Fundamental groups of log configuration spaces and the cuspidalization problem

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Abstract

In the present paper, we study the cuspidalization problem of the fundamental group of a curve by means of the log geometry of the log configuration space, which is a natural compactification of the usual configuration space of the curve. The goal of this paper is to show that the fundamental group of the configuration space is generated by the images from morphisms from a group extension of the fundamental groups of the configuration spaces of lower dimension, and that the fundamental group of the configuration space can be partially reconstructed from a collection of data concerning the fundamental groups of the configuration spaces of lower dimension.

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0 Introduction

In this paper, we consider the *cuspidalization problem* of the fundamental group of a curve. Let X be a smooth, proper, geometrically connected curve of genus $g \geq 2$ over a field K whose (not necessarily positive) characteristic we denote by p.

Problem 0.1. *Let $U \hookrightarrow X$ be an open subscheme of X. Then can one reconstruct the* (*arithmetic*) *fundamental group*

$$\pi_1(U)$$

of U from the (*arithmetic*) *fundamental group $\pi_1(X)$ of X?*

More "generally",

Problem 0.2. *Let r be a natural number. Then can one reconstruct the* (*arithmetic*) *fundamental group*

$$\pi_1(U_{(r)})$$

of the r-th configuration space $U_{(r)}$ of X (*i.e., the open subscheme of the r-th product of X* [*over K*] *whose complement consists of the diagonals "$D_{(r)\{i,j\}} = \{(x_1, \cdots, x_r) \mid x_i = x_j\}$"* ($i \neq j$)) *from the* (*arithmetic*) *fundamental group $\pi_1(X)$ of X?*

In this paper, we study Problem 1.2 by means of the log geometry of the *log configuration scheme* of X, which is a natural compactification of $U_{(r)}$.

Let $\overline{\mathcal{M}}_{g,r}^{\log}$ be the log stack obtained by equipping the moduli stack $\overline{\mathcal{M}}_{g,r}$ of r-pointed stable curves of genus g whose r sections are equipped with an ordering with the log structure associated to the divisor with normal crossings which parametrizes singular curves. Then, for a natural number r, we define the (r-th) log configuration scheme $X_{(r)}^{\log}$ as the fiber product

$$\operatorname{Spec} K \times_{\overline{\mathcal{M}}_{g,0}^{\log}} \overline{\mathcal{M}}_{g,r}^{\log},$$

where the (1-)morphism $\operatorname{Spec} K \to \overline{\mathcal{M}}_{g,0}^{\log}$ is the classifying (1-)morphism determined by the curve $X \to \operatorname{Spec} K$, and the (1-)morphism $\overline{\mathcal{M}}_{g,r}^{\log} \to \overline{\mathcal{M}}_{g,0}^{\log}$ is the (1-)morphism obtained by forgetting the sections. Note that the interior of $X_{(r)}^{\log}$ (i.e., the largest open subset of the underlying scheme of $X_{(r)}^{\log}$ on which the log structure is trivial) is the *usual* (*r-th*) *configuration space* $U_{(r)}$ of X, and that the natural inclusion $U_{(r)} \hookrightarrow X_{(r)}^{\log}$ induces an isomorphism of the *geometrically maximal pro-prime to p quotient* of $\pi_1(U_{(r)})$ (i.e., the quotient of $\pi_1(U_{(r)})$ by the kernel of the natural surjection $\pi_1(U_{(r)} \times_K K^{\mathrm{sep}}) \to \pi_1(U_{(r)} \times_K K^{\mathrm{sep}})^{(\Sigma)}$, where $\pi_1(U_{(r)} \times_K K^{\mathrm{sep}})^{(\Sigma)}$ is the maximal pro-prime to p quotient) with the *geometrically maximal pro-prime to p quotient* of $\pi_1(X_{(r)}^{\log})$.

Let Σ be a (non-empty) set of prime numbers. We shall denote by $\Pi_{X_{(r)}}^{\log}$ the geometrically maximal pro-Σ quotient of $\pi_1(X_{(r)}^{\log})$, by $\Pi_{\mathbb{P}_K}^{\log}$ the geometrically maximal pro-Σ quotient of the log fundamental group of the log scheme

$\mathbb{P}^{\log}_K$ obtained by equipping the projective line $\mathbb{P}^1_K$ with the log structure associated to the divisor $\{0,1,\infty\} \subseteq \mathbb{P}^1_K$, and by G_K the absolute Galois group of K. Then the first main result of this paper is as follows (cf. Theorem 2.5):

Theorem 0.3. *Let $r \geq 3$ be an integer. Then there exist extensions*

$$\Pi_1,\ \Pi_3$$

of $\Pi^{\log}_{X_{(r-1)}}$ *by* $\widehat{\mathbb{Z}}^{(\Sigma)}(1)$*, an extension*

$$\Pi_2$$

of $\Pi^{\log}_{X_{(r-2)}} \times_{G_K} \Pi^{\log}_{\mathbb{P}^1_K}$ *by* $\widehat{\mathbb{Z}}^{(\Sigma)}(1)$*, and continuous homomorphisms*

$$\Pi_i \longrightarrow \Pi^{\log}_{X_{(r)}} \quad (1 \leq i \leq 3)$$

over G_K *such that the morphism*

$$\Pi^{\mathcal{G}}_{X_{(r)}} \stackrel{\mathrm{def}}{=} \varinjlim(\Pi_1 \leftarrow \{1\} \rightarrow \Pi_2 \leftarrow \{1\} \rightarrow \Pi_3) \longrightarrow \Pi^{\log}_{X_{(r)}}$$

induced by the morphisms $\Pi_i \to \Pi^{\log}_{X_{(r)}}$ *is surjective, where the inductive limit is taken in the category of profinite groups.*

Note that Theorem 0.3 can be regarded as a logarithmic analogue of [7], Remark 1.2.

We shall denote by $p^{\log}_{X_{(r)}i} : X^{\log}_{(r+1)} \to X^{\log}_{(r)}$ the morphism induced by the (1-)morphism $\overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ obtained by forgetting the i-th section. Then the second main result of this paper is as follows (cf. Theorem 2.16):

Theorem 0.4. *Let $r \geq 2$ be an integer. Moreover, we assume that*

$$\Sigma = \begin{cases} \text{the set of all prime numbers or } \{l\} & \text{if } p = 0 \\ \{l\} & \text{if } p \geq 2\,. \end{cases}$$

If the collection of data consisting of the profinite groups $\Pi^{\log}_{X_{(k)}}$ $(0 \leq k \leq r)$*, the profinite group* $\Pi^{\log}_{\mathbb{P}}$*, the surjections* $\Pi^{\log}_{X_{(k)}} \to \Pi^{\log}_{X_{(k-1)}}$ $(2 \leq k \leq r)$ *induced by the* $p^{\log}_{X_{(k-1)}i}$*'s* $(2 \leq k \leq r,\ 1 \leq i \leq k)$*, the morphisms* $\Pi_X \to G_K$ *and* $\Pi^{\log}_{\mathbb{P}} \to G_K$ *induced by the respective structure morphisms, and some data concerning the log fundamental groups of the irreducible components of the divisor at infinity (i.e., the divisor with normal crossings which defines the log structure) of* $X^{\log}_{(r)}$ *is given, then we can "reconstruct" the profinite group*

$$\Pi^{\mathcal{G}}_{X_{(r+1)}}$$

defined in Theorem 0.3 and morphisms

$$q_{X_{(r)i}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \longrightarrow \Pi^{\log}_{X_{(r)}} \ \ (1 \leq i \leq r+1)$$

such that $q_{X_{(r)i}}$ *factors as the composite*

$$\Pi^{\mathcal{G}}_{X_{(r+1)}} \longrightarrow \Pi^{\log}_{X_{(r+1)}} \overset{\text{via } p^{\log}_{X_{(r)}i}}{\longrightarrow} \Pi^{\log}_{X_{(r)}},$$

where the first morphism is the morphism obtained in Theorem 0.3.

In Theorem 0.4, we use the terminology "reconstruct" as a sort of "abbreviation" for the somewhat lengthy but mathematically precise formulation given in the statement of Theorem 2.16.

By Theorem 0.3 and Theorem 0.4, if one can also reconstruct group-theoretically *the kernel of the surjection* $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ (which appears in the above composite), then, by taking the quotient by this kernel, one can reconstruct the profinite group $\Pi^{\log}_{X_{(r+1)}}$ (cf. Proposition 2.15, (ii)). However, unfortunately, reconstruction of this kernel is not performed in this paper. Moreover, it seems to the author that if such a reconstruction should prove to be possible, it is likely that the method of reconstruction of this kernel should depend on the "arithmetic" of K in an essential way.

This paper is organized as follows:

In Section 1, we consider the scheme-theoretic and log scheme-theoretic properties of log configuration schemes. Moreover, we study the geometry of the divisor at infinity of $X^{\log}_{(r)}$ in more detail.

In Section 2, we consider the reconstruction of the fundamental groups of higher dimensional log configuration schemes.

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Notation

Symbols:

We shall denote by $\mathbb{Z}$ the set of rational integers, by $\mathbb{N}$ the set of rational integers $n \geq 0$, by $\mathbb{Q}$ the set of rational numbers and by $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$.

Subscripts:

For a ring A (respectively, a scheme X), we shall denote by A_{red} (respectively, X_{red}) the quotient ring by the ideal of all nilpotent elements of A (respectively, the reduced closed subscheme of X associated to X). For a ring A, we shall denote by A^* the group of unity of A. For a field k, we shall use the notation k^{sep} to denote a separable closure of k. For a monoid P, (respectively, a sheaf of monoids $\mathcal{P}$) we shall denote by P^{gp} the group associated to P (respectively, $\mathcal{P}^{\mathrm{gp}}$ the sheaf of groups associated to $\mathcal{P}$). For a group G, we shall denote by G^{ab} the abelianization of G.

Log schemes:

For a log scheme $X^{\log}$, we shall denote by $\mathcal{M}_X$ the sheaf of monoids that defines the log structure of $X^{\log}$.

Let $\mathcal{P}$ be a property of schemes [for example, "quasi-compact", "connected", "normal", "regular"] (respectively, morphisms of schemes [for example, "proper", "finite", "étale", "smooth"]). Then we shall say that a log scheme (respectively, a morphism of log schemes) satisfies $\mathcal{P}$ if the underlying scheme (respectively, the underlying morphism of schemes) satisfies $\mathcal{P}$.

For a log scheme $X^{\log}$ (respectively, a morphism $f^{\log}$ of log schemes), we shall denote by X the underlying scheme (respectively, by f the underlying morphism of schemes). For fs log schemes $X^{\log}$, $Y^{\log}$ and $Z^{\log}$, we shall denote by $X^{\log} \times_{Y^{\log}} Z^{\log}$ the fiber product of $X^{\log}$ and $Z^{\log}$ over $Y^{\log}$ in the category of fs log schemes. In general, the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is not $X \times_Y Z$. However, since strictness (a morphism $f^{\log} : X^{\log} \to Y^{\log}$ is called *strict* if the induced morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ on X is an isomorphism) is stable under base-change in the category of *arbitrary* log schemes, if $X^{\log} \to Y^{\log}$ is strict, then the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is $X \times_Y Z$. Note that since the natural morphism from the saturation of a fine log scheme to the original fine log scheme is finite, properness and finiteness are stable under fs base-change.

If there exist both schemes and log schemes in a commutative diagram, then we regard each scheme in the diagram as the log scheme obtained by equipping the scheme with the trivial log structure.

Terminologies:

We shall assume that the underlying topological space of a *connected* scheme is not empty. In particular, if a morphism is geometrically connected, then it is surjective.

Let Σ be a set of prime numbers, and n an integer. Then we shall say that n is a Σ-*integer* if the prime divisors of n are in Σ. Let Γ be a profinite group. Then we shall refer to the quotient

$$\varprojlim \Gamma/H$$

(where the projective limit is over all open normoal subgroups $H \subseteq \Gamma$ whose orders are Σ-integers) as the *maximal pro-Σ quotient* of Γ. We shall denote by $\Gamma^{(\Sigma)}$ the maximal pro-Σ quotient of Γ.

We shall refer to the largest open subset (possibly empty) of the underlying scheme of an fs log scheme on which the log structure is trivial as the *interior* of the fs log scheme. We shall refer to a Kummer log étale (respectively, finite Kummer log étale) morphism of fs log schemes as a *ket* morphism (respectively, a *ket covering*).

Let $X^{\log}$ and $Y^{\log}$ be log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes. Then we shall refer to the quotient of $\mathcal{M}_X$ by the image of the morphism $(f^{\log})^*\mathcal{M}_Y \to \mathcal{M}_X$ induced by $f^{\log}$ as the *relative characteristic sheaf* of $f^{\log}$. Moreover, we shall refer to the relative characteristic sheaf of the morphism $X^{\log} \to X$ induced by the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$ as the *characteristic sheaf* of $X^{\log}$.

1 Log configuration schemes

In this Section, we define the *log configuration scheme* of a curve over a field and consider the geometry of such log configuration schemes.

Throughout this Section, we shall denote by X a smooth, proper, geometrically connected curve of genus $g \geq 2$ over a field K whose (not necessarily positive) characteristic we denote by p, by $\mathbb{P}_K^{\log}$ the log scheme obtained by equipping $\mathbb{P}_K^1$ with the log structure associated to the divisor $\{0, 1, \infty\} \subseteq \mathbb{P}_K^1$, and by $U_{\mathbb{P}}$ the interior of $\mathbb{P}_K^{\log}$.

Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of r-pointed stable curves of genus g whose r sections are equipped with an ordering, and $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing smooth curves ([6]). Then $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ is a divisor with normal crossings in $\overline{\mathcal{M}}_{g,r}$ ([6], Theorem 2.7). Let us write $\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_{g,0}$ and $\mathcal{M}_g = \mathcal{M}_{g,0}$. By considering the (1-)morphism $p^{\mathcal{M}}_{(r)r+1} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ obtained by forgetting the $(r+1)$-st section, we obtain a natural isomorphism of $\overline{\mathcal{M}}_{g,r+1}$ with the universal r-pointed stable curve over $\overline{\mathcal{M}}_{g,r}$ ([6], Corollary 2.6). Now we have a natural action of $\mathcal{S}_r$ (where $\mathcal{S}_r$ is the symmetric group on r letters) on $\overline{\mathcal{M}}_{g,r}$ which is given by permuting the sections. For $1 \leq i \leq r$, we shall denote by $p^{\mathcal{M}}_{(r)i} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ the (1-)morphism obtained by forgetting the i-th section.

Let us denote by $\overline{\mathcal{M}}_{g,r}^{\log}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$.

Since the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ preserves the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$, the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ extends to an action on $\overline{\mathcal{M}}_{g,r}^{\log}$.

First, we define the *log configuration scheme* $X_{(r)}^{\log}$ as follows:

Definition 1.1. We define $X_{(r)}$ by the following (1-)commutative diagram

$$\begin{array}{ccc} X_{(r)} & \longrightarrow & \overline{\mathcal{M}}_{g,r} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} K & \underset{[X/K]}{\longrightarrow} & \overline{\mathcal{M}}_g \, , \end{array}$$

where the bottom horizontal arrow $\operatorname{Spec} K \overset{[X/K]}{\to} \overline{\mathcal{M}}_g$ is the classifying (1-)morphism determined by the curve $X \to \operatorname{Spec} K$, the right-hand vertical arrow $\overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_g$ the (1-)morphism obtained by forgetting the sections, and the (1-)commutative diagram is cartesian in the (2-)category of stacks. Since $\overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_g$ is representable, $X_{(r)}$ is a scheme. We shall denote by $X_{(r)}^{\log}$ the fs log scheme obtained by equipping $X_{(r)}$ with the log structure induced by the log structure of $\overline{\mathcal{M}}_{g,r}^{\log}$. We shall denote by $U_{X_{(r)}}$ the interior of $X_{(r)}^{\log}$, and by $D_{X_{(r)}}$ the complement of $U_{X_{(r)}}$ of $X_{(r)}$. Note that, by definition, the scheme $U_{X_{(r)}}$ is isomorphic to the usual r-th configuration space of X. For simplicity, we shall write $U_{(r)}$ (respectively, $D_{(r)}$) instead of $U_{X_{(r)}}$ (respectively, $D_{X_{(r)}}$) when there is no danger of confusion. By the definition of $X_{(r)}$ (respectively, $X_{(r)}^{\log}$), the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ (respectively, $\overline{\mathcal{M}}_{g,r}^{\log}$) induces an action on $X_{(r)}$ (respectively, $X_{(r)}^{\log}$).

As is well-known, the pull-back of the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ via the (1-)morphism $p_{(r)r+1}^{\mathcal{M}} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ is a subdivisor of the divisor $\overline{\mathcal{M}}_{g,r+1} \setminus \mathcal{M}_{g,r+1}$ (cf. [6], the proof of Theorem 2.7). Thus, there exists a unique (1-)morphism $p_{(r)r+1}^{\mathcal{M}\log} : \overline{\mathcal{M}}_{g,r+1}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ whose underlying morphism is the (1-)morphism $p_{(r)r+1}^{\mathcal{M}}$. Moreover, for an integer $1 \leq i \leq r$, since the composite of the automorphism of $\overline{\mathcal{M}}_{g,r+1}$ determined by the action of

$$(1, 2, \cdots, r+1) \mapsto (1, 2, \cdots, i-1, \overset{i-\mathrm{th}}{r+1}, i, i+1, \cdots, r) \in \mathcal{S}_{r+1}$$

and $p_{(r)r+1}^{\mathcal{M}}$ coincides with the (1-)morphism $p_{(r)i}^{\mathcal{M}}$, the (1-)morphism $p_{(r)i}^{\mathcal{M}}$ also extends to a (1-)morphism $\overline{\mathcal{M}}_{g,r+1}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$. We shall denote this (1-)morphism by $p_{(r)i}^{\mathcal{M}\log}$.

The (1-)morphism $p_{(r)i}^{\mathcal{M}} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ (respectively, $p_{(r)i}^{\mathcal{M}\log} : \overline{\mathcal{M}}_{g,r+1}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$) determines a morphism $X_{(r+1)} \to X_{(r)}$ (respectively, $X_{(r+1)}^{\log} \to X_{(r)}^{\log}$).

We denote this morphism by $p_{X_{(r)}i}$ (respectively, $p^{\log}_{X_{(r)}i}$). Thus, we obtain the following (1-)cartesian diagrams:

$$\begin{array}{ccc} X_{(r+1)} & \xrightarrow{p_{X_{(r)}i}} & X_{(r)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,r+1} & \xrightarrow[p^{\mathcal{M}}_{(r)i}]{} & \overline{\mathcal{M}}_{g,r} \end{array} \qquad \begin{array}{ccc} X^{\log}_{(r+1)} & \xrightarrow{p^{\log}_{X_{(r)}i}} & X^{\log}_{(r)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}^{\log}_{g,r+1} & \xrightarrow[p^{\mathcal{M}\log}_{(r)i}]{} & \overline{\mathcal{M}}^{\log}_{g,r} . \end{array}$$

Note that, by the definition of a stable curve, $p_{X_{(r)}i}$ is proper, flat, geometrically connected, and geometrically reduced. For simplicity, we shall write $p_{(r)i}$ (respectively, $p^{\log}_{(r)i}$) instead of $p_{X_{(r)}i}$ (respectively, $p^{\log}_{X_{(r)}i}$) when there is no danger of confusion.

Definition 1.2.

(i) Let $1 \leq i \leq r$ be an integer. Then we shall denote by

$$\mathrm{pr}^{\log}_{X_{(r)}i} : X^{\log}_{(r)} \longrightarrow X$$

the composite

$$p^{\log}_{X_{(1)}2} \circ p^{\log}_{X_{(2)}2} \circ \cdots \circ p^{\log}_{X_{(r-i-1)}2} \circ p^{\log}_{X_{(r-i)}2} \circ p^{\log}_{X_{(r-i+1)}1} \circ \cdots \circ p^{\log}_{X_{(r-2)}1} \circ p^{\log}_{X_{(r-1)}1} ,$$

and by $\mathrm{pr}_{X_{(r)}i}$ the underlying morphism of schemes of $\mathrm{pr}^{\log}_{X_{(r)}i}$. For simplicity, we shall write $\mathrm{pr}^{\log}_{(r)i}$ (respectively, $\mathrm{pr}_{(r)i}$) instead of $\mathrm{pr}^{\log}_{X_{(r)}i}$ (respectively, $\mathrm{pr}_{X_{(r)}i}$) when there is no danger of confusion.

(ii) Let $1 \leq i < j \leq r$ be integers. Then we shall denote by

$$\mathrm{pr}^{\log}_{X_{(r)}i,j} : X^{\log}_{(r)} \longrightarrow X^{\log}_{(2)}$$

the composite

$$p^{\log}_{X_{(2)}3} \circ p^{\log}_{X_{(3)}3} \circ \cdots \circ p^{\log}_{X_{(r-j)}3} \circ p^{\log}_{X_{(r-j+1)}3} \circ p^{\log}_{X_{(r-j+2)}2} \circ \cdots$$

$$\cdots \circ p^{\log}_{X_{(r-i-1)}2} \circ p^{\log}_{X_{(r-i)}2} \circ p^{\log}_{X_{(r-i+1)}1} \circ \cdots \circ p^{\log}_{X_{(r-2)}1} \circ p^{\log}_{X_{(r-1)}1} ,$$

and by $\mathrm{pr}_{X_{(r)}i,j}$ the underlying morphism of schemes of $\mathrm{pr}^{\log}_{X_{(r)}i,j}$. For simplicity, we shall write $\mathrm{pr}^{\log}_{(r)i,j}$ (respectively, $\mathrm{pr}_{(r)i,j}$) instead of $\mathrm{pr}^{\log}_{X_{(r)}i,j}$ (respectively, $\mathrm{pr}_{X_{(r)}i,j}$) when there is no danger of confusion.

Remark 1.3. Let $1 \leq i \leq r$ (respectively, $1 \leq i < j \leq r$) be an integer (respectively, integers). Then, by the definiitons of $\mathrm{pr}_{(r)i}$ (respectively, $\mathrm{pr}_{(r)i,j}$), the restriction of $\mathrm{pr}_{(r)i}$ (respectively, $\mathrm{pr}_{(r)i,j}$) to $U_{(r)}$ coincides with the composite

$$U_{(r)} \hookrightarrow \overbrace{X \times_K \cdots \times_K X}^{r} \xrightarrow{\mathrm{pr}_i} X$$

(respectively, factors through $U_{(2)}$, the resulting morphism $U_{(r)} \to U_{(2)}$ coincides with the composite

$$U_{(r)} \hookrightarrow \overbrace{X \times_K \cdots \times_K X}^{r} \xrightarrow{\mathrm{pr}_{i,j}} U_{(2)}) \,.$$

Next, let us consider the scheme-theoretic and log scheme-theoretic properties of $X^{\log}_{(r)}$ in more detail.

Proposition 1.4. *$X_{(r)}$ is connected.*

Proof. Since $X_{(0)} = \mathrm{Spec}\, K$ is connected, and the $p_{(r)i}$'s are proper and geometrically connected, it follows immediately that $X_{(r)}$ is connected. □

Proposition 1.5. *$p^{\log}_{(r)i}$ is log smooth. In particular, since* $\mathrm{Spec}\, K$ (*equipped with the trivial log structure*) *is log regular, $X^{\log}_{(r)}$ is log regular.*

Proof. The assertion for $p^{\log}_{(r)r+1}$ follows from the fact that the (1-)morphism $p^{\mathcal{M}\log}_{(r)r+1} : \overline{\mathcal{M}}^{\log}_{g,r+1} \to \overline{\mathcal{M}}^{\log}_{g,r}$ is log smooth. (See [5], Section 4.) Since $p^{\log}_{(r)i}$ is a composite of an automorphism of $X^{\log}_{(r)}$ (obtained by permuting of the sections) and $p^{\log}_{(r)r+1}$, $p^{\log}_{(r)i}$ is also log smooth. □

Remark 1.6. By Propositions 1.4; 1.5 and [4], Proposition A.10, $U_{(r)} \hookrightarrow X^{\log}_{(r)}$ induces a natural equivalence between the Galois category of ket coverings over $X^{\log}_{(r)}$ and the Galois category of coverings over $U_{(r)}$ tamely ramified along the divisor with normal crossings $D_{(r)} \subseteq X_{(r)}$. In particular, $\pi_1^{\mathrm{tame}}(X_{(r)}, D_{(r)}) \simeq \pi_1(X^{\log}_{(r)})$. (Concerning $\pi_1^{\mathrm{tame}}(X_{(r)}, D_{(r)})$, see [3], Corollary 2.4.4.)

Proposition 1.7. *Let $\overline{x}^{\log} \to X^{\log}_{(r)}$ be a strict geometric point. Then, for any integer $1 \leq i \leq r+1$, the following sequence is exact:*

$$\varprojlim \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda}) \xrightarrow{s} \pi_1(X^{\log}_{(r+1)}) \xrightarrow{\pi_1(p^{\log}_{(r)i})} \pi_1(X^{\log}_{(r)}) \longrightarrow 1\,.$$

Here, the projective limit is over all reduced covering points $\overline{x}^{\log}_{\lambda} \to \overline{x}^{\log}$, and s is induced by the natural morphism $X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda} \to X^{\log}_{(r+1)}$.

Proof. This follows immediately from Propositions 1.4; 1.5 and [4], Theorem 2.3. □

Proposition 1.8. *Let $S^{\log}$ be a log regular fs log scheme, and $\overline{s} \to S$ a geometric point of S. If the stalk $(\mathcal{M}_S/\mathcal{O}_S^*)_{\overline{s}}$ of the characteristic sheaf of $S^{\log}$ at $\overline{s} \to S$ is isomorphic to $\mathbb{N}^{\oplus n}$ for some $n \in \mathbb{N}$, then S is regular at the image of $\overline{s} \to S$, and the log structure of $S^{\log}$ is given by a divisor with normal crossings around the image of $\overline{s} \to S$.*

Proof. We take a clean chart $\alpha : \mathbb{N}^{\oplus n} \to \mathcal{O}_{S,\overline{s}}$ of $S^{\log}$ at $\overline{s} \to S$, and write $f_i \stackrel{\text{def}}{=} \alpha(e_i) \in \mathcal{O}_{S,\overline{s}}$ (where $e_i = (0, \cdots, 0, \overset{\text{i-th}}{1}, 0, \cdots, 0) \in \mathbb{N}^{\oplus n}$). Then, by the definition of log regularity, the following assertions are satisfied:

(i) $\mathcal{O}_{S,\overline{s}}/(f_1, \ldots, f_n)$ is regular.

(ii) $(d \stackrel{\text{def}}{=})\dim \mathcal{O}_{S,\overline{s}} = \dim (\mathcal{O}_{S,\overline{s}}/(f_1, \ldots, f_n)) + n$.

Thus, there exist elements $f_{n+1}, \ldots, f_d$ of $\mathcal{O}_{S,\overline{s}}$ such that $f_1, \ldots, f_d$ generate the maximal ideal of $\mathcal{O}_{S,\overline{s}}$. Therefore, $\mathcal{O}_{S,\overline{s}}$ is regular, and the log structure of $S^{\log}$ is given by the divisor with normal crossings defined by $f_1 \cdots f_n \in \mathcal{O}_{S,\overline{s}}$. □

Proposition 1.9. *$X_{(r)}$ is regular, and the log structure of $X^{\log}$ is given by a divisor with normal crossings.*

Proof. Since the natural morphism $X_{(r)}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ is strict, for any geometric point $\overline{x} \to X_{(r)}$, the stalk $(\mathcal{M}_{X_{(r)}}/\mathcal{O}_{X_{(r)}}^*)_{\overline{x}}$ of characteristic sheaf of $X_{(r)}^{\log}$ at $\overline{x} \to X_{(r)}$ is isomorphic to $\mathbb{N}^{\oplus n}$ for some $n \in \mathbb{N}$. Thus, the assertion follows immediately from Proposition 1.8. □

Definition 1.10. Let $r \geq 2$ be a natural number, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$ equipped with an ordering. Then we shall denote by

$$(C_{(r)I} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1};\ s_1, \cdots, s_r : X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \longrightarrow C_{(r)I})$$

the r-pointed stable curve of genus g (whose r sections are equipped with an ordering) obtained by applying the clutching (1-)morphism ([6], Definition 3.8)

$$\beta_{0,g,I,\{1,2,\cdots,r\}\setminus I} : \overline{\mathcal{M}}_{0,I^{\#}+1} \times \overline{\mathcal{M}}_{g,r-I^{\#}+1} \to \overline{\mathcal{M}}_{g,r}$$

(where $\{1, 2, \cdots, r\} \setminus I$ is equipped with the natural ordering) to the $(I^{\#}+1)$-pointed stable curve of genus 0

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}$$

obtained by base-changing the universal curve $\overline{\mathcal{M}}_{0,I^{\#}+2} \to \overline{\mathcal{M}}_{0,I^{\#}+1}$ over $\overline{\mathcal{M}}_{0,I^{\#}+1}$ and the $(r - I^{\#} + 1)$-pointed stable curve of genus g

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}$$

obtained by base-changing $X_{(r-I^{\#}+2)} \overset{p_{X_{(r-I^{\#}+1)}}{}^{r-I^{\#}+2}}{\to} X_{(r-I^{\#}+1)}$. [Note that "the clutching locus" of

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}$$

$$(\text{respectively, } X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1})$$

is the $(I^{\#} + 1)$-st (respectively, $(r - I^{\#} + 1)$-st) section [cf. [6], Definition 3.8].]

Then it is immediate that the classifying (1-)morphism $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to \overline{\mathcal{M}}_{g,r}$ of this curve factors through $X_{(r)}$, and this morphism $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to X_{(r)}$ is a closed immersion (since it is a proper monomorphism). We shall denote by $\delta_{X_{(r)}I}$ this closed immersion, by $D_{X_{(r)}I}$ the scheme-theoretic image of $\delta_{X_{(r)}I}$, by $D^{\log}_{X_{(r)}I}$ the log scheme obtained by equipping $D_{X_{(r)}I}$ with the log structure induced by the log structure of $X^{\log}_{(r)}$, and by $\delta^{\log}_{X_{(r)}I} : D^{\log}_{X_{(r)}I} \to X^{\log}_{(r)}$ the strict closed immersion whose underlying morphism is $\delta_{X_{(r)}I}$. Note that, by the construction of $D_{X_{(r)}I}$, the closed subscheme $D_{X_{(r)}I} \subseteq X_{(r)}$ does not depend on the imposed ordering of I. For simplicity, we shall write $D_{(r)I}$ (respectively, $D^{\log}_{(r)I}$; respectively, $\delta_{(r)I}$; respectively, $\delta^{\log}_{(r)I}$) instead of $D_{X_{(r)}I}$ (respectively, $D^{\log}_{X_{(r)}I}$; respectively, $\delta_{X_{(r)}I}$; respectively, $\delta^{\log}_{X_{(r)}I}$) when there is no danger of confusion.

Remark 1.11. Let $r \geq 2$ be a natural number, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2. By the definition of $D_{(r)I}$, $D_{(r)I}$ is irreducible. (Indeed, the log smoothness of the morphism $p^{\log}_{(s)s+1} : X^{\log}_{(s+1)} \to X^{\log}_{(s)}$ and the (1-)morphism $\overline{\mathcal{M}}^{\log}_{0,t+1} \to \overline{\mathcal{M}}^{\log}_{0,t}$ [obtained by forgetting the $(t + 1)$-st section] [s, $t \in \mathbb{N}$] imply the log regularity [hence, in particular, the normality of the underlying scheme] of $X^{\log}_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^{\#}+1}$; moreover, by a similar argument to the argument used in the proof of Proposition 1.4, $D_{(r)I}$ is connected, hence, [in light of the normality just observed] irreducible.) Thus, $D_{(r)I}$ is an *irreducible component* of $D_{(r)}$. Moreover, $D_{(r)} = \bigcup_I D_{(r)I}$. (Indeed, if the image of a geometric point $\overline{x} \to X_{(r)}$ lies on $D_{(r)}$, then by considering the curve which corresponds to the composite $\overline{x} \to X_{(r)} \to \overline{\mathcal{M}}_{g,r}$, there exists a subset I of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2 such that the image of the geometric point $\overline{x} \to X_{(r)}$ lies on $D_{(r)I}$.) Therefore, the log structure of $X_{(r)}^{\log}$ is the log structure associated to the divisor with normal crossings

$$\bigcup_{I^{\#} \geq 2} D_{(r)I} \subseteq X_{(r)} \,,$$

i.e., if we denote by $\mathcal{M}(D_{(r)I})$ the log structure on $X_{(r)}$ associated to the divisor $D_{(r)I} \subseteq X_{(r)}$, then the log structure of $X_{(r)}^{\log}$ is

$$\sum_{I^{\#} \geq 2} \mathcal{M}(D_{(r)I})$$

(cf. [4], Definition 4.6).

Proposition 1.12. *Let $r \geq 2$ be a natural number, I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$, and $1 \leq i \leq r+1$ an integer.*

(*i*) *The closed subscheme of $X_{(r+1)}$ determined by the composite of the natural closed immersions* (*defined in Definition 1.10*)

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}$$

is $D_{(r+1)I \cup \{r+1\}}$.

(*ii*) *The closed subscheme of $X_{(r+1)}$ determined by the composite of the natural closed immersions* (*defined in Definition 1.10*)

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}$$

is $D_{(r+1)I}$.

(*iii*) *The inverse image of $D_{(r)I} \subseteq X_{(r)}$ via $p_{(r)i}$ is $D_{(r+1)(I\cup\{r+1\})^{\sigma_i}} \cup D_{(r+1)I^{\sigma_i}}$, where*

$$\sigma_i = ((1, 2, \cdots, r+1) \mapsto (1, 2, \cdots, i-1, \overset{i-\mathrm{th}}{r+1}, i, i+1, \cdots, r)) \in \mathcal{S}_{r+1} \,,$$

and $I^{\sigma_i} = \{\sigma_i(k) \mid k \in I\}$.

(*iv*) *The closed subscheme $D_{(r+1)\{i,j\}} \subseteq X_{(r+1)}$ $(j \neq i)$ is the image of a section of $p_{(r)i}$.*

Proof. First, we prove assertion (i). By the definition of the r-pointed stable curve

$$(C_{(r)I} \longrightarrow D_{(r)I};\ s_1, \cdots, s_r : D_{(r)I} \longrightarrow C_{(r)I}) \,,$$

the $(r+1)$-pointed stable curve determined by the closed immersion $C_{(r)I} \hookrightarrow X_{(r+1)}$ is obtained as the stabilization ([6], Definition 2.3) of the r-pointed stable curve of genus g

$$(C_{(r)I} \times_{D_{(r)I}} C_{(r)I} \xrightarrow{\mathrm{pr}_1} C_{(r)I};\ \widetilde{s}_1, \cdots, \widetilde{s}_r : C_{(r)I} \longrightarrow C_{(r)I} \times_{D_{(r)I}} C_{(r)I}),$$

(where $\widetilde{s}_i$ is the section obtained by base-changing s_i) with the extra section obtained as the diagnal morphism $C_{(r)I} \to C_{(r)I} \times_{D_{(r)I}} C_{(r)I}$. Therefore, since the operation of stabilization commutes with base-change, the closed immersion in question

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}$$

determines the $(r+1)$-pointed stable curve obtained as the stabilization of the r-pointed stable curve of genus g

$$((X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} C_{(r)I} \xrightarrow{\mathrm{pr}_1} X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2};$$

$$s'_1, \cdots, s'_r : X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow (X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} C_{(r)I}) \quad (*_1)$$

(where s'_i is the section obtained by base-changing s_i) with the extra section induced by the diagonal morphism of $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$ over $D_{(r)I}$. On the other hand, since the operation of clutching commutes with the base-change, the r-pointed stable curve of genus g $(*_1)$ is obtained by applying the clutching (1-)morphism $\beta_{0,g,I,\{1,2,\cdots,r\}\setminus I}$ to the $(I^{\#}+1)$-pointed stable curve

$$(X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} (X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \xrightarrow{\mathrm{pr}_1} X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \quad (*_2)$$

obtained by base-changing the $(I^{\#}+1)$-pointed stable curve $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \to D_{(r)I}$ defined in Definition 1.10 and the $(r - I^{\#} + 1)$-pointed stable curve

$$(X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} (X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}) \xrightarrow{\mathrm{pr}_1} X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \quad (*_3)$$

obtained by base-changing the $(r-I^{\#}+1)$-pointed stable curve $X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to D_{(r)I}$ defined in Definition 1.10. Note that then, by definition, the stable curve $(*_3)$ is isomorphic to the $(r - I^{\#} + 1)$-pointed stable curve

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$$

obtained by base-changing the $(r - I^{\#} + 1)$-pointed stable curve $X_{(r-I^{\#}+2)} \overset{p_{(r-I^{\#}+1)r-I^{\#}+2}}{\to} X_{(r-I^{\#}+1)}$. Moreover, since the image of the extra section of the r-pointed stable curve of genus g $(*_1)$ lies on the stable curve $(*_2)$, the $(r+1)$-pointed stable curve determined by the closed immersion in question is the $(r+1)$-pointed stable curve obtained by applying the clutching (1-)morphism $\beta_{0,g,I\cup\{r+1\},\{1,2,\cdots,r+1\}\setminus(I\cup\{r+1\})}$ to the $(I^{\#}+2)$-pointed stable curve

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+3} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$$

obtained by base-changing the universal curve $\overline{\mathcal{M}}_{0,I^{\#}+3} \to \overline{\mathcal{M}}_{0,I^{\#}+2}$ over $\overline{\mathcal{M}}_{0,I^{\#}+2}$ and the $(r-I^{\#}+1)$-pointed stable curve

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$$

obtained by base-changing the $(r-I^{\#}+1)$-pointed stable curve $X_{(r-I^{\#}+2)} \stackrel{p_{(r-I^{\#}+1)r-I^{\#}+2}}{\to} X_{(r-I^{\#}+1)}$. This completes the proof of assertion (i).

Assertion (ii) follows from a similar argument to the argument used in the proof of assertion (i).

Assertion (iii) follows from assertion (i) and (ii), together with the fact that $p_{(r)i}$ coincides with the composite of the automorphism of $X_{(r+1)}$ determined by $\sigma_i \in \mathcal{S}_{r+1}$ and $p_{(r)r+1}$.

Finally, we prove assertion (iv). By the definition of $D_{(r+1)\{j,r+1\}}$, the composite

$$D_{(r+1)\{j,r+1\}} \stackrel{\delta_{(r+1)\{j,r+1\}}}{\longrightarrow} X_{(r+1)} \stackrel{p_{(r)r+1}}{\longrightarrow} X_{(r)}$$

is the classifying morphism of the r-pointed stable curve $X_{(r+1)} \stackrel{p_{(r)r+1}}{\to} X_{(r)}$. Thus, the composite $p_{(r)r+1} \circ \delta_{(r+1)\{j,r+1\}}$ is an isomorphism. This completes the proof of the assertion in the case where $i = r+1$. In general, the assertion follows from the fact that $p_{(r)i}$ coincides with the composite of the automorphism of $X_{(r+1)}$ determined by $\sigma_i \in \mathcal{S}_{r+1}$ and $p_{(r)r+1}$. $\square$

Remark 1.13. Let $r \geq 2$ and $1 \leq i \leq r+1$ be natural numbers, and σ_i the element of $\mathcal{S}_{r+1}$ defined in Proposition 1.12, (iii). Then one may verify easily that the image of the k-th section $(1 \leq k \leq r)$ of the r-pointed stable curve $p_{(r)r+1} : X_{(r+1)} \to X_{(r)}$ is $D_{(r+1)\{k,r+1\}}$ (see Proposition 1.12, (iv)). Therefore, by taking the composite of the sections of the r-pointed stable curve $p_{(r)r+1} : X_{(r+1)} \to X_{(r)}$ and the automorphism of $X_{(r+1)}$ determined by σ_i, we obtain a r-pointed stable curve $p_{(r)i} : X_{(r+1)} \to X_{(r)}$ such that the image of the k-th section $(1 \leq k \leq r)$ is

$$\begin{cases} D_{(r+1)\{k,i\}} & (\text{if } k \leq i-1) \\ D_{(r+1)\{i,k+1\}} & (\text{if } i \leq k)\,. \end{cases}$$

Thus, in particular, if $j \neq j'$ then $D_{(r+1)\{i,j\}} \cap D_{(r+1)\{i,j'\}}$ is empty. Moreover, we obtain

$$D_{(r+1)} = \bigcup_{j \neq i} D_{(r+1)\{i,j\}} \cup p_{(r)i}^{-1} D_{(r)} \,.$$

(See the proof of [6], Theorem 2.7. Note that the restriction of $S_{g,n+1}^{i,n+1}$ in the proof of [6], Theorem 2.7 to $X_{(n+1)}$ is $D_{(n+1)\{i,n+1\}}$.) On the other hand, the morphism $p_{(r)i}^{\log} : X_{(r+1)}^{\log} \to X_{(r)}^{\log}$ factors through the log scheme $(X_{(r+1)}, p_{(r)i}^{-1} D_{(r)})^{\log}$ obtained by equipping $X_{(r+1)}$ with the log structure associated to the divisor with normal crossings $p_{(r)i}^{-1} D_{(r)}$, the morphism

$$(X_{(r+1)}, p_{(r)i}^{-1} D_{(r)})^{\log} \to X_{(r)}^{\log}$$

is log smooth, and the morphism $X_{(r+1)}^{\log} \to (X_{(r+1)}, p_{(r)i}^{-1} D_{(r)})^{\log}$ is obtained by "forgetting" the portion of the log structure of $X_{(r+1)}^{\log}$ defined by the divisors determined by the sections $D_{(r+1)\{i,j\}} \subseteq X_{(r+1)}$ $(j \neq i)$ (i.e., $\Sigma_{j \neq i} \mathcal{M}(D_{(r+1)\{i,j\}})$).

Lemma 1.14. *Let $r \geq 3$ be a natural number, and $i = 1$ or 2. Then the composite*

$$D_{(r)\{i,i+1\}}^{\log} \overset{\delta_{(r)\{i,i+1\}}^{\log}}{\longrightarrow} X_{(r)}^{\log} \overset{p_{(r-1)i}^{\log}}{\longrightarrow} X_{(r-1)}^{\log}$$

coincides with the composite

$$D_{(r)\{i,i+1\}}^{\log} \overset{\delta_{(r)\{i,i+1\}}^{\log}}{\longrightarrow} X_{(r)}^{\log} \overset{p_{(r-1)i+1}^{\log}}{\longrightarrow} X_{(r-1)}^{\log} \,.$$

Moreover, this is a morphism of type $\mathbb{N}$.

Proof. The assersion that $p_{(r-1)i}^{\log} \circ \delta_{(r)\{i,i+1\}}^{\log}$ coincides with $p_{(r-1)i+1}^{\log} \circ \delta_{(r)\{i,i+1\}}^{\log}$ follows from the fact that $p_{(r-1)i+1}^{\log}$ coincides with the composite of the automorphism of $X_{(r)}^{\log}$ determined by

$$\sigma = ((1, 2, \cdots, r) \mapsto (1, 2, \cdots, i-1, i+1, i, i+2, \cdots, r)) \in \mathcal{S}_r$$

and $p_{(r-1)i}^{\log}$, together with the fact that the restriction of the automorphism of $X_{(r)}^{\log}$ determined by σ to the closed subscheme $D_{(r)\{i,i+1\}}^{\log}$ is the identity morphism of $D_{(r)\{i,i+1\}}^{\log}$.

Now $p_{(r-1)i} \circ \delta_{(r)\{i,i+1\}}$ is an isomorphism by Proposition 1.12, (iv). Moreover, since $p_{(r-1)i}^{\log} \circ \delta_{(r)\{i,i+1\}}^{\log}$ is obtained by "forgetting" the portion of the log structure of $D_{(r)\{i,i+1\}}^{\log}$ that originates from

$$D_{(r)\{i,i+1\}} \subseteq X_{(r)}$$

(i.e., $\mathcal{M}(D_{(r)\{i,i+1\}}) \mid_{D_{(r)\{i,i+1\}}}$) (see Remark 1.13), the composite $p^{\log}_{(r-1)i} \circ \delta^{\log}_{(r)\{i,i+1\}}$ is a morphism of type $\mathbb{N}$. $\square$

Definition 1.15. Let $r \geq 3$ be a natural number, and $i = 1$ or 2. Then we shall denote by $a^{\log}_{X_{(r)}\{i,i+1\}}$ the composite

$$D^{\log}_{X_{(r)}\{i,i+1\}} \xrightarrow{\delta^{\log}_{X_{(r)}\{i,i+1\}}} X^{\log}_{(r)} \xrightarrow{p^{\log}_{X_{(r-1)}i}} X^{\log}_{(r-1)},$$

and by $a_{X_{(r)}\{i,i+1\}}$ the underlying morphism of schemes of $a^{\log}_{X_{(r)}\{i,i+1\}}$. By Lemma 1.14, $a^{\log}_{X_{(r)}\{i,i+1\}}$ is a morphism of type $\mathbb{N}$.

We shall denote by $\mathcal{L}_{X_{(r)}\{i,i+1\}}$ the invertible sheaf on $D_{X_{(r)}\{i,i+1\}}$ which corresponds to $a^{\log}_{X_{(r)}\{i,i+1\}}$ under the bijection ι in [4], Theorem 4.13. Note that, by the definition of ι and the proof of Lemma 1.14, $\mathcal{L}_{X_{(r)}\{i,i+1\}}$ is isomorphic to the conormal sheaf of $D_{X_{(r)}\{i,i+1\}}$ in $X_{(r)}$ (cf. [4], Remark 4.14).

We shall denote by $U_{X_{(r)}\{i,i+1\}}$ the open subscheme of $D_{X_{(r)}\{i,i+1\}}$ determined by the open immersion

$$U_{X_{(r-1)}} \hookrightarrow X_{(r-1)} \overset{a^{-1}_{X_{(r)}\{i,i+1\}}}{\xrightarrow{\sim}} D_{X_{(r)}\{i,i+1\}}.$$

For simplicity, we shall write $a^{\log}_{(r)\{i,i+1\}}$ (respectively, $a_{(r)\{i,i+1\}}$; respectively, $\mathcal{L}_{(r)\{i,i+1\}}$; respectively, $U_{(r)\{i,i+1\}}$) instead of $a^{\log}_{X_{(r)}\{i,i+1\}}$ (respectively, $a_{X_{(r)}\{i,i+1\}}$; respectively, $\mathcal{L}_{X_{(r)}\{i,i+1\}}$; respectively, $U_{X_{(r)}\{i,i+1\}}$) when there is no danger of confusion.

Definition 1.16. Let $r \geq 3$ be a natural number, and $I = \{1,2\}$, $\{2,3\}$ or $\{1,3\}$. Then we shall denote by $D_{X_{(r)}I:\{1,2,3\}}$ the closed subscheme $D_{X_{(r)}I} \cap D_{X_{(r)}\{1,2,3\}}$ of $D_{X_{(r)}I}$ and $D_{X_{(r)}\{1,2,3\}}$. For simplicity, we shall write $D_{(r)I:\{1,2,3\}}$ instead of $D_{X_{(r)}I:\{1,2,3\}}$ when there is no danger of confusion.

Lemma 1.17. *Let $r \geq 3$ be a natural number. Then the composite*

$$D_{(r)\{1,2,3\}} \xrightarrow{\delta_{(r)\{1,2,3\}}} X_{(r)} \xrightarrow{p_{(r-1)1}} X_{(r-1)}$$

factors through $D_{(r-1)\{1,2\}}$. Moreover, this resulting morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ determines a trivial $\mathbb{P}^1$-bundle over $D_{(r-1)\{1,2\}}$, and $D_{(r)\{1,2\}:\{1,2,3\}}$, $D_{(r)\{2,3\}:\{1,2,3\}}$, and $D_{(r)\{1,3\}:\{1,2,3\}}$ determine sections of this $\mathbb{P}^1$-bundle.

Proof. The assertion that the composite $p_{(r-1)1} \circ \delta_{(r)\{1,2,3\}}$ factors through $D_{(r-1)\{1,2\}}$ follows from the fact that the inverse image of $D_{(r-1)\{1,2\}} \hookrightarrow X_{(r-1)}$

via $p_{(r-1)1}$ is $D_{(r)\{2,3\}} \cup D_{(r)\{1,2,3\}}$ (Proposition 1.12, (iii)). Moreover, by the proof of Proposition 1.12, (i), the resulting morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ determined by $p_{(r-1)1} \circ \delta_{(r)\{1,2,3\}}$ is isomorphic to the stable curve

$$X_{(r-2)} \times_K \overline{\mathcal{M}}_{0,4} \longrightarrow X_{(r-2)} \times_K \overline{\mathcal{M}}_{0,3}$$

obtained by base-changing the universal curve $\overline{\mathcal{M}}_{0,4} \to \overline{\mathcal{M}}_{0,3}$ over $\overline{\mathcal{M}}_{0,3}$; thus, the resulting morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ determines a trivial $\mathbb{P}^1$-bundle. The assertion that $D_{(r)\{1,2\}:\{1,2,3\}}$, $D_{(r)\{2,3\}:\{1,2,3\}}$, and $D_{(r)\{1,3\}:\{1,2,3\}}$ determine sections of this $\mathbb{P}^1$-bundle follows from the fact that by the definition of the operation of clutching and Remark 1.13, the images of the 1-st and 2-nd sections of the resulting morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ are $D_{(r)\{1,2\}:\{1,2,3\}}$ and $D_{(r)\{1,3\}:\{1,2,3\}}$, respectively, together with the fact that by Proposition 1.12, (iii), the image of the 3-rd section (i.e., "the clutching locus" of the stable curve determined by the closed immersion $\delta_{(r-1)\{1,2\}}$) is $D_{(r)\{1,2,3\}} \cap D_{(r)\{2,3\}} = D_{(r)\{2,3\}:\{1,2,3\}}$. $\square$

Definition 1.18. Let $r \geq 3$ be a natural number. Then we shall denote by $b_{X_{(r)}\{1,2,3\}}$ the isomorphism $D_{X_{(r)}\{1,2,3\}} \xrightarrow{\sim} X_{(r-2)} \times_K \mathbb{P}^1_K$ such that

- the composite

$$D_{X_{(r)}\{1,2,3\}} \overset{b_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} X_{(r-2)} \times_K \mathbb{P}^1_K \xrightarrow{\mathrm{pr}_1} X_{(r-2)}$$

  coincides with the composite

$$D_{X_{(r)}\{1,2,3\}} \longrightarrow D_{X_{(r-1)}\{1,2\}} \overset{a_{X_{(r-1)}\{1,2\}}}{\underset{\sim}{\longrightarrow}} X_{(r-2)}\,,$$

  where the first morphism is the morphism determined by $p_{X_{(r-1)}1} \circ \delta_{X_{(r)}\{1,2,3\}}$ (cf. Lemma 1.17); and

- the closed subscheme of $D_{X_{(r)}\{1,2,3\}}$ determined by the closed immersion

$$X_{(r-2)} \times_K \{0\} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}}$$

$$\text{(respectively, } X_{(r-2)} \times_K \{1\} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}}\,;$$

$$\text{respectively, } X_{(r-2)} \times_K \{\infty\} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}})$$

  is $D_{X_{(r)}\{1,2\}:\{1,2,3\}}$ (respectively, $D_{X_{(r)}\{2,3\}:\{1,2,3\}}$; respectively, $D_{X_{(r)}\{1,3\}:\{1,2,3\}}$).

We shall denote by $U_{X_{(r)}\{1,2,3\}}$ the open subscheme of $D_{X_{(r)}\{1,2,3\}}$ determined by the open immersion

$$U_{X_{(r-2)}} \times_K U_{\mathbb{P}} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}} .$$

For simplicity, we shall write $b_{(r)\{1,2,3\}}$ (respectively, $U_{(r)\{1,2,3\}}$) instead of $b_{X_{(r)}\{1,2,3\}}$ (respectively, $U_{X_{(r)}\{1,2,3\}}$) when there is no danger of confusion.

Lemma 1.19. *Let $r \geq 3$ be a natural number. Then the isomorphism $b_{(r)\{1,2,3\}} : D_{(r)\{1,2,3\}} \overset{\sim}{\to} X_{(r-2)} \times_K \mathbb{P}^1_K$ extends to a unique morphism of log schemes $D^{\log}_{(r)\{1,2,3\}} \to X^{\log}_{(r-2)} \times_K \mathbb{P}^{\log}_K$ of type $\mathbb{N}$.*

Proof. It is immediate that if $b_{(r)\{1,2,3\}}$ extends to such a morphism, then it is unique. Thus, it is enough to show that $b_{(r)\{1,2,3\}}$ extends to such a morphism.

By Remark 1.13, the morphism $D^{\log}_{(r)\{1,2,3\}} \to X^{\log}_{(r-2)} \times_K \mathbb{P}^1_K$ determined by the composite

$$D^{\log}_{(r)\{1,2,3\}} \overset{\text{via } p^{\log}_{(r-1)1} \circ \delta^{\log}_{(r)\{1,2,3\}}}{\longrightarrow} D^{\log}_{(r-1)\{1,2\}} \overset{a^{\log}_{(r-1)\{1,2\}}}{\longrightarrow} X^{\log}_{(r-2)} \qquad (*)$$

and the composite

$$D^{\log}_{(r)\{1,2,3\}} \to D_{(r)\{1,2,3\}} \overset{b_{(r)\{1,2,3\}}}{\underset{\sim}{\to}} X_{(r-2)} \times_K \mathbb{P}^1_K \overset{\mathrm{pr}_2}{\to} \mathbb{P}^1_K$$

is obtained by "forgetting" the portion of the log structure of $D^{\log}_{(r)\{1,2,3\}}$ defined by $D_{(r)\{1,2\}:\{1,2,3\}}$, $D_{(r)\{2,3\}:\{1,2,3\}}$ and $D_{(r)\{1,3\}:\{1,2,3\}}$ (i.e., $\mathcal{M}(D_{(r)\{1,2\}:\{1,2,3\}} + D_{(r)\{2,3\}:\{1,2,3\}} + D_{(r)\{1,3\}:\{1,2,3\}})$) and the portion of the log structure of $D^{\log}_{(r)\{1,2,3\}}$ that originates from $D_{(r)\{1,2,3\}} \subseteq X_{(r)}$ (i.e., $\mathcal{M}(D_{(r)\{1,2,3\}}) |_{D_{(r)\{1,2,3\}}}$). Therefore, the morphism $D^{\log}_{(r)\{1,2,3\}} \longrightarrow X^{\log}_{(r-2)} \times_K \mathbb{P}^{\log}_K$ determined by the above composite $(*)$ and the composite

$$D^{\log}_{(r)\{1,2,3\}} \longrightarrow D'^{\log}_{(r)\{1,2,3\}} \longrightarrow \mathbb{P}^{\log}_K$$

(where $D'^{\log}_{(r)\{1,2,3\}}$ is the log scheme obtained by equipping $D_{(r)\{1,2,3\}}$ with the log structure associated to the divisors

$$D_{(r)\{1,2\}:\{1,2,3\}}, \; D_{(r)\{2,3\}:\{1,2,3\}} \text{ and } D_{(r)\{1,3\}:\{1,2,3\}} \subseteq D_{(r)\{1,2,3\}} ,$$

the first morphism is the natural morphism obtained by "forgetting" the portion of the log structure of $D^{\log}_{(r)\{1,2,3\}}$ that originates from the divisors other than

$$D_{(r)\{1,2\}:\{1,2,3\}}, \; D_{(r)\{2,3\}:\{1,2,3\}} \text{ and } D_{(r)\{1,3\}:\{1,2,3\}} \subseteq D_{(r)\{1,2,3\}} ,$$

[among the divisors of the form $D_{(r)I} |_{D_{(r)\{1,2,3\}}}$ [where $I \subseteq \{1, 2, \cdots, r\}$ of cardinarity ≥ 2]] and the second morphism is the strict morphism induced by the natural morphism

$$D_{(r)\{1,2,3\}} \overset{b_{(r)\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} X_{(r-2)} \times_K \mathbb{P}^1_K \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1_K \;)$$

is an extension of $b_{(r)\{1,2,3\}}$ of the desired type. □

Definition 1.20. Let $r \geq 3$ be a natural number. Then we shall denote by $b^{\log}_{X_{(r)}\{1,2,3\}}$ the morphism

$$D^{\log}_{X_{(r)}\{1,2,3\}} \longrightarrow X^{\log}_{(r-2)} \times_K \mathbb{P}^{\log}_K ,$$

obtained in Lemma 1.19. Note that this is a morphism of type $\mathbb{N}$ by Lemma 1.19.

We shall denote by $\mathcal{L}_{X_{(r)}\{1,2,3\}}$ the invertible sheaf on $D_{X_{(r)}\{1,2,3\}}$ which corresponds to the morphism $b^{\log}_{X_{(r)}\{1,2,3\}}$ under the bijection ι in [4], Theorem 4.13. Note that, by the definition of ι and the proof of Lemma 1.19, $\mathcal{L}_{X_{(r)}\{1,2,3\}}$ is isomorphic to the conormal sheaf of $D_{X_{(r)}\{1,2,3\}}$ in $X_{(r)}$ (cf. [4], Remark 4.14). For simplicity, we shall write $b^{\log}_{(r)\{1,2,3\}}$ (respectively, $\mathcal{L}_{(r)\{1,2,3\}}$) instead of $b^{\log}_{X_{(r)}\{1,2,3\}}$ (respectively, $\mathcal{L}_{X_{(r)}\{1,2,3\}}$) when there is no danger of confusion.

Lemma 1.21. *Let $r \geq 2$ be a natural number.*

(*i*) $\mathcal{L}_{(r+1)\{1,2\}} |_{U_{(r+1)\{1,2\}}} \simeq (p_{(r)i} |_{U_{(r+1)\{1,2\}}})^* \mathcal{L}_{(r)\{1,2\}}$ *for* $i \neq 1,\ 2$.

(*ii*) $\mathcal{L}_{(r+1)\{2,3\}} |_{U_{(r+1)\{2,3\}}} \simeq (p_{(r)1} |_{U_{(r+1)\{2,3\}}})^* \mathcal{L}_{(r)\{1,2\}} \simeq (p_{(r)i} |_{U_{(r+1)\{2,3\}}})^* \mathcal{L}_{(r)\{2,3\}}$ *for* $i \neq 1,\ 2,\ 3$.

(*iii*) $\mathcal{L}_{(r+1)\{1,2,3\}} |_{U_{(r+1)\{1,2,3\}}} \simeq (p_{(r)j} |_{U_{(r+1)\{1,2,3\}}})^* \mathcal{L}_{(r)\{1,2\}} \simeq (p_{(r)i} |_{U_{(r+1)\{1,2,3\}}})^* \mathcal{L}_{(r)\{1,2,3\}}$ *for* $j = 1,\ 2,\ 3$ *and* $i \neq 1,\ 2,\ 3$.

Proof. First, we prove assertion (i). It follows from the fact that $\mathcal{L}_{(r)\{1,2\}}$ is the conormal sheaf of $D_{(r)\{1,2\}}$ in $X_{(r)}$, together with the flatness of $p_{(r)i}$ that $p^*_{(r)i}\mathcal{L}_{(r)\{1,2\}}$ is naturally isomorphic to the conormal sheaf of the closed subscheme of $X_{(r+1)}$ obtained as the fiber product of

$$\begin{array}{ccc} & & D_{(r)\{1,2\}} \\ & & \big\downarrow{\scriptstyle \delta_{(r)\{1,2\}}} \\ X_{(r+1)} & \xrightarrow{p_{(r)i}} & X_{(r)}\,. \end{array}$$

Thus, by Proposition 1.12, (iii), and the fact that $\mathcal{L}_{(r+1)\{1,2\}}$ is the conormal sheaf of $D_{(r+1)\{1,2\}}$ in $X_{(r+1)}$, together with the fact that the intersection of

$D_{(r+1)\{1,2\}}$ and $D_{(r+1)\{1,2,i\}}$ is contained in $D_{(r+1)\{1,2\}} \setminus U_{(r+1)\{1,2\}}$, the restriction of $p^*_{(r)i}\mathcal{L}_{(r)\{1,2\}}$ to $U_{(r+1)\{1,2\}}$ is naturally isomorphic to $\mathcal{L}_{(r+1)\{1,2\}} |_{U_{(r+1)\{1,2\}}}$. This completes the proof of (i).

Assertions (ii) and (iii) follow from a similar argument to the argument used in the proof of (i). □

2 Reconstruction of the fundamental groups of higher dimensional log configuration schemes

We continue with the notation of the preceding Section. Let Σ be a (non-empty) set of prime numbers, and l a prime number that is invertible in K. (Thus, it makes sense to speak of Σ-integers.) Then we shall say that Σ is K-*innocuous* if

$$\Sigma = \begin{cases} \text{the set of all prime numbers or } \{l\} & \text{if } p = 0 \\ \{l\} & \text{if } p \geq 2\,. \end{cases}$$

We shall fix a separable closure K^{sep} of K and denote by G_K the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K. Moreover, we shall denote by Λ the maximal pro-Σ quotient of $\widehat{\mathbb{Z}}(1)$.

Definition 2.1.

(i) Let r be a positive natural number. We shall denote by $\Pi^{\log}_{X_{(r)}}$ the quotient of $\pi_1(X^{\log}_{(r)})$ by the closed normal subgroup

$$\mathrm{Ker}(\pi_1(X^{\log}_{(r)} \times_K K^{\mathrm{sep}}) \to \pi_1(X^{\log}_{(r)} \times_K K^{\mathrm{sep}})^{(\Sigma)})$$

and write Π_X for $\Pi^{\log}_{X_{(1)}}$. For simplicity, we shall write $\Pi^{\log}_{(r)}$ instead of $\Pi^{\log}_{X_{(r)}}$ when there is no danger of confusion.

(ii) Let $r \geq 2$ be a natural number, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2. We shall denote by $\Pi^{\log}_{X_{(r)}I}$ the quotient of $\pi_1(D^{\log}_{X_{(r)}I})$ by the closed normal subgroup

$$\mathrm{Ker}(\pi_1(D^{\log}_{X_{(r)}I} \times_K K^{\mathrm{sep}}) \to \pi_1(D^{\log}_{X_{(r)}I} \times_K K^{\mathrm{sep}})^{(\Sigma)})\,.$$

For simplicity, we shall write $\Pi^{\log}_{(r)I}$ instead of $\Pi^{\log}_{X_{(r)}I}$ when there is no danger of confusion.

(iii) We shall denote by $\Pi^{\log}_{\mathbb{P}_K}$ the quotient of $\pi_1(\mathbb{P}^{\log}_K)$ by the closed normal subgroup

$$\mathrm{Ker}(\pi_1(\mathbb{P}^{\log}_K \times_K K^{\mathrm{sep}}) \to \pi_1(\mathbb{P}^{\log}_K \times_K K^{\mathrm{sep}})^{(\Sigma)})\,.$$

For simplicity, we shall write $\Pi^{\log}_{\mathbb{P}}$ instead of $\Pi^{\log}_{\mathbb{P}_K}$ when there is no danger of confusion.

Definition 2.2. Let $r \geq 3$ be a natural number. We shall denote by $\mathcal{G}^{\log}_{X_{(r)}}(\Sigma)$ the graph of groups defined as follows:

$$\mathcal{G}^{\log}_{(r)}(\Sigma) = (\overset{\Pi^{\log}_{X_{(r)}\{1,2\}}}{\bullet} \underset{\{1\}}{-} \overset{\Pi^{\log}_{X_{(r)}\{1,2,3\}}}{\bullet} \underset{\{1\}}{-} \overset{\Pi^{\log}_{X_{(r)}\{2,3\}}}{\bullet})\,.$$

Here, $\{1\}$ is the trivial group; the symbols "$\bullet$" (respectively, "$-$") denote the vertices (respectively, the edges) of the underlying graphs; and the group that lies above a vertex (respectively, below an edge) denotes the group that corresponds to the vertex (respectively, edge). We shall denote by $\Pi^{\mathcal{G}}_{X_{(r)}}$ the profinite group

$$\varinjlim(\Pi^{\log}_{X_{(r)}\{1,2\}} \longleftarrow \{1\} \longrightarrow \Pi^{\log}_{X_{(r)}\{1,2,3\}} \longleftarrow \{1\} \longrightarrow \Pi^{\log}_{X_{(r)}\{2,3\}})\,,$$

where the inductive limit is taken in the category of profinite groups. For simplicity, we shall write $\mathcal{G}^{\log}_{(r)}(\Sigma)$ (respectively, $\Pi^{\mathcal{G}}_{(r)}$) instead of $\mathcal{G}^{\log}_{X_{(r)}}(\Sigma)$ (respectively, $\Pi^{\mathcal{G}}_{X_{(r)}}$) when there is no danger of confusion.

Definition 2.3. Let G be a group. Then we shall denote by $G_\bullet$ the graph of groups whose underlying graph has one vertex that corresponds to G and no edges.

Definition 2.4. Let $r \geq 3$ be an integer.

(i) We shall denote by

$$f^{\log}_{X_{(r)}}(\Sigma) : \mathcal{G}^{\log}_{X_{(r)}}(\Sigma) \longrightarrow (\Pi^{\log}_{X_{(r)}})_\bullet$$

(cf. Definition 2.3) the morphism of graphs of groups determined by the morphisms $D^{\log}_{X_{(r)}I} \overset{\delta^{\log}_{X_{(r)}I}}{\to} X^{\log}_{(r)}$ ($I = \{1,2\}$, $\{2,3\}$, and $\{1,2,3\}$). For simplicity, we shall shall write $f^{\log}_{(r)}(\Sigma)$ instead of $f^{\log}_{X_{(r)}}(\Sigma)$ when there is no danger of confusion.

(ii) Let $I = \{1,2\}$, $\{2,3\}$, or $\{1,2,3\}$. Then, by the definition of $\mathcal{G}^{\log}_{X_{(r)}}(\Sigma)$, we have a natural morphism of graphs of groups

$$(\Pi^{\log}_{X_{(r)}I})_{\bullet} \longrightarrow \mathcal{G}^{\log}_{X_{(r)}}(\Sigma)\,.$$

We shall denote this morphism by $\delta^{\mathcal{G}\log}_{X_{(r)}I}$.

First, we will show the following theorem.

Theorem 2.5. *For a set of prime numbers Σ (which is not necessary K-innocuous), $f^{\log}_{(r)}(\Sigma)$ induces a surjection $\Pi^{\mathcal{G}}_{(r)} \to \Pi^{\log}_{(r)}$.*

Proof. First, we prove the assertion in the case where Σ is the set of all prime numbers. Since the morphism $p^{\log}_{(r-1)3}\mid_{D^{\log}_{(r)\{2,3\}}} = a^{\log}_{(r)\{2,3\}} : D^{\log}_{(r)\{2,3\}} \to X^{\log}_{(r-1)}$ is a morphism of type $\mathbb{N}$, the composite

$$\Pi^{\log}_{(r)\{2,3\}} \overset{\text{via } \delta^{\mathcal{G}\log}_{X_{(r)}\{2,3\}}}{\longrightarrow} \Pi^{\mathcal{G}}_{(r)} \overset{\text{via } f^{\log}_{(r)}(\Sigma)}{\longrightarrow} \Pi^{\log}_{(r)} \overset{\text{via } p^{\log}_{(r-1)3}}{\longrightarrow} \Pi^{\log}_{(r-1)}$$

is surjective ([4], Lemma 4.5). Thus, the morphism

$$\Pi^{\mathcal{G}}_{(r)} \longrightarrow \Pi^{\log}_{(r-1)}$$

induced by the composite of $p^{\log}_{(r-1)3} \circ f^{\log}_{(r)}(\Sigma)$ is surjective. In particular, it is enough to show that the image of the morphism $\Pi^{\mathcal{G}}_{(r)} \to \Pi^{\log}_{(r)}$ induced by $f^{\log}_{(r)}(\Sigma)$ generates the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$. Let $\overline{x}^{\log} \to X^{\log}_{(r-1)}$ be a strict geometric point of $X^{\log}_{(r-1)}$ such that the image of the underlying morphism of schemes of $\overline{x}^{\log} \to X^{\log}_{(r-1)}$ lies on $U_{(r-1)\{1,2\}}$. Then it follows from Proposition 1.7 that the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$ is generated by the image of the natural morphism $\pi_1(X^{\log}_{(r)\overline{x}^{\log}}) \to \Pi^{\log}_{(r)}$, where $X^{\log}_{(r)\overline{x}^{\log}}$ is the log scheme determined by the base-change of $p^{\log}_{(r-1)3} : X^{\log}_{(r)} \to X^{\log}_{(r-1)}$ via $\overline{x}^{\log} \to X^{\log}_{(r-1)}$. Let $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}$ (respectively, $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}$) be the log scheme determined by the base-change of $p^{\log}_{(r-1)3}\mid_{D^{\log}_{(r)\{1,2\}}} : D^{\log}_{(r)\{1,2\}} \to X^{\log}_{(r-1)}$ (respectively, $p^{\log}_{(r-1)3}\mid_{D^{\log}_{(r)\{1,2,3\}}} : D^{\log}_{(r)\{1,2,3\}} \to X^{\log}_{(r-1)}$) via $\overline{x}^{\log} \to X^{\log}_{(r-1)}$; $D^{\log}_{(r)\{1,2\}:\{1,2,3\}\overline{x}^{\log}}$ the fiber product $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}} \times_{X^{\log}_{(r)}} D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}} (= D^{\log}_{(r)\{1,2\}\overline{x}^{\log}} \times_{X^{\log}_{(r)\overline{x}^{\log}}} D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}})$; $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}}$ the graph of groups defined by

$$\mathcal{G}^{\log}_{(r)\overline{x}^{\log}} = (\overset{\pi_1(D^{\log}_{(r)\{1,2\}\overline{x}^{\log}})}{\bullet} \underset{\pi_1(D^{\log}_{(r)\{1,2\}:\{1,2,3\}\overline{x}^{\log}})}{-\!\!-\!\!-} \overset{\pi_1(D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}})}{\bullet})\,;$$

and $\pi_1(\mathcal{G}^{\log}_{(r)\overline{x}^{\log}})$ the group definied by

$$\varinjlim(\pi_1(D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}) \longleftarrow \pi_1(D^{\log}_{(r)\{1,2\}:\{1,2,3\}\overline{x}^{\log}}) \longrightarrow \pi_1(D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}))$$

(where the inductive limit is taken in the category of profinite groups). Then the natural strict closed immersions $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}} \to X^{\log}_{(r)\overline{x}^{\log}}$ and $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}} \to X^{\log}_{(r)\overline{x}^{\log}}$ (note that, by construction, the underlying schemes of $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}$ and $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}$ are the irreducible components of the underlying scheme of $X^{\log}_{(r)\overline{x}^{\log}}$) induce a morphism of graphs of groups $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}} \to \pi_1(X^{\log}_{(r)\overline{x}^{\log}})_\bullet$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}^{\log}_{(r)\overline{x}^{\log}} & \longrightarrow & \pi_1(X^{\log}_{(r)\overline{x}^{\log}})_\bullet \\ \downarrow & & \downarrow \\ \mathcal{G}^{\log}_{(r)}(\Sigma) & \xrightarrow[f^{\log}_{(r)}(\Sigma)]{} & (\Pi^{\log}_{(r)})_\bullet\,. \end{array}$$

Now since the underlying schemes of $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}$ and $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}$ are the irreducible components of the underlying scheme of $X^{\log}_{(r)\overline{x}^{\log}}$, if we naturally regard $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}}$ as a *graph of anabelioids* (cf. [10]), then the underlying graph of the graph of anabelioids determined as the *pull-back* of a ket covering $Y^{\log} \to X^{\log}_{(r)\overline{x}^{\log}}$ of $X^{\log}_{(r)\overline{x}^{\log}}$ via the morphism $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}} \to \pi_1(X^{\log}_{(r)\overline{x}^{\log}})_\bullet$ coincides with the dual graph of the pointed stable curve Y_{red}. Thus, it follows that $\pi_1(\mathcal{G}^{\log}_{(r)\overline{x}^{\log}}) \to \pi_1(X^{\log}_{(r)\overline{x}^{\log}})$ is surjective. Therefore, since the image of $\pi_1(X^{\log}_{(r)\overline{x}^{\log}}) \to \Pi^{\log}_{(r)}$ generates the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$, the image of $\Pi^{\mathcal{G}}_{(r)}$ in $\Pi^{\log}_{(r)}$ via the morphism induced by $f^{\log}_{(r)}(\Sigma)$ generates the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$. This completes the proof of the desired surjectivity in the case where Σ is the set of all prime numbers.

In the general case, the assertion follows immediately from the assertion in the case where Σ is the set of all prime numbers. □

Remark 2.6. Theorem 2.5 can be regarded as a logarithmic analogue of [7], Remark 1.2.

In the rest of this Section, we assume that

Σ *is K-innocuous.*

Next, we prove fundamental facts concerning the fundamental groups of the log configuration schemes.

Lemma 2.7.

(*i*) *The natural morphism* $U_{(r)} \to X^{\log}_{(r)}$ *induces a natural isomorphism* $\pi_1(U_{(r)})^{(\Sigma)} \xrightarrow{\sim} \Pi^{\log}_{(r)}$, *where* $\pi_1(U_{(r)})^{(\Sigma)}$ *is the quotient of* $\pi_1(U_{(r)})$ *by the closed normal subgroup*

$$\mathrm{Ker}(\pi_1(U_{(r)} \times_K K^{\mathrm{sep}}) \to \pi_1(U_{(r)} \times_K K^{\mathrm{sep}})^{(\Sigma)}) \,.$$

(*ii*) *The natural morphism* $U_{(r)\{1,2,3\}} \to X^{\log}_{(r)} \times_K \mathbb{P}^{\log}_K$ *induces a natural isomorphism* $\pi_1(U_{(r)\{1,2,3\}})^{(\Sigma)} \xrightarrow{\sim} \Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathbb{P}}$, *where* $\pi_1(U_{(r)\{1,2,3\}})^{(\Sigma)}$ *is the quotient of* $\pi_1(U_{(r)\{1,2,3\}})$ *by the closed normal subgroup*

$$\mathrm{Ker}(\pi_1(U_{(r)\{1,2,3\}} \times_K K^{\mathrm{sep}}) \to \pi_1(U_{(r)\{1,2,3\}} \times_K K^{\mathrm{sep}})^{(\Sigma)}) \,.$$

(*iii*) *Let* $1 \leq i \leq r+1$ *be an integer, and* $\overline{x} \to X_{(r)}$ *a geometric point of* $X_{(r)}$ *whose image lies on* $U_{(r)}$. *Then the cartesian diagram*

$$\begin{array}{ccc} X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x} & \longrightarrow & \overline{x} \\ \downarrow & & \downarrow \\ X^{\log}_{(r+1)} & \xrightarrow[p^{\log}_{(r)i}]{} & X^{\log}_{(r)} \end{array}$$

induces the following exact sequence:

$$1 \longrightarrow \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})^{(\Sigma)} \longrightarrow \Pi^{\log}_{(r+1)} \xrightarrow{\text{via } p^{\log}_{(r)i}} \Pi^{\log}_{(r)} \longrightarrow 1 \,.$$

(*iv*) *For a profinite group* Γ (*respectively, a scheme* S), *we shall denote by* $\mathcal{S}(\Gamma)$ (*respectively,* $S_{\text{ét}}$) *the classifying site of* Γ, (*i.e., the site defined by considering the category of finite sets equipped with a continuous action of* Γ [*and coverings given by surjections of such sets*]) (*respectively, the étale site of* S). *Then we have natural morphisms of sites*

$$(U_{(r)})_{\text{ét}} \longrightarrow \mathcal{S}(\pi_1(U^{\log}_{(r)})^{(\Sigma)}) \longrightarrow \mathcal{S}(\Pi^{\log}_{(r)}) \,.$$

Let A *be a finite* $\Pi^{\log}_{(r)}$*-module whose order is a* Σ*-integer, and* n *an integer. Then the natural morphisms*

$$\mathrm{H}^n(\Pi^{\log}_{(r)}, A) \longrightarrow \mathrm{H}^n(\pi_1(U^{\log}_{(r)})^{(\Sigma)}, A) \longrightarrow \mathrm{H}^n_{\text{ét}}(U_{(r)}, \mathcal{F}_A)$$

induced by the above morphisms of sites are isomorphisms, where $\mathcal{F}_A$ *is the locally constant sheaf on* $U_{(r)}$ *determined by* A.

(*v*) *Let* A *be a finite* $\Pi_{(r)}^{\log} \times_{G_K} \Pi_{\mathbb{P}}^{\log}$*-module whose order is a* Σ*-integer, and* n *an integer. Then the natural morphisms of sites*

$$(U_{(r)\{1,2,3\}})_{\text{ét}} \longrightarrow \mathcal{S}(\pi_1(U_{(r)\{1,2,3\}}^{\log})^{(\Sigma)}) \longrightarrow \mathcal{S}(\Pi_{(r)}^{\log} \times_{G_K} \Pi_{\mathbb{P}}^{\log})$$

induce isomorphisms

$$\mathrm{H}^n(\Pi_{(r)}^{\log} \times_{G_K} \Pi_{\mathbb{P}}^{\log}, A) \xrightarrow{\sim} \mathrm{H}^n(\pi_1(U_{(r)\{1,2,3\}}^{\log})^{(\Sigma)}, A) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(U_{(r)\{1,2,3\}}, \mathcal{F}_A)\,,$$

where $\mathcal{F}_A$ *is the locally constant sheaf determined by* A*.*

Proof. First, we prove (i). It is immediate that we may assume that K is separably closed. Let $V \to U_{(r)}$ be a Galois covering whose order is a Σ-integer (i.e., a Galois covering determined by an open normal subgroup of $\pi_1(U_{(r)}^{\log})^{(\Sigma)} = \pi_1(U_{(r)}^{\log})^{(\Sigma)}$), $Y \to X_{(r)}$ the normalization of $X_{(r)}$ in V, and $\overline{\eta} \to X_{(r)}$ a geometric point over the generic point of an irreducible component of $D_{(r)} = X_{(r)} \setminus U_{(r)} \subseteq X_{(r)}$. Then it follows from the Galoisness of $V \to U_{(r)}$ and the fact that the order of $V \to U_{(r)}$ is prime to p (whenever $p \geq 2$) that the base-change $Y \times_{X_{(r)}} \operatorname{Spec} \mathcal{O}_{X_{(r)},\overline{\eta}} \to \operatorname{Spec} \mathcal{O}_{X_{(r)},\overline{\eta}}$ is a tamely ramified covering (along the unique closed point of $\operatorname{Spec} \mathcal{O}_{X_{(r)},\overline{\eta}}$). Thus, by the log purity theorem ([8], Theorem 3.3. cf. also [4], Remark 1.10), $Y \to X_{(r)}$ extends to a ket covering $Y^{\log} \to X_{(r)}^{\log}$. In particular, $\pi_1(U_{(r)}^{\log})^{(\Sigma)} \to \Pi_{(r)}^{\log}$ is injective, hence an isomorphism.

Next, we prove (ii). By [4], Proposition 2.4, (ii), the natural morphism $\pi_1(X_{(r)}^{\log} \times_K \mathbb{P}_K^{\log}) \to \pi_1(X_{(r)}^{\log}) \times_{G_K} \pi_1(\mathbb{P}_K^{\log})$ is an isomorphism. Moreover, it is immediate that we may assume that K is separably closed. Therefore, by taking pro-Σ completions, $\pi_1(X_{(r)}^{\log} \times_K \mathbb{P}_K^{\log})^{(\Sigma)} \xrightarrow{\sim} (\pi_1(X_{(r)}^{\log}) \times \pi_1(\mathbb{P}_K^{\log}))^{(\Sigma)} \xrightarrow{\sim} \Pi_{(r)}^{\log} \times \Pi_{\mathbb{P}}^{\log}$. On the other hand, by a similar argument to the argument used in the proof of (i), we obtain an isomorphism $\pi_1(U_{(r)\{1,2,3\}})^{(\Sigma)} \xrightarrow{\sim} \pi_1(X_{(r)}^{\log} \times_K \mathbb{P}_K^{\log})^{(\Sigma)}$. This completes the proof of (ii).

Next, we prove (iii). To prove (iii), we may assume that K is separably closed field. Moreover, if Σ is the set of all prime numbers, then this follows from [7], Lemma 2.4, together with (i). Thus, we may assume that $\Sigma = \{l\}$ for a prime number l which is invertible in K. By [12], Proposition 2.7, we have an exact sequence

$$1 \longrightarrow \pi_1(U)^{(\Sigma)} \longrightarrow \pi_1(U_{(r+1)})^{(')} \xrightarrow{\text{via } p_{(r)i}^{\log}} \pi_1(U_{(r)}) \longrightarrow 1\,,$$

where U is the interior of $X_{(r+1)}^{\log} \times_{X_{(r)}^{\log}} \overline{x}$, and the profinite group $\pi_1(U_{(r+1)})^{(')}$ is the quotient of $\pi_1(U_{(r+1)})$ by the kernel of the natural surjection

$$\pi_1(U) \longrightarrow \pi_1(U)^{(\Sigma)}\,.$$

Now, by a similar argument to the argument used in the proof of (i), the group $\pi_1(U)^{(\Sigma)}$ is naturally isomorphic to $\pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})^{(\Sigma)}$. By the exactness of

$$1 \longrightarrow \pi_1(U)^{(\Sigma)} \longrightarrow \pi_1(U_{(r+1)})^{(\prime)} \overset{\text{via } p^{\log}_{(r)i}}{\longrightarrow} \pi_1(U_{(r)}) \longrightarrow 1\,,$$

it is enough to show that the outer representation

$$\pi_1(U_{(r)}) \longrightarrow \mathrm{Out}(\pi_1(U)^{(\Sigma)})$$

induced by the above sequence factors through $\pi_1(U_{(r)})^{(\Sigma)}$ ([1], Proposition 3). On the other hand, if we denote by U^{cpt} a (unique) compactification of U, then the following hold:

(i) If we denote by $\mathrm{Out}^*(\pi_1(U)^{(\Sigma)})$ the subgroup of $\mathrm{Out}(\pi_1(U)^{(\Sigma)})$ whose elements preserve the kernel of the surjection $\pi_1(U)^{(\Sigma)} \to \pi_1(U^{\mathrm{cpt}})^{(\Sigma)}$, then the outer representation $\pi_1(U_{(r)}) \to \mathrm{Out}(\pi_1(U)^{(\Sigma)})$ factors through $\mathrm{Out}^*(\pi_1(U)^{(\Sigma)})$. (This follows from the existence of the "compactification" of $p_{(r)r+1}\,|_{U_{(r+1)}}$

$$\begin{array}{ccc}
U_{(r+1)} & \xrightarrow{p_{(r)r+1}|_{U_{(r+1)}} \times \mathrm{pr}_{(r+1)r+1}|_{U_{(r+1)}}} & U_{(r)} \times_K X \\
{\scriptstyle p_{(r)r+1}|_{U_{(r+1)}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\
U_{(r)} & = & U_{(r)}\,.)
\end{array}$$

(ii) The kernel of the natural morphism

$$\mathrm{Out}^*(\pi_1(U)^{(\Sigma)}) \longrightarrow \mathrm{Aut}((\pi_1(U^{\mathrm{cpt}})^{(\Sigma)})^{\mathrm{ab}})$$

is pro-Σ. (This follows from [7], Lemma 3.1, (i).)

Therefore, it is enough to show that the natural representation

$$\pi_1(U_{(r)}) \longrightarrow \mathrm{Aut}((\pi_1(U^{\mathrm{cpt}})^{(\Sigma)})^{\mathrm{ab}})$$

induced by the above outer representation factors through $\pi_1(U_{(r)})^{(\Sigma)}$. Now this is immediate. This completes the proof of assertion (iii).

Next, we prove (iv). The assertion that the first morphism is an isomorphism follows immediately from (i). Let $\overline{x} \to X_{(r)}$ be a geometric point of $X_{(r)}$ whose image lies on $U_{(r)}$. Then, by considering the Hochschild-Serre spectral sequence ([11], Theorem 2.1.5) associated to the exact sequence obtained in (iii)

$$1 \longrightarrow \pi \longrightarrow \Pi^{\log}_{(r+1)} \overset{\text{via } p^{\log}_{(r)r+1}}{\longrightarrow} \Pi^{\log}_{(r)} \longrightarrow 1$$

(where $\pi = \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})^{(\Sigma)})$ and the Leray spectral sequence associated to the morphism $p_{(r)r+1} \mid_{U_{(r+1)}}$, we obtain the following morphism of spectral sequences:

$$\begin{array}{ccccc} E_2^{p,q} & = & \mathrm{H}^p(\Pi^{\log}_{(r)}, \mathrm{H}^q(\pi, A)) \Longrightarrow \mathrm{H}^{p+q}(\Pi^{\log}_{(r+1)}, A) & = & E^{p+q} \\ \downarrow & & & & \downarrow \\ E_2^{'\,p,q} & = & \mathrm{H}^p_{\text{ét}}(U_{(r)}, \mathbb{R}^q(p_{(r)r+1} \mid_{U_{(r)}})_* \mathcal{F}_A) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(U_{(r+1)}, \mathcal{F}_A) & = & E^{'\,p+q}. \end{array}$$

Now, by considering the "compactification" of $p_{(r)r+1} \mid_{U_{(r+1)}}$

$$\begin{array}{ccc} U_{(r+1)} & \xrightarrow{p_{(r)r+1}|_{U_{(r+1)}} \times \mathrm{pr}_{(r+1)r+1}|_{U_{(r+1)}}} & U_{(r)} \times_K X \\ {\scriptstyle p_{(r)r+1}|_{U_{(r+1)}}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} \\ U_{(r)} & = & U_{(r)}, \end{array}$$

it follows that the sheaf $\mathbb{R}^q(p_{(r)r+1} \mid_{U_{(r)}})_* \mathcal{F}_A$ is locally constant and constructible ([2], Corollary 10.3); moreover, the $\Pi_{(r+1)}$-module $(\mathbb{R}^q(p_{(r)r+1} \mid_{U_{(r)}})_* \mathcal{F}_A)_{\overline{x}}$ is naturally isomorphic to $\mathrm{H}^q(U, \mathcal{F}_A \mid_U)$ ([2], Theorem 7.3). Therefore, it is enough to show that the natural morphism

$$\mathrm{H}^n(\pi, A) \longrightarrow \mathrm{H}^n_{\text{ét}}(U, \mathcal{F}_A \mid_U)$$

is an isomorphism, where U is the interior of $X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}$. Thus, one then verifies immediately that it is enough to verify that every étale cohomology class of U (with coefficients in $\mathcal{F}_A \mid_U$) vanishes upon pull-back to some (connected) finite étale Σ-covering $V \to U$. Moreover, by passing to an appropriate U, we may assume that $\mathcal{F}_A \mid_U$ is trivial. Then the vanishing assertion in question is immediate (respectively, a tautology) for $n = 0$ (respectively, $n = 1$). Moreover, the vanishing assertion in question is immediate for $n \geq 3$ by [2], Theorem 9.1. If U is *affine*, then since $\mathrm{H}^n_{\text{ét}}(U, \mathcal{F}_A \mid_U)$ vanishes for $n = 2$ ([2], Theorem 9.1), the assertion is immediate. If U is proper, then it is enough to take $V \to U$ so that the degree of $V \to U$ annihilates A (cf., e.g., the discussion at the bottom of [2], p. 136).

Finally, we prove (v). The assertion that the first morphism is an isomorphism follows from (i). Moreover, by a similar argument to the argument used in the proof of (iv), the second morphism is also an isomorphism. □

Remark 2.8.

(i) By Lemma 2.7, (iv), (v), together with a similar argument to the argument used in [9], Lemma 4.3, any invertible sheaf on $X^{\log}_{(r)}$ or $X^{\log}_{(r)} \times_K \mathbb{P}^{\log}_K$ satisfies the condition $(*)$ in [4], Proposition 4.22.

(ii) By (i) and Lemma 2.7, (iv), (v), the equivalence class of the extension of $\Pi^{\log}_{(r)}$ (respectively, $\Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathbb{P}}$) associated to an invertible sheaf $\mathcal{L}$ on $X_{(r)}$ (respectively, $X_{(r)} \times_K \mathbb{P}^1_K$) (cf. [4], Definition 4.23) depends only on the (étale-theoretic) first Chern class of $\mathcal{L} \mid_{U_{(r)}}$ (respectively, $\mathcal{L} \mid_{U_{(r)} \times_K U_{\mathbb{P}}}$). In particular, for example, the extension

$$1 \longrightarrow \Lambda \longrightarrow \Pi^{\log}_{(r+1)\{1,2\}} \overset{a^{\log}_{(r+1)\{1,2\}}}{\longrightarrow} \Pi^{\log}_{(r)} \longrightarrow 1$$

of $\Pi^{\log}_{(r)}$ by Λ (i.e., the extension of $\Pi^{\log}_{(r)}$ associated to $(a^{-1}_{(r+1)\{1,2\}})^* \mathcal{L}_{(r+1)\{1,2\}}$) is isomorphic to the extension of $\Pi^{\log}_{(r)}$ by Λ associated to the invertible sheaf $(a^{-1}_{(r+1)\{1,2\}})^*(p_{(r)i} \mid_{D_{(r+1)\{1,2\}}})^*(\mathcal{L}_{(r)\{1,2\}})$ ($i \neq 1, 2$) (cf. Lemma 1.21, (i)).

Lemma 2.9.

(i) Let $r \geq 2$ be an integer and $2 \leq i \leq r$ an integer. Then the following diagram is cartesian:

$$\begin{array}{ccc}
\Pi^{\log}_{(r+1)\{1,2\}} & \xrightarrow{\text{via } p^{\log}_{(r)i+1}} & \Pi^{\log}_{(r)\{1,2\}} \\
{\scriptstyle \text{via } a^{\log}_{(r+1)\{1,2\}}} \downarrow & & \downarrow {\scriptstyle \text{via } a^{\log}_{(r)\{1,2\}}} \\
\Pi^{\log}_{(r)} & \xrightarrow[\text{via } p^{\log}_{(r-1)i}]{} & \Pi^{\log}_{(r-1)} .
\end{array}$$

(ii) Let $r \geq 2$ be an integer. Then the following diagram is cartesian:

$$\begin{array}{ccc}
\Pi^{\log}_{(r+1)\{2,3\}} & \xrightarrow{\text{via } p^{\log}_{(r)1}} & \Pi^{\log}_{(r)\{1,2\}} \\
{\scriptstyle \text{via } a^{\log}_{(r+1)\{2,3\}}} \downarrow & & \downarrow {\scriptstyle \text{via } a^{\log}_{(r)\{1,2\}}} \\
\Pi^{\log}_{(r)} & \xrightarrow[\text{via } p^{\log}_{(r-1)1}]{} & \Pi^{\log}_{(r-1)} .
\end{array}$$

(iii) Let $r \geq 3$ be an integer and $3 \leq i \leq r$ an integer. Then the following diagram is cartesian:

$$\begin{array}{ccc}
\Pi^{\log}_{(r+1)\{2,3\}} & \xrightarrow{\text{via } p^{\log}_{(r)i+1}} & \Pi^{\log}_{(r)\{2,3\}} \\
{\scriptstyle \text{via } a^{\log}_{(r+1)\{2,3\}}} \downarrow & & \downarrow {\scriptstyle \text{via } a^{\log}_{(r)\{2,3\}}} \\
\Pi^{\log}_{(r)} & \xrightarrow[\text{via } p^{\log}_{(r-1)i}]{} & \Pi^{\log}_{(r-1)} .
\end{array}$$

(*iv*) *Let $r \geq 2$ be an integer, and $j = 1$, 2, or 3. Then the following diagram is cartesian:*

$$\begin{array}{ccc} \Pi^{\log}_{(r+1)\{1,2,3\}} & \xrightarrow{\text{via } p^{\log}_{(r)j}} & \Pi^{\log}_{(r)\{1,2\}} \\ {\scriptstyle \text{via } b^{\log}_{(r+1)\{1,2,3\}}}\downarrow & & \downarrow{\scriptstyle \text{via } a^{\log}_{(r)\{1,2\}}} \\ \Pi^{\log}_{(r-1)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} & \xrightarrow[\text{via pr}_1]{} & \Pi^{\log}_{(r-1)} . \end{array}$$

(*v*) *Let $r \geq 3$ be an integer and $2 \leq i \leq r-1$ be an integer. Then the following diagram is cartesian:*

$$\begin{array}{ccc} \Pi^{\log}_{(r+1)\{1,2,3\}} & \xrightarrow{\text{via } p^{\log}_{(r)i+2}} & \Pi^{\log}_{(r)\{1,2,3\}} \\ {\scriptstyle \text{via } b^{\log}_{(r)\{1,2,3\}}}\downarrow & & \downarrow{\scriptstyle \text{via } b^{\log}_{(r)\{1,2,3\}}} \\ \Pi^{\log}_{(r-1)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} & \xrightarrow[\text{via } p_{(r-2)i} \times \mathrm{id}_{\mathbb{P}^{\log}}]{} & \Pi^{\log}_{(r-2)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} . \end{array}$$

Proof. First, we prove assertion (i). By Remark 2.8, (ii), the extension

$$1 \longrightarrow \Lambda \longrightarrow \Pi^{\log}_{(r+1)\{1,2\}} \xrightarrow{\text{via } a^{\log}_{(r)\{1,2\}}} \Pi^{\log}_{(r)} \longrightarrow 1$$

of $\Pi^{\log}_{(r)}$ by Λ is isomorphic to the extension of $\Pi^{\log}_{(r)}$ associated to

$$(a^{-1}_{(r+1)\{1,2\}})^* (p_{(r)j} \mid_{D_{(r+1)\{1,2\}}})^* \mathcal{L}_{(r)\{1,2\}}$$

($j \neq 1$, 2). On the other hand, by the commutativity of the diagram

$$\begin{array}{ccccc} X_{(r)} & \xleftarrow[\sim]{a_{(r+1)\{1,2\}}} & D_{(r+1)\{1,2\}} & \xrightarrow{\delta_{(r+1)\{1,2\}}} & X_{(r+1)} \\ {\scriptstyle p_{(r-1)i}}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{(r)i+1}} \\ X_{(r-1)} & \xleftarrow[\sim]{a_{(r)\{1,2\}}} & D_{(r)\{1,2\}} & \xrightarrow[\delta_{(r)\{1,2\}}]{} & X_{(r)} \end{array}$$

(cf. the definition of "$a_{(*)\{1,2\}}$" in Definition 1.15) implies that $(a^{-1}_{(r+1)\{1,2\}})^* (p_{(r)i+1} \mid_{D_{(r+1)\{1,2\}}})^* \mathcal{L}_{(r)\{1,2\}}$ is naturally isomorphic to $p^*_{(r-1)i} (a^{-1}_{(r)\{1,2\}})^* \mathcal{L}_{(r)\{1,2\}}$. Therefore, the fiber product of

$$\begin{array}{ccc} & & \Pi^{\log}_{(r)\{1,2\}} \\ & & \downarrow{\scriptstyle \text{via } a^{\log}_{(r)\{1,2\}}} \\ \Pi^{\log}_{(r)} & \xrightarrow[\text{via } p^{\log}_{(r-1)i}]{} & \Pi^{\log}_{(r-1)} , \end{array}$$

is isomorphic to the extension of $\Pi^{\log}_{(r)}$ associated to $(a^{-1}_{(r+1)\{1,2\}})^*(p_{(r)i+1}\ |_{D_{(r+1)\{1,2\}}})^*\mathcal{L}_{(r)\{1,2\}}$; thus, by Lemma 1.21, (i) (cf. also the argument in Remark 2.8, (ii)), this fiber product is isomorphic to $\Pi^{\log}_{(r+1)\{1,2\}}$.

Assertion (ii) (respectively, (iii); respectively, (iv); respectively, (v)) follows from a similar argument to the argument used in the proof of assertion (i), Lemma 1.21, (ii) (respectively, (ii); respectively, (iii); respectively, (iii)) (cf. also the argument in Remark 2.8, (ii)), together with the commutativity of the following diagram:

$$
\begin{array}{ccccc}
X_{(r)} & \xleftarrow[\sim]{a_{(r+1)\{2,3\}}} & D_{(r+1)\{2,3\}} & \xrightarrow{\delta_{(r+1)\{2,3\}}} & X_{(r+1)} \\
\downarrow{\scriptstyle p_{(r-1)1}} & & \downarrow & & \downarrow{\scriptstyle p_{(r)1}} \\
X_{(r-1)} & \xleftarrow[a_{(r)\{1,2\}}]{\sim} & D_{(r)\{1,2\}} & \xrightarrow[\delta_{(r)\{1,2\}}]{} & X_{(r)}
\end{array}
$$

(cf. the definitions of "$a_{(*)\{1,2\}}$" and "$a_{(*)\{2,3\}}$" in Definition 1.15) (respectively,

$$
\begin{array}{ccccc}
X_{(r)} & \xleftarrow[\sim]{a_{(r+1)\{2,3\}}} & D_{(r+1)\{2,3\}} & \xrightarrow{\delta_{(r+1)\{2,3\}}} & X_{(r+1)} \\
\downarrow{\scriptstyle p_{(r-1)i}} & & \downarrow & & \downarrow{\scriptstyle p_{(r)i+1}} \\
X_{(r-1)} & \xleftarrow[a_{(r)\{2,3\}}]{\sim} & D_{(r)\{2,3\}} & \xrightarrow[\delta_{(r)\{2,3\}}]{} & X_{(r)}
\end{array}
$$

[cf. the definition of "$a_{(*)\{2,3\}}$" in Definition 1.15]; respectively,

$$
\begin{array}{ccccc}
X_{(r-1)} \times_K \mathbb{P}^1_K & \xleftarrow[\sim]{b_{(r+1)\{1,2,3\}}} & D_{(r+1)\{1,2,3\}} & \xrightarrow{\delta_{(r+1)\{1,2,3\}}} & X_{(r+1)} \\
\downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow & & \downarrow{\scriptstyle p_{(r)j}} \\
X_{(r-1)} & \xleftarrow[a_{(r)\{1,2\}}]{\sim} & D_{(r)\{1,2\}} & \xrightarrow[\delta_{(r)\{1,2\}}]{} & X_{(r)}\ ,
\end{array}
$$

[cf. the definitions of "$a_{(*)\{1,2\}}$" and "$b_{(*)\{1,2,3\}}$" in Definition 1.15 and Definition 1.18]; respectively,

$$
\begin{array}{ccccc}
X_{(r-1)} \times_K \mathbb{P}^1_K & \xleftarrow[\sim]{b_{(r+1)\{1,2,3\}}} & D_{(r+1)\{1,2,3\}} & \xrightarrow{\delta_{(r+1)\{1,2,3\}}} & X_{(r+1)} \\
\downarrow{\scriptstyle p_{(r-2)i}\times \mathrm{id}_{\mathbb{P}^1_K}} & & \downarrow & & \downarrow{\scriptstyle p_{(r)i+2}} \\
X_{(r-2)} \times_K \mathbb{P}^1_K & \xleftarrow[b_{(r)\{1,2,3\}}]{\sim} & D_{(r)\{1,2,3\}} & \xrightarrow[\delta_{(r)\{1,2,3\}}]{} & X_{(r)}
\end{array}
$$

[cf. the definition of "$b_{(*)\{1,2,3\}}$" in Definition 1.18]). □

Lemma 2.10.

(*i*) *Let* $r \geq 2$ *be an integer, and* $I = \{i, i{+}1\}$ $(i = 1,\, 2)$. *Then the following diagram is cartesian:*

$$\begin{array}{ccc} \Pi^{\log}_{(r)I} & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r)i,i+1}} & \Pi^{\log}_{(2)\{1,2\}} \\ {\scriptstyle \text{via } a^{\log}_{(r)I}} \downarrow & & \downarrow {\scriptstyle \text{via } a^{\log}_{(2)\{1,2\}}} \\ \Pi^{\log}_{(r-1)} & \xrightarrow[\text{via } \mathrm{pr}^{\log}_{(r-1)i}]{} & \Pi_X\ . \end{array}$$

(*ii*) *Let* $r \geq 3$ *be an integer. Then the following diagram is cartesian:*

$$\begin{array}{ccccc} \Pi^{\log}_{(r)\{1,2,3\}} & & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r)1,2}} & & \Pi^{\log}_{(2)\{1,2\}} \\ {\scriptstyle \text{via } b^{\log}_{(r)\{1,2,3\}}} \downarrow & & & & \downarrow {\scriptstyle \text{via } a^{\log}_{(2)\{1,2\}}} \\ \Pi^{\log}_{(r-2)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} & \xrightarrow[\mathrm{pr}_1]{} & \Pi^{\log}_{(r-2)} & \xrightarrow[\text{via } \mathrm{pr}^{\log}_{(r-1)1}]{} & \Pi_X\ . \end{array}$$

Proof. Assertion (i) (respectively, assertion (ii)) follows immediately from Lemma 2.9, (i), (ii) (respectively, (i), (ii), and (iv)), by induction on r. $\square$

Definition 2.11.

(i) Let $r \geq 2$ be an integer, and $I = \{i, i+1\}$ $(i = 1,\, 2)$. Then, by Lemma 2.10, (i), the morphism $\Pi^{\log}_{X_{(r)}I} \to \Pi^{\log}_{X_{(r-1)}}$ induced by $a^{\log}_{X_{(r)}I}$ and the morphism $\Pi^{\log}_{X_{(r)}I} \to \Pi^{\log}_{X_{(2)}\{1,2\}}$ induced by $\mathrm{pr}^{\log}_{X_{(r)}i,i+1}$ induces an isomorphism $\Pi^{\log}_{X_{(r)}I} \xrightarrow{\sim} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$. We shall denote this isomorphism by $\alpha^{\log}_{X_{(r)}I}$. For simplicity, we shall write $\alpha^{\log}_{(r)I}$ instead of $\alpha^{\log}_{X_{(r)}I}$ when there is no danger of confusion.

(ii) Let $r \geq 3$ be an integer. Then, by Lemma 2.10, (ii), the morphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \to \Pi^{\log}_{X_{(r-2)}} \times_{G_K} \Pi^{\log}_{\mathbb{P}_K}$ induced by $b^{\log}_{X_{(r)}\{1,2,3\}}$ and the morphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \to \Pi^{\log}_{X_{(2)}\{1,2\}}$ induced by $\mathrm{pr}^{\log}_{X_{(r)}1,2}$ induces an isomorphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-2)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$. We shall denote this isomorphism by $\beta^{\log}_{X_{(r)}\{1,2,3\}}$. For simplicity, we shall write $\beta^{\log}_{(r)\{1,2,3\}}$ instead of $\beta^{\log}_{X_{(r)}\{1,2,3\}}$ when there is no danger of confusion.

Definition 2.12. Let $* = 0$, 1 or ∞, and $D \subseteq \pi_1(\mathbb{P}^{\log}_K)$ the decompositon group at $* \in \mathbb{P}^1_K$ (well-defined up to conjugation by an element of $\pi_1(\mathbb{P}^{\log}_K)$). Then we shall refer to the quotient of D by the kernel of the composite

$$D \hookrightarrow \pi_1(\mathbb{P}^{\log}_K) \longrightarrow \Pi^{\log}_{\mathbb{P}}$$

as the *pro-*($\underline{\Sigma}$) *decomposition group at* $* \in \mathbb{P}^1_K$.

Next, we will define the collection of data used in the *reconstruction of the fundamental groups of higher dimensional log configuration schemes* performed in Theorem 2.16 below.

Definition 2.13. Let $r \geq 2$ be an integer.

(i) We shall denote by $\mathcal{D}_X(\Sigma)$, or $\mathcal{D}_{X_{(1)}}(\Sigma)$ the collection of data consisting of

- the profinite groups

$$\Pi^{\log}_{X_{(2)}},\ \Pi^{\log}_{X_{(2)}\{1,2\}},\ \Pi_X,\ G_K,\ \text{and}\ \Pi^{\log}_{\mathbb{P}_K};$$

- the morphisms

$$\Pi^{\log}_{X_{(2)}} \overset{\text{via } p^{\log}_{X_{(1)}i}}{\longrightarrow} \Pi_X\ (i = 1,\ 2),$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\text{via } \delta^{\log}_{X_{(2)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(2)}},$$

  and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K,$$

$$\Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K; \text{ and}$$

- the subgroups

$$\mathfrak{D}^{\log}_{K*} \subseteq \Pi^{\log}_{\mathbb{P}_K}$$

  determined by the pro-($\underline{\Sigma}$) decomposition groups $\mathfrak{D}^{\log}_{K*}$ at $* \in \mathbb{P}^1_K$ ($* = 0$, 1 and ∞).

(ii) We shall denote by $\mathcal{D}_{X_{(r)}}(\Sigma)$ the collection of data consisting of

- the profinite groups

$$\Pi^{\log}_{X_{(k)}}\ (1 \leq k \leq r+1),\ \Pi^{\log}_{X_{(2)}\{1,2\}},\ G_K,\ \text{and}\ \Pi^{\log}_{\mathbb{P}_K};$$

- the morphisms

$$\Pi^{\log}_{X_{(k)}} \overset{\text{via } p^{\log}_{X_{(k-1)}i}}{\longrightarrow} \Pi^{\log}_{X_{(k-1)}} \ (2 \leq k \leq r+1,\ 1 \leq i \leq k),$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\text{via } a^{\log}_{X_{(2)}\{1,2\}}}{\longrightarrow} \Pi_X \ ,$$

and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K \ ,$$

$$\Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K \ ;$$

- the composites

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\alpha^{\log}_{X_{(r)}\{1,2\}})^{-1}}{\underset{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{1,2\}} \overset{\text{via } \delta^{\log}_{X_{(r+1)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \overset{\text{via } \mathrm{pr}^{\log}_{X_{(r)}1}}{\to} \Pi_X$),

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\alpha^{\log}_{X_{(r)}\{2,3\}})^{-1}}{\underset{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{2,3\}} \overset{\text{via } \delta^{\log}_{X_{(r+1)}\{2,3\}}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \overset{\text{via } \mathrm{pr}^{\log}_{X_{(r)}2}}{\to} \Pi_X$) and

$$\Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\beta^{\log}_{X_{(r)}\{1,2,3\}})^{-1}}{\underset{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{1,2,3\}} \overset{\text{via } \delta^{\log}_{X_{(r+1)}\{1,2,3\}}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r-1)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r-1)}} \overset{\text{via } \mathrm{pr}^{\log}_{X_{(r-1)}1}}{\to} \Pi_X$); and
- the subgroups

$$\mathfrak{D}^{\log}_{K*} \subseteq \Pi^{\log}_{\mathbb{P}_K}$$

determined by the pro-($\underline{\Sigma}$) decomposition groups $\mathfrak{D}^{\log}_{K*}$ at $* \in \mathbb{P}^1_K$ ($* = 0$, 1 and ∞).

(iii) We shall denote by $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ the collection of data consisting of

- the profinite groups

$$\Pi^{\mathcal{G}}_{X_{(r+1)}},\ \Pi^{\log}_{X_{(k)}}\ (1 \leq k \leq r),\ \Pi^{\log}_{X_{(2)}\{1,2\}},\ G_K,\ \text{and}\ \Pi^{\log}_{\mathbb{P}_K}\ ;$$

- the morphisms

$$\Pi^{\mathcal{G}}_{X_{(r+1)}} \stackrel{\text{via } p^{\log}_{X_{(r)}i} \circ f^{\log}_{X_{(r)}}(\Sigma)}{\longrightarrow} \Pi^{\log}_{X_{(r)}}\ (1 \leq i \leq r+1),$$

$$\Pi^{\log}_{X_{(k)}} \stackrel{\text{via } p^{\log}_{X_{(k-1)}i}}{\longrightarrow} \Pi^{\log}_{X_{(k-1)}}\ (2 \leq k \leq r,\ 1 \leq i \leq k),$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \stackrel{\text{via } a^{\log}_{X_{(2)}\{1,2\}}}{\longrightarrow} \Pi_X\ ,$$

  and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K\ ,$$

$$\Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K\ ;$$

- the composites

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \stackrel{(\alpha^{\log}_{X_{(r)}\{1,2\}})^{-1}}{\stackrel{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{1,2\}} \stackrel{\text{via } \delta^{\mathcal{G}\log}_{X_{(r+1)}\{1,2\}}}{\longrightarrow} \Pi^{\mathcal{G}}_{X_{(r+1)}}$$

  (where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \stackrel{\text{via } \mathrm{pr}^{\log}_{X_{(r)}1}}{\to} \Pi_X$),

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \stackrel{(\alpha^{\log}_{X_{(r)}\{2,3\}})^{-1}}{\stackrel{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{2,3\}} \stackrel{\text{via } \delta^{\mathcal{G}\log}_{X_{(r+1)}\{2,3\}}}{\longrightarrow} \Pi^{\mathcal{G}}_{X_{(r+1)}}$$

  (where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \stackrel{\text{via } \mathrm{pr}^{\log}_{X_{(r)}2}}{\to} \Pi_X$) and

$$\Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \stackrel{(\beta^{\log}_{X_{(r)}\{1,2,3\}})^{-1}}{\stackrel{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{1,2,3\}} \stackrel{\text{via } \delta^{\mathcal{G}\log}_{X_{(r+1)}\{1,2,3\}}}{\longrightarrow} \Pi^{\mathcal{G}}_{X_{(r+1)}}$$

  (where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r-1)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r-1)}} \stackrel{\text{via } \mathrm{pr}^{\log}_{X_{(r-1)}1}}{\to} \Pi_X$); and

- the subgroups
$$\mathfrak{D}^{\log}_{K*} \subseteq \Pi^{\log}_{\mathbb{P}_K}$$
determined by the pro-(Σ) decomposition groups $\mathfrak{D}^{\log}_{K*}$ at $* \in \mathbb{P}^1_K$ ($* = 0$, 1 and ∞).

In the following, let Y be a smooth, proper, geometrically connected curve of genus $g_Y \geq 2$ over a field L, and $\mathbb{P}^{\log}_L$ the log scheme obtained by equipping $\mathbb{P}^1_L$ with the log structure associated to the divisor $\{0, 1, \infty\} \subseteq \mathbb{P}^1_L$. Moreover, we shall fix a separable closure L^{sep} of L and denote by G_L the absolute Galois group $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ of L.

Definition 2.14. Let $r \geq 2$ be an integer. Let Σ_Y be a (non-empty) set of prime numbers that is L-innocuous.

(i) We shall refer to isomorphisms

$$\phi^{\Pi^{\log}_{(k)}}_{(1)} : \Pi^{\log}_{X_{(k)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(k)}} \ (k = 1, 2);$$

$$\phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(1)} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} \, ;$$

$$\phi^{G}_{(1)} : G_K \xrightarrow{\sim} G_L \, ; \text{ and}$$

$$\phi^{\Pi^{\log}_{\mathbb{P}}}_{(1)} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$

which are compatible with the morphisms and subgroups given in the definitions of $\mathcal{D}_X(\Sigma)$ and $\mathcal{D}_Y(\Sigma_Y)$ as an *isomorphism of* $\mathcal{D}_X(\Sigma)$ *with* $\mathcal{D}_Y(\Sigma_Y)$.

(ii) We shall refer to isomorphisms

$$\phi^{\Pi^{\log}_{(k)}}_{(r)} : \Pi^{\log}_{X_{(k)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(k)}} \ (1 \leq k \leq r+1);$$

$$\phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(r)} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} \, ;$$

$$\phi^{G}_{(r)} : G_K \xrightarrow{\sim} G_L \, ; \text{ and}$$

$$\phi^{\Pi^{\log}_{\mathbb{P}}}_{(r)} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$

which are compatible with the morphisms and subgroups given in the definitions of $\mathcal{D}_{X_{(r)}}(\Sigma)$ and $\mathcal{D}_{Y_{(r)}}(\Sigma_Y)$ as an *isomorphism of* $\mathcal{D}_{X_{(r)}}(\Sigma)$ *with* $\mathcal{D}_{Y_{(r)}}(\Sigma_Y)$.

(iii) We shall refer to isomorphisms

$$\phi_{(r)}^{\mathcal{G}^{\log}_{(r+1)}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}} ;$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(k)}} : \Pi^{\log}_{X_{(k)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(k)}} \ (1 \leq k \leq r);$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} ;$$

$$\phi_{(r)}^{\mathcal{G}\,G} : G_K \xrightarrow{\sim} G_L \,; \text{ and}$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$

which are compatible with the morphisms and subgroups given in the definitions of $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ and $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ as an *isomorphism of* $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ *with* $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$.

Proposition 2.15. *Let $r \geq 2$ be an integer, and Σ_X (respectively, Σ_Y) a set of prime numbers that is innocuous in K (respectively, L). Let $\phi^{\mathcal{G}}_{(r)}$: $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ be an isomorphism. Then the following hold:*

(*i*) *There exists an isomorphism $F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)}) : \mathcal{D}_{X_{(r-1)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}_{Y_{(r-1)}}(\Sigma_Y)$. Moreover, the correspondence*

$$\phi^{\mathcal{G}}_{(r)} \mapsto F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})$$

is functorial.

(*ii*) *If $\phi_{(r)}^{\mathcal{G}^{\log}_{(r+1)}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ induces an isomorphism of the kernel of the morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ induced by $f^{\log}_{X_{(r+1)}}(\Sigma)$ with the kernel of the morphism $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$ induced by $f^{\log}_{Y_{(r+1)}}(\Sigma)$, then there exists an isomorphism $F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)}) : \mathcal{D}_{X_{(r)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}_{Y_{(r)}}(\Sigma_Y)$. Moreover, the correspondence*

$$\phi^{\mathcal{G}}_{(r)} \mapsto F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})$$

is functorial.

Proof. First, we prove assertion (i). If we write

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(k)}} \stackrel{\text{def}}{=} \phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(k)}} \ (1 \leq k \leq r)\,,$$

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(2)\{1,2\}}} \stackrel{\text{def}}{=} \phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}}\,,$$

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{G} \stackrel{\text{def}}{=} \phi^{\mathcal{G}\,G}_{(r)}\,, \text{ and}$$

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{\mathbb{P}}} \stackrel{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}}_{(r)}\,,$$

then we obtain an isomorphism $F^{\check{\mathcal{G}}}_{-1}(\phi_{(r)})$ of the desired type.

Next, we prove Assertion (ii). We denote by N_X (respectively, N_Y) the kernel of the morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$) induced by $f^{\log}_{X_{(r+1)}}(\Sigma)$ (respectively, $f^{\log}_{Y_{(r+1)}}(\Sigma)$). Then, by the assumption, the isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ induces an isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}|_{N_X} : N_X \xrightarrow{\sim} N_Y$. Therefore, the isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}$ induces an isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}/N : \Pi^{\mathcal{G}}_{X_{(r+1)}}/N_X \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}/N_Y$. Since the morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$) induced by $f^{\log}_{X_{(r+1)}}(\Sigma)$ (respectively, $f^{\log}_{Y_{(r+1)}}(\Sigma)$) is surjective (Theorem 2.5), we obtain that $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}/N : \Pi^{\log}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r+1)}}$. Therefore, if we write

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(r+1)}} \stackrel{\text{def}}{=} \phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}/N : \Pi^{\log}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r+1)}}\,,$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(k)}} \stackrel{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{(k)}}_{(r)}\ (1 \le k \le r)\,,$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(2)\{1,2\}}} \stackrel{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}}_{(r)}\,,$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{G} \stackrel{\text{def}}{=} \phi^{\mathcal{G}\,G}_{(r)}\,, \text{ and}$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{\mathbb{P}}} \stackrel{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}}_{(r)}\,,$$

then we obtain an isomorphism $F^{\check{\mathcal{G}}}(\phi_{(r)})$ of the desired type. □

Theorem 2.16. *Let $r \ge 2$ be an integer, and Σ_X (respectively, Σ_Y) a set of prime numbers that is K-innocuous (respectively, L-innocuous). Let $\phi_{(r-1)} : \mathcal{D}_{X_{(r-1)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}_{Y_{(r-1)}}(\Sigma_Y)$ be an isomorphism. Then there exists an isomorphism $F^{\mathcal{G}}_{+1}(\phi_{(r-1)}) : \mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ such that*

$$F^{\check{\mathcal{G}}}_{-1}(F^{\mathcal{G}}_{+1}(\phi_{(r-1)})) = \phi_{(r-1)}\,,$$

and, moreover, the isomorphism $F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}^{\log}_{(r+1)}}$ arises from an isomorphism of graphs of groups of $\mathcal{G}^{\log}_{X_{(r+1)}}(\Sigma_X)$ with $\mathcal{G}^{\log}_{Y_{(r+1)}}(\Sigma_Y)$. Moreover, the correspondence

$$\phi_{(r-1)} \mapsto F^{\mathcal{G}}_{+1}(\phi_{(r-1)})$$

is functorial.

Proof. First, we define a profinite groups $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}}$, $\Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}}$, and $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}}$, $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}}$, and $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}}$) as follows:

(i) $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} \stackrel{\text{def}}{=} \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}} \stackrel{\text{def}}{=} \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$), where the morphism implicit in the fiber product $\Pi_{X_{(r)}} \to \Pi_X$ (respectively, $\Pi_{Y_{(r)}} \to \Pi_Y$) is the morphism induced by $\mathrm{pr}^{\log}_{X_{(r)}1}$ (respectively, $\mathrm{pr}^{\log}_{Y_{(r)}1}$) (cf. Lemma 2.10, (i)).

(ii) $\Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} \stackrel{\text{def}}{=} \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}} \stackrel{\text{def}}{=} \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$), where the morphism implicit in the fiber product $\Pi_{X_{(r)}} \to \Pi_X$ (respectively, $\Pi_{Y_{(r)}} \to \Pi_Y$) is the morphism induced by $\mathrm{pr}^{\log}_{X_{(r)}2}$ (respectively, $\mathrm{pr}^{\log}_{Y_{(r)}2}$) (cf. Lemma 2.10, (i)).

(iii) $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} \stackrel{\text{def}}{=} \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}} \stackrel{\text{def}}{=} \Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$), where the morphism implicit in the fiber product $\Pi_{X_{(r-1)}} \to \Pi_X$ (respectively, $\Pi_{Y_{(r-1)}} \to \Pi_Y$) is the morphism induced by $\mathrm{pr}^{\log}_{X_{(r-1)}1}$ (respectively, $\mathrm{pr}^{\log}_{Y_{(r-1)}1}$) (cf. Lemma 2.10, (ii)).

Then we define a profinite group "$\Pi^{\mathcal{G}}_{X_{(r+1)}}$" (respectively, "$\Pi^{\mathcal{G}}_{Y_{(r+1)}}$") as the inductive limit of the diagram

$$\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} \longleftarrow \{1\} \longrightarrow \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} \longleftarrow \{1\} \longrightarrow \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}}$$

(respectively,

$$\Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}} \longleftarrow \{1\} \longrightarrow \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}} \longleftarrow \{1\} \longrightarrow \Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}})$$

(cf. Definition 2.2, Lemma 2.10). Moreover, for an integer $1 \leq i \leq r+1$, we define a "projection" $q_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $q_{Y_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r)}}$) as follows:

(i) If $i = 1$ or 2, then we define a morphism $q^{\{1,2\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $q^{\{1,2\}}_{Y_{(r)}i} : \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}} = \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \to \Pi^{\log}_{Y_{(r)}}$) as the first projection (cf. Lemma 2.10, (i)). If $i \geq 3$, then we define a morphism $q^{\{1,2\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$

(respectively, $q^{\{1,2\}}_{Y_{(r)}i} : \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}} = \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \to \Pi^{\log}_{Y_{(r)}}$) as the composite

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\text{via } p^{\log}_{X_{(r-1)}i-1} \times \mathrm{id}_{D^{\log}_{X_{(2)}\{1,2\}}}} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{X_{(r)}\{1,2\}})^{-1}} \Pi^{\log}_{X_{(r)}\{1,2\}} \xrightarrow{\text{via } \delta^{\log}_{X_{(r)}\{1,2\}}} \Pi^{\log}_{X_{(r)}}$$

(respectively,

$$\Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \xrightarrow{\text{via } p^{\log}_{Y_{(r-1)}i-1} \times \mathrm{id}_{D^{\log}_{Y_{(2)}\{1,2\}}}} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{Y_{(r)}\{1,2\}})^{-1}} \Pi^{\log}_{Y_{(r)}\{1,2\}} \xrightarrow{\text{via } \delta^{\log}_{Y_{(r)}\{1,2\}}} \Pi^{\log}_{Y_{(r)}})$$

(cf. Lemmas 2.9, (i); 2.10, (i)).

(ii) We define a morphism $q^{\{2,3\}}_{X_{(r)}1} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{2,3\}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $q^{\{2,3\}}_{Y_{(r)}1} : \Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}} = \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{2,3\}} \to \Pi^{\log}_{Y_{(r)}}$) as the composite

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\text{via } p^{\log}_{X_{(r-1)}1} \times \mathrm{id}_{D^{\log}_{X_{(2)}\{1,2\}}}} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{X_{(r)}\{1,2\}})^{-1}} \Pi^{\log}_{X_{(r)}\{1,2\}} \xrightarrow{\text{via } \delta^{\log}_{X_{(r)}\{1,2\}}} \Pi^{\log}_{X_{(r)}}$$

(respectively,

$$\Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \xrightarrow{\text{via } p^{\log}_{Y_{(r-1)}1} \times \mathrm{id}_{D^{\log}_{Y_{(2)}\{1,2\}}}} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{Y_{(r)}\{1,2\}})^{-1}} \Pi^{\log}_{Y_{(r)}\{1,2\}} \xrightarrow{\text{via } \delta^{\log}_{Y_{(r)}\{1,2\}}} \Pi^{\log}_{Y_{(r)}})$$

(cf. Lemmas 2.9, (ii); 2.10, (i)). If $i = 2$ or 3, then we define a morphism $q^{\{2,3\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $q^{\{2,3\}}_{Y_{(r)}i} : \Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}} = \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \to \Pi^{\log}_{Y_{(r)}}$) as the first projection (cf. Lemma 2.10, (i)). If $i \geq 4$, then we define a morphism $q^{\{2,3\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $q^{\{2,3\}}_{Y_{(r)}i} : \Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}} = \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \to \Pi^{\log}_{Y_{(r)}}$) as the composite

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{2,3\}} \xrightarrow{\text{via } p^{\log}_{X_{(r-1)}i-1} \times \mathrm{id}_{D^{\log}_{X_{(2)}\{1,2\}}}} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{X_{(r)}\{2,3\}})^{-1}} \Pi^{\log}_{X_{(r)}\{2,3\}} \xrightarrow{\text{via } \delta^{\log}_{X_{(r)}\{2,3\}}} \Pi^{\log}_{X_{(r)}}$$

(respectively,

$$\Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \xrightarrow{\text{via } p^{\log}_{Y_{(r-1)}i-1} \times \mathrm{id}_{D^{\log}_{Y_{(2)}\{1,2\}}}} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{Y_{(r)}\{2,3\}})^{-1}} \Pi^{\log}_{Y_{(r)}\{1,2\}} \xrightarrow{\text{via } \delta^{\log}_{Y_{(r)}\{2,3\}}} \Pi^{\log}_{Y_{(r)}})$$

(cf. Lemmas 2.9, (iii); 2.10, (i)).

(iii) If $i = 1$, 2, or 3, then we define a morphism $q^{\{1,2,3\}}_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} = \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $q^{\{1,2,3\}}_{Y_{(r)}i} : \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}} = \Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \to \Pi^{\log}_{Y_{(r)}}$) as the composite

$$\Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\text{projection}} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{X_{(r)}\{1,2\}})^{-1}} \Pi^{\log}_{X_{(r)}\{1,2\}} \xrightarrow{\text{via } \delta^{\log}_{X_{(r)}\{1,2\}}} \Pi^{\log}_{X_{(r)}}$$

(respectively,

$$\Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \xrightarrow{\text{projection}} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$$

$$\xrightarrow{(\alpha^{\log}_{Y_{(r)}\{1,2\}})^{-1}} \Pi^{\log}_{Y_{(r)}\{1,2\}} \xrightarrow{\text{via } \delta^{\log}_{Y_{(r)}\{1,2\}}} \Pi^{\log}_{Y_{(r)}})$$

(cf. Lemmas 2.9, (iv)) If $i \geq 4$, then we define a morphism $q^{\{1,2,3\}}_{X_{(r)}i}$: $\Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} = \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $q^{\{1,2,3\}}_{Y_{(r)}i} : \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}} = \Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \to \Pi^{\log}_{Y_{(r)}}$) as the composite

$$\Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{2,3\}} \xrightarrow{\text{via } \mathrm{id}_{\mathbb{P}^{\log}_K} \times p^{\log}_{X_{(r-1)}i-1} \times \mathrm{id}_{D^{\log}_{X_{(2)}\{1,2\}}}} \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-2)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$$

$$\xrightarrow{(\beta^{\log}_{X_{(r)}\{1,2,3\}})^{-1}} \Pi^{\log}_{X_{(r)}\{1,2,3\}} \xrightarrow{\text{via } \delta^{\log}_{X_{(r)}\{1,2,3\}}} \Pi^{\log}_{X_{(r)}}$$

(respectively,

$$\Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{2,3\}} \xrightarrow{\text{via } \mathrm{id}_{\mathbb{P}^{\log}_L} \times p^{\log}_{Y_{(r-1)}i-1} \times \mathrm{id}_{D^{\log}_{Y_{(2)}\{1,2\}}}} \Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-2)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$$

$$\xrightarrow{(\beta^{\log}_{Y_{(r)}\{1,2,3\}})^{-1}} \Pi^{\log}_{Y_{(r)}\{1,2,3\}} \xrightarrow{\text{via } \delta^{\log}_{Y_{(r)}\{1,2,3\}}} \Pi^{\log}_{Y_{(r)}}$$

(cf. Lemmas 2.9, (v); 2.10, (ii)).

These morphisms $q^{\{1,2\}}_{X_{(r)}i}$, $q^{\{2,3\}}_{X_{(r)}i}$, and $q^{\{1,2,3\}}_{X_{(r)}i}$ (respectively, $q^{\{1,2\}}_{Y_{(r)}i}$, $q^{\{2,3\}}_{Y_{(r)}i}$, and $q^{\{1,2,3\}}_{Y_{(r)}i}$) induce a morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r)}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r)}}$). We denote this morphism by $q_{X_{(r)}i}$ (respectively, $q_{Y_{(r)}i}$).

Next, we define an isomorphism $\phi^{\mathcal{G}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ as follows:

(i) we define an isomorphism

$$\phi^{\mathcal{G}_{\{1,2\}}} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} = \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2\}}$$

as

$$\phi^{\Pi^{\log}_{(r)}}_{(r-1)} \times_{\phi^{\Pi}_{(r-1)}} \phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(r-1)}.$$

(ii) we define an isomorphism

$$\phi^{\mathcal{G}_{\{2,3\}}} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{2,3\}} = \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} = \Pi^{\mathcal{G}}_{Y_{(r+1)}\{2,3\}}$$

as

$$\phi^{\Pi^{\log}_{(r)}}_{(r-1)} \times_{\phi^{\Pi}_{(r-1)}} \phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(r-1)}.$$

(iii) we define an isomorphism

$$\phi^{\mathcal{G}_{\{1,2,3\}}} : \Pi^{\mathcal{G}}_{X_{(r+1)}\{1,2,3\}} = \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$$

$$\xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} = \Pi^{\mathcal{G}}_{Y_{(r+1)}\{1,2,3\}}$$

as

$$\phi^{\Pi^{\log}_{\mathbb{P}}}_{(r-1)} \times_{\phi^{G}_{(r-1)}} \phi^{\Pi^{\log}_{(r-1)}}_{(r-1)} \times_{\phi^{\Pi}_{(r-1)}} \phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(r-1)}.$$

These isomorphisms $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{1,2,3\}}}$, $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{1,2,3\}}}$, and $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{1,2,3\}}}$ induce an isomorphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$. We denote this isomorphism by $\phi^{\mathcal{G}}$.

Then, by the constructions, for any $1 \leq i \leq r+1$, the following diagram commutes:

$$\begin{array}{ccc} \Pi^{\mathcal{G}}_{X_{(r+1)}} & \xrightarrow[\sim]{\phi^{\mathcal{G}}} & \Pi^{\mathcal{G}}_{X_{(r+1)}} \\ {\scriptstyle q_{X_{(r)i}}}\downarrow & & \downarrow{\scriptstyle q_{Y_{(r)i}}} \\ \Pi^{\log}_{X_{(r)}} & \underset{\phi_{(r-1)}^{\Pi^{\log}_{(r)}}}{\xrightarrow[\sim]{}} & \Pi^{\log}_{Y_{(r)}} . \end{array}$$

Therefore, the isomorphisms

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}^{\log}_{(r+1)}} \stackrel{\text{def}}{=} \phi^{\mathcal{G}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}} ;$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,\Pi^{\log}_{(k)}} \stackrel{\text{def}}{=} \phi_{(r-1)}^{\Pi^{\log}_{(k)}} : \Pi^{\log}_{X_{(k)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(k)}} \ (1 \leq k \leq r);$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}} \stackrel{\text{def}}{=} \phi_{(r-1)}^{\Pi^{\log}_{(2)\{1,2\}}} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} ;$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,G} \stackrel{\text{def}}{=} \phi^{G}_{(r-1)} : G_K \xrightarrow{\sim} G_L \text{ ; and}$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}} \stackrel{\text{def}}{=} \phi_{(r-1)}^{\Pi^{\log}_{\mathbb{P}}} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$

form an isomorphism $F^{\mathcal{G}}_{+1}(\phi_{(r-1)})$ of $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma_X)$ with $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ of the desired type. □

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