Time Evolution with and without Remote Past (I): Noise Driven Automorphisms of Compact Abelian Groups

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1 Introduction

Usually the time evolution is discussed from the present to the future or from the past, precisely, from some fixed initial time in the past, to the present or to the future. But we sometimes consider the time evolution from the remote past to the remote future as in the theory of stationary stochastic processes. In the present paper we consider time evolutions governed by noise driven automorphism on compact abelian groups and give a necessary and sufficient condition for the time evolution to admit remote past. It turns out that *to admit the remote past is fairly restrictive.*

Let G be a compact abelian group and φ be an automorphism of G. Consider the random time evolution governed by a stochastic equation

$$\eta_n = \varphi(\eta_{n-1}) + \xi_n \qquad (n \in \mathbb{Z}) \tag{1.1}$$

on G where (ξ_n) is a noise in the sense that (a) the random variables ξ_n are mutually independent and subject to a common probability distribution, say μ, and (b) for each n the random variable ξ_n is independent of the random variables η_k with $k < n$.

The point is that ξ_n and η_n are indexed for all integer n, including negative n. We say that the time evolution admits remote past if there exists a solution (ξ_n, η_n) of (1.1).

It is immediate (cf. [1]) to see that the above equation (1.1) is reduced to the convolution equation

$$\lambda_n = \mu_n * \lambda_{n-1} \qquad (n \in \mathbb{Z}) \tag{1.2}$$

where λ_n's are unknown probability measures on G which stand for the probability distribution of $\varphi^{-n}\eta_n$ and μ_n's are known probability measures which come from $\varphi^{-n}\xi_n$. The equation (1.2) for arbitrarily given μ_n's will be discussed in Section 3. We remark that the equation (1.2) always admits the *trivial solution* $(\lambda_n^{\mathrm{Haar}})$ where each $\lambda_n^{\mathrm{Haar}}$ is the normalized Haar measure on G.

Let Γ stand for the characteristic group of G.

Theorem 1.1. *Assume that the noise is stationary and the automorphism is the identity:*

$$\mu_n = \mu \textit{ for any } n \in \mathbb{Z} \textit{ and } \varphi = \mathrm{id}. \tag{1.3}$$

Set

$$\Gamma_\mu = \{\chi \in \Gamma : |\mu(\chi)| = 1\} \tag{1.4}$$

and

$$G_\mu = \{g \in G : \chi(g) = 1 \textit{ for all } \chi \in \Gamma_\mu\}. \tag{1.5}$$

Then there exists a unique element $\alpha(\mu)$ in G/G_μ such that solutions (λ_n) of (1.2) are characterized by the following two properties:
(a) *Each λ_n is G_μ-invariant.*
(b) *The projections $\widehat{\lambda}_n$ of λ_n to G/G_μ evolves by the Weyl transformation $\tau_{\alpha(\mu)}$:*

$$\widehat{\lambda}_n = \tau_{\alpha(\mu)}\widehat{\lambda}_{n-1} \qquad (n \in \mathbb{Z}) \tag{1.6}$$

where $\tau_\alpha\beta = \beta + \alpha$ for $\alpha, \beta \in G/G_\mu$.

To illustrate the idea, we give examples in the case where $G = \mathbb{T}^1$. Here we identify $\mathbb{T}$ with the interval $[0, 1)$.

Example 1.2 (cf. [4, Section 7, Lemma 5]). For $p = 1, 2, \ldots$, we denote $\mathbb{Z}_p = \{k/p : k = 0, 1, \ldots, p-1\}$.
(i) Assume that $\mu(x + \mathbb{Z}_p) < 1$ for any $p = 1, 2, \ldots$ and any $x \in \mathbb{T}$. Then $G_\mu = \mathbb{T}$. In this case the solution of (1.2) is only the trivial solution.
(ii) Assume that $\mu(\{x\}) = 1$ for some $x \in \mathbb{T}$. Then $G_\mu = \{0\}$ and $\alpha(\mu) = \{x\}$. In this case every solution (λ_n) of (1.2) evolves by the translation by x.
(iii) Otherwise, we can choose $x \in \mathbb{T}$ and $p = 2, 3, \ldots$ such that p is minimum among the pairs (x, p) with $\mu(x + \mathbb{Z}_p) = 1$. Then $G_\mu = \mathbb{Z}_p$ and $\alpha(\mu) = x + \mathbb{Z}_p$.

In particular, we consider the case where the support of μ consists of two points: $\mu = p_1\delta_{x_1} + p_2\delta_{x_2}$ for some $p_1, p_2 > 0$, $p_1 + p_2 = 1$ and $x_1, x_2 \in \mathbb{T}$, $x_1 \neq x_2$.
(i)′ Assume that $x_2 - x_1$ is rational under identification $G \simeq [0, 1)$. We can express $x_2 - x_1 = r/p$ for some $p = 2, 3, \ldots$ and $r \in \mathbb{N}$ where p and r are coprime. Then $G_\mu = \mathbb{Z}_p$ and $\alpha(\mu) = x_1 + \mathbb{Z}_p$ $(= x_2 + \mathbb{Z}_p)$.
(ii)′ Otherwise, $G_\mu = \mathbb{T}$.

The following observation shows that the existence of nontrivial solution is fairly restrictive and is surprising at least to the authors.

Theorem 1.3. *Assume that G has a countable basis. Let μ be a probability measure on G and φ an automorphism of G. Set*

$$\Gamma_\mu = \left\{ \chi \in \Gamma : \prod_{k=m}^{\infty} |\mu(\chi \circ \varphi^k)| > 0 \text{ for some } m \right\} \tag{1.7}$$

and

$$G_\mu = \{g \in G : \chi(g) = 1 \text{ for all } \chi \in \Gamma_\mu\}. \tag{1.8}$$

Then there exists an element $\alpha(\mu) \in G/G_\mu$ such that $\mu(\cap_{\chi\in\Gamma_\mu} W^s(a, \chi, \varphi)) = 1$ for any $a \in \alpha(\mu)$ where

$$W^s(a, \chi, \varphi) = \left\{ x \in G : \lim_{k\to\infty} \chi(\varphi^k x)/\chi(\varphi^k a) = 1 \right\}. \tag{1.9}$$

Theorem 1.4. *Under the same assumption and notations as in Theorem 1.3, the following statements hold:*

(a) *Every solution* (λ_n) *of* (1.2) *consists of* G_μ*-invariant measures* λ_n.
(b) *There exists a sequence* (ν_n) *of* G_μ*-invariant probability measures on* G *such that every extramal solution* (λ_n) *of the convolution equation* (1.2) *corresponds to a unique element* $\gamma \in G/G_\mu$ *by the relation*

$$\lambda_n = \nu_n * \delta_{-\sum_{j=0}^{-n-1} \varphi^j a + c} \qquad (n \in \mathbb{Z}) \tag{1.10}$$

where a *and* c *are arbitrary elements of the cosets* $\alpha(\mu)$ *and* γ*, respectively, and the sum* $\sum_{j=0}^{-n-1}$ *is interpreted as 0 for* $n = 0$ *and* $-\sum_{j=-n}^{-1}$ *for positive* n.

In Section 5 we give a further property in the special case where G is a finite-dimensional torus and $\Gamma_\mu = \Gamma$.

The equation (1.1) has a background in the theory of stochastic differential equation of the probability theory. Roughly speaking, a stochastic differential equation is a differential equation driven by a noise process. Hence one may ask whether the randomness of a solution is exhausted by that of the noise process, namely, in the terminology of the discrete-time equation (1.1), whether each η_n is expressed as a function of $\xi_n, \xi_{n-1}, \dots$ up to a null set. If the answer is yes, the solution is called *strong*, and otherwise *nonstrong*. While for ordinary differential equations any solution is always strong, for stochastic differential equations several examples have been known since 1960's which possess nonstrong solutions.

Tsirelson ([3]) has constructed such an example by considering the discrete-time equation (1.1) when $G = \mathbb{T}$. Yor ([4]) has studied the discrete-time equation for general noise laws and determined the necessary and sufficient condition on the noise law for presence/absence of strong solutions and for uniqueness/nonuniqueness of the solution. Recently Akahori et. al. ([1]) have studied the equation (1.1) for general compact groups G and proved that the extremal set of solutions is homeomorphic to the quotient space G/H for some subgroup H of G.

We must also refer to a work of Brossard and Leuridan ([2]). They have studied rather general Markov chains and investigate the uniqueness problem and the behavior of sample paths at the remote past. But they imposed a restrictive assumption that the one-step transition probability is absolute continuous with respect to a measure; Consequently, the case which involves the Weyl transform is excluded.

2 The simple case

In this section we assume that the noise is stationary and the automorphism is the identity:

$$\mu_n = \mu \text{ for any } n \in \mathbb{Z} \text{ and } \varphi = \mathrm{id}. \tag{2.1}$$

Then the convolution equation (1.2) takes the form

$$\lambda_n = \mu * \lambda_{n-1} \qquad (n \in \mathbb{Z}). \tag{2.2}$$

Then we obtain

$$\lambda_n(\chi) = \mu(\chi)^m \lambda_{n-m}(\chi) \qquad n \in \mathbb{Z},\ m \in \mathbb{N},\ \chi \in \Gamma. \tag{2.3}$$

Definition 2.1. Set

$$\Gamma_\mu = \{\chi \in \Gamma : |\mu(\chi)| = 1\}. \tag{2.4}$$

Lemma 2.2. *If $|\mu(\chi)| < 1$, then $\lambda_n(\chi) = 0$ for any $n \in \mathbb{Z}$.*

Proof. By (2.3), we have $|\lambda_n(\chi)| = |\mu(\chi)|^m |\lambda_{n-m}(\chi)|$. Since $|\lambda_{n-m}(\chi)| \le \lambda_{n-m}(|\chi|) \le 1$, we obtain $|\lambda_n(\chi)| \le |\mu(\chi)|^m$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Letting m tend to infinity, we obtain $\lambda_n(\chi) = 0$. □

Lemma 2.3. *Let λ be a probability measure on G and Γ_0 be a subset of Γ. Assume that $\lambda(\chi) = 0$ whenever $\chi \notin \Gamma_0$. Then, the measure λ is G_0-invariant where G_0 is the annihilator of Γ_0:*

$$G_0 = \{x \in G : \chi(x) = 1 \text{ for all } \chi \in \Gamma_0\}. \tag{2.5}$$

Proof. Let T_g be the translation by $g \in G_0$. Then, $(T_g\lambda)(\chi) = \chi(g)\lambda(\chi) = \lambda(\chi)$ if $\chi \in \Gamma_0$ by the definition of G_0. Otherwise, $(T_g\lambda)(\chi) = \chi(g)\lambda(\chi) = 0 = \lambda(\chi)$ by the assumption on λ. Hence, $T_g\lambda = \lambda$. □

Let us denote the annihilator of Γ_μ by G_μ.

Lemma 2.4. *If $|\mu(\chi)| = 1$, then $\chi(x)$ is constant μ-a.e. In particular, $\chi(x) = \mu(\chi)$ for μ-a.e. x.*

Proof. Since $|\chi(x)| = 1$ μ-a.e., one obtains

$$\begin{aligned} 0 &\le \int_X |\chi(x) - \mu(\chi)|^2 \mu(dx) \\ &\le \int_X \int_X |\chi(x) - \chi(y)|^2 \mu(dx)\mu(dy) = 2(1 - |\mu(\chi)|^2). \end{aligned} \tag{2.6}$$

Hence, if $|\mu(\chi)| = 1$, then $\chi(x) = \mu(\chi)$ μ-a.e. □

Proposition 2.5. *The following statements hold:*
(i) *Γ_μ is a subgroup of the character group Γ.*
(ii) *For any $\chi_1, \chi_2 \in \Gamma_\mu$,*

$$\mu(\chi_1\overline{\chi_2}) = \mu(\chi_1)\overline{\mu(\chi_2)}. \tag{2.7}$$

In other words, the restriction $\mu|_{\Gamma_\mu}$ is a character of Γ_μ.
(iii) *There exists a unique element α_μ in G/G_μ such that $\mu(\chi) = \chi(a)$ for any $a \in \alpha_\mu$ and any character $\chi \in \Gamma_\mu$.*

Proof. Let $\chi_1, \chi_2 \in \Gamma_\mu$. By Lemma 2.4, we see that $\chi_1(x) = \mu(\chi_1)$ and $\chi_2(x) = \mu(\chi_2)$ for μ-a.e. $x \in G$. Then we have $(\chi_1\overline{\chi_2})(x) = \mu(\chi_1)\overline{\mu(\chi_2)}$ for μ-a.e. $x \in G$, and, hence, we obtain $\mu(\chi_1\overline{\chi_2}) = \mu(\chi_1)\overline{\mu(\chi_2)}$. This implies (ii) and also (i).

Note that Γ_μ is identified with the character group of G/G_μ. By Pontryagin's duality theorem, the character $\mu|_{\Gamma_\mu}$ of Γ_μ obtained in Proposition 2.5 can be identified with an element of G/G_μ. We identify it with a coset $\alpha(\mu)$. Then, for any $a \in \alpha(\mu)$ and any character $\chi \in \Gamma_\mu$, we obtain $\mu(\chi) = \chi(a)$. □

Proof of Theorem 1.1. We already proved (a) in Lemmas 2.2 and 2.3.

It then follows from Proposition 2.5 (iii) and from a similar argument in the proof of Lemma 2.3 that $\lambda_n(\chi) = \chi(a)\lambda_{n-1}(\chi)$ for all $n \in \mathbb{Z}$, $a \in \alpha(\mu)$ and $\chi \in \Gamma$. Consequently, $\lambda_n = T_a\lambda_{n-1}$. Since each λ_n is G_μ-invariant, we obtain (b). □

Remark 2.6. Assume, in addition, that G has a countable basis. Since each λ_n is G_μ-invariant, there exists a probability measure $\widehat{\lambda}_n$ on the quotient group G/G_μ such that

$$\int_G \lambda_n(dx)f(x) = \int_{G/G_\mu} \widehat{\lambda}_n(dh) \int_{G_\mu} \nu(dy)f(h.y) \tag{2.8}$$

for any continuous function f on G where ν is the normalized Haar measure on G_μ and $h.y$ stands for an element of $h \in G/G_\mu$ identified with a coset.

3 Nonstationary noise

In this section we continue to assume that $\varphi = \mathrm{id}$ but we consider the case where μ_n does depend on n. Now the convolution equation (1.2) takes the original form

$$\lambda_n = \mu_n * \lambda_{n-1} \qquad n \in \mathbb{Z} \tag{3.1}$$

and, hence, we obtain

$$\lambda_n(\chi) = \mu_n(\chi)\mu_{n-1}(\chi)\cdots\mu_{n-m+1}(\chi)\lambda_{n-m}(\chi) \qquad n \in \mathbb{Z},\ m \in \mathbb{N},\ \chi \in \Gamma. \tag{3.2}$$

Lemma 3.1. *If $\prod_{k=1}^{\infty} \mu_{-k}(\chi) = 0$, then $\lambda_n(\chi) = 0$ for any $n \in \mathbb{Z}$.*

Proof. By (3.2), we have $|\lambda_n(\chi)| = \prod_{k=1}^{m-1} |\mu_{n-k}(\chi)||\lambda_{n-m}(\chi)|$. Since $|\lambda_{n-m}(\chi)| \leq \lambda_{n-m}(|\chi|) \leq 1$, we obtain $|\lambda_n(\chi)| \leq \prod_{k=1}^{m-1} |\mu_{n-k}(\chi)|$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Letting m tend to infinity, we obtain $\lambda_n(\chi) = 0$. □

Definition 3.2. Set

$$\Gamma_\mu = \left\{ \chi \in \Gamma : \prod_{k=m}^{\infty} |\mu_{-k}(\chi)| > 0 \text{ for some } m \right\}. \tag{3.3}$$

Remark 3.3. In the case considered in Section 2 the two definitions of Γ_μ given by (2.4) and (3.3) coincide.

Lemma 3.4. *The inequality $\prod_{k=m}^{\infty} |\mu_{-k}(\chi)| > 0$ holds if and only if $\mu_{-k}(\chi) \neq 0$ for any $k \geq m$ and*

$$\sum_{k=m}^{\infty} \int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy)|\chi(x) - \chi(y)|^2 < \infty. \tag{3.4}$$

Proof. Notice that

$$\int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy)|\chi(x) - \chi(y)|^2 = 2(1 - |\mu_{-k}(\chi)|^2). \tag{3.5}$$

Hence the assertion follows from the fact that the infinite product $\prod_{k=1}^{\infty} c_k$ of $0 \leq c_k \leq 1$ converges to a positive limit if and only if c_k's are positive and $\sum_{k=1}^{\infty}(1 - c_k) < \infty$. □

Proposition 3.5. Γ_μ *is a subgroup of the character group* Γ.

Proof. Let $\chi_1, \chi_2 \in \Gamma_\mu$. Then it follows from Lemma 3.4 that, for sufficiently large m,

$$
\begin{aligned}
(3.6) \qquad & \sum_{k=m}^{\infty} \int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy) |(\chi_1\overline{\chi_2})(x) - (\chi_1\overline{\chi_2})(y)|^2 \\
& \le \sum_{k=m}^{\infty} \int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy) 2 \left\{ |\chi_1(x) - \chi_1(y)|^2 + |\chi_2(x) - \chi_2(y)|^2 \right\} < \infty.
\end{aligned}
$$

Moreover,

$$
(3.7) \qquad \sum_{k=m}^{\infty} \int_G \mu_{-k}(dx) |(\chi_1\overline{\chi_2})(x) - \mu_{-k}(\chi_1\overline{\chi_2})|^2 < \infty.
$$

Hence, $\mu_{-k}(\chi_1\overline{\chi_2}) \neq 0$ except for finitely many k. Consequently, again by Lemma 3.4 we conclude that $\chi_1\overline{\chi_2} \in \Gamma_\mu$. Since we assume that G is compact, the character group Γ is discrete. Hence, an algebraic subgroup of Γ is a (topological) subgroup. □

Remark 3.6. Lemma 2.3 in the previous section works here, too. Thus, each λ_n of a solution (λ_n) is G_μ-invariant.

Here we stop the preliminary discussion on nonstationary noises and we proceed in the next section to the case where the noise is stationary and the automorphism is arbitrary.

4 General case

Let φ be an automorphism of a compact abelian group G. We assume that the noise (ξ_n) is stationary so that the random variables ξ_n's are independent and subject to a common probability distribution μ. So we consider the stochastic equation (1.1) stated in Section 1 where μ_n is the probability distribution of $\varphi^{-n}\xi_n$ and λ_n is the probability distribution of $\varphi^{-n}\eta_n$. We denote by $\widetilde{\lambda}_n$ the probability distribution of η_n itself.

The automorphism φ of G induces an automorphism φ^* of the character group Γ: $\varphi^*\chi(x) = \chi(\varphi x)$, $x \in G$, $\chi \in \Gamma$. For $\chi \in \Gamma$ define

$$
(4.1) \qquad W_2^s(\chi, \varphi) = \left\{ (x, y) \in G \times G : \sum_{k=0}^{\infty} |(\varphi^{*n}\chi)(x) - (\varphi^{*n}\chi)(y)|^2 < \infty \right\}
$$

and, for $x \in G$,

$$
\begin{aligned}
(4.2) \qquad W_2^s(x; \chi, \varphi) &= \{ y \in G : (x, y) \in W_2^s(\chi, \varphi) \} \\
&= \left\{ y \in G : \sum_{k=0}^{\infty} |(\varphi^{*n}\chi)(y) - (\varphi^{*n}\chi)(x)|^2 < \infty \right\}.
\end{aligned}
$$

We may call it the ℓ^2*-stable set of* x *in the direction* χ *with respect to* φ.

Remark 4.1. We have the obvious relation $W_2^s(x;\chi,\varphi) \subset W^s(x;\chi,\varphi)$ where $W^s(x;\chi,\varphi)$ is defined in (1.9).

Lemma 4.2. *The set $W_2^s(0;\chi,\varphi)$ is a φ-invariant subgroup of G.*

Proof. Obvious. □

Now Lemma 3.4, Proposition 3.5 and Remark 3.6 can be restated as follows.

Proposition 4.3. *Assume that (λ_n) solves the equation (1.2). Then the following statements hold.*
(i) *If $\chi \in \Gamma_\mu$, then, $(\mu \otimes \mu)(W_2^s(\chi,\varphi)) = 1$.*
(ii) *Γ_μ is φ^*-invariant.*
(iii) *G_μ is a φ-invariant subgroup.*
(iv) *λ_n is G_μ-invariant and so is $\widetilde{\lambda}_n$.*

Proof. (ii)-(iv) are obvious restatements. To see (i) it suffices to note that

$$\begin{aligned}(4.3)\qquad &\int_G\int_G \mu(dx)\mu(dy)\sum_{k=m}^{\infty}|(\varphi^{*k}\chi)(x)-(\varphi^{*k}\chi)(y)|^2\\ &=\sum_{k=m}^{\infty}\int_G\int_G \mu_{-k}(dx)\mu_{-k}(dy)|\chi(x)-\chi(y)|^2.\end{aligned}$$

□

Remark 4.4. If $\varphi = \mathrm{id}$, then, $W_2^s(x;\chi,\varphi) = \{y : \chi(y) = \chi(x)\}$. Hence, if we assume, in addition, that G is metrizable or that Γ_μ is coutable, then we can apply Fubini's theorem and the assertion (i) of Proposition 4.2 shows

$$(4.4)\qquad (\mu\otimes\mu)\{(x,y)\in G : \chi(x)=\chi(y) \text{ for all } \chi\in\Gamma_\mu\} = 1.$$

This implies that the support of μ consists of a single coset in G/G_μ, which is nothing but the element $\alpha(\mu)$ introduced in Section 2.

Proof of Theorem 1.2. Obvious from (i) of Proposition 4.2. □

Proof of Theorem 1.3. Let (λ_n) be a solution of (1.2) and $a \in \alpha(\mu)$. Set

$$(4.5)\qquad \mu_n^\circ = T_{-\varphi^{-n}a}\mu_n \qquad (n\in\mathbb{Z})$$

and

$$(4.6)\qquad \lambda_n^\circ = T_{\sum_{j=0}^{-n-1}\varphi^j a}\lambda_n \qquad (n\in\mathbb{Z})$$

for each n. Here we interpret $\sum_{j=0}^{-1} = 0$ and $\sum_{j=0}^{-n-1} = -\sum_{j=-n}^{-1}$ for positive n. Then they satisfy

$$(4.7)\qquad \lambda_n^\circ = \mu_n^\circ * \lambda_{n-1}^\circ \qquad (n\in\mathbb{Z}),$$

and, hence,

$$\lambda_n^\circ = \mu_n^\circ * \mu_{n-1}^\circ * \cdots * \mu_{n-k}^\circ * \lambda_{n-k-1}^\circ \qquad (n \in \mathbb{Z},\ k \in \mathbb{N}). \tag{4.8}$$

Recall that the totality of probability measures on a compact metrizable space is compact in the weak topology. Here we say that $\mu_n \to \mu$ weakly if $\mu_n(f) \to \mu(f)$ for any continuous function f. This topology is called the *weak* topology* in the context of the functional analysis.

Now we can choose an increasing sequence of integers $m_j \to \infty$ such that the weak limit

$$\nu_n^\circ = \lim_{j\to\infty} \mu_n^\circ * \mu_{n-1}^\circ * \cdots * \mu_{n-m_j}^\circ \qquad (n \in \mathbb{Z}) \tag{4.9}$$

exists. Since

$$\nu_n^\circ(\chi) = \lim_{j\to\infty} \prod_{i=0}^{m_j} \mu_{n-i}^\circ(\chi) \qquad (n \in \mathbb{Z}) \tag{4.10}$$

for any $\chi \in \Gamma$, we see that $\nu_n^\circ(\chi)$ for each $n \in \mathbb{Z}$ is not equal to 0 for any $\chi \in \Gamma_\mu$ and is equal to 0 for any $\chi \notin \Gamma_\mu$. Thus we conclude that ν_n° is G_μ-invariant by Lemma 2.3. Note that (ν_n) is not uniquely determined from (μ_n), but, for each choice of a sequence m_j, the limit $(\nu_n(\chi))$ is uniquely determined up to a multiplicative constant of modulus 1.

Take a limit point of the sequence $(\lambda_{-m_j}^\circ)$ and denote it by $\lambda_{-\infty}^\circ$. Recall that $\mu_n \to \mu$ and $\nu_n \to \nu$ weakly imply $\mu_n * \nu_n \to \mu * \nu$ weakly. In fact, it is obvious that the product measure $\mu_n \otimes \nu_n \to \mu \otimes \nu$ weakly and that the pullback of any continuous function under the map $(x, y) \mapsto x + y$ is again a continuous function on the product space. Letting k tend to infinity in (4.8), we obtain

$$\lambda_n^\circ = \nu_n^\circ * \lambda_{-\infty}^\circ. \tag{4.11}$$

Then (λ_n) is expressed as

$$\begin{aligned} \lambda_n &= T_{-\sum_{j=0}^{-n-1} \varphi^j a}(\nu_n^\circ * \lambda_{-\infty}^\circ) \\ &= \nu_n * T_{-\sum_{j=0}^{-n-1} \varphi^j a}(\lambda_{-\infty}^\circ). \end{aligned} \tag{4.12}$$

Here we denote $\nu_n = T_{-\sum_{j=0}^{-n-1} \varphi^j a}(\nu_n^\circ)$, which is also G_μ-invariant.

Consequently, an extremal solution (λ_n) is expressed as

$$\lambda_n = \nu_n * \delta_{-\sum_{j=0}^{-n-1} \varphi^j a + c} \tag{4.13}$$

for some $c \in G$ and the element c is unique modulo G_μ.

□

5 The case of toral automorphisms

By the definition the probability measures ν_n° satisfy the equation

$$\nu_n^\circ = \mu_n^\circ * \nu_{n-1}^\circ \qquad n \in \mathbb{Z}. \tag{5.1}$$

Consequently, if we take a sequence of independent random variables ξ_n° subject to μ_n°, then, we may formally understand that each ν_n° is the probability distribution of the infinite sum $\sum_{k=-\infty}^n \xi_k^\circ$.

In some special cases the convergence of $\sum_{k=-\infty}^n \xi_k^\circ$ is justified and an explicit formula for the solution (η_n) of the equation (1.1) is obtained.

Theorem 5.1. *Let G be a finite dimensional torus, say $G = \mathbb{T}^d$ and assume that $\Gamma_\mu = \mathbb{Z}^d$. Let d be a distance in $\mathbb{T}^d$. Then*

$$\sum_{k=-n}^{\infty} \varphi^k(\xi_{-k}^\circ) \text{ converges almost surely} \tag{5.2}$$

and

$$E\left[\sum_{k=-n}^{\infty} d(\varphi^k(\xi_{-k}^\circ), 0)^2\right] < \infty \tag{5.3}$$

for each $n \in \mathbb{Z}$. Moreover, the extremal solution (η_n) is given by the formula

$$\varphi^{-n}(\eta_n) = c + \sum_{k=-n}^{\infty} \varphi^k(\xi_{-k}^\circ) + \sum_{k=0}^{-n-1} \varphi^k(a) \qquad \text{for } n \in \mathbb{Z} \tag{5.4}$$

with $c \in \mathbb{T}^d$ and $a \in \alpha(\mu)$ where $\alpha(\mu)$ is defined in Theorem 1.4.

Lemma 5.2. *There exists a constant r with $0 < r < 1$ such that*

$$\begin{aligned} \cap_{\chi\in\Gamma} W^s(a;\chi,\varphi) &= \cap_{\chi\in\Gamma} W_2^s(a;\chi,\varphi) \\ &= \left\{x \in G : d(\varphi^k(x), \varphi^k(a)) \le Cr^n \text{ for any } k \in \mathbb{N} \text{ and for some constant } C\right\}. \end{aligned} \tag{5.5}$$

Proof. Let us identify $\mathbb{T}^d$ with the unit cube $[-1/2, 1/2)^d$ in $\mathbb{R}^d$ and measures on $\mathbb{T}^d$ with those on $[-1/2, 1/2)^d$. Then the automorphism φ is regarded as an automorphism on $[-1/2, 1/2)^d$ and is defined by a matrix A as

$$\varphi(x) \quad = \quad Ax \bmod \mathbb{Z}^d. \tag{5.6}$$

Under the identification stated above, $\varphi^k(x) \to 0$ as $k \to \infty$ in $\mathbb{T}^d$ if and only if $A^k x \to 0$ as $k \to \infty$ in $\mathbb{R}^d$. Since A is a finite dimensional matrix, it means that the vector x in $\mathbb{R}^d$ belongs to the linear span of eigenvectors of A corresponding to eigenvalues of modulus less than 1. Take a constant r which is less than 1 and is greater than the maximum modulus of such eigenvalues. Then for any norm $\|\cdot\|$ there holds the inequality $\|A^k x\| \le Cr^k$ for some constant C. Consequently, for any distance d on $\mathbb{T}^d$ there holds the inequality $d(0, \varphi^k(x)) \le Cr^k$ for some constant C depnending on x (which may be different from the previous C). Hence follows the desired assertion. □

Remark 5.3. If $x \neq 0$ and $A^k x \to 0$ as $k \to \infty$, then $\|A^{-k}x\| \to \infty$ as $k \to \infty$ but the converse is not true.

To prove Theorem 5.1 we want to apply a well-known convergence theorem: if X_k, $k = 0, 1, \ldots$, are independent $\mathbb{R}^d$-valued random variables and if they are square integrable with $\sum_{k=0}^{\infty} E[\|X_k - E[X_k]\|^2] < \infty$, then $\sum_{k=0}^{\infty}(X_k - E[X_k])$ converges almost surely and in L^2 sense. Here appear two obstacles:
(a) Absence of the notion of mean for group elements.
(b) The sequence $\chi(\varphi^k(x)) - \int_G \mu(dy)\chi(\varphi^k(y))$ is square summable for μ-a.e. x but is generally not summable.

Lemma 5.2 above shows that the assumptions of Theorem 5.1 eliminates (b). Indeed, the sequence $\chi(\varphi^k(x)) - \int_G \mu(dy)\chi(\varphi^k(y))$ decreases exponentially.

Lemma 5.4. *Under the identification of* $\mathbb{T}^d$ *with* $[-1/2, 1/2)^d$*, we obtain*

$$E\left[\sum_{k=0}^{\infty} \|\varphi^k(\xi_{-k}^{\circ})\|^2\right] < \infty. \tag{5.7}$$

Proof. We start with the following restatement of (3.4):

$$\int_G \mu(dx) E\left[\sum_{k=0}^{\infty} |\chi(\varphi^k(\xi_{-k})) - \chi(\varphi^k(x))|^2\right] < \infty. \tag{5.8}$$

Thus, for μ-a.e. x,

$$E\left[\sum_{k=0}^{\infty} |\chi(\varphi^k(\xi_{-k})) - \chi(\varphi^k(x))|^2\right] < \infty. \tag{5.9}$$

Thererfore, $\xi_{-k}^{\circ} = \xi_{-k} - \varphi^k(a)$ satisfies

$$E\left[\sum_{k=0}^{\infty} |\chi(\varphi^k(\xi_{-k}^{\circ})) - 1|^2\right] < \infty. \tag{5.10}$$

Now let χ_j, $j = 1, 2, \ldots, d$, be the standard generators of Γ:

$$\chi_j(x) = \exp(2\pi\sqrt{-1}x_j) \text{ for } x = (x_1, \ldots, x_d) \in [-1/2, 1/2)^d. \tag{5.11}$$

Note that $|\exp(2\pi\sqrt{-1}x) - 1| \geq c|x|$ for $x \in [-1/2, 1/2)$. Hence it follows from (5.10) with $\chi = \chi_j$, $j = 1, 2, \ldots, d$, in $\mathbb{T}^d$ that

$$E\left[\sum_{k=0}^{\infty} \|\varphi^k(\xi_{-k}^{\circ})\|^2\right] < \infty \tag{5.12}$$

in $\mathbb{R}^d$. □

Proof of Theorem 5.1. It follows from Lemma 5.4 that

$$E\left[\sum_{k=0}^{\infty} \left\|\varphi^k(\xi^\circ_{-k}) \ - E[\varphi^k(\xi^\circ_{-k})]\right\|^2\right] < \infty. \tag{5.13}$$

Hence, the sum

$$\sum_{k=0}^{\infty} \left\{\varphi^k(\xi^\circ_{-k}) - E[\varphi^k(\xi^\circ_{-k})]\right\} \tag{5.14}$$

converges almost surely. On the other hand, we already know that

$$\sum_{k=0}^{\infty} (E[\varphi^k(\xi^\circ_{-k})] - 1) \tag{5.15}$$

converges absolutely. Consequently, the sum $\sum_{k=0}^{\infty} \varphi^k(\xi^\circ_{-k})$ converges almost surely. □

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