Rank functions of strict cg-matroids

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Abstract

A matroid-like structure defined on a convex geometry, called a cg-matroid, is defined by S. Fujishige, G. A. Koshevoy, and Y. Sano in [9]. A cg-matroid whose rank function is naturally defined is called a strict cg-matroid. In this paper, we give characterizations of strict cg-matroids by their rank functions.

1. Introduction

A matroid is one of the most important structures in combinatorial optimization. Many researchers have studied and extended the matroid theory. Dunstan, Ingleton, and Welsh [3] introduced the concept of a *supermatroid* defined on a poset in 1972 as a generalization of the concept of an ordinary matroid ([14]; also see [13] and [10]). In 1980 Faigle [6] considered a geometric structure on a poset (a special case of a supermatroid), and Tardos [12] showed a matroid-type intersection theorem for distributive supermatroids in 1990. A distributive supermatroid is also called a *poset matroid*. Peled and Srinivasan [11] considered a matroid-type independent matching problem for poset matroids in 1993. Moreover, in 1993 and 1998 Barnabei, Nicoletti, and Pezzoli [1, 2] studied poset matroids in terms of the poset structure of the ground set.

In [9], S. Fujishige, G. A. Koshevoy, and Y. Sano generalized poset matroids by considering convex geometries, instead of posets, as underlying combinatorial structures on which they define matroid-like structures, called *cg-matroids*. For a cg-matroid they defined independent sets, bases, and other related concepts, and examined their combinatorial structural properties. They have shown characterizations of the families of bases, independent sets, and spanning sets of cg-matroids. It is shown that cg-matroids are not special cases of supermatroids.

They also considered a special class of cg-matroids, called *strict cg-matroids*, for which rank functions are naturally defined, and they show the equivalence of the concept of a strict cg-matroid and that of a supermatroid defined on the lattice of closed sets of a convex geometry. (See Figure 1.)

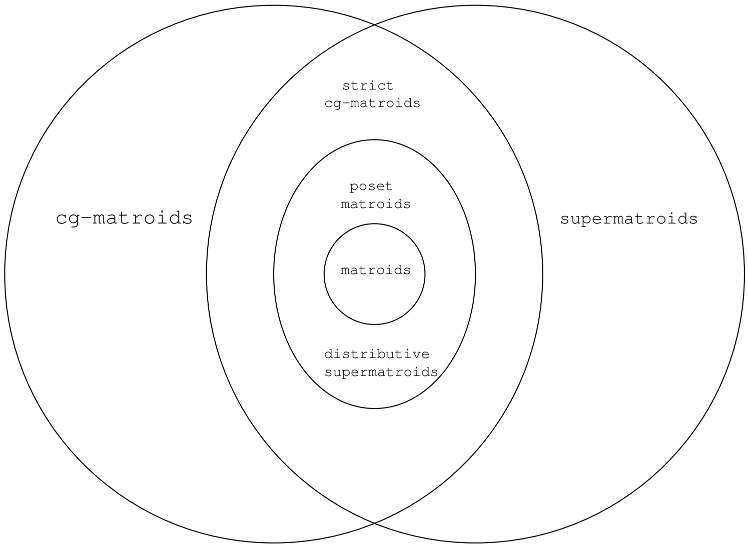

Figure 1: Generalizations of matroids.

The rank functions of strict cg-matroids were defined. And they have shown some properties which the rank function satisfy. But it was unknown to characterize strict cg-matroids in terms of rank functions.

In this paper, we give characterizations of the rank functions of strict cg-matroids. Our main results are as follows. Let $\mathbb{Z}_+$ be the set of nonnegative integers.

Theorem 1.1. *Let $(E, \mathcal{F})$ be a convex geometry and $\rho : \mathcal{F} \to \mathbb{Z}_+$ be a function on $\mathcal{F}$. Then ρ is the rank function of a strict cg-matroid on $(E, \mathcal{F})$ if and only if ρ satisfies the following properties.*

(RL0) $\rho(\emptyset) = 0$.

(RL1) $X \in \mathcal{F},\ e \in \mathrm{ex}^*(X) \implies \rho(X) \leq \rho(X \cup \{e\}) \leq \rho(X) + 1$.

(RGE) (Global Extension Property)
For any $X, Y \in \mathcal{F}$ such that $X \subseteq Y$ and $\rho(X) = |X| < \rho(Y)$,
there exists $Z \in \mathcal{F}$ such that $X \subsetneq Z \subseteq Y$ and $\rho(Z) = |Z| = \rho(Y)$. $\square$

Theorem 1.2. *Let $(E, \mathcal{F})$ be a convex geometry and $\rho : \mathcal{F} \to \mathbb{Z}_+$ be a function on $\mathcal{F}$. Then ρ is the rank function of a strict cg-matroid on $(E, \mathcal{F})$ if and only if ρ satisfies the following properties.*

(RG0) $0 \leq \rho(X) \leq |X|$ for any $X \in \mathcal{F}$.

(RG1) $X, Y \in \mathcal{F}, X \subseteq Y \Longrightarrow \rho(X) \leq \rho(Y)$.

(RGS) (Global Submodularity)
For any $X, Y \in \mathcal{F}$ such that $X \cup Y \in \mathcal{F}$,
$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$.

(RLE) (Local Extension Property)
For any $X, Y \in \mathcal{F}$ such that $X \subseteq Y$ and $\rho(X) = |X| < \rho(Y)$,
there exists $e \in \mathrm{ex}^*(X) \cap Y$ such that $\rho(X \cup \{e\}) = \rho(X) + 1$. □

This paper is organized as follows. In Section 2, we give definitions and some preliminaries on convex geometries, matroids on convex geometries (cg-matroids), and strict cg-matroids. In Section 3, we give the proofs of Theorem 1.1 and Theorem 1.2, and make some remarks.

2. Definitions and Preliminaries

2.1. Convex geometries

First, we define a convex geometry which is a fundamental combinatorial structure defined on a finite set. (See [4].)

Definition 2.1 (Convex geometries). Let E be a nonempty finite set and $\mathcal{F}$ be a family of subsets of E. The pair $(E, \mathcal{F})$ is called a *convex geometry* on E if it satisfies the following three conditions:

(F0) $\emptyset,\ E \in \mathcal{F}$.

(F1) $X, Y \in \mathcal{F} \Longrightarrow X \cap Y \in \mathcal{F}$.

(F2) $\forall X \in \mathcal{F} \setminus \{E\}$, $\exists e \in E \setminus X$: $X \cup \{e\} \in \mathcal{F}$.

The set E is called the *ground set* of the convex geometry $(E, \mathcal{F})$, and each member of $\mathcal{F}$ is called a *closed set*. It should be noted that Condition (F2) is equivalent to the following condition:

(F2)$'$ Every maximal chain $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = E$ in $\mathcal{F}$ has length $n = |E|$. □

Next, we define operators associated with the convex geometry $(E,\mathcal{F})$.

Definition 2.2 (Closure operators). For a convex geometry $(E,\mathcal{F})$, we define $\tau : 2^E \to \mathcal{F}$, called the *closure operator* of $(E,\mathcal{F})$, by

$$\tau(X) = \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\} \quad (X \in 2^E). \tag{2.1}$$

That is, $\tau(X)$ is the unique minimal closed set containing X. □

Definition 2.3 (Extreme-point operators). For a convex geometry $(E,\mathcal{F})$, we define dual operators, ex and ex^*. The first operator $\mathrm{ex} : \mathcal{F} \to 2^E$, called the *extreme-point operator* of $(E,\mathcal{F})$, is defined by

$$\mathrm{ex}(X) = \{e \in X \mid X \setminus \{e\} \in \mathcal{F}\} \quad (X \in \mathcal{F}). \tag{2.2}$$

An element in $\mathrm{ex}(X)$ is called an *extreme point* of X.

The second operator $\mathrm{ex}^* : \mathcal{F} \to 2^E$, called the *co-extreme-point operator* of $(E,\mathcal{F})$, is defined by

$$\mathrm{ex}^*(X) = \{e \in E \setminus X \mid X \cup \{e\} \in \mathcal{F}\} \quad (X \in \mathcal{F}). \tag{2.3}$$

An element in $\mathrm{ex}^*(X)$ is called a *co-extreme point* of X. □

2.2. Matroids on convex geometries (cg-matroids)

Let $(E,\mathcal{F})$ be a convex geometry on E with a family $\mathcal{F}$ of closed sets. Let $\tau : 2^E \to \mathcal{F}$ be the closure operator of the convex geometry $(E,\mathcal{F})$, and $\mathrm{ex} : \mathcal{F} \to 2^E$ be the extreme-point operator of the convex geometry $(E,\mathcal{F})$.

2.2.1. Bases

First, we give the definition of a cg-matroid.

Definition 2.4 (Matroids on convex geometries). For a convex geometry $(E,\mathcal{F})$ and a family $\mathcal{B} \subseteq \mathcal{F}$, suppose that $\mathcal{B}$ satisfies the following three conditions:

(B0) $\mathcal{B} \neq \emptyset$.

(B1) $B_1, B_2 \in \mathcal{B}\,,\ B_1 \subseteq B_2 \implies B_1 = B_2$.

(BM) (Middle Base Property)
For any $B_1, B_2 \in \mathcal{B}$ and $X, Y \in \mathcal{F}$ with $X \subseteq B_1$, $B_2 \subseteq Y$, and $X \subseteq Y$, there exists $B \in \mathcal{B}$ such that $X \subseteq B \subseteq Y$.

Then we call $(E,\mathcal{F};\mathcal{B})$ a *matroid on the convex geometry* $(E,\mathcal{F})$ or a *cg-matroid* for short. Each $B \in \mathcal{B}$ is called a *base*, and $\mathcal{B}$ the *family of bases* of the cg-matroid $(E,\mathcal{F};\mathcal{B})$. □

The family of bases satisfies the following.

Theorem 2.5 ([9]). *For any cg-matroid $(E, \mathcal{F}; \mathcal{B})$, all the bases in $\mathcal{B}$ have the same cardinality, i.e.,*

$(\mathrm{B1})'$ $B_1, B_2 \in \mathcal{B} \implies |B_1| = |B_2|$. □

In [9], Fujishige, Koshevoy, and Sano have shown a characterization of the family of bases of a cg-matroid by 'Exchange Property' as follows.

Theorem 2.6 ([9]). *Let $(E, \mathcal{F})$ be a convex geometry and $\mathcal{B} \subseteq \mathcal{F}$ be a subfamily of $\mathcal{F}$. Then, $\mathcal{B}$ is the family of bases of a cg-matroid on $(E, \mathcal{F})$ if and only if $\mathcal{B}$ satisfies* (B0) *and* (BE).

(BE) (Exchange Property)
For any $B_1, B_2 \in \mathcal{B}$ and any $e_1 \in \mathrm{ex}(\tau(B_1 \cup B_2)) \cap \mathrm{ex}(B_1) \setminus B_2$,
there exists $e_2 \in \tau(B_1 \cup B_2) \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$. □

2.2.2. Independent sets

We define a family of independent sets for a cg-matroid, similarly as for ordinary matroids.

Definition 2.7 (Independent sets). For a cg-matroid $(E, \mathcal{F}; \mathcal{B})$ with a family $\mathcal{B}$ of bases, we put

$$\mathcal{I}(\mathcal{B}) = \{X \in \mathcal{F} \mid X \subseteq B \text{ for some } B \in \mathcal{B}\}. \tag{2.4}$$

Each element in $\mathcal{I}(\mathcal{B})$ is called an *independent set* of the cg-matroid $(E, \mathcal{F}; \mathcal{B})$, and $\mathcal{I} = \mathcal{I}(\mathcal{B})$ is called the *family of independent sets* of the cg-matroid $(E, \mathcal{F}; \mathcal{B})$. □

In [9], they have also shown a characterization of the family of independent sets of a cg-matroid. For a family $\mathcal{I} \subseteq \mathcal{F}$, we put

$$\mathcal{B}(\mathcal{I}) = \{X \in \mathcal{F} \mid X \in \mathcal{I} : \text{ maximal } \}. \tag{2.5}$$

Theorem 2.8 ([9]). *The family $\mathcal{I} = \mathcal{I}(\mathcal{B})$ of independent sets of a cg-matroid $(E, \mathcal{F}; \mathcal{B})$ with a family $\mathcal{B}$ of bases satisfies the following three properties.*

(I0) $\emptyset \in \mathcal{I}$.

(I1) $I_1 \in \mathcal{F},\ I_2 \in \mathcal{I},\ I_1 \subseteq I_2 \implies I_1 \in \mathcal{I}$.

(IA) (Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ and I_2 being maximal in $\mathcal{I}$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Conversely, if a family $\mathcal{I} \subseteq \mathcal{F}$ satisfies the above three conditions, then $(E, \mathcal{F}; \mathcal{B}(\mathcal{I}))$ is a cg-matroid with a family $\mathcal{B}(\mathcal{I})$ of bases. □

2.3. Strict cg-matroids

In this subsection, we consider a special class of cg-matroids, called strict cg-matroids.

2.3.1. Independent sets

Definition 2.9 (Strict cg-matroids). Let $(E, \mathcal{F})$ be a convex geometry. If $\mathcal{I} \subseteq \mathcal{F}$ satisfies (I0), (I1), and the Strict Augmentation Property (IsA), then we call $(E, \mathcal{F}; \mathcal{I})$ a *strict cg-matroid* with a family $\mathcal{I}$ of independent sets.

(I0) $\emptyset \in \mathcal{I}$.

(I1) $I_1 \in \mathcal{F},\ I_2 \in \mathcal{I},\ I_1 \subseteq I_2 \implies I_1 \in \mathcal{I}$.

(IsA) (Strict Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$. □

Theorem 2.10 ([9]). *Let $(E, \mathcal{F})$ be a convex geometry. Suppose that a family $\mathcal{I} \subseteq \mathcal{F}$ satisfies* (I0) *and* (I1). *Then the Strict Augmentation Property* (IsA) *is equivalent to one of the following properties.*

(ILA) (Local Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| + 1 = |I_2|$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

(IS) *For each $X \in \mathcal{F}$, all the maximal elements of $\mathcal{I}^{(X)} \equiv \{X \cap I \mid I \in \mathcal{I}\}$ have the same cardinality (as subsets of E).* □

Axioms (I0), (I1), and (IS) are exactly those for what is called a *supermatroid* [3] when restricted on the lattices of closed sets of convex geometries. Hence the above theorem establish the following.

Theorem 2.11 ([9]). *The concept of a strict cg-matroid is equivalent to that of a supermatroid on the lattice of closed sets of a convex geometry.* □

2.3.2. Rank functions (of strict cg-matroids)

In [9], the rank functions of strict cg-matroids are defined as follows.

Definition 2.12 (Rank functions of strict cg-matroids). Let $(E, \mathcal{F}; \mathcal{I})$ be a strict cg-matroid with a family $\mathcal{I}$ of independent sets. Define a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ by

$$\rho(X) = \max\{|I| \mid I \in \mathcal{I}, I \subseteq X\} \quad (X \in \mathcal{F}). \tag{2.6}$$

We call the function ρ the *rank function* of the strict cg-matroid $(E, \mathcal{F}; \mathcal{I})$. We call $\rho(X)$ the *rank* of X for $X \in \mathcal{F}$. □

In [9], they studied properties of the rank functions, and they have shown the following theorems. See Theorem 1.1 for (RL0) and (RL1), and Theorem 1.2 for (RG0), (RG1), and (RGS).

Theorem 2.13 ([9]). *The rank function* $\rho : \mathcal{F} \to \mathbb{Z}_+$ *of a strict cg-matroid* $(E, \mathcal{F}; \mathcal{I})$ *with a family* $\mathcal{I}$ *of independent sets satisfies properties* (RL0)*,* (RL1)*, and* (RLS)*.*

(RLS) (Local Submodularity)
For any $X \in \mathcal{F}$ and $e_1, e_2 \in \mathrm{ex}^*(X)$ such that $X \cup \{e_1, e_2\} \in \mathcal{F}$,
if $\rho(X) = \rho(X \cup \{e_1\}) = \rho(X \cup \{e_2\})$, then $\rho(X) = \rho(X \cup \{e_1, e_2\})$. □

Theorem 2.14 ([9]). *The rank function* $\rho : \mathcal{F} \to \mathbb{Z}_+$ *of a strict cg-matroid* $(E, \mathcal{F}; \mathcal{I})$ *with a family* $\mathcal{I}$ *of independent sets satisfies properties* (RG0)*,* (RG1)*, and* (RGS)*.* □

In ordinary matroid theory, both the local conditions (RL0), (RL1), and (RLG), and the global conditions (RG0), (RG1), and (RGS) characterize the rank functions of matroids. But, for strict cg-matroids, these properties do not characterize the rank functions of strict cg-matroids. The following example tells us this fact.

Example 2.15 ([9]). Let $E = \{1, 2, 3, 4\}$. Consider a tree with a vertex set E and an edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ that forms a path of length three. Let $(E, \mathcal{F})$ be the tree shelling of the tree, i.e., $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. (See Figure 2.) Define a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ as follows: $\rho(\emptyset) = 0$, $\rho(\{1\}) = \rho(\{2\}) = \rho(\{3\}) = \rho(\{4\}) = \rho(\{2, 3\}) = 1$, $\rho(\{1, 2\}) = \rho(\{3, 4\}) = \rho(\{1, 2, 3\}) = \rho(\{2, 3, 4\}) = 2$, $\rho(\{1, 2, 3, 4\}) = 3$. Then the function $\rho : \mathcal{F} \to \mathbb{Z}_+$ satisfies both the conditions (RL0), (RL1), and (RLS), and the conditions (RG0), (RG1), and (RGS). But we have $\mathcal{I}(\rho) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\}$, where $\mathcal{I}(\rho)$ is defined by (3.1) in Section 3, and the obtained $\mathcal{I}(\rho)$ is not a family of independent sets of a strict cg-matroid on $(E, \mathcal{F})$. □

It was an open problem to give a characterization of the rank functions of strict cg-matroids.

3. Main Results

In this section, we give the proofs Theorem 1.1 and Theorem 1.2.

First we show the necessary conditions (only-if part).

Proposition 3.1. *The rank function* $\rho : \mathcal{F} \to \mathbb{Z}_+$ *of a strict cg-matroid* $(E, \mathcal{F}; \mathcal{I})$ *with a family* $\mathcal{I}$ *of independent sets satisfies the following property.*

(RLE) (Local Extension Property)
For any $X, Y \in \mathcal{F}$ such that $X \subseteq Y$ and $\rho(X) = |X| < \rho(Y)$,
there exists $e \in \mathrm{ex}^*(X) \cap Y$ such that $\rho(X \cup \{e\}) = \rho(X) + 1$.

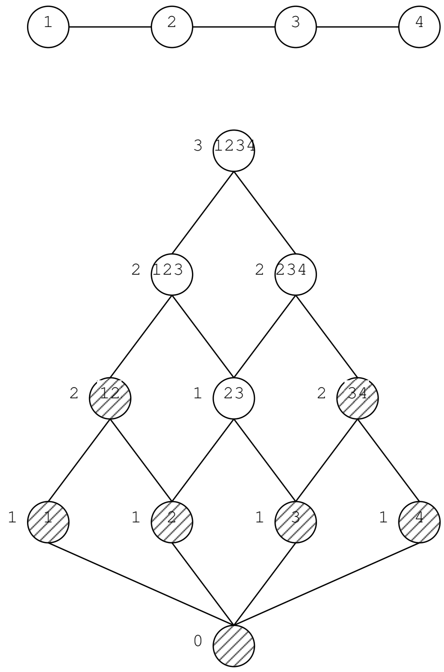

Figure 2: A path of length three and its tree shelling.

Proof. Take any $X, Y \in \mathcal{F}$ such that $X \subseteq Y$ and $\rho(X) = |X| < \rho(Y)$. Then, from the definition of the rank function, we have $X \in \mathcal{I}$. Let $I_Y \in \mathcal{F}$ be an independent set such that $I_Y \in \mathcal{I}$, $I_Y \subseteq Y$, and $\rho(Y) = |I_Y|$. Here $X, I_Y \in \mathcal{I}$ and $|X| = \rho(X) < \rho(Y) = |I_Y|$ hold. Hence, from the Strict Augmentation Property (IsA), there exists $e \in \tau(X \cup I_Y) \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$. Since $X \subseteq Y$ and $I_Y \subseteq Y$ imply $\tau(X \cup I_Y) \subseteq Y$ and since $X \cup \{e\} \in \mathcal{F}$, we have $e \in \mathrm{ex}^*(X) \cap Y$. Moreover, since $X \cup \{e\} \in \mathcal{I}$, we have $\rho(X \cup \{e\}) = |X \cup \{e\}| = |X| + 1 = \rho(X) + 1$. Hence the Local Extension Property (RLE) holds. □

Proposition 3.2. *The rank function $\rho : \mathcal{F} \to \mathbb{Z}_+$ of a strict cg-matroid $(E, \mathcal{F}; \mathcal{I})$ with a family $\mathcal{I}$ of independent sets satisfies the following property.*

(RGE) (Global Extension Property)
For any $X, Y \in \mathcal{F}$ such that $X \subseteq Y$ and $\rho(X) = |X| < \rho(Y)$,
there exists $Z \in \mathcal{F}$ such that $X \subsetneq Z \subseteq Y$ and $\rho(Z) = |Z| = \rho(Y)$.

Proof. Take any $X, Y \in \mathcal{F}$ such that $X \subseteq Y$ and $\rho(X) = |X| < \rho(Y)$. We will show (RGE) by induction on $k := \rho(Y) - \rho(X)$. First, we consider the case when $k = 1$. Then, from Proposition 3.1, we get $e \in \mathrm{ex}^*(X) \cap Y$ such that $\rho(X \cup \{e\}) = \rho(X) + 1$. Put $Z = X \cup \{e\}$. Then Z satisfies $Z \in \mathcal{F}$, $X \subsetneq Z \subseteq Y$, and $\rho(Z) = |Z| = \rho(Y)$. Hence (RGE) holds for $k = 1$.

Next, suppose that (RGE) holds for $k = n(\geq 1)$, and consider the case when $k = n + 1$. From Proposition 3.1, as well as when $k = 1$, we get $e \in \mathrm{ex}^*(X) \cap Y$ such that $\rho(X \cup \{e\}) = \rho(X) + 1$. Put $X' = X \cup \{e\}$. Then $X' \in \mathcal{F}$, $X' \subseteq Y$, and $\rho(X') = |X'| = \rho(X) + 1 < \rho(Y)$ hold, and also $\rho(Y) - \rho(X') = n$ holds. Using the assumption of induction, we can easily see that (RGE) holds for $k = n + 1$.

Thus the Global Extension Property (RGE) holds. $\square$

Next, we show the sufficient conditions (if part).

For any convex geometry $(E, \mathcal{F})$ and any function $\rho : \mathcal{F} \to \mathbb{Z}_+$, we put

$$\mathcal{I}(\rho) = \{X \in \mathcal{F} \mid \rho(X) = |X|\}. \tag{3.1}$$

Theorem 3.3. *Let $(E, \mathcal{F})$ be a convex geometry. Suppose that a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ satisfies properties* (RL0), (RL1), *and* (RGE). *Then $(E, \mathcal{F}; \mathcal{I}(\rho))$ is a strict cg-matroid with the family $\mathcal{I}(\rho)$ of independent sets.*

Proof. We will show that $\mathcal{I}(\rho)$ satisfies properties (I0), (I1), and (IsA).

From (RL0), we have $\rho(\emptyset) = 0 = |\emptyset|$. Hence $\emptyset \in \mathcal{I}(\rho)$ and (I0) holds.

Take $I_1 \in \mathcal{F}$ and $I_2 \in \mathcal{I}(\rho)$ such that $I_1 \subseteq I_2$. Then $\rho(I_2) = |I_2|$. We will show $\rho(I_1) = |I_1|$. If $I_1 = I_2$ then (I1) holds, so we suppose that $I_1 \subsetneq I_2$. Consider a maximal chain in $\mathcal{F}$ which contains I_1 and I_2. $\emptyset \subsetneq \ldots \subsetneq I_1 \subsetneq \ldots \subsetneq I_2 \subsetneq \ldots \subsetneq E$. From (RL1), we must have $\rho(I_1) = |I_1|$ since $\rho(I_2) = |I_2|$. Thus (I1) holds.

Next we will show (IsA). Take $I_1, I_2 \in \mathcal{I}(\rho)$ such that $|I_1| < |I_2|$. (In the property (RGE), we consider $X = I_1$, $Y = \tau(I_1 \cup I_2)$.) Here $I_1, \tau(I_1 \cup I_2) \in \mathcal{F}$, $I_1 \subseteq \tau(I_1 \cup I_2)$, and $\rho(I_1) = |I_1| < |I_2| = \rho(I_2) \leq \rho(\tau(I_1 \cup I_2))$ holds. (The last inequality follows from (RL1).) It follows from (RGE) that there exists $Z \in \mathcal{F}$ such that $I_1 \subsetneq Z \subseteq \tau(I_1 \cup I_2)$ and $\rho(Z) = |Z| = \rho(\tau(I_1 \cup I_2))$. Then $Z \in \mathcal{I}(\rho)$ and there exists $e \in \mathrm{ex}^*(I_1)$ such that $I_1 \cup \{e\} \subseteq Z$. If $Z = I_1 \cup \{e\}$, then this implies that (IsA) holds. If $Z \supsetneq I_1 \cup \{e\}$, then from (I1) we have $I_1 \cup \{e\} \in \mathcal{I}(\rho)$. And we have $e \in Z \setminus I_1 \subseteq \tau(I_1 \cup I_2) \setminus I_1$. Thus (IsA) holds. $\square$

Now we have a proof of Theorem 1.1.

Proof of Theorem 1.1. The present theorem follows from Theorem 2.13, Proposition 3.2, and Theorem 3.3. $\square$

Remark 3.4. Since the Local Extension Property (RLE) is apparently stronger than the Global Extension Property (RGE), the rank functions of strict cg-matroids are also characterized by only the local conditions (RL0), (RL1), and (RLE). $\square$

Remark 3.5. Although the submodularity of rank functions is very important in ordinary matroid theory, the submodularity does not appear explicitly in Theorem 1.1. But we can show that the three conditions (RL0), (RL1), and (RGE) imply the Local Submodularity (RLS) directly. $\square$

Theorem 3.6. *Let $(E, \mathcal{F})$ be a convex geometry. Suppose that a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ satisfies properties* (RG0)*,* (RG1)*,* (RGS) *and* (RLE)*. Then $(E, \mathcal{F}; \mathcal{I}(\rho))$ is a strict cg-matroid with the family $\mathcal{I}(\rho)$ of independent sets.*

Proof. We will show that $\mathcal{I}(\rho)$ satisfies properties (I0), (I1), and (IsA).

From (RG0), we have $0 \leq \rho(\emptyset) \leq |\emptyset| = 0$, i.e., $\rho(\emptyset) = |\emptyset|$. Hence $\emptyset \in \mathcal{I}(\rho)$, and (I0) holds.

Take any $I_1 \in \mathcal{F}$ and $I_2 \in \mathcal{I}(\rho)$ such that $I_1 \subseteq I_2$. Then $\rho(I_2) = |I_2|$. We will show $\rho(I_1) = |I_1|$. If $I_1 = I_2$ then (I1) holds, so we suppose that $I_1 \subsetneq I_2$. Put $k := |I_2|$, $l := |I_1|(< k)$. Since $\emptyset, I_2 \in \mathcal{I}(\rho)$, $\emptyset \subseteq I_2$, and $\rho(\emptyset) = 0 < \rho(I_2)$ hold, using (RLE) repeatedly, we have a chain in $\mathcal{I}(\rho)$ as follows.

$\emptyset = I_{2,0} \subsetneq I_{2,1} \subsetneq ... \subsetneq I_{2,k-1} \subsetneq I_{2,k} = I_2$,

where $I_{2,j} := \{e_1, ..., e_j\} \in \mathcal{I}(\rho)$ for $j = 1, ..., k$ and $I_{2,0} := \emptyset$.
Since $I_1 \subseteq I_2$, we can denote $I_1 = \{e_{i_1}, ..., e_{i_l}\}$ where $1 \leq i_1 < ... < i_l \leq k$. And we put $I_{1,j} := \{e_{i_1}, ..., e_{i_j}\}$ for $j = 1, ..., l$ and $I_{1,0} := \emptyset$. Then, for each $j = 1, ..., l$, we have that for $I_{1,j} \in \mathcal{F}$ and $I_{2,i_j-1} \in \mathcal{F}$, $I_{1,j} \cap I_{2,i_j-1} = I_{1,j-1} (\in \mathcal{F})$ and $I_{1,j} \cup I_{2,i_j-1} = I_{2,i_j} \in \mathcal{F}$. Therefore, from (RGS), we have $\rho(I_{1,j}) + \rho(I_{2,i_j-1}) \geq \rho(I_{1,j-1}) + \rho(I_{2,i_j})$. Since $\rho(I_{2,j}) = j$, we have $\rho(I_{1,j-1}) + 1 \leq \rho(I_{1,j})$ for $j = 1, ..., l$. From these inequalities with $\rho(I_{1,0}) = 0$, we have $l \leq \rho(I_1)$. Also we have $\rho(I_1) \leq |I_1| = l$ from (RG0). Hence we have $\rho(I_1) = |I_1| (= l)$, i.e., $I_1 \in \mathcal{I}(\rho)$. Hence (I1) holds.

Finally, we will show (IsA). Take $I_1, I_2 \in \mathcal{I}(\rho)$ such that $|I_1| < |I_2|$. (In the property (RLE), we consider $X = I_1$, $Y = \tau(I_1 \cup I_2)$.) Here $I_1, \tau(I_1 \cup I_2) \in \mathcal{F}$, $I_1 \subseteq \tau(I_1 \cup I_2)$, and $\rho(I_1) = |I_1| < |I_2| = \rho(I_2) \leq \rho(\tau(I_1 \cup I_2))$ hold. (The last inequality follows from (RG1).) From (RLE), there exists $e \in \mathrm{ex}^*(I_1) \cap \tau(I_1 \cup I_2) \subseteq \tau(I_1 \cup I_2) \setminus I_1$ such that $\rho(I_1 \cup \{e\}) = \rho(I_1) + 1 = |I_1| + 1 = |I_1 \cup \{e\}|$, i.e., $I_1 \cup \{e\} \in \mathcal{I}(\rho)$. Hence (IsA) holds. $\square$

Now we have a proof of Theorem 1.2.

Proof of Theorem 1.2. The present theorem follows from Theorem 2.14, Proposition 3.1, and Theorem 3.6. $\square$

Remark 3.7. It should be noted that, in Theorem 3.6, the Local Extension Property (RLE) cannot be replaced by the Global Extension Property (RGE). $\square$

An example of Remark 3.7 is given as follows.

Example 3.8. Let $E = \{1, 2, 3, 4\}$ be a linearly ordered set on four elements with order relations $1 < 2 < 3 < 4$, and $(E, \mathcal{F})$ be a poset shelling of the poset $(E, <)$, i.e., $\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. Define a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ by $\rho(\emptyset) = 0$, $\rho(\{1\}) = 1$, $\rho(\{1, 2\}) = 1$, $\rho(\{1, 2, 3\}) = 3$, $\rho(\{1, 2, 3, 4\}) = 3$.

Then ρ satisfies (RG0), (RG1), (RGS), and (RGE). (But ρ does not satisfy either (RLE) or (RL1).)

Now, $\mathcal{I}(\rho) = \{\emptyset, \{1\}, \{1, 2, 3\}\}$. And then $(E, \mathcal{F}; \mathcal{I}(\rho))$ is not a strict cg-matroid because $\mathcal{I}(\rho)$ does not satisfy property (I1). (See Figure 3.) $\square$

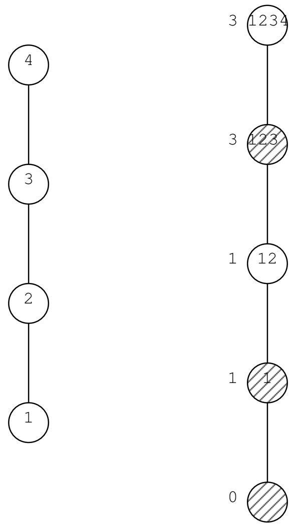

Figure 3: A poset and its poset shelling.

Remark 3.9. It should also be noted that Theorem 3.6, requires the Global Submodularity (RGS). $\square$

Example 3.10. Let $E = \{1, 2, 3\}$ be a poset on three elements with partial order relations $1 < 3, 2 < 3$, and $(E, \mathcal{F})$ be a poset shelling of the poset $(E, <)$, i.e., $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$. Define a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ by $\rho(\emptyset) = 0$, $\rho(\{1\}) = 1$, $\rho(\{2\}) = 0$, $\rho(\{1, 2\}) = 2$, $\rho(\{1, 2, 3\}) = 2$.

Then ρ satisfies (RG0), (RG1), and (RLE). (So ρ also satisfies (RL0) and (RGE).) But ρ does not satisfy either (RGS) or (RL1).

Now $\mathcal{I}(\rho) = \{\emptyset, \{1\}, \{1, 2\}\}$. And then $(E, \mathcal{F}; \mathcal{I}(\rho))$ is not a strict cg-matroid because $\mathcal{I}(\rho)$ does not satisfy property (I1). (See Figure 4.) $\square$

From Theorem 1.1 and Theorem 1.2, for a convex geometry $(E, \mathcal{F})$ and a function $\rho : \mathcal{F} \to \mathbb{Z}_+$ which satisfies conditions (RL0), (RL1), and (RGE), or the conditions (RG0), (RG1), (RGS), and (RLE), we call $(E, \mathcal{F}; \rho)$ a *strict cg-matroid* with a rank function ρ.

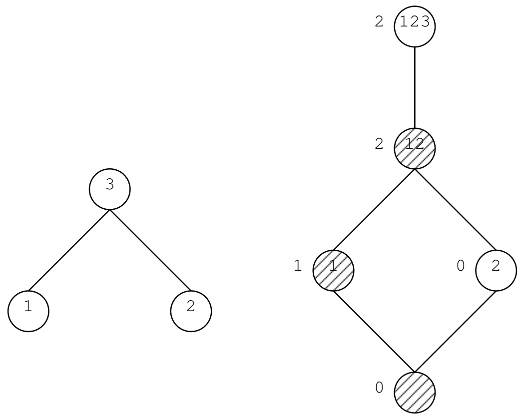

Figure 4: A poset and its poset shelling.

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References

[1] M. Barnabei, G. Nicoletti, and L. Pezzoli: Matroids on partially ordered sets, *Advances in Applied Mathematics* **21** (1998) 78–112.

[2] M. Barnabei, G. Nicoletti, and L. Pezzoli: The symmetric exchange property for poset matroids, *Advances in Mathematics* **102** (1993) 230–239.

[3] F. D. J. Dunstan, A. W. Ingleton, and D. J. A. Welsh: Supermatroids, *Combinatorics* (*Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972*) (1972) 72–122.

[4] P. H. Edelman and R. E. Jamison: The theory of convex geometries, *Geometriae Dedicata* **19** (1985) 247–270.

[5] J. Edmonds: Submodular functions, matroids, and certain polyhedra. *Proceedings of the Calgary International Conference on Combinatorial Structures and Their Applications* (R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., Gordon and Breach, New York, 1970) 69–87; also in: *Combinatorial Optimization—Eureka, You Shrink!* (M. Jünger, G. Reinelt and G. Rinaldi, eds., Lecture Notes in Computer Science **2570**, Springer, Berlin, 2003) 11–26.

[6] U. Faigle: Geometries on partially ordered sets, *Journal of Combinatorial Theory, Ser. B* **28** (1980) 26–51.

[7] U. Faigle: On supermatroids with submodular rank function, *Algebraic methods in graph theory, Vol. I (Szeged, 1978)*, Colloq. Math. Soc. János Bolyai **25** (North-Holland, Amsterdam, 1981) 149–158.

[8] S. Fujishige: *Submodular Functions and Optimization*, Second Edition, Annals of Discrete Mathematics **58** (Elsevier, Amsterdam, 2005).

[9] S. Fujishige, G. A. Koshevoy, and Y. Sano: Matroids on convex geometries (cg-matroids), *preprint* RIMS-**1539**, RIMS, Kyoto University, April 2006.

[10] J. Oxley: *Matroid Theory* (Oxford University Press, Oxford, 1992).

[11] U. N. Peled and M. K. Srinivasan: Poset matching—a distributive analog of independent matching, *Discrete Mathematics* **114** (1993) 403–424.

[12] É. Tardos: An intersection theorem for supermatroids, *Journal of Combinatorial Theory. Series B* **50** (1990) 150–159.

[13] D. J. A. Welsh: *Matroid Theory* (Academic Press, London, 1976).

[14] H. Whitney: On the abstract properties of linear dependence, *American Journal of Mathematics* **57** (1935) 509–533.