu_\tau(V_\eta) = \mu_{\tau,n+1}(W_\eta),$$

where $\mu_{\tau,n+1}$ is Milnor invariant $\mu(\tau(1), ..., \tau(k-1), \tau(k+1), ..., \tau(n), k, n+1)$ and where W_η denotes the $(n+1)$-component string link obtained from V_η by taking, as the $(n+1)^{th}$ component, a parallel copy of the k^{th} component (so that the k^{th} and the $(n+1)^{th}$ components of W_η have linking number zero). Now recall that $V_\eta \cong (\mathbf{1}_n)_{G_\eta}$, where G_η is a C_n^a-tree as shown in Figure 4.2. So $W_\eta \cong (\mathbf{1}_{n+1})_{\tilde{G}_\eta}$, where $\tilde{G}_\eta$ is a C_n^a-tree obtained from G_η by replacing each leaf intersecting the k^{th} component of $\mathbf{1}_n$ with a leaf intersecting components k and $n+1$ as depicted in Figures 4.3 and 4.4.

If $k \neq l$, then $\tilde{G}_\eta$ contains exactly one leaf f intersecting both the k^{th} and the $(n+1)^{th}$ components of $\mathbf{1}_{n+1}$.

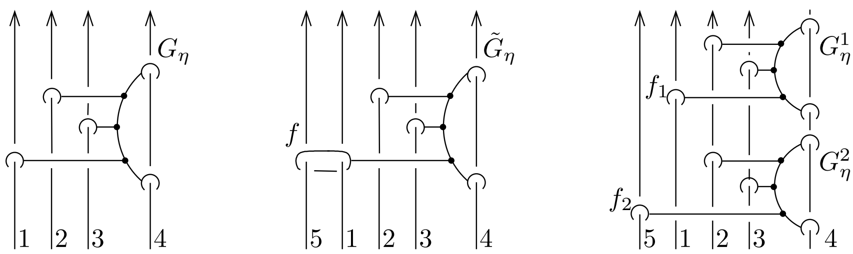

FIGURE 4.3. Here (and in subsequent figures) we fix, for simplicity, $n = 4$, $k = 1$, $l = 4$ and η is the cyclic permutation $(231) \in S_3$

By Lemma 2.4, we have

$$(\mathbf{1}_{n+1})_{\tilde{G}_\eta} \sim_{C_{n+1}} (\mathbf{1}_{n+1})_{G_\eta^1} \cdot (\mathbf{1}_{n+1})_{G_\eta^2},$$

where G_η^i denotes the simple C_n-tree for $\mathbf{1}_{n+1}$ obtained from $\tilde{G}_\eta$ by replacing f by f_i as shown in Figure 4.3 $(i = 1, 2)$. By Lemmas 3.1 and 3.3, $\mu_\tau(V_\eta)$ is thus equal to $\mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G_\eta^1}) + \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G_\eta^2})$. It follows from Lemma 3.2 that $\mu_\tau(V_\eta) = 0$.

Now suppose that $k = l$. Then $\tilde{G}_\eta$ contains two leaves intersecting both the k^{th} and the $(n+1)^{th}$ components of $\mathbf{1}_{n+1}$. By Lemma 2.4, we obtain

$$(\mathbf{1}_{n+1})_{\tilde{G}_\eta} \sim_{C_{n+1}} (\mathbf{1}_{n+1})_{G_\eta^1} \cdot (\mathbf{1}_{n+1})_{G_\eta^2} \cdot (\mathbf{1}_{n+1})_{G_\eta^3} \cdot (\mathbf{1}_{n+1})_{G_\eta^4},$$

where, for $1 \leq i \leq 4$, G_η^i is a simple C_n-tree for $\mathbf{1}_{n+1}$ as depicted in Figure 4.4. By Lemmas 3.1, 3.2 and 3.3, it follows that

$$\mu_\tau(V_\eta) = \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G_\eta^3}) + \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G_\eta^4}).$$

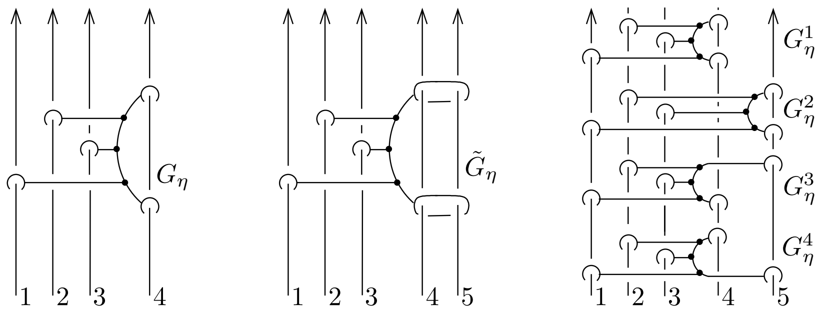

FIGURE 4.4

Observe that the closure of each of these two string links is a copy of Milnor's link [15, Fig. 7]. By a formula of Milnor [15, pp. 190], we obtain $\mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^3_\eta}) = \delta_{\tau,\eta}$, and $\mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^4_\eta}) = 0$, where δ denotes Kronecker's symbol. So we obtain that

$$\mu_\tau(V_\eta) = \delta_{\tau,\eta}.$$

Moreover, it follows from Lemmas 3.3 and 2.2 that $\mu_\tau((V_\eta)^{-1}) = -\delta_{\tau,\eta}$.

Now consider the string link $V_{\eta\rho}$. By the same arguments as above, we have that $\mu_\tau(V_{\eta\rho}) = \mu_\tau((V_{\eta\rho})^{-1}) = 0$ if $k \neq l$. If $k = l$, it follows from the same arguments as above that

$$\mu_\tau(V_{\eta\rho}) = \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^1_{\eta\rho}}) + \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^2_{\eta\rho}}),$$

where $G^1_{\eta\rho}$ and $G^2_{\eta\rho}$ are two simple C^a_n-trees for $\mathbf{1}_{n+1}$ as depicted in Figure 4.5.

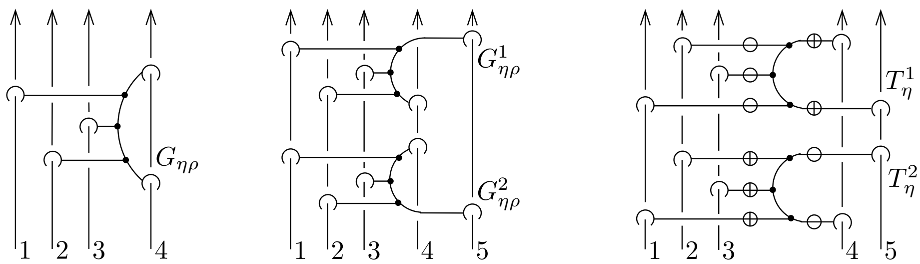

FIGURE 4.5

By Lemma 2.1 and isotopy, $(\mathbf{1}_{n+1})_{G^i_{\eta\rho}}$ is C_{k+1}-equivalent to $(\mathbf{1}_{n+1})_{T^i_\eta}$, where T^i_η is as shown in Figure 4.5, $i = 1, 2$. By Lemma 2.2, we thus obtain

$$\mu_\tau(V_{\eta\rho}) = (-1)^{n-1}\delta_{\tau,\eta}.$$

We conclude that

$$\mu_\tau(L_1 \cdot ... \cdot L_p) = \sum_{1\leq i\leq p} \mu_\tau(L_i) = n_\tau(L) + (-1)^{n-1}n'_\tau(L).$$

□

Remark 4.4. Observe that we obtain the following as a byproduct of the proof of Proposition 4.3. Consider the n-component Brunnian link B_τ represented in Figure 4.6, for some $\tau \in \mathcal{B}(k)$. B_τ is the closure of the n-component string link V_τ considered above. We showed that, for $1 \leq l \leq n$ and $\eta \in \mathcal{B}(l)$,

$$\overline{\mu}_\eta(B_\tau) = \mu_\eta(B_\tau) = \delta_{\eta,\tau}.$$

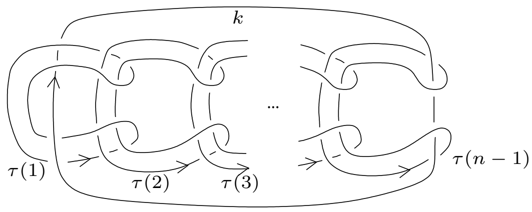

FIGURE 4.6. The link B_τ.

We conclude this section by showing that the string links V_τ and $V_{\tau\rho}$ are linearly independent in $BSL(n)/C_{n+1}$.

Proposition 4.5. *For any integer k in* $\{1, ..., n\}$ $(n \geq 3)$ *and any element* $\tau \in \mathcal{B}(k)$, *we have* $V_\tau \nsim_{C_{n+1}} V_{\tau\rho}$ *nor* $(V_{\tau\rho})^{-1}$.

Remark 4.6. In contrast to the lemma above, we will see in the proof of Proposition 5.1 that $\mathrm{cl}(V_\tau) \sim_{C_{n+1}} \mathrm{cl}(V_{\tau\rho})$ or $\mathrm{cl}((V_{\tau\rho})^{-1})$.

Proof. Consider a diagram of an n-component string link L. L lives in a copy of $D^2 \times I$ standardly embedded in S^3. The *origin* (resp. *terminal*) of the i^{th} component of L is the starting point (resp. ending point) of the component, according to the orientation of L. We can construct a knot $K_\tau(L)$ in S^3 as follows.

Connect the terminals of the k^{th} and the $\tau(1)^{th}$ components by an arc a_1 in $S^3 \setminus (D^2 \times I)$. Next, connect the origins of the $\tau(1)^{th}$ and the $\tau(2)^{th}$ components by an arc a_2 in $S^3 \setminus (D^2 \times I)$ disjoint from a_1, then the terminals of the $\tau(2)^{th}$ and the $\tau(3)^{th}$ components by an arc a_3 in $S^3 \setminus (D^2 \times I)$ disjoint from $a_1 \cup a_2$. Repeat this construction until reaching the last component, the $\tau(n-1)^{th}$ component, and connect the terminal or the origin (depending on whether n is even or odd) to the origin of the k^{th} component by an arc a_n in $S^3 \setminus (D^2 \times I)$ disjoint from $\bigcup_{1 \leq i \leq n-1} a_i$. The arcs are chosen so that, if a_i and a_j $(i < j)$ meet in the diagram of L, then a_i overpasses a_j. The orientation of K_τ is the one induced from the k^{th} component. An example is given in Figure 4.7 for the case $n = 4$, $k = 4$ and $\tau = (231) \in S_3$.

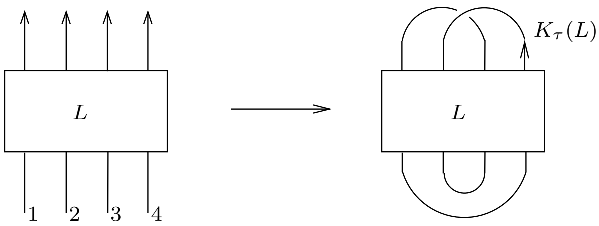

FIGURE 4.7. The knot $K_\tau(L)$.

It follows immediately from the above construction and [9, Thm. 1.4] that

$$P_0^{(n)}(K_\tau(V_\tau); 1) = \pm n! 2^n \text{ and } P_0^{(n)}(K_\tau(V_{\tau\rho}); 1) = P_0^{(n)}(K_\tau((V_{\tau\rho})^{-1}); 1) = 0,$$

where $P_l^{(k)}(K; 1)$ denotes the k^{th} derivative of the coefficient polynomial $P_k(K; t)$ of z^k in the HOMFLY polynomial $P(K; t, z)$ of a link K, evaluated in 1. The result then follows from [6, Cor. 6.8] and the fact that $P_0^{(n)}(K; 1)$ is a Goussarov-Vassiliev invariant of degree $\leq n$ [10]. $\square$

5. C_{n+1}-moves for n-component Brunnian links

In this section we prove Theorems 1.4 and 1.5. Let us begin with stating the following link version of Proposition 4.3.

Proposition 5.1. *Let L be an n-component Brunnian link. Then*

$$L \sim_{C_{n+1}} \mathrm{cl}\left(\prod_{\sigma \in S_{n-2}} (T_\sigma)^{\mu_\sigma(L)} \cdot \prod_{1\le k\le n} L'_k \right),$$

where, for each i ($1 \le i \le n$),

$$L'_k := \prod_{\tau_k \in \mathcal{B}(k)} (V_{\tau_k})^{\mu_\tau(L'_1 \cdot \ldots \cdot L'_p)}.$$

Proof. By Proposition 4.3, L is C_{n+1}-equivalent to the closure of the string link

$$l = \prod_{\sigma \in S_{n-2}} ((\mathbf{1}_n)_{T_\sigma})^{\mu_\sigma(L)} \cdot \prod_{1\le k\le n} \prod_{\tau \in \mathcal{B}(k)} ((\mathbf{1}_n)_{G_\tau})^{n_\tau(L)} \cdot ((\mathbf{1}_n)_{G_{\tau\rho}})^{n'_\tau(L)}, \tag{5.1}$$

where $n_\tau(L)$ and $n'_\tau(L)$ are two integers satisfying (4.2). Denote by F the union of all the tree claspers involved in (5.1), that is $l = (\mathbf{1}_n)_F$.

For some $k \in \{1, ..., n\}$, $\tau \in \mathcal{B}(k)$, let G be a copy of the simple C_n-tree $G_{\tau\rho}$ in F. Let f be a leaf of G which intersects k^{th} component of $\mathbf{1}_n$ (Figure 5.1). When we close the k^{th} component of $\mathbf{1}_n$, we can slide f over leaves of the components of $F \setminus G$ until we obtain the C_n-tree G' of Figure 5.1. Denote by F' the union of tree claspers obtained from F by this operation. By Lemma 2.1, we have $\mathrm{cl}((\mathbf{1}_n)_F) \sim_{C_{n+1}} \mathrm{cl}((\mathbf{1}_n)_{F'})$.

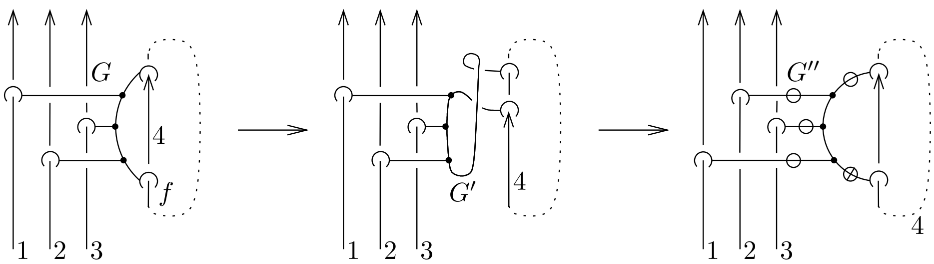

Figure 5.1

By Lemma 2.1 and isotopy, $(\mathbf{1}_n)_{G'}$ is C_{n+1}-equivalent to $(\mathbf{1}_n)_{G''}$, where G'' is the C_n-tree depicted in Figure 5.1. G'' differs from a copy of G_τ by $(n+1)$ half-twists on its edges. It thus follows from Lemma 2.2 that

$$\mathrm{cl}((\mathbf{1}_n)_{G_\tau} \cdot (\mathbf{1}_n)_{G_{\tau\rho}}) \sim_{C_{n+1}} \begin{cases} \mathrm{cl}(\mathbf{1}_n) & \text{if } n \text{ is even,} \\ \mathrm{cl}(((\mathbf{1}_n)_{G_\tau})^2) & \text{if } n \text{ is odd.} \end{cases}$$

L is thus C_{n+1}-equivalent to the closure of the string link

$$\prod_{\sigma \in S_{n-2}} ((\mathbf{1}_n)_{T_\sigma})^{\mu_\sigma(L)} \cdot \prod_{1\le k\le n} \prod_{\tau \in \mathcal{B}(k)} ((\mathbf{1}_n)_{G_\tau})^{n_\tau(L) + (-1)^{n-1} n'_\tau(L)}.$$

The result follows from (4.2). □

5.1. **The link-homotopically trivial links case: Proof of Theorem 1.4.**

Proof of Theorem 1.4. By Proposition 4.1, if an n-component Brunnian link B is link-homotopically trivial, then $\mu_\sigma(B) = 0$ for all $\sigma \in S_{n-2}$. For all $\tau \in \mathcal{B}(k)$, $k = 1, ..., n$, $\mu_\tau(B)$ is thus a well-defined integer, which satisfies $\mu_\tau(B) = \mu_\tau(L(B))$ for any string link $L(B)$ whose closure is B. By Proposition 5.1, we have

$$B \sim_{C_{n+1}} \mathrm{cl}\left(\prod_{1\leq k\leq n}\prod_{\tau\in\mathcal{B}(k)} (V_\tau)^{\mu_\tau(B)}\right).$$

The result follows immediately. □

5.2. **The 3-component links case: Proof of Theorem 1.5.**

Proof of Theorem 1.5. The 'only if' part of Theorem 1.5 is an immediate consequence of Lemma 3.1. Here we prove the 'if' part.

Let L be a 3-component Brunnian link. By Proposition 5.1, we have

$$L \sim_{C_4} \mathrm{cl}(L_0 \cdot L_1 \cdot L_2 \cdot L_3), \quad \text{with} \quad L_0 := B^{\mu_L(123)} \text{ and } L_p := V_p^{n_p} \ (p = 1, 2, 3), \tag{5.2}$$

where B and V_p $(p = 1, 2, 3)$ are 3-component string links obtained from $\mathbf{1}_3$ by surgery along a C_2-tree and C_3-trees as shown in Figure 5.2 respectively, and where $n_k = \mu_{L_1\cdot L_2\cdot L_3}(ijkk)$ with $\{i, j, k\} = \{1, 2, 3\}$ and $i < j$. Note that $\mu_L(123) = \overline{\mu}_L(123)$ since L is Brunnian.

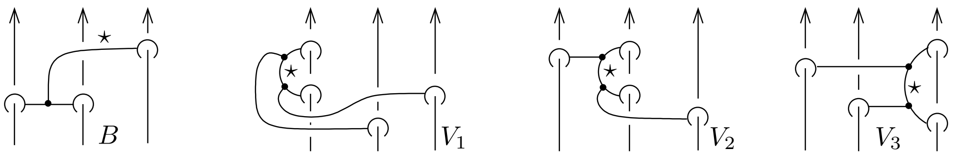

FIGURE 5.2. Here B^{-1} (resp. V_p^{-1}, $1 \leq p \leq 3$) is defined as obtained from B (resp. V_p, $1 \leq p \leq 3$) by a positive half-twist on the edge marked by a $\star$.

We now make an observation. Consider a union Y of k parallel copies of a simple C_2^a-tree for the 3-component unlink $U = U_1 \cup U_2 \cup U_3$, and perform an isotopy as illustrated in Figure 5.3. Denote by Y' the resulting union of C_2-trees. By [6, Prop. 4.5], Y' can be deformed into Y by a sequence of k C_3-moves, corresponding to k parallel copies of a simple C_3-tree intersecting twice U_i and once U_j and U_k. So by

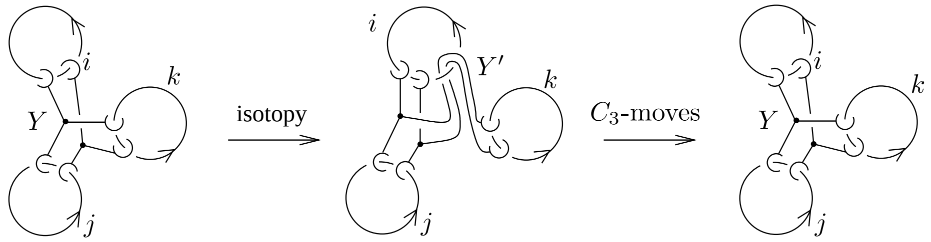

FIGURE 5.3

Lemma 2.2, U_Y is C_4-equivalent to $\mathrm{cl}\left((\mathbf{1}_n)_Y \cdot (\mathbf{1}_n)_{V_i}^{\pm k}\right)$.[2] Note that for any union F of C_3-trees, $U_{Y\cup F} \sim_{C_4} \mathrm{cl}((\mathbf{1}_n)_{Y\cup F} \cdot (\mathbf{1}_n)_{V_i}^{\pm k})$.

[2]Here, abusing notations, we still denote by Y a union of k simple C_2-trees for $\mathbf{1}_3$ such that $\mathrm{cl}((\mathbf{1}_3)_Y) \cong U_Y$.

This observation implies that the n_k $(k = 1, 2, 3)$ in (5.2) are changeable up to $|\mu_L(123)|$. So we can suppose that, for all $k = 1, 2, 3$, n_k satisfies

$$0 \leq n_k < |\mu_L(123)|. \tag{5.3}$$

Now by [11] we have, for all $\{i, j, k\} = \{1, 2, 3\}$,

$$\mu_L(ijkk) \equiv \mu_{\mathrm{cl}(L_0)}(ijkk) + \mu_{\mathrm{cl}(L_1 \cdot L_2 \cdot L_3)}(ijkk) \bmod \mu_L(123).$$

By Lemma 3.3, we have

$$\mu_{\mathrm{cl}(L_0)}(ijkk) \equiv 0 \bmod \mu_L(123)$$

and

$$\mu_{\mathrm{cl}(L_1 \cdot L_2 \cdot L_3)}(ijkk) \equiv \sum_{1 \leq p \leq 3} n_p \mu_{\mathrm{cl}(V_p)}(ijkk) \bmod \mu_L(123).$$

As seen in Remark 4.4, we have $\mu_{\mathrm{cl}(V_p)}(ijkk) = \delta_{p,k}$. It follows that

$$\mu_L(ijkk) \equiv n_k \bmod \mu_L(123). \tag{5.4}$$

Consider 3-component Brunnian links L and L' such that $\overline{\mu}_L(123) = \overline{\mu}_{L'}(123)$ and $\overline{\mu}_L(ijkk) = \overline{\mu}_{L'}(ijkk)$ for $(i, j, k) = (1, 2, 3)$, $(1, 3, 2)$ and $(2, 3, 1)$. It follows from (5.2), (5.4) and (5.3) that $L \sim_{C_4} L'$. This completes the proof. □

5.3. **Minimal string link.** Let L be an n-component Brunnian link in S^3. Denote by $\mathcal{L}(L)$ the set of all n-component string links l such that $\mathrm{cl}(l) = L$.

By Proposition 4.3, for each $l \in \mathcal{L}(L)$ there exists $l' \in SL(n)$ such that l is C_{n+1}-equivalent to a string link of the form $\prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(l)} \cdot l'$.

Put any total order on the set $\mathcal{B} := \bigcup_{1 \leq k \leq n} \mathcal{B}(k)$ and fix it. We denote by τ_i, $i = 1, ..., m$, the elements of $\mathcal{B}$ according to this total order. For all $l \in \mathcal{L}(L)$, $\tau \in \mathcal{B}$, set $\alpha_\tau(l) := \mu_\tau(l')$. For each element $l \in \mathcal{L}(L)$, we can thus define a vector

$$v_l := (|\alpha_{\tau_1}(l)|, ..., |\alpha_{\tau_k}(l)|, ..., |\alpha_{\tau_m}(l)|, -\alpha_{\tau_1}(l), ..., -\alpha_{\tau_k}(l), ..., -\alpha_{\tau_m}(l)).$$

Set $\mathcal{V}_L = \{v_l \mid l \in \mathcal{L}(L)\}$. Define $L_{\min}$ to be the element $l \in \mathcal{L}(L)$ such that $v_l = \min \mathcal{V}_L$ (for the natural lexicographical order on $\mathcal{V}$). It follows from Proposition 5.1 that L is C_{n+1}-equivalent to the closure of $L_{\min}$. So we have the following.

Proposition 5.2. *Two n-component Brunnian links L and L' are C_{n+1}-equivalent if and only if $\overline{\mu}_\sigma(L) = \overline{\mu}_\sigma(L')$ for all $\sigma \in S_{n-1}$ and* $\min \mathcal{V}_L = \min \mathcal{V}_{L'}$.

In subsection 5.2, if we take $-|\mu_L(123)|/2 < n_k < (|\mu_L(123)| - 1)/2$ instead of inequality (5.3), then we have explicitly $L_{\min}$ for a 3-component Brunnian link L. In general, it is a problem to determine $L_{\min}$ from L.

6. C_4-equivalence for links

In this section we prove Theorem 1.1 and Proposition 1.3. The first subsection provides a lemma which is the main new ingredient for the proofs of these two results.

6.1. **The index lemma.** Let T be a simple C_k-tree for an n-component link L. The *index* of T is the collection of all integers i such that T intersects the i^{th} component of L, counted with multiplicities. For example, a simple C_3-tree of index $\{2, 3^{(2)}, 5\}$ for L intersects twice component 3 and once components 2 and 5 (and is disjoint from all other components of L).

Lemma 6.1. *Let T be a simple C_k-tree $(k \geq 3)$ of index $\{i, j^{(k)}\}$ for an n-component link L, $1 \leq i \neq j \leq n$. Then $L_T \sim_{C_{k+1}} L$.*

In order to prove this lemma, we need the notion of graph clasper introduced in [6, §8.2]. A *graph clasper* is defined as an embedded connected surface which is decomposed into leaves, nodes and bands as in Definition 1, but which is not necessarily a disk. A graph clasper may contain loops. The degree of a graph clasper G is defined as half of the number of nodes and leaves (which coincides with the usual degree if G is a tree clasper). We call a degree k graph clasper a C_k*-graph.* A C_k-graph for a link L is *simple* if each of its leaves intersects L at one point.

Recall from [6, §8.2] that the STU relation holds for graph clasper.

Lemma 6.2. *Let G_S, G_T and G_U be three C_k-graphs for $\mathbf{1}_n$ which differ only in a small ball as depicted in Figure 6.1. Then $(\mathbf{1}_n)_{G_S} \sim_{C_{k+1}} (\mathbf{1}_n)_{G_T} \cdot (\mathbf{1}_n)_{G_U}$.*

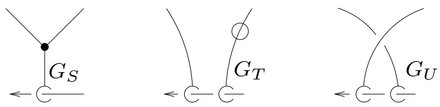

FIGURE 6.1. The STU relation for C_k-graphs.

It should be noted that, in contrast with the diagram case, this STU relation only holds among *connected* claspers. Note also that it differs by a sign from the STU relation for unitrivalent diagrams.

Lemma 6.3. *Let C be a simple C_k-graph for an n-component link L in S^3, which intersects a certain component of L exactly once. If C contains a loop (that is, if C is not a C_k-tree), then $L_C \sim_{C_{k+1}} L$.*

Proof. Suppose that C intersect the i^{th} component of L exactly once. By [6] and Lemma 2.1, there exists a union F of tree claspers for $\mathbf{1}_n$ and a simple C_k-tree G for $\mathbf{1}_n$ containing a loop and intersecting the i^{th} component once, such that

$$L_C \cong \mathrm{cl}\left((\mathbf{1}_n)_F \cdot (\mathbf{1}_n)_G\right).$$

Consider the unique leaf f of G intersecting the i^{th} component. This leaf f is connected to a loop γ of G by a path P of edges and nodes. We proceed by induction on the number n of nodes in P.

If $n = 0$, that is if f is connected to γ by a single edge, apply Lemma 6.2 at this edge. The result then follows from Lemmas 2.1 and 2.2, by the arguments similar to those in the proof of Proposition 5.1.

For an arbitrary $n \geq 1$, apply the IHX relation at the edge of P which is incident to γ. By Lemma 2.3,[3] we obtain $(\mathbf{1}_n)_G \sim_{C_{k+1}} (\mathbf{1}_n)_{G'} \cdot (\mathbf{1}_n)_{G''}$, where G' and G'' are C_k-graphs each of which has a unique leaf intersecting the i^{th} component connected to a loop by a path with $(n-1)$ nodes. By the induction hypothesis, we thus have $(\mathbf{1}_n)_{G'} \sim_{C_{k+1}} \mathbf{1}_n \sim_{C_{k+1}} (\mathbf{1}_n)_{G''}$. □

Proof of Lemma 6.1. Let T be a simple C_k-tree of index $\{i, j^{(k)}\}$ for an n-component link L, $1 \leq i \neq j \leq n$. By several applications of Lemmas 6.2, 6.3, 2.1 and 2.2, one can easily verify that $L_T \sim_{C_{k+1}} L_{T'}$, where T' is a simple C_k-tree of index $\{i, j^{(k)}\}$ for L which contains two leaves as depicted in Figure 6.2. By applying the IHX and STU relations, we have $L_{T'} \sim_{C_{k+1}} L_{T''}$, where T'' is a C_k-graph for L as illustrated in Figure 6.2. T'' clearly satisfies the hypothesis of Lemma 6.3. We thus have $L_T \sim_{C_{k+1}} L_{T''} \sim_{C_{k+1}} L$. □

[3]Strictly speaking, we cannot apply Lemma 2.3 here, as G is not a C_k-tree. However, similar relations hold among C_k-graphs [6, §8.2].

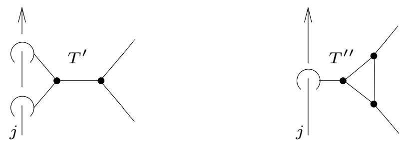

FIGURE 6.2

6.2. **Proof of Theorem 1.1.** We can now prove Theorem 1.1. We only need to prove the 'if' part of the statement.

Proof of Theorem 1.1. Let L be a C_3-trivial n-component link. Consider an n-component string link l such that its closure is L and $l \sim_{C_3} \mathbf{1}_n$. By Lemmas 2.1, 2.2 and 2.3, and the same arguments as those used in the proof of Proposition 5.1, we have that

$$l \sim_{C_4} l_0 \cdot l_1 \cdot l_2 \cdot l_3 \cdot l_4,$$

where

- $l_0 = \prod_i (\mathbf{1}_n)_{U_i}$, where U_i is union of simple C_3-trees of index $\{i^{(4)}\}$ contained in a regular neighborhood of the i^{th} component of $\mathbf{1}_n$; $1 \le i \le n$.
- $l_1 = \prod_{i<j} \left((\mathbf{1}_n)_{X_{ij}}\right)^{x_{ij}}$, where X_{ij} is the simple C_3-tree of index $\{i^{(2)}, j^{(2)}\}$ represented in Figure 6.3, and where $x_{ij} \in \mathbb{Z}$.
- $l_2 = \prod_{i<j;k} \left((\mathbf{1}_n)_{Y_{ijk}}\right)^{y_{ijk}}$, where Y_{ijk} is the simple C_3-tree of index $\{i, j, k^{(2)}\}$ represented in Figure 6.3.
- $l_3 = \prod_{i\neq j<k<l} \left((\mathbf{1}_n)_{Z_{ijkl}}\right)^{z_{ijkl}}$, where Z_{ijkl} is the simple C_3-tree of index $\{i, j, k, l\}$ represented in Figure 6.3.
- l_4 is obtained from $\mathbf{1}_n$ by surgery along simple C_3-trees with index of the form $\{i, j^{(3)}\}$; $1 \le i \neq j \le n$.

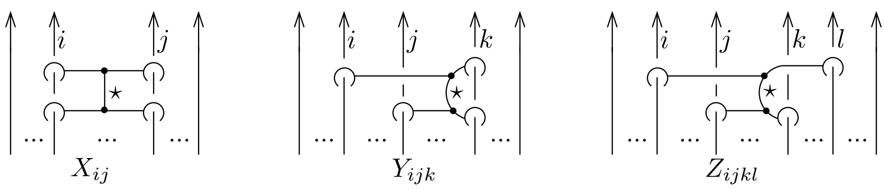

FIGURE 6.3. Here X_{ij}^{-1} (resp. Y_{ijk}^{-1}, Z_{ijkl}^{-1}) is defined as obtained from X_{ij} (resp. Y_{ijk}, Z_{ijkl}) by a positive half-twist on the edge marked by a $\star$.

As an immediate consequence of Lemma 6.1, we thus have

$$L = \mathrm{cl}(l) \sim_{C_4} \mathrm{cl}(l_0 \cdot l_1 \cdot l_2 \cdot l_3).$$

It follows from standard computations (see preceding sections) that

$$\begin{aligned} \overline{\mu}_L(i,i,j,j) &= \mu_{l_1}(i,i,j,j) = x_{ij} && \text{for all } 1 \le i < j \le n, \\ \overline{\mu}_L(i,j,k,k) &= \mu_{l_2}(i,j,k,k) = y_{ijk} && \text{for all } 1 \le i < j \le n,\ 1 \le k \le n, \\ \overline{\mu}_L(i,j,k,l) &= \mu_{l_3}(i,j,k,l) = z_{ijkl} && \text{for all } 1 \le i \neq j < k < l \le n. \end{aligned}$$

Now, consider another C_3-trivial n-component link L', such that L and L' satisfy assertions (1) and (2) in the statement of Theorem 1.1. By the same construction as above and (1), we have

$$L' \sim_{C_4} \mathrm{cl}(l'_0 \cdot l_1 \cdot l_2 \cdot l_3).$$

Here $l'_0 = \prod_i (\mathbf{1}_n)_{U'_i}$, where U'_i is union of simple C_3-trees of index $\{i^{(4)}\}$ contained in a regular neighborhood of the i^{th} component of $\mathbf{1}_n$ ($1 \le i \le n$). Denote respectively by $(l_0)_i$ and $(l'_0)_i$ the i^{th} component of l_0 and l'_0. By (2) and [6, Thm. 6.18], we have $(l_0)_i \sim_{C_4} (l'_0)_i$ for all i in $\{1, ..., n\}$. We thus have $l_0 \sim_{C_4} l'_0$, which implies the result. □

6.3. **Proof of Proposition 1.3.** It suffices to show that two 2-component links L and L' which are not distinguished by Vassiliev invariants of order ≤ 3 are C_4-equivalent (the converse is well-known).

Proof of Proposition 1.3. By [17, Thm. 1.5], L' can be obtained from L by a sequence of surgeries along

(1) C_4-trees,
(2) simple C_3-trees with index $\{i, j^{(3)}\}$, $\{i, j\} = \{1, 2\}$.

By Lemma 6.1, each surgery of type (2) can be achieved by surgery along C_4-trees. It follows that $L \sim_{C_4} L'$. □

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Research Institute for Mathematical Sciences (RIMS), Kyoto University, Oiwake-cho, Kitashirakawa, Sakyo-ku,, Kyoto 606-8502, Japan
E-mail address: meilhan@kurims.kyoto-u.ac.jp

Tokyo Gakugei University, Department of Mathematics, Koganeishi, Tokyo 184-8501, Japan
E-mail address: yasuhara@u-gakugei.ac.jp